

# A solution to the fermion doubling problem for supersymmetric theories on the transverse lattice

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## Abstract

Species doubling is a problem that infects most numerical methods that use a spatial lattice. An understanding of species doubling can be found in the Nielsen-Ninomiya theorem which gives a set of conditions that require species doubling. The transverse lattice approach to solving field theories, which has at least one spatial lattice, fails one of the conditions of the Nielsen-Ninomiya theorem nevertheless one still finds species doubling for the standard Lagrangian formulation of the transverse lattice. We will show that the Supersymmetric Discrete Light Cone Quantization (SDLCQ) formulation of the transverse lattice does not have species doubling.

# 1 Introduction

When one formulates a theory with chiral fermions on a spatial lattice, one of the most notorious obstacles is the Nielsen-Ninomiya theorem[1] which gives a set of conditions that require species doubling. In our transverse lattice formulation of field theory we use both a spatial lattice and a momentum lattice. The transverse lattice formulations usually has some non-local interaction(s) which voids the Nielsen-Ninomiya theorem however it still seems to have the species doubling problem [2].

Recently, the authors proposed a super Yang-Mills (SYM) model in 2+1 dimensions on a transverse lattice with one exact supersymmetry [3]. It is well known that in the standard Lagrangian formulation of SYM on the transverse lattice one finds a fermion species doubling problem. We will show however that we are free from species doubling when one uses Supersymmetric Discrete Light Cone Quantization (SDLCQ). This is yet another demonstration of value of maintaining an exact supersymmetry in the numerical approximation. Of course two popular methods of dealing with the doubling, staggered fermions [4] and the Wilson term [5], work for the lagrangian formulation of SYM theories on a transverse lattice. In addition Chakrabarti, De and Harindranath recently proposed the use of the forward and backward derivatives to remove the species doubling on the light front transverse lattice [6]. However those methods badly break the supersymmetry and it is unclear how many of the unique properties of supersymmetry persist. While our approach can only be used for the transverse lattice formulation of supersymmetric theories, it resolves the doubling problem automatically.

This paper is organized as follows. In Section 2 we will see that the species doubling arises in the standard Lagrangian formulation of the transverse lattice, but can be resolved when one applies the method proposed by Ref. [6]. In Section 3 we show that in the SDLCQ formulation of the transverse lattice we do not have any species doubling. In section 4 we discuss some general reasons for this result and give the generalization to 3+1 dimensions.

## 2 Fermion species doubling problem on a transverse lattice

To focus on the fermion species doubling problem of the transverse lattice [3], let us consider fermion fields only by setting the coupling  $g = 0$  and the link variables  $M, M^\dagger = 1$ . For this theory one spatial dimension is discretized on a spatial lattice. We work in the light cone coordinates so that  $x^\pm \equiv (x^0 \pm x^1)/\sqrt{2}$  with  $x^\pm = x_\mp$  and  $x^\perp \equiv x^2 = -x_2$  is the dimension that is discretized on the spatial lattice. The Lagrangian is given by

$$\mathcal{L} = \sum_i \text{tr} \left[ \bar{\Psi}_i \gamma^\mu \partial_\mu \Psi_i + \frac{i}{2a} \bar{\Psi}_i \gamma^\perp (\Psi_{i+1} - \Psi_{i-1}) \right],$$

where  $i$  is the site index, the trace has been taken with respect to the color indices,  $\mu = \pm$ , and  $a$  is the lattice spacing. The gamma matrices are defined to be  $\gamma^0 = \sigma^2$ ,  $\gamma^1 = i\sigma_1$ , and  $\gamma^\perp = i\sigma^3$  with  $\gamma^\pm \equiv (\gamma^0 \pm \gamma^1)/\sqrt{2}$ . For  $\Psi_i = 2^{-1/4} \begin{pmatrix} \psi_i \\ \chi_i \end{pmatrix}$  we find the

equation of motion

$$\partial_- \chi_i = \frac{1}{2\sqrt{2}a}(\psi_{i+1} - \psi_{i-1}).$$

Inverting the light cone spatial derivative, we eliminate the non-dynamical field  $\chi_i$  from  $\mathcal{L}$  and get

$$\mathcal{L} = \sum_i \text{tr} \left[ i\psi_i \partial_+ \psi_i + \frac{i}{8a^2}(\psi_{i+1} - \psi_{i-1}) \partial_-^{-1}(\psi_{i+1} - \psi_{i-1}) \right].$$

Note that the second term is non-local. This is sufficient to avoid the Nielsen-Ninomiya theorem. The equation of motion for  $\psi_i$  is

$$\partial_+ \psi_i = \frac{1}{8a^2} \partial_-^{-1}(\psi_{i+2} - 2\psi_i + \psi_{i-2}). \quad (1)$$

We substitute the Fourier transformed form of  $\psi_i$ ,

$$\psi_j(x) = \int_{-\pi/a}^{\pi/a} dk^\perp \int_0^\infty dk^+ dk^- e^{i(k^+ x^- + k^- x^+ - k^\perp (aj))} \tilde{\psi}_j(k),$$

into Eq. (1) to find a dispersion relation

$$k^- = \frac{1}{2k^+} \left( \frac{\sin k^\perp a}{a} \right)^2. \quad (2)$$

Clearly, in the continuum limit where  $a \rightarrow 0$ , we find finite energy not only at  $k^\perp \approx 0$ , but also at  $k^\perp \approx \pm\pi/a$  for  $-\pi/a < k^\perp < \pi/a$ , yielding *extra* unwanted fermion species, that is, the notorious fermion species doubling problem.

Let us point out that the same equation of motion and thus the same dispersion relation follow if one uses Heisenberg equation of motion  $i\partial_+ \psi_{i,rs}(x) = [\psi_{i,rs}(x), P^-]$ . This is the approach we will use in the next section. In this calculation we use the equal (light cone) time anticommutation relation  $\{\psi_{i,rs}(x^-), \psi_{j,pq}(y^-)\} = \delta(x^- - y^-) \delta_{ij} \delta_{rp} \delta_{sq} / 2a$ , where we've explicitly written out the color indices  $r, s, p, q$  and

$$P^- \equiv a \sum_i \int dx^- T^{+-} = a \sum_i \int dx^- \text{tr} \left[ -\frac{i}{8a^2}(\psi_{i+1} - \psi_{i-1}) \partial_-^{-1}(\psi_{i+1} - \psi_{i-1}) \right].$$

$T^{\mu\nu}$  is the stress-energy tensor.

One might wonder what happens if we tried another difference operator, for instance, the forward/backward derivative in place of the symmetric derivative. Answering this question is instructive since the authors of Ref. [6] have found no fermion doubling for chiral fermions if one uses forward and backward derivatives on the light front transverse lattice. Following their procedure, we get in terms of  $\psi_i$  and  $\chi_i$

$$\begin{aligned} \mathcal{L} &= \sum_i \text{tr} \left[ i\psi_i \partial_+ \psi_i + i\chi_i \partial_- \chi_i - \frac{i}{\sqrt{2}a} (\chi_i(\psi_{i+1} - \psi_i) + \psi_i(\chi_i - \chi_{i-1})) \right] \\ &= \sum_i \text{tr} \left[ i\psi_i \partial_+ \psi_i + i\chi_i \partial_- \chi_i - \frac{\sqrt{2}i}{a} \chi_i(\psi_{i+1} - \psi_i) \right]. \end{aligned}$$

This yields

$$\partial_- \chi_i = \frac{1}{\sqrt{2}a}(\psi_{i+1} - \psi_i)$$

and

$$\mathcal{L} = \sum_i \text{tr} \left[ i\psi_i \partial_+ \psi_i + \frac{i}{2a^2} (\psi_{i+1} - \psi_i) \partial_-^{-1} (\psi_{i+1} - \psi_i) \right].$$

From this we find a dispersion relation

$$k^- = \frac{1}{2k^+} \left( \frac{\sin \frac{k^+ a}{2}}{a/2} \right)^2.$$

In the continuum limit we find a finite energy only at  $k^\perp \approx 0$ , meaning that we do not have the doubling problem. Hence, we found that the method to remove the doubling proposed in Ref. [6] works even for adjoint fermions.

### 3 Transverse lattice with SDLCQ

In Ref. [3] we proposed a discrete transverse lattice formulation of the supercharge  $Q^-$ , which gives the correct continuum form and the  $P^-$  obtained from SUSY algebra  $\{Q^-, Q^-\} = 2\sqrt{2}P^-$  also gives the correct continuum form. With this  $P^-$  in hand, following the same procedure we did in the previous section, we set  $g = 0$  and  $M, M^\dagger = 1$  to see whether we suffer from the fermion doubling problem. This  $P^-$  is given by

$$P^- = a \sum \int dx^- \text{tr} \left[ -\frac{i}{2a^2} (\psi_{i+1} - \psi_i) \partial_-^{-1} (\psi_{i+1} - \psi_i) \right].$$

Heisenberg equation of motion yields

$$i\partial_+ \psi_{i,rs} = [\psi_{i,rs}, P^-] = \frac{i}{2a^2} \partial_-^{-1} (\psi_{i+1} - 2\psi_i + \psi_{i-1})_{rs}.$$

Hence, it follows that

$$k^- = \frac{1}{2k^+} \left( \frac{\sin \frac{k^+ a}{2}}{a/2} \right)^2.$$

Notice, remarkably, that we have a finite energy *only* at  $k^\perp \approx 0$ , so that we are *free from the species doubling problem with SDLCQ*.

A word of caution is due here. This  $P^-$  happens to be the same as the one obtained in Ref. [6], where the authors used the forward and backward derivatives however we get  $P^-$  in a completely different way.

### 4 Discussion

We reviewed the known result that one suffers from a species doubling problem in the transverse lattice Lagrangian formalism with the symmetric derivative in spite of the fact that our adjoint fermions interact non-locally. We applied the method of removing the doubling proposed by the authors of Ref. [6] originally for chiral fermions and found that it works as well even for adjoint fermions. We then showed that we do not suffer from species doubling in the SDLCQ formulation of the transverse lattice [3].

While we did the calculation in 2+1 dimensions, we should note that this doubling persists in 3+1 dimensions. The authors have been working on the extension of the

model in Ref. [3] to a 3+1 dimensional model with transverse lattices in two spatial directions. We have found [7] that the standard transverse Lagrangian formulation leads to the following dispersion relation,

$$k^- = \frac{1}{2k^+} \left[ \left( \frac{\sin k_1^\perp a}{a} \right)^2 + \left( \frac{\sin k_2^\perp a}{a} \right)^2 \right],$$

where  $k_i^\perp$  is the  $i$ -th transverse momentum. For a model with SDLCQ formulation of the transverse lattice,

$$k^- = \frac{1}{2k^+} \left[ \left( \frac{\sin \frac{k_1^\perp a}{2}}{a/2} \right)^2 + \left( \frac{\sin \frac{k_2^\perp a}{2}}{a/2} \right)^2 \right].$$

Again, we do not have any species doubling with SDLCQ.

In Ref. [3] we found that the color of physical states must be contracted at each site. However, this constraint was derived in the standard Lagrangian formalism, which suffers from the doubling problem. Therefore, one might ask if there is any change in the physical constraint due to the doubling problem. We believe the answer is no. The reason is the following. The physical constraint we found in [3] comes from the equation of motion  $\frac{\delta \mathcal{L}}{\delta A_i^-} - \partial_+ \frac{\delta \mathcal{L}}{\delta (\partial_+ A_i^-)} = 0$ , where  $A_i^-$  is the “-” component of the gauge field  $A_i^\mu$  residing at the  $i$ -th site. However, this equation of motion has nothing to do with the terms involving the difference between fermions at different sites, which are the cause of the doubling. Hence, even if we made some change(s) in the standard Lagrangian e.g. by adding a Wilson term to fix the doubling problem, we would not see any change in the equation of motion which leads to the physical constraint.

It seems that SUSY algebra by itself resolves the species doubling problem. This is indeed expected since we do not have any doubling problems in boson sector and SUSY requires that the number of degrees of freedom be the same for bosons and fermions. In general it is difficult to maintain exact SUSY on a lattice, but it appears that if it is achieved, then it automatically solves the species doubling problem. Clearly SDLCQ is one of a class of promising approaches in the attempt to put a SYM theory on a spatial lattice [8].

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