

# On-shell improvement of the massive Wilson quark action<sup>\*)</sup>

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## Abstract

We review a relativistic approach to the heavy quark physics in lattice QCD by applying a relativistic  $O(a)$  improvement to the massive Wilson quark action on the lattice. After explaining how power corrections of  $m_Q a$  can be avoided and remaining uncertainties are reduced to be of order  $(a\Lambda_{\text{QCD}})^2$ , we demonstrate a determination of four improvement coefficients in the action up to one-loop level in a mass dependent way. We also show a perturbative determination of mass dependent renormalization factors and  $O(a)$  improvement coefficients for the vector and axial vector currents. Some preliminary results of numerical simulations are also presented.

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<sup>\*)</sup> presented by Y. Kuramashi

## §1. Introduction

Lattice QCD should allow quantitative predictions for the heavy quark physics from first principles. Up to now, however, most approaches have based on the nonrelativistic effective theory, with which the continuum limit can not be taken.<sup>1)</sup> In this report we review a new relativistic approach to the heavy quark physics in lattice QCD developed in a series of publications.<sup>2), 3), 4), 5), 6), 7), 8), 9)</sup>

We consider a relativistic  $O(a)$  improvement to deal with the heavy quarks on the lattice.<sup>2)</sup> We discuss cutoff effects by extending the on-shell improvement programme<sup>10), 11), 12), 13)</sup> from massless to massive case. An important finding is that in our formulation leading cutoff effects of order  $(m_Q a)^n$  are absorbed in the definition of renormalization factors for the quark mass and the wave function. After removing the next-leading cutoff effects of  $O((m_Q a)^n a \Lambda_{\text{QCD}})$  with four parameters in the quark action properly adjusted in a  $m_Q a$  dependent way, we are left with at most  $O((a \Lambda_{\text{QCD}})^2)$  errors. We show a determination of the four parameters in the quark action up to one-loop level for various improved gauge actions.<sup>4)</sup> They are determined free from the infrared divergences, once their tree level values are correctly tuned in the  $m_Q a$  dependent way.

We also make the  $O(a)$  improvement of the vector and axial vector currents at the one-loop level.<sup>5)</sup> We give a general discussion about what kind of improvement operators are required from the symmetries allowed on the lattice, in which the Euclidean rotational symmetry is violated because of  $m_Q a$  corrections. We consider both the heavy-heavy and heavy-light cases.

This report is organized as follows. In Sec. 2 we consider a relativistic  $O(a)$  improvement to handle the heavy quarks on the lattice avoiding large  $m_Q a$  corrections. The four parameters in the quark action are determined up to one-loop level in Sec. 3. In Sec. 4 we explicitly show the  $O(a)$  improvement of the axial vector current up to one-loop level. With the use of the  $O(a)$  improved quark action and axial vector current, we make some numerical studies in quenched QCD focusing on restoration of the space-time symmetry. Their results are presented in Sec. 5. Our conclusions are summarized in Sec. 6.

## §2. On-shell improvement of the massive Wilson quark action

We consider a relativistic  $O(a)$  improvement to control  $m_Q a$  corrections for the heavy quarks on the lattice. The basic idea is to apply the on-shell improvement programme,<sup>10), 11), 12), 13)</sup> which has been developed in the small mass case, to the heavy quarks on the lattice. This method allows us to obtain the physical quantities in the continuum limit without requiring

harsh condition  $m_Q a \ll 1$  that is not achievable in near future.

Before going into details, we first remark on the on-shell improvement. As explicitly stated in Ref. 13), the on-shell improvement is meant to improve the correlation functions in which local composite fields are separated by non-zero physical distances. All on-shell quantities (particle energies, scattering amplitudes, normalized matrix elements of local composite fields between particle states, etc.) are extracted from these correlation functions.<sup>13)</sup> It should be stressed that the on-shell quantities are not restricted to the spectral ones.

Let us consider general cutoff effects for the heavy quarks on the lattice, where the heavy quark mass  $m_Q$  is allowed to be much heavier than  $\Lambda_{\text{QCD}}$ . Under the condition that  $m_Q \gg \Lambda_{\text{QCD}}$  and  $m_Q a \sim O(1)$ , we assume that the leading cutoff effects are

$$f_0(m_Q a) > f_1(m_Q a) a \Lambda_{\text{QCD}} > f_2(m_Q a) (a \Lambda_{\text{QCD}})^2 > \dots, \quad (2.1)$$

where  $f_i(m_Q a)$  ( $i \geq 0$ ) are smooth and continuous all over the range of  $m_Q a$  and have Taylor expansions at  $m_Q a = 0$  with sufficiently large convergence radii beyond  $m_Q a = 1$ , taking  $f_0(0) = 0$  and  $f_i(0) \sim O(1)$  for  $i \geq 1$ . The essential point in this assumption on cutoff effects is that the cutoff effects of  $O(a \Lambda_{\text{QCD}})$  in the chiral limit are still  $O(a \Lambda_{\text{QCD}})$  even if the quark mass is increased. This means that our power counting is not based on the nonrelativistic effective theory. To control the scaling violation effects we want to remove the cutoff effects up to  $f_1(m_Q a) a \Lambda_{\text{QCD}}$  by adding the counter terms to the lattice quark action with the on-shell improvement. If  $m_Q a$  is small enough, the remaining  $f_2(m_Q a) (a \Lambda_{\text{QCD}})^2$  contributions can be removed by extrapolating the numerical data at several lattice spacings to the continuum limit. Otherwise, in case of sufficiently small lattice spacing, the  $O((a \Lambda_{\text{QCD}})^2)$  errors can be neglected.

We first search for the relevant counter terms required in the on-shell improvement. Listed below are the allowed operators under the requirement of the gauge, axis interchange and other various discrete symmetries on the lattice, where the chiral symmetry is not imposed. According to the work of Ref. 14), all the operators with dimension up to six are given by

$$\text{dim.3 : } \mathcal{O}_3(x) = \bar{q}(x)q(x), \quad (2.2)$$

$$\text{dim.4 : } \mathcal{O}_4(x) = \bar{q}(x)\not{D}q(x), \quad (2.3)$$

$$\text{dim.5 : } \mathcal{O}_{5a}(x) = \bar{q}(x)D_\mu^2 q(x), \quad (2.4)$$

$$\mathcal{O}_{5b}(x) = i\bar{q}(x)\sigma_{\mu\nu}F_{\mu\nu}q(x), \quad (2.5)$$

$$\text{dim.6 : } \mathcal{O}_{6a}(x) = \bar{q}(x)\gamma_\mu D_\mu^3 q(x), \quad (2.6)$$

$$\mathcal{O}_{6b}(x) = \bar{q}(x)D_\mu^2\not{D}q(x), \quad (2.7)$$

$$\mathcal{O}_{6c}(x) = \bar{q}(x)\not{D}D_\mu^2 q(x), \quad (2.8)$$

$$\mathcal{O}_{6d}(x) = i\bar{q}(x)\gamma_\mu[D_\nu, F_{\mu\nu}]q(x), \quad (2.9)$$

$$\mathcal{O}_{6e}(x) = \bar{q}(x)\not{D}^3q(x), \quad (2.10)$$

$$\mathcal{O}_{6f}(x) = \bar{q}(x)\Gamma q(x)\bar{q}(x)\Gamma q(x), \quad (2.11)$$

where  $\Gamma = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$ . These operators lead to a following generic form of the quark action on the isotropic lattice:

$$S_q^{\text{imp}} = \sum_x \left[ c_3\mathcal{O}_3(x) + c_4\mathcal{O}_4(x) + \sum_{i=a,b} c_{5i}\mathcal{O}_{5i}(x) + \sum_{i=a,\dots,f} c_{6i}\mathcal{O}_{6i}(x) + \dots \right], \quad (2.12)$$

where  $c_3, \dots, c_{6f}$  are functions of the bare gauge coupling  $g$  and the power corrections of  $m_Q a$ . At  $m_Q = 0$  the contributions of dimension 6 operators are of order  $(a\Lambda_{\text{QCD}})^2$ , which are negligible for the  $O(a)$  improvement. However, we are now interested in the case of  $m_Q \gg \Lambda_{\text{QCD}}$  and  $m_Q a \sim O(1)$ .

We first point out that, regardless of the magnitude of  $m_Q a$ , the  $m_Q a$  corrections to the quark mass term and the kinetic term can be absorbed in the renormalizations of the quark mass  $Z_m$  and the wave function  $Z_q$ . For the sake of convenience we choose  $c_3 = m_0$  and  $c_4 = 1$ .

In the next step we reduce the number of basis operators with the aid of the classical field equations. It is easily found that  $\mathcal{O}_{5a}$ ,  $\mathcal{O}_{6b}$ ,  $\mathcal{O}_{6c}$  and  $\mathcal{O}_{6e}$  can be related to the quark mass term or the kinetic term. In the on-shell improvement these operators are redundant and can be eliminated from the action of eq. (2.12). The operator  $\mathcal{O}_{5a}$ , however, is used to avoid the species doubling and the value of its coefficient  $c_{5a}$  is given by hand.

The remaining operators are  $\mathcal{O}_{5b}$ ,  $\mathcal{O}_{6a}$ ,  $\mathcal{O}_{6d}$  and  $\mathcal{O}_{6f}$ . It is easy to see that  $\mathcal{O}_{6d}$  and  $\mathcal{O}_{6f}$  are  $O((\Lambda_{\text{QCD}}a)^2)$ , and therefore are irrelevant for the  $O(a)$  improvement. The operator  $\mathcal{O}_{5b}$  is the so-called clover term, for which the nonperturbative method to determine the coefficient  $c_{5b}$  in the massless limit is already established.<sup>13)</sup> However, the contributions of  $(m_Q a)^n \mathcal{O}_{5b}$  ( $n \geq 1$ ) cannot be neglected in the present condition that allows  $m_Q a \sim O(1)$ . For  $O(a\Lambda_{\text{QCD}})$  improvement the coefficient  $c_{5b}$  has to be adjusted in the mass dependent way.

Our main concern is  $\mathcal{O}_{6a}$ . Under the condition that  $\partial_0 q(x) \sim m_Q q(x)$  and  $\partial_i q(x) \sim p_i q(x)$  with  $m_Q \gg |p_i|$  and  $m_Q a \sim O(1)$ , we have to treat the time and space components of  $\mathcal{O}_{6a}$  in a different way: They follow different power counting. Actually, the contribution of the time component  $\bar{q}(x)\gamma_0 D_0^3 q(x)$ , which is  $O((\Lambda_{\text{QCD}}a)^2)$  in the massless limit, is not negligible any more in the present condition. It can be related to other lower dimensional operators by the equation of motion:

$$a^2 \bar{q}(x)\gamma_0 D_0^3 q(x) = -\frac{1}{a}(m_Q a)^3 \bar{q}(x)q(x)$$

$$\begin{aligned}
& -(m_Q a)^2 \bar{q}(x) \gamma_i D_i q(x) \\
& + a(m_Q a) \bar{q}(x) D_i^2 q(x) + O((\Lambda_{\text{QCD}} a)^2).
\end{aligned} \tag{2.13}$$

This relation tells us that the contribution of  $\bar{q}(x) \gamma_0 D_0^3 q(x)$  is expressed by the lower dimensional operators multiplied by the power corrections of  $m_Q a$ , so that the coefficients of the space derivative terms in  $\mathcal{O}_4$  and  $\mathcal{O}_{5a}$  become different from those of the time derivative terms. On the other hand, the contribution of the space component of  $\mathcal{O}_{6a}$  is to be  $O((\Lambda_{\text{QCD}} a)^2)$ , which is negligible for the  $O(a)$  improvement. It is essential to note that the Lorentz non-covariant terms like  $\mathcal{O}_{6a}$  yield the difference of magnitude between the time and space components due to finite  $m_Q a$  corrections.

The generalization of the above argument to any operators with higher dimensions makes the discussion more transparent. Let us consider an arbitrary operator with  $4+k$  dimension,  $a^k \mathcal{O}_{4+k}$ , where we write the lattice spacing  $a$  explicitly. The operator  $\mathcal{O}_{4+k}$  contains  $l$  pairs of  $\bar{q}$  and  $q$  and  $n$  covariant derivatives  $D_\mu$  with  $4+k = 3 \times l + n$ . Using the classical field equation, some (but not all) of covariant derivatives can be replaced by the quark mass  $m_Q$ . For  $l \geq 2$  the largest possible power of the scaling violation is  $(m_Q a)^n (a \Lambda_{\text{QCD}})^{3l-4}$ . Therefore the operators which contain four or more quarks are irrelevant for the  $O(a \Lambda_{\text{QCD}})$  improvement. All the relevant contributions come from the quark bilinear operators. With the aid of the classical field equations, they can be reduced to

$$(m_Q a)^n a^{-1} \bar{q}(x) q(x) \tag{2.14}$$

$$(m_Q a)^{n-1} \bar{q}(x) \gamma_0 D_0 q(x), \quad (m_Q a)^{n-1} \sum_i \bar{q}(x) \gamma_i D_i q(x) \tag{2.15}$$

$$(m_Q a)^{n-2} a \bar{q}(x) D_0^2 q(x), \quad (m_Q a)^{n-2} a \sum_i \bar{q}(x) D_i^2 q(x) \tag{2.16}$$

$$(m_Q a)^{n-2} a i \sum_i \bar{q}(x) \sigma_{0i} F_{0i} q(x), \quad (m_Q a)^{n-2} a i \sum_{ij} \bar{q}(x) \sigma_{ij} F_{ij} q(x), \tag{2.17}$$

for  $n \geq 0$ . The time and space components of  $\mathcal{O}_4$  and  $\mathcal{O}_{5a,5b}$  should be treated separately in case of finite  $m_Q a$ , where the space-time asymmetry reflects the contributions of the higher dimensional operators that break the rotational symmetry. Now we know that the seven operators are needed for the  $O(a \Lambda_{\text{QCD}})$  improvement. Since three coefficients among these seven operators can be absorbed in  $Z_m$ ,  $Z_q$  and the Wilson parameter  $r_t$  for the time derivative of  $\mathcal{O}_{5a}$  as already explained, the remaining four coefficients have to be actually tuned.

In conclusion, at all order of  $m_Q a$ , the generic quark action is written as

$$S_q^{\text{imp}} = \sum_x \left[ m_0 \bar{q}(x) q(x) + \bar{q}(x) \gamma_0 D_0 q(x) + \nu \sum_i \bar{q}(x) \gamma_i D_i q(x) \right]$$

$$\begin{aligned}
& -\frac{r_t a}{2} \bar{q}(x) D_0^2 q(x) - \frac{r_s a}{2} \sum_i \bar{q}(x) D_i^2 q(x) \\
& -\frac{i g a}{2} c_E \sum_i \bar{q}(x) \sigma_{0i} F_{0i} q(x) - \frac{i g a}{4} c_B \sum_{i,j} \bar{q}(x) \sigma_{ij} F_{ij} q(x) \Big], \quad (2.18)
\end{aligned}$$

where we are allowed to choose  $r_t = 1$  and the four parameters  $\nu$ ,  $r_s$ ,  $c_E$  and  $c_B$  are to be adjusted. In general these parameters have the form that  $X = \sum_n X^{(n)}(g^2)(m_Q a)^n$  with  $X = \nu$ ,  $r_s$ ,  $c_E$  and  $c_B$ , and  $X^{(0)}$  should agree with the one in the massless  $O(a)$  improved theory:  $\nu^{(0)} = 1$ ,  $r_s^{(0)} = r_t = 1$ ,  $c_E^{(0)} = c_B^{(0)} = c_{\text{SW}}$ .<sup>13)</sup> Note that  $\nu = 1 + O((m_Q a)^2)$  and  $r_s = r_t + O(m_Q a)$  since the space-time asymmetry arises from Lorentz non-covariant terms such as  $\mathcal{O}_{6a}$  via the on-shell reduction of eq.(2.13) accompanied by power corrections of  $m_Q a$ . In brief the differences between  $\nu$  and 1,  $r_t$  and  $r_s$ ,  $c_E$  and  $c_B$  reflect the contributions of Lorentz non-covariant terms with higher dimensions.

We find that a similar quark action for the heavy quarks has been proposed in Ref. 15). The important difference is that the parameter  $r_s$  is redundant in their formulation, while it should be tuned in ours. In this sense our action is equal to theirs with a special choice of parameters. The reason of discrepancy in the number of relevant operators between in our formulation and that in Ref. 15) is explained in detail in Ref. 2). We explicitly show in the next section that the parameter  $r_s$  actually needs to be adjusted to reproduce the correct on-shell quark-quark scattering amplitude.

### §3. Determination of the improvement parameters in the quark action

#### 3.1. Tree level

The four improvement parameters in the quark action of eq.(2.18) are determined such that  $O(a)$  cutoff effects in on-shell quantities (particle energies, scattering amplitudes, normalized matrix elements of local composite fields between particle states, etc.) are removed. We employ the on-shell quark-quark scattering amplitude to determine the parameters  $\nu$ ,  $r_s$ ,  $c_E$  and  $c_B$  at tree level, which are adjusted to reproduce the continuum form of the scattering amplitude removing the  $m_Q a$  corrections,<sup>2)</sup>

$$\begin{aligned}
T = & -g^2 (T^A)^2 \bar{u}(p') \gamma_\mu u(p) D_{\mu\nu}(p-p') \bar{u}(q') \gamma_\nu u(q) \\
& -g^2 (T^A)^2 \bar{u}(q') \gamma_\mu u(p) D_{\mu\nu}(p-q') \bar{u}(p') \gamma_\nu u(q) \\
& + O((p_i a)^2, (q_i a)^2, (p'_i a)^2, (q'_i a)^2), \quad (3.1)
\end{aligned}$$

where  $p$ ,  $q$  denote the incoming quark momenta and  $p'$ ,  $q'$  for the outgoing quark ones.  $D_{\mu\nu}$  denotes the gluon propagator. This improvement procedure follows the previous work<sup>16)</sup>

that determined the  $c_{\text{SW}} = c_B = c_E$  parameter up to one-loop level in the massless case. At the tree level the quark-quark-gluon vertex is written as

$$\left(\bar{u}(p')\Lambda_0^{(0)}(p,p')u(p)\right)_{\text{latt}} = Z_q^{(0)} (\bar{u}(p')i\gamma_0u(p))_{\text{cont}} + O((p_ia)^2, (p'_ia)^2), \quad (3.2)$$

$$\left(\bar{u}(p')\Lambda_k^{(0)}(p,p')u(p)\right)_{\text{latt}} = Z_q^{(0)} (\bar{u}(p')i\gamma_ku(p))_{\text{cont}} + O((p_ia)^2, (p'_ia)^2), \quad (3.3)$$

for

$$\begin{aligned} \Lambda_0^{(0)}(p,p') &= i\gamma_0 \cos\left(\frac{p_0+p'_0}{2}\right) + r_t \sin\left(\frac{p_0+p'_0}{2}\right) \\ &\quad + \frac{c_E^{(0)}}{2} \cos\left(\frac{p_0-p'_0}{2}\right) \sum_l \sigma_{0l} \sin(p_l-p'_l), \end{aligned} \quad (3.4)$$

$$\begin{aligned} \Lambda_k^{(0)}(p,p') &= i\nu^{(0)}\gamma_k \cos\left(\frac{p_k+p'_k}{2}\right) + r_s^{(0)} \sin\left(\frac{p_k+p'_k}{2}\right) \\ &\quad + \frac{c_E^{(0)}}{2} \cos\left(\frac{p_k-p'_k}{2}\right) \sigma_{k0} \sin(p_0-p'_0) \\ &\quad + \frac{c_B^{(0)}}{2} \cos\left(\frac{p_k-p'_k}{2}\right) \sum_{l \neq k} \sigma_{kl} \sin(p_l-p'_l), \end{aligned} \quad (3.5)$$

where the spinor on the lattice is given by

$$u(p) = \begin{pmatrix} \phi \\ \frac{\nu\vec{p}\cdot\vec{\sigma}}{N(p)}\phi \end{pmatrix} + O((p_ia)^2), \quad (3.6)$$

with  $N(p) = (-i)\sin(p_0) + m_0 + r_t(1 - \cos(p_0)) + r_s \sum_i(1 - \cos(p_i))$ . The  $O(a)$  improvement condition yields<sup>2)</sup>

$$\nu^{(0)} = \frac{\sinh(m_p^{(0)})}{m_p^{(0)}}, \quad (3.7)$$

$$r_s^{(0)} = \frac{\cosh(m_p^{(0)}) + r_t \sinh(m_p^{(0)})}{m_p^{(0)}} - \frac{\sinh(m_p^{(0)})}{m_p^{(0)2}}, \quad (3.8)$$

$$c_E^{(0)} = r_t \nu^{(0)}, \quad (3.9)$$

$$c_B^{(0)} = r_s^{(0)}, \quad (3.10)$$

where  $m_p^{(0)}$  is the tree-level pole mass explained below. It is now shown that the four parameters are uniquely determined from the on-shell quark-quark scattering amplitude. This is an evidence we actually need four improvement parameters in the quark action of eq.(2.18).

It is instructive to show that the  $\nu$  and  $r_s$  parameters are also determined from the quark propagator which is obtained by inverting the Wilson-Dirac operator in eq.(2-18),

$$S_q^{-1}(p) = i\gamma_0 \sin(p_0) + \nu i \sum_i \gamma_i \sin(p_i) + m_0 + r_t(1 - \cos(p_0)) + r_s \sum_i (1 - \cos(p_i)), \quad (3.11)$$

At the tree level the parameters are adjusted such that the above quark propagator reproduces the correct relativistic form:<sup>2)</sup>

$$S_q(p) = \frac{1}{Z_q^{(0)}} \frac{-i\gamma_0 p_0 - i \sum_i \gamma_i p_i + m_p^{(0)}}{p_0^2 + \sum_i p_i^2 + m_p^{(0)2}} + (\text{no pole terms}) + O((p_i a)^2) \quad (3.12)$$

around the pole.  $Z_q^{(0)}$  and  $m_p^{(0)}$  are extracted with  $p_i = 0$ ,

$$m_p^{(0)} = \log \left| \frac{m_0 + r_t + \sqrt{m_0^2 + 2r_t m_0 + 1}}{1 + r_t} \right|, \quad (3.13)$$

$$Z_q^{(0)} = \cosh(m_p^{(0)}) + r_t \sinh(m_p^{(0)}). \quad (3.14)$$

Imposing finite spatial momenta the parameter  $\nu$  is determined by demanding the correct relativistic spinor structure on  $S_q^{-1}(p)$  of eq.(3-11). Comparing the coefficients of  $\gamma_0$  and  $\gamma_i$  in the numerator we obtain

$$\nu^{(0)} = \frac{\sinh(m_p^{(0)})}{m_p^{(0)}}. \quad (3.15)$$

The parameter  $r_s$  is determined such that the correct dispersion relation is reproduced:

$$E^2 = m_p^2 + \sum_i p_i^2 + O(p_i^4). \quad (3.16)$$

The result is

$$r_s^{(0)} = \frac{\cosh(m_p^{(0)}) + r_t \sinh(m_p^{(0)})}{m_p^{(0)}} - \frac{\sinh(m_p^{(0)})}{m_p^{(0)2}} \quad (3.17)$$

$$= \frac{1}{m_p^{(0)}} (Z_q^{(0)} - \nu^{(0)}). \quad (3.18)$$

It should be noted that the values of  $\nu^{(0)}$  and  $r_s^{(0)}$  are exactly the same as those determined from the on-shell quark-quark scattering amplitude. This is not an accident: The correct relativistic form of quark propagator yields the correct relativistic form of Dirac spinor required in the calculation of matrix elements. Since it is simpler to treat the conditions on the quark propagator, we employ them to determine the one-loop contributions to  $\nu$  and  $r_s$  in the next subsection.



### 3.2. One-loop level

The one-loop contributions to the quark self-energy come from the rainbow and the tadpole diagrams, which are written as

$$g^2 \Sigma(p, m_0) = g^2 \left[ i\gamma_0 \sin p_0 B_0(p, m_0) + \nu i \sum_i \gamma_i \sin p_i B_i(p, m_0) + C(p, m_0) \right].$$

Incorporating this contribution, the inverse quark propagator up to the one-loop level is written as

$$\begin{aligned} S_q^{-1}(p, m) &= i\gamma_0 \sin p_0 [1 - g^2 B_0(p, m)] + \nu i \sum_i \gamma_i \sin p_i [1 - g^2 B_i(p, m)] + m \\ &\quad + 2r_t \sin^2 \left( \frac{p_0}{2} \right) + 2r_s \sum_i \sin^2 \left( \frac{p_i}{2} \right) - g^2 \hat{C}(p, m), \end{aligned} \quad (3.19)$$

where we redefine the quark mass as

$$m = m_0 - g^2 C(p=0, m=0), \quad (3.20)$$

$$\hat{C}(p, m) = C(p, m) - C(p=0, m=0). \quad (3.21)$$

With this definition the inverse quark propagator satisfies the on-shell condition for the massless quark up to the one-loop level :  $S_q^{-1}(p_0=0, p_i=0, m=0) = 0$ .

The  $\nu$  and  $r_s$  parameters are determined by employing the same improvement condition as the tree level. The parameter  $\nu$  is determined from the relativistic spinor structure in  $S_q^{-1}$  of eq.(3.19) at the pole.

$$\nu [1 - g^2 B_i(p^*, m)] = \frac{\sinh(m_p)}{m_p} [1 - g^2 B_0(p^*, m)], \quad (3.22)$$

where  $p^* \equiv (p_0 = im_p, p_i = 0)$ . We show the quark mass dependences of  $\nu^{(1)}/\nu^{(0)}$  over the range  $0 \leq m_p^{(0)} \leq 10$  for the plaquette and the Iwasaki gauge actions<sup>18)</sup> in Fig. 1(a). The solid lines depict the results of the interpolation with a rational expression. The errors are within symbols. The parameter  $r_s$  is determined from the relativistic dispersion relation as done at the tree level. The  $m_p^{(0)}$  dependences of  $r_s^{(1)}/r_s^{(0)}$  for the plaquette and the Iwasaki gauge actions are shown in Fig. 1(b).

Let us turn to the one-loop calculation of  $c_E$  and  $c_B$ . Recently the authors have shown the validity of the conventional perturbative method, which uses the fictitious gluon mass as an infrared regulator,<sup>17)</sup> to determine the clover coefficient  $c_{\text{SW}}$  up to the one-loop level in the massless case from the on-shell quark-quark scattering amplitude.<sup>3)</sup> We extend this calculation to the massive case. According to Ref. 16), it is sufficient to improve each on-shell

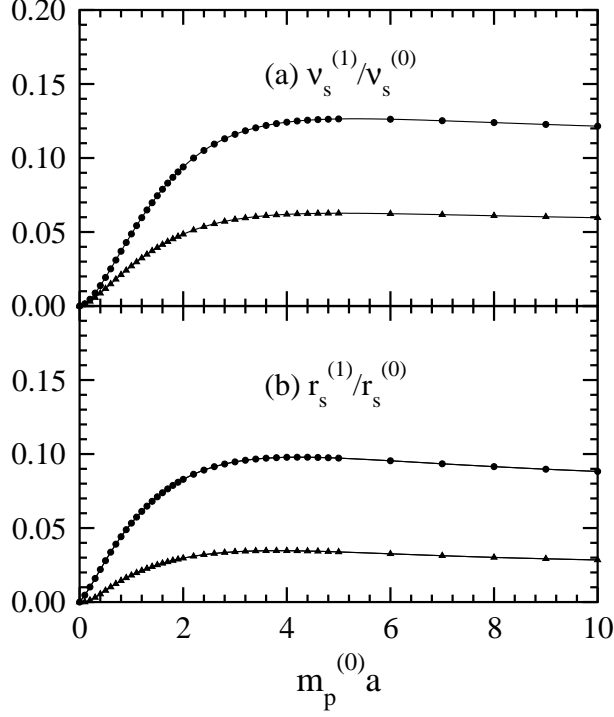


Fig. 1. (a)  $\nu^{(1)}/\nu^{(0)}$  and (b)  $r_s^{(1)}/r_s^{(0)}$  as a function of  $m_p^{(0)}$ . Circles denote the plaquette gauge action and triangles for the Iwasaki gauge action.

quark-quark-gluon vertex individually. To determine the one-loop coefficients  $c_E^{(1)}$  and  $c_B^{(1)}$  we need six types of diagrams shown in Fig. 2. We first consider to calculate  $c_B^{(1)}$ . Without the space-time symmetry the general form of the off-shell vertex function at the one-loop level is written as

$$\begin{aligned}
\Lambda_k^{(1)}(p, q, m) &= \sum_{i=a, \dots, f} \Lambda_k^{(1-i)}(p, q, m) \\
&= \gamma_k F_1^k + \gamma_k \{ \not{p} F_2^k + \not{p}_0 F_3^k \} + \{ \not{q} F_4^k + \not{q}_0 F_5^k \} \gamma_k \\
&\quad + \not{q} \gamma_k \not{p} F_6^k + \not{q} \gamma_k \not{p}_0 F_7^k + \not{q}_0 \gamma_k \not{p} F_8^k + \gamma_k \not{p}_0 \not{p} F_9 + \not{q} \not{q}_0 \gamma_k F_{10}^k \\
&\quad + (p_k + q_k) [H_1^k + \not{p} H_2^k + \not{q} H_3^k + \not{q} \not{p} H_4^k] \\
&\quad + (p_k - q_k) [G_1^k + \not{p} G_2^k + \not{q} G_3^k + \not{q} \not{p} G_4^k] + O(a^2), \tag{3.23}
\end{aligned}$$

where  $\Lambda_k(p, q, m) = \Lambda_k^{(0)}(p, q, m) + g^2 \Lambda_k^{(1)}(p, q, m) + O(g^4)$  and  $\not{p} = \sum_{\alpha=0}^3 p_\alpha \gamma_\alpha$ ,  $\not{q} = \sum_{\alpha=0}^3 q_\alpha \gamma_\alpha$ ,  $\not{p}_0 = p_0 \gamma_0$ ,  $\not{q}_0 = q_0 \gamma_0$ . The coefficients  $F_i^k$  ( $i = 1, \dots, 10$ ),  $G_i^k$  ( $i = 1, 2, 3, 4$ ) and  $H_i^k$  ( $i = 1, 2, 3, 4$ ) are functions of  $p^2$ ,  $q^2$ ,  $p \cdot q$  and  $m$ . From the charge conjugation symmetry they have to satisfy the following condition:  $F_2^k = F_4^k$ ,  $F_3^k = F_5^k$ ,  $F_7^k = F_8^k$ ,  $F_9^k = F_{10}^k$ ,  $H_2^k = H_3^k$ ,  $G_1^k = G_4^k = 0$  and  $G_2^k = -G_3^k$ . Sandwiching  $\Lambda_k^{(1)}(p, q, m)$  by the on-shell quark states  $u(p)$  and  $\bar{u}(q)$ , which satisfy  $\not{p}u(p) = im_p u(p)$  and  $\bar{u}(q)\not{q} = im_p \bar{u}(q)$ , the matrix element

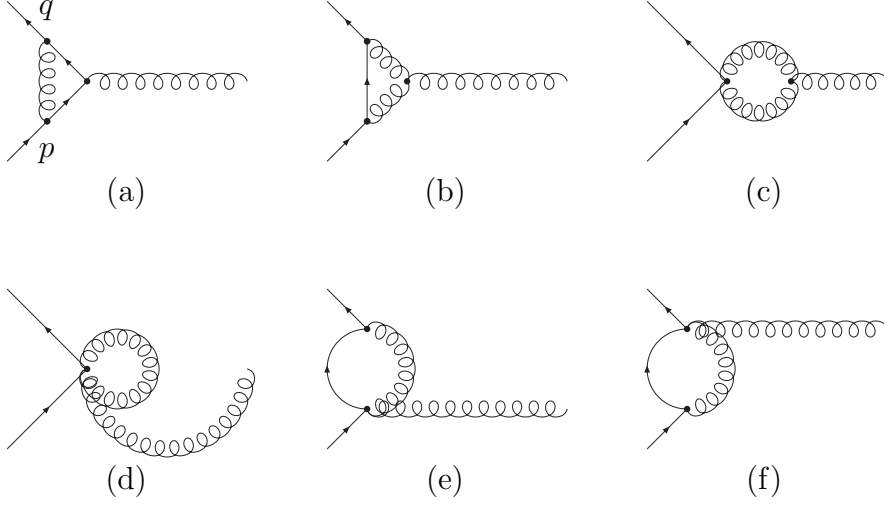


Fig. 2. Quark-quark-gluon vertices at one-loop level.  $p$  ( $q$ ) is incoming (outgoing) quark momentum.

is reduced to

$$\begin{aligned}
& \bar{u}(q)A_k^{(1)}(p, q, m)u(p) \\
&= \bar{u}(q)\gamma_k u(p) \{F_1^k + im_p(F_2^k + F_4^k) - m_p^2 F_6^k\} \\
&+ (p_k + q_k)\bar{u}(q)u(p) \{H_1^k + im_p(H_2^k + H_3^k) - m_p^2 H_4^k\} \\
&+ (p_k - q_k)\bar{u}(q)u(p) \{G_1^k + im_p(G_2^k + G_3^k) - m_p^2 G_4^k\} + O(a^2), \tag{3.24}
\end{aligned}$$

where we use  $F_3^k = F_5^k$ ,  $F_7^k = F_8^k$  and  $F_9^k = F_{10}^k$ . (Note that we can replace  $m_p$  with  $m_p^{(0)}$  in the one-loop diagrams.) The first term in the right hand side contributes to the renormalization factor of the quark-quark-gluon vertex, which is equal to  $Z_q^{(0)}$  at the tree level. With the use of  $G_1^k = G_4^k = 0$  and  $G_2^k = -G_3^k$ , we find that the last term of eq.(3.24) vanishes: this term is not allowed from the charge conjugation symmetry. It is also possible to numerically check  $G_1^k + im_p(G_2^k + G_3^k) - m_p^2 G_4^k = 0$ .

The relevant term for the determination of  $c_B$  is the third one, which can be extracted by setting  $p = p^* \equiv (p_0 = im_p, p_i = 0)$  and  $q = q^* \equiv (q_0 = im_p, q_i = 0)$  in eq.(3.23):

$$\begin{aligned}
& H_1^k + im_p(H_2^k + H_3^k) - m_p^2 H_4^k \Big|_{p=p^*, q=q^*} \\
&= \frac{1}{8} \text{Tr} \left[ \left\{ \frac{\partial}{\partial p_k} + \frac{\partial}{\partial q_k} \right\} A_k^{(1)}(p^*, q^*, m)(\gamma_0 + 1) \right] \\
&- \frac{1}{8} \text{Tr} \left[ \left\{ \frac{\partial}{\partial p_i} - \frac{\partial}{\partial q_i} \right\} A_k^{(1)}(p^*, q^*, m)(\gamma_0 + 1)\gamma_i \gamma_k \right]^{i \neq k}, \tag{3.25}
\end{aligned}$$

where we have used the fact that  $F^k$ ,  $G^k$  and  $H^k$  are functions of  $p^2$ ,  $q^2$  and  $p \cdot q$ , so that

$$\left. \frac{\partial F_j^k}{\partial p_i} \right|_{p=p^*, q=q^*} = \left. \frac{\partial F_j^k}{\partial q_i} \right|_{p=p^*, q=q^*} = 0, \quad (3.26)$$

$$\left. \frac{\partial H_l^k}{\partial p_i} \right|_{p=p^*, q=q^*} = \left. \frac{\partial H_l^k}{\partial q_i} \right|_{p=p^*, q=q^*} = 0, \quad (3.27)$$

$$\left. \frac{\partial G_l^k}{\partial p_i} \right|_{p=p^*, q=q^*} = \left. \frac{\partial G_l^k}{\partial q_i} \right|_{p=p^*, q=q^*} = 0 \quad (3.28)$$

with  $j = 1, \dots, 10$ ,  $l = 1, 2, 3, 4$  and  $i = 1, 2, 3$ .

We should remark that the third term in eq.(3.24) contains both the lattice artifact of  $O(p_k a, q_k a)$  and the physical contribution of  $O(p_k/m, q_k/m)$ . The parameter  $c_B$  is determined to eliminate the lattice artifacts of  $O(p_k a, q_k a)$ :

$$\begin{aligned} \frac{c_B^{(1)} - r_s^{(1)}}{2} &= [H_1^k + im_p(H_2^k + H_3^k) - m_p^2 H_4^k]_{p=p^*, q=q^*}^{\text{latt}} \\ &\quad - Z_q^{(0)} [H_1^k + im_p(H_2^k + H_3^k) - m_p^2 H_4^k]_{p=p^*, q=q^*}^{\text{cont}}, \end{aligned} \quad (3.29)$$

where we take account of the tree-level expression for the quark-quark-gluon vertex in eq.(3.3) and eq.(3.51) of Ref. 2). We show the quark mass dependences of  $c_B^{(1)}/c_B^{(0)}$  over the range  $0 \leq m_p^{(0)} \leq 10$  for the plaquette and the Iwasaki gauge actions in Fig. 3(a). The solid lines represent the interpolation with a rational expression. The errors are within symbols.

The calculation for  $c_E^{(1)}$  is done in a similar way as for  $c_B^{(1)}$ : we determine  $c_E^{(1)}$  to remove the  $O(a)$  contribution from the on-shell matrix element  $\bar{u}(q)\Lambda_0^{(1)}(p, q, m)u(p)$ . The quark mass dependences of  $c_E^{(1)}/c_E^{(0)}$  are also shown in Fig. 3(b).

Before closing this section we should remark an important feature in the calculation of  $c_E^{(1)}$  and  $c_B^{(1)}$ . The infrared divergences originating from Figs. 2 (a), (b), (c) contain both the lattice artifacts and the physical contributions. The former exactly cancels out if and only if the four parameters  $\nu^{(0)}$ ,  $r_s^{(0)}$ ,  $c_B^{(0)}$  and  $c_E^{(0)}$  are properly tuned. In Ref. 4) we demonstrate it by explicitly writing down the infrared behaviors of the one-loop diagrams. This is another evidence that the tree-level improvement is correctly implemented in Ref. 2). It may be also instructive to mention the massless case,<sup>16),3)</sup> where we find a similar situation: The infrared divergences originating from Figs. 2 (a), (b), (c), (e), (f) cancel out if and only if  $c_{\text{SW}}^{(0)}$  is correctly adjusted at the tree level, namely  $c_{\text{SW}}^{(0)} = 1$ .

#### §4. $O(a)$ improvement of the axial vector currents

We consider the on-shell  $O(a)$  improvement of the axial vector current both for the heavy-heavy and heavy-light cases. The discussion for the vector case is in parallel with the axial

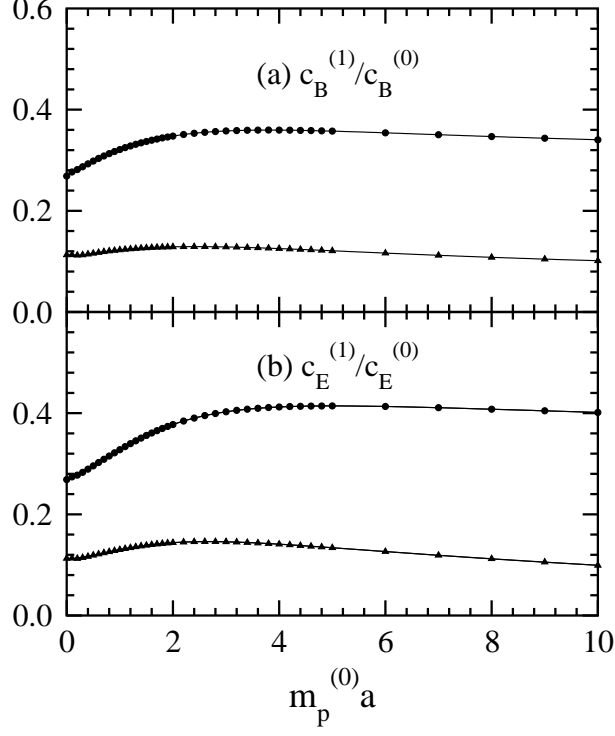


Fig. 3. (a)  $c_B^{(1)}/c_B^{(0)}$  and (b)  $c_E^{(1)}/c_E^{(0)}$  as a function of  $m_p^{(0)}$ . Circles denote the plaquette gauge action and triangles for the Iwasaki gauge action.

vector case. (See Ref. 5) for details.)

The renormalized operator with the  $O(a)$  improvement is given by

$$\begin{aligned}
A_\mu^{\text{latt},R}(x) = Z_{A_\mu}^{\text{latt}} & \left[ \bar{q}(x)\gamma_\mu\gamma_5 Q(x) - g^2 c_{A_\mu}^+ \partial_\mu^+ \{ \bar{q}(x)\gamma_5 Q(x) \} - g^2 c_{A_\mu}^- \partial_\mu^- \{ \bar{q}(x)\gamma_5 Q(x) \} \right. \\
& \left. + g^2 c_{A_\mu}^L \{ \vec{\partial}_i \bar{q}(x) \} \gamma_i \gamma_\mu \gamma_5 Q(x) - g^2 c_{A_\mu}^H \bar{q}(x) \gamma_\mu \gamma_5 \gamma_i \{ \vec{\partial}_i Q(x) \} + O(g^4) \right], \quad (4.1)
\end{aligned}$$

where we assume that the Euclidean space-time rotational symmetry is not retained on the lattice. The coefficients  $Z_{A_\mu}^{\text{latt}}$  and  $c_{A_\mu}^{(+,-,H,L)}$  are functions of the quark masses  $m_Q$  and  $m_q$ . With the aid of equation of motion we are allowed to set  $c_{A_0}^H = c_{A_0}^L = 0$ . In the special case of  $m_Q = m_q$ ,  $c_{A_\mu}^- = 0$  and  $c_{A_\mu}^H = -c_{A_\mu}^L$  are derived from the charge conjugation symmetry. We also note that all the improvement coefficients except  $c_{A_\mu}^+$  vanish in the limit of  $m_Q = m_q = 0$ . We determine  $Z_{A_\mu}^{\text{latt}}$  and  $c_{A_\mu}^{(+,-,H,L)}$  at the one-loop level for both heavy-heavy and heavy-light cases. In the following we are restricted to the time component of the axial vector current because of the limitation of space. As for the space component refer to Ref. 5).

The general form of the off-shell vertex function at the one-loop level on the lattice is given by

$$A_{05}^{(1)}(p, q, m_{p1}, m_{p2}) = \gamma_0 \gamma_5 F_1^{05} + \gamma_0 \gamma_5 \not{p} F_2^{05} + \not{q} \gamma_0 \gamma_5 F_3^{05} + \not{q} \gamma_0 \gamma_5 \not{p} F_4^{05}$$

$$\begin{aligned}
& +(p_0 - q_0) [\gamma_5 G_1^{05} + \gamma_5 \not{p} G_2^{05} + \not{q} \gamma_5 G_3^{05} + \not{q} \gamma_5 \not{p} G_4^{05}] \\
& +(p_0 + q_0) [\gamma_5 H_1^{05} + \gamma_5 \not{p} H_2^{05} + \not{q} \gamma_5 H_3^{05} + \not{q} \gamma_5 \not{p} H_4^{05}] + O(a^2),
\end{aligned} \tag{4.2}$$

where  $p$  denotes incoming quark momentum and  $q$  for outgoing one. The coefficients  $F^{05}$ ,  $G^{05}$ ,  $H^{05}$  are functions of  $p^2$ ,  $q^2$ ,  $p \cdot q$ ,  $m_{p1}$  and  $m_{p2}$ .

Sandwiching eq.(4.2) by the on-shell quark states  $u(p)$  and  $\bar{u}(q)$ , which satisfy  $\not{p}u(p) = im_{p1}u(p)$  and  $\bar{u}(q)\not{q} = im_{p2}\bar{u}(q)$ , the matrix element is reduced to

$$\begin{aligned}
& \bar{u}(q) \Lambda_{05}^{(1)}(p, q, m_{p1}, m_{p2}) u(p) \\
& = \bar{u}(q) \gamma_0 \gamma_5 u(p) \{ F_1^{05} + im_{p1} F_2^{05} + im_{p2} F_3^{05} - m_{p1} m_{p2} F_4^{05} \} \\
& \quad + (p_0 - q_0) \bar{u}(q) u(p) \{ G_1^{05} + im_{p1} G_2^{05} + im_{p2} G_3^{05} - m_{p1} m_{p2} G_4^{05} \} \\
& \quad + (p_0 + q_0) \bar{u}(q) \gamma_5 u(p) \{ H_1^{05} + im_{p1} H_2^{05} + im_{p2} H_3^{05} - m_{p1} m_{p2} H_4^{05} \} + O(a^2),
\end{aligned} \tag{4.3}$$

where the coefficients are summarized as

$$X_{05} = F_1^{05} + im_{p1} F_2^{05} + im_{p2} F_3^{05} - m_{p1} m_{p2} F_4^{05}, \tag{4.4}$$

$$Y_{05} = G_1^{05} + im_{p1} G_2^{05} + im_{p2} G_3^{05} - m_{p1} m_{p2} G_4^{05}, \tag{4.5}$$

$$Z_{05} = H_1^{05} + im_{p1} H_2^{05} + im_{p2} H_3^{05} - m_{p1} m_{p2} H_4^{05}. \tag{4.6}$$

Since the above coefficients contain both the lattice artifacts and the physical contributions which remain in the continuum, we have to isolate the lattice artifacts in order to determine the improvement coefficients in eq.(4.1). The improvement coefficients are given by

$$\Delta_{\gamma_0 \gamma_5} = (X_{05})^{\text{latt}} - (X_{05})^{\text{cont}}, \tag{4.7}$$

$$ic_{A_0}^+ = (Y_{05})^{\text{latt}} - (Y_{05})^{\text{cont}}, \tag{4.8}$$

$$ic_{A_0}^- = (Z_{05})^{\text{latt}} - (Z_{05})^{\text{cont}}, \tag{4.9}$$

where the continuum contributions are obtained by employing the naive dimensional regularization (NDR) with the modified minimal subtraction scheme ( $\overline{\text{MS}}$ ). Note that  $c_{A_0}^- = 0$  for  $m_{p1} = m_{p2}$  from the charge conjugation symmetry.

Here we should remark an important point.  $(X_{05})^{\text{latt}}, (Y_{05})^{\text{latt}}, (Z_{05})^{\text{latt}}$  are process-dependent: They are allowed to have different finite values and infrared divergences, which are regularized by the fictitious gluon mass in our calculation, for decay and scattering processes. However, once we subtract the continuum counter part, the process dependences are canceled out and we are left with finite constants, namely the improvement coefficients  $\Delta_{\gamma_0 \gamma_5}, c_{A_0}^+, c_{A_0}^-$ .

Combining  $\Delta_{\gamma_\mu \gamma_5}$  and the wave function renormalization factors, we obtain the renormalization factor of the axial vector currents:

$$\frac{Z_{A_\mu}^{\text{latt}}}{Z_{A_\mu}^{\text{cont}}} = \sqrt{Z_{Q,\text{latt}}^{(0)}(m_{p1}^{(0)})} \sqrt{Z_{q,\text{latt}}^{(0)}(m_{p2}^{(0)})} (1 - g^2 \Delta_{A_\mu}) \quad (4.10)$$

with

$$Z_{Q,\text{latt}}^{(0)}(m_{p1}^{(0)}) = \cosh(m_{p1}^{(0)}) + r_t \sinh(m_{p1}^{(0)}) \quad (4.11)$$

$$Z_{q,\text{latt}}^{(0)}(m_{p2}^{(0)}) = \cosh(m_{p2}^{(0)}) + r_t \sinh(m_{p2}^{(0)}) \quad (4.12)$$

$$\Delta_{A_\mu} = \Delta_{\gamma_\mu \gamma_5} - \frac{\Delta_Q}{2} - \frac{\Delta_q}{2}, \quad (4.13)$$

where  $\Delta_{Q,q}$  are found in Ref. 4).

Employing a set of special momentum assignments  $p = p^* \equiv (p_0 = im_{p1}, p_i = 0)$  and  $q = q_s^* \equiv (q_0 = im_{p2}, q_i = 0)$  or  $q = q_d^* \equiv (q_0 = -im_{p2}, q_i = 0)$ , where subscripts  $s$  and  $d$  represent the scattering and the decay respectively, we extract the relevant coefficients  $X_{05}, Y_{05}, Z_{05}$  for  $A_0$  from the off-shell vertex function (4.2):

$$X_{05}^s = \frac{1}{4} \text{Tr} \left[ A_{05}^{(1)} \gamma_5 \gamma_0 + im_{p1} \frac{\partial}{\partial p_k} A_{05}^{(1)} (1 + \gamma_0) \gamma_k \gamma_5 + im_{p2} \frac{\partial}{\partial q_k} A_{05}^{(1)} (1 + \gamma_0) \gamma_k \gamma_5 \right]_{p=p^*, q=q_s^*}, \quad (4.14)$$

$$X_{05}^s + im_{p1} (Y_{05}^s + Z_{05}^s) - im_{p2} (Y_{05}^s - Z_{05}^s) = \frac{1}{4} \text{Tr} \left[ A_{05}^{(1)} \gamma_5 (1 + \gamma_0) + 2im_{p1} \frac{\partial}{\partial p_k} A_{05}^{(1)} (1 + \gamma_0) \gamma_k \gamma_5 \right]_{p=p^*, q=q_s^*}, \quad (4.15)$$

$$X_{05}^d - im_{p1} (Y_{05}^d + Z_{05}^d) - im_{p2} (Y_{05}^d - Z_{05}^d) = -\frac{1}{4} \text{Tr} \left[ A_{05}^{(1)} (1 + \gamma_0) \gamma_5 \right]_{p=p^*, q=q_d^*}, \quad (4.16)$$

where superscripts  $s$  and  $d$  in  $X_{05}, Y_{05}, Z_{05}$  represent their momentum assignments. and  $F^{05}, G^{05}$  and  $H^{05}$  are functions of  $p^2, q^2$  and  $p \cdot q$  resulting in

$$\left. \frac{\partial F^{05}}{\partial p_i} \right|_{p=p^*, q=q^*} = \left. \frac{\partial F^{05}}{\partial q_i} \right|_{p=p^*, q=q^*} = 0, \quad (4.17)$$

$$\left. \frac{\partial H^{05}}{\partial p_i} \right|_{p=p^*, q=q^*} = \left. \frac{\partial H^{05}}{\partial q_i} \right|_{p=p^*, q=q^*} = 0, \quad (4.18)$$

$$\left. \frac{\partial G^{05}}{\partial p_i} \right|_{p=p^*, q=q^*} = \left. \frac{\partial G^{05}}{\partial q_i} \right|_{p=p^*, q=q^*} = 0 \quad (4.19)$$

with  $i = 1, 2, 3$ .

The set of improvement coefficients are determined from eqs.(4.7–4.9). Assuming that the  $m_{p2}a$  corrections are negligible, we evaluate the improvement coefficients, except  $c_{A_0}^{(+,-)}$ ,

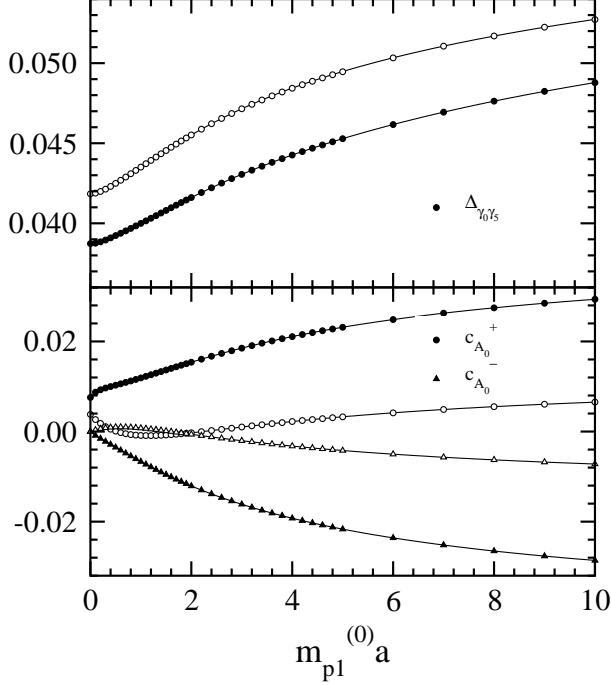


Fig. 4.  $\Delta_{\gamma_0 \gamma_5}, c_{A_0}^{(+,-)}$  for heavy-light case as a function of  $m_{p1}^{(0)}$ . Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.

as a function of  $m_{p1}^{(0)}$  with  $m_{p2}^{(0)} = 0$ . In eqs.(4.15), (4.16) we find that  $Y_{05} - Z_{05}$  are not determined if we set  $m_{p2}^{(0)} = 0$ . Therefore one should extrapolate data at non-zero  $m_{p2}^{(0)}$  to  $m_{p2}^{(0)} = 0$ . We however keep  $m_{p2}^{(0)} = 0.0001$  in our calculation to determine  $c_{A_0}^{(+,-)}$  since the difference between the value at  $m_{p2}^{(0)} = 0.0001$  and the one extrapolated to  $m_{p2}^{(0)} = 0$  is less than 1 %. In Fig. 4 we show numerical results of  $\Delta_{\gamma_0 \gamma_5}, c_{A_0}^{(+,-)}$  for the heavy-light case. The solid lines denote the interpolation with a rational expression. The errors are within symbols. We can find the heavy-heavy case in Fig. 9 of Ref. 5).

We also make a brief comment on our recent work of the  $O(a)$  improvement for the heavy-light vector and axial vector currents with relativistic heavy and domain-wall light quarks.<sup>6)</sup> The most important feature in this calculation is that the renormalization and the improvement coefficients of the heavy-light vector current agree with those of the axial vector current. We have shown that this is indebted to the exact chiral symmetry for the light quark irrespective of the heavy quark action. We have also presented how to implement the on-shell improvement on the massive domain-wall quark action, which is required to cancel out the infrared divergences generating from the one-loop vertex corrections.



## §5. Numerical studies in quenched QCD

Now we have achieved the  $O(a)$  improvement both for the quark action and the axial vector current, the next step is to check the effectiveness of the improvement. Our numerical studies are carried out in quenched QCD focusing on the restoration of the space-time symmetry for the heavy-heavy and heavy-light meson systems. We investigate the dispersion relation of moving mesons and the difference of the pseudoscalar meson decay constants extracted from the temporal and spatial components of axial vector currents.

We take the clover action with non-perturbative  $c_{\text{SW}}^{\text{NP} (19), (20)}$  for light quarks. As for the heavy quarks, we replace the massless contribution in  $c_E^{(1)}$  and  $c_B^{(1)}$  of the improved heavy quark action with that of the non-perturbative one as

$$c_B = \{c_B^{\text{PT}}(m_Q a) - c_B^{\text{PT}}(0)\} + c_{\text{SW}}^{\text{NP}}, \quad (5.1)$$

$$c_E = \{c_E^{\text{PT}}(m_Q a) - c_E^{\text{PT}}(0)\} + c_{\text{SW}}^{\text{NP}}, \quad (5.2)$$

where the superscript PT represents "perturbative" value. With this replacement  $O(a)$  errors are completely removed at  $m_Q = 0$ . This is required by a consistency between the light and heavy quarks. We generated 200 configurations with the plaquette and the Iwasaki gauge actions on a  $24^3 \times 48$  lattice at  $a^{-1} \approx 2$  GeV. We employ three light quark masses corresponding to  $m_{\text{PS}}/m_V \sim 0.56 - 0.77$  and four heavy quark masses covering the charm quark mass. For comparison we make another simulation with the clover action both for the heavy and light quarks.

To investigate the dispersion relation of heavy-heavy and heavy-light mesons, it is convenient to define the effective speed of light as

$$c_{\text{eff}} = \sqrt{\frac{E(p_s)^2 - E(0)^2}{p_s^2}}, \quad (5.3)$$

where  $E(p_s)$  is the pseudoscalar meson energy with the spatial momentum  $p_s$ . This quantity is supposed to be unity in the continuum limit, which means the restoration of relativistic dispersion relation. In Figs. 5(a), (b) we plot numerical results as a function of the meson mass for heavy-heavy(H-H) and heavy-light(H-L) systems. We observe that around  $J/\Psi$  mass in the heavy-heavy system the deviation from  $c_{\text{eff}} = 1$  is equal to or larger than about 10% for the clover heavy quark action, while less than 4% for the  $O(a)$  improved heavy quark action.

We also measure the space-time asymmetry from the difference of the pseudoscalar meson decay constants determined from the temporal and spatial components of axial vector

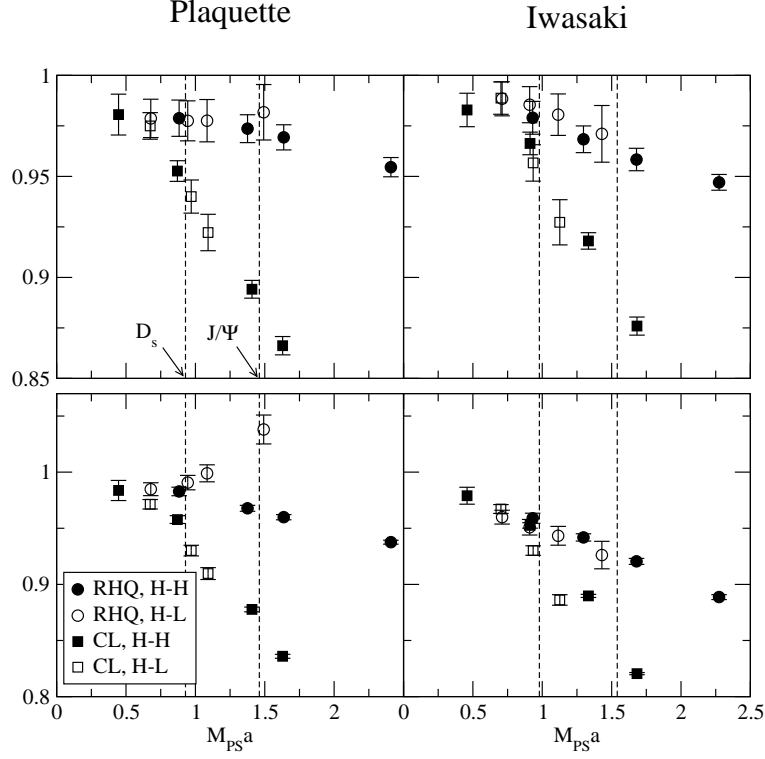


Fig. 5. Effective speed of light(top) and pseudoscalar meson decay constants as a function of the meson masses for the plaquette and the Iwasaki gauge actions. RHQ denotes the  $O(a)$  improved heavy quark action and CL represents the clover action for heavy quarks.

currents:

$$R \equiv i \frac{\langle 0 | A_k^R | PS \rangle}{\langle 0 | A_4^R | PS \rangle} \cdot \frac{E}{|p_s|}, \quad (5.4)$$

which should become unity in the continuum limit. Results are plotted in Figs.5(c), (d). We find that the asymmetry reaches about 10 ~ 20% for the clover action around  $J/\Psi$  mass in the heavy-heavy system, while it is less than 7% for the improved action. These observations allow us to conclude that the improved action clearly reduces the errors caused by  $m_Q a$ .

## §6. Conclusion and perspective

We have proposed a relativistic  $O(a)$  improvement to the heavy quarks on the lattice. The idea is based on the relativistic on-shell improvement with the finite  $m_Q a$  corrections. We have shown that the cutoff effects can be reduced to  $O((a\Lambda_{\text{QCD}})^2)$  putting the  $(m_Q a)^n$  corrections on the renormalization factors of the quark mass  $Z_m$  and wave function  $Z_q$ . Our relativistic approach has the strong point over the nonrelativistic ones: the finer the lattice

spacing becomes, the better the approach works. This is a desirable feature because we can take the full advantage of configurations with finer lattice spacing generated to control the cutoff effects on the light hadron physics.

In the next step we have determined the  $O(a)$  improvement coefficients,  $\nu$ ,  $r_s$ ,  $c_B$  and  $c_E$  in the quark action up to one-loop level for the various improved gauge actions. While  $\nu$  and  $r_s$  are determined from the quark propagator, we use the on-shell quark-quark scattering amplitude for  $c_B$  and  $c_E$ . The  $m_Q a$  dependences are examined by making the perturbative calculations done in a  $m_Q a$  dependent way. Employing the conventional perturbative method with the fictitious gluon mass as an infrared regulator we have shown that the parameters  $\nu$ ,  $r_s$ ,  $c_B$  and  $c_E$  in the action are determined free from the infrared divergences. This is achieved if and only if the tree-level values for  $\nu$ ,  $r_s$ ,  $c_B$  and  $c_E$  are properly adjusted.

We have also made the  $O(a)$  improvement of the vector and the axial vector currents up to one-loop level. Our calculation is carried out both for the heavy-heavy and the heavy-light cases with the various gauge actions. It is explicitly shown that the renormalization and improvement coefficients of the heavy-light vector current agree with those of the axial vector current, once we impose the exact chiral symmetry for the light quark.

Given the  $O(a)$  improved quark action and axial vector current, we can check their effectiveness by investigating the restoration of the space-time symmetry for the heavy-heavy and heavy-light meson systems. We have focused on two quantities: the dispersion relation and the difference of the pseudoscalar meson decay constants determined from the temporal and spatial components of axial vector currents. Our results show clear improvement for both quantities.

Up to now the  $O(a)$  improvement of the massive Wilson quark action works well: We do not encounter any theoretical contradiction in the improvement procedure of the quark action and the vector and axial vector currents up to one-loop level, and numerical studies clearly show the effectiveness of improvement. These encouraging results lead us to the next step. Our ongoing projects are the  $O(a)$  improvement of the four-fermi operators up to one-loop level and numerical simulations with two- and three-flavors of dynamical quarks. We also plan a detailed scaling study in quenched QCD and a nonperturbative determination of the improvement coefficients.

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## References

- 1) G. P. Lepage and B. A. Thacker, Nucl. Phys. **B** (Proc. Suppl.) **4** (1988), 199.  
B. A. Thacker and G. P. Lepage, Phys. Rev. **D43** (1991), 196. G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, *ibid.* **46** (1992), 4052.
- 2) S. Aoki, Y. Kuramashi and S. Tominaga, Prog. Theor. Phys. **109** (2003) 383.
- 3) S. Aoki and Y. Kuramashi, Phys. Rev. **D68** (2003) 094019.
- 4) S. Aoki, Y. Kayaba and Y. Kuramashi, Nucl. Phys. **B** (in press), hep-lat/0309161.
- 5) S. Aoki, Y. Kayaba and Y. Kuramashi, Nucl. Phys. **B689** (2004) 127.
- 6) N. Yamada, S. Aoki and Y. Kuramashi, hep-lat/0407031.
- 7) S. Aoki, Y. Kuramashi and S. Tominaga, Nucl. Phys. **B** (Proc. Suppl.) **106** (2002) 349.
- 8) S. Aoki and Y. Kuramashi, Nucl. Phys. **B** (Proc. Suppl.) **119** (2003) 583.
- 9) S. Aoki, Y. Kayaba and Y. Kuramashi, Nucl. Phys. **B** (Proc. Suppl.) **129** (2004) 352.
- 10) K. Symanzik, in *Mathematical Problems In Theoretical Physics*, eds. R. Schrader *et al.*, Lecture Notes in Physics, Vol. 153 (Springer, New York, 1982).
- 11) K. Symanzik, Nucl. Phys. **B226** (1983), 187; 205.
- 12) M. Lüscher and P. Weisz, Commun. Math. Phys. **97** (1985), 59.
- 13) M. Lüscher, S. Sint, R. Sommer and P. Weisz Nucl. Phys. **B478** (1996), 365.
- 14) B. Sheikholeslami and R. Wohlert, Nucl. Phys. **B259** (1985), 572.
- 15) A. X. El-Khadra, A. S. Kronfeld and P. B. Mackenzie, Phys. Rev. **D55** (1997) 3933.
- 16) R. Wohlert, Improved continuum limit lattice action for quarks, DESY preprint 87-069 (1987), unpublished.
- 17) S. Aoki, K. Nagai, Y. Taniguchi and A. Ukawa, Phys. Rev. **D58** (1998) 074505;  
Y. Taniguchi and A. Ukawa, Phys. Rev. **D58** (1998) 114503.
- 18) Y. Iwasaki, preprint, UTHEP-118 (Dec. 1983), unpublished.
- 19) M. Lüscher, S. Sint, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. **B491** (1997), 323.
- 20) CP-PACS Collaboration, S. Aoki *et al.*, in preparation.