

Twisted mass Lattice QCD*

R. Frezzotti^a

^aI.N.F.N. Milano and University of Milano Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy

I review the main theoretical properties and some recent analytical and numerical investigations of the formulations of lattice QCD with chirally twisted Wilson quarks (also known as twisted mass lattice QCD).

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In this contribution I review lattice formulations based on so-called twisted mass (tm) Wilson quarks, focusing on the general properties and problems that in my opinion are most relevant for on-going studies and applications to physics².

The simplest of these formulations, which corresponds to QCD with one mass degenerate quark doublet, was introduced in ref. [1] as a way to get rid of the spurious quark zero modes that plague standard Wilson fermions and is referred to as twisted mass lattice QCD (tmLQCD).

1. $N_f = 2$ twisted mass lattice QCD

Following refs. [1,2,3], a sensible lattice formulation of QCD with an $SU_f(2)$ flavour doublet of mass degenerate quarks is given by the action $S = S_g[U] + S_F^{(\omega)}[\psi, \bar{\psi}, U]$, where $S_g[U]$ stands for any discretization of the YM action and

$$S_F^{(\omega)}[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\psi}(x) \left[\gamma \cdot \tilde{\nabla} + e^{-i\omega\gamma_5\tau_3} W_{\text{cr}}(r) + m_q \right] \psi(x), \quad (1)$$

$$\gamma \cdot \tilde{\nabla} \equiv \frac{1}{2} \sum_{\mu} \gamma_{\mu} (\nabla_{\mu}^* + \nabla_{\mu}), \quad (2)$$

$$W_{\text{cr}}(r) \equiv -a^r \sum_{\mu} \nabla_{\mu}^* \nabla_{\mu} + M_{\text{cr}}(r). \quad (3)$$

Here M_{cr} denotes the critical mass, m_q is the (yet unrenormalized) offset quark mass (such that, once M_{cr} has been set to the appropriate value, $m_{\pi}^2 \sim m_q$ as $m_q \rightarrow 0$), while $r \in [-1, 1]$ and $\omega \in [-\pi, \pi)$ characterize the specific Wilson-type UV regularization. As discussed below (see sect. 1.3), M_{cr} is, up to $O(a)$ uncertainties, independent of ω . With the whole $O(a)$ and numerical uncertainty on the critical point brought onto $M_{\text{cr}}(r)$, one can treat ω and m_q (besides r) as exactly known parameters.

For $\omega = 0$ the familiar action for LQCD with two mass degenerate Wilson quarks is recovered. As long as $\omega \neq 0$ and $m_q \neq 0$, no zero modes of the (two-flavour) Dirac operator in eq. (1), which I denote by D_F , can occur on any gauge configuration, because [4]

$$\text{Det}[D_F] = \det[Q^2 + m_q^2 \sin^2 \omega], \quad (4)$$

$$Q \equiv \gamma_5 [\gamma \cdot \tilde{\nabla} + W_{\text{cr}}(r) + m_q \cos \omega] = Q^{\dagger}. \quad (5)$$

The property (4) solves the problems related to exceptional configurations in the quenched approximation as well as to MD instabilities due to exceptionally small eigenvalues of Q^2 in HMC-like algorithms for unquenched simulations.

The extension of unquenched simulation algorithms of the HMC and multiboson type to the action (1) –or eq. (12) below– is straightforward [5]. Preliminary results from on-going simulations of tmLQCD seem to hint at a numerical cost comparable to that of simulations with staggered fermions [6].

1.1. Symmetries and reflection positivity

The symmetries of $N_f = 2$ tmLQCD are discussed in detail in refs. [1,3]. Here I recall only the main points arising from those analyses.

The chirally twisted Wilson term in the action (1) breaks flavour chiral symmetry in such a way that,

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²The reader is urged to consult the cited papers for more details. For several aspects this write-up should be regarded as an introduction (or a comment) to those works.

if $m_q = 0$, three symmetry generators are preserved (see [3], eq. (4.7)) for any ω -value. Correspondingly, there exist three exactly conserved isotriplet lattice currents (see [3], eqs. (4.8)–(4.9)). One of them is the neutral vector current, while the other two are mixtures of the charged axial and vector currents (they get purely axial at $\omega = \pm\pi/2$ and vector at $\omega = 0, \pi$).

Charge conjugation symmetry remains exact, but, owing to the chiral twisting of the Wilson term, all single-axis inversions leave the action invariant only if combined with the discrete isospin rotations $\mathcal{T}_{1,2}$

$$\mathcal{T}_{1,2} : \psi(x) \rightarrow i\tau_{1,2}\psi(x), \quad \bar{\psi}(x) \rightarrow -i\bar{\psi}(x)\tau_{1,2}. \quad (6)$$

or, alternatively, a sign change of the twist angle ω . In particular this remark holds for the physical parity transformation (here $x_P \equiv (-\mathbf{x}, t)$)

$$\mathcal{P} : \begin{cases} U_0(x) \rightarrow U_0(x_P) \\ U_k(x) \rightarrow U_k^\dagger(x_P - a\hat{k}), \quad k = 1, 2, 3 \\ \psi(x) \rightarrow \gamma_0\psi(x_P) \\ \bar{\psi}(x) \rightarrow \bar{\psi}(x_P)\gamma_0 \end{cases} \quad (7)$$

For later use I note the non-anomalous symmetry [3]

$$\mathcal{R}_5^d \equiv \mathcal{R}_5 \times \mathcal{D}_d, \quad (8)$$

where

$$\mathcal{R}_5 : \begin{cases} \psi(x) \rightarrow \gamma_5\psi(x) \\ \bar{\psi}(x) \rightarrow -\bar{\psi}(x)\gamma_5, \end{cases} \quad (9)$$

$$\mathcal{D}_d : \begin{cases} U_\mu(x) \rightarrow U_\mu^\dagger(-x - a\hat{\mu}) \\ \psi(x) \rightarrow e^{3i\pi/2}\psi(-x) \\ \bar{\psi}(x) \rightarrow e^{3i\pi/2}\bar{\psi}(-x). \end{cases} \quad (10)$$

Correlation functions evaluated with the tmLQCD action enjoy link reflection positivity for all values of $r \in [-1, 1]$ and $\omega \in [-\pi, \pi]$ [7], as well as site reflection positivity provided $|r| = 1$ and $|8r + 2aM_{\text{cr}}(r) + 2am_q \cos \omega| > 6$ [1,3].

1.2. Renormalizability

The tmLQCD fermionic action (1) is written in what is usually called the ‘‘physical basis’’ [3], with m_q real (and positive). By a change of basis,

$$\chi = e^{-i\omega\gamma_5\tau_3/2}\psi, \quad \bar{\chi} = \bar{\psi}e^{-i\omega\gamma_5\tau_3/2}, \quad (11)$$

the action takes the form considered in ref. [1],

$$S_F^{(\omega)}[\chi, \bar{\chi}, U] = a^4 \sum_x \bar{\chi}(x) \left[\gamma \cdot \tilde{\nabla} + \right. \\ \left. - a\frac{r}{2} \sum_\mu \nabla_\mu^* \nabla_\mu + m_0 + im_2\gamma_5\tau_3 \right] \chi(x), \quad (12)$$

$$m_0 = M_{\text{cr}} + m_q \cos \omega, \quad m_2 = m_q \sin \omega. \quad (13)$$

All the symmetry properties described above or in refs. [1,3] can be expressed in the quark basis (11). In particular I note the spurionic symmetries of the action (12)

$$\mathcal{R}_5 \times (r \rightarrow -r) \times (m_0 \rightarrow -m_0) \times (m_2 \rightarrow -m_2), \quad (14)$$

$$\tilde{\mathcal{P}} \times (m_2 \rightarrow -m_2), \quad (15)$$

with \mathcal{R}_5 form invariant under (11) and

$$\tilde{\mathcal{P}} : \begin{cases} U_0(x) \rightarrow U_0(x_P) \\ U_k(x) \rightarrow U_k^\dagger(x_P - a\hat{k}), \quad k = 1, 2, 3 \\ \chi(x) \rightarrow \gamma_0\chi(x_P) \\ \bar{\chi}(x) \rightarrow \bar{\chi}(x_P)\gamma_0 \end{cases} \quad (16)$$

Standard symmetry and power counting arguments imply [1] that (up to terms irrelevant as $a \rightarrow 0$) the most general action for two-flavour tmLQCD is just of the form $S_g[U] + S_F^{(\omega)}[\chi, \bar{\chi}, U]$, with $S_g[U] \propto 1/g_0^2$ and $S_F^{(\omega)}$ given in eq. (12). Invariance under $\mathcal{P} \times \mathcal{T}_{1,2}$ (see eqs. (7) and (6)) rules out parity odd pure gauge terms ($\propto \text{tr}[F\tilde{F}]$ as $a \rightarrow 0$). While g_0^2 and m_2 need only multiplicative renormalization, m_0 undergoes additive and multiplicative renormalization [1].

The continuum flavour chiral WTI’s of QCD with two mass degenerate quarks can be implemented (up to $\mathcal{O}(a)$) for all ω -values [1,3], with the renormalized current quark mass given by

$$\hat{m}_q = Z_P^{-1} Z_M(\omega) m_q, \\ Z_M(\omega) = [Z_P^2 Z_{S^0}^{-2} \cos^2 \omega + \sin^2 \omega]^{1/2}. \quad (17)$$

Z_P and Z_{S^0} are the (mass-independent scheme) renormalization constants of $\bar{\chi}\gamma_5\tau_a\chi$ and $\bar{\chi}\chi$.

1.3. Critical mass

For given values of g_0^2 and r , the appropriate value of M_{cr} can be determined by adjusting m_0 so as to enforce one of the properties (chiral WTI’s, pions with minimal mass) dictated by flavour chiral symmetry³. In the quark basis (11)

³In perturbation theory all the conditions of this type defining M_{cr} lead to a unique result: in fact $\mathcal{O}(ap)$ terms can be singled out and removed, since all momentum scales p are controllable at fixed g_0^2 . Non-perturbatively this is no longer the case, because there is no way of letting $a\Lambda_{QCD} \rightarrow 0$ at fixed g_0^2 . At $g_0^2 > 0$ different definitions of M_{cr} in general differ by amounts of order $a\Lambda_{QCD}$.

and for any m_2 -value, a sensible condition defining M_{cr} is given e.g. by (index $b = 1, 2$ only)

$$\sum_{\mathbf{x}} \partial_{\mu} \langle (\bar{\chi} \gamma_{\mu} \gamma_5 \tau_b \chi)(x) \mathcal{O}(y, \dots) \rangle |_{m_0 = M_{\text{cr}}} = 0, \quad (18)$$

with \mathcal{O} a conveniently chosen multilocal operator and $x \neq \{y, \dots\}$. Owing to the symmetry (15), numerical estimates of M_{cr} corresponding to different values of m_2 differ by $\mathcal{O}(am_2^2)$ from each other. Since, by taking e.g. $\mathcal{O} = (\bar{\chi} \gamma_5 \tau_b \chi)(y)$, the parameters m_q and ω enter the condition (18) only through m_2 , one sees that, up to irrelevant $\mathcal{O}(a)$ terms, M_{cr} is independent of m_q and ω .

In infinite volume, correlation functions and derived quantities may depend (e.g. if chiral symmetry is spontaneously broken) also on the path along which the critical point $(m_0, m_2) = (M_{\text{cr}}, 0)$ is approached, and thus on ω . However, since renormalization is a local procedure, any possible ω -dependence of whatever determination of M_{cr} based on infinite volume quantities is necessarily limited to irrelevant $\mathcal{O}(a\Lambda_{\text{QCD}})$ contributions.

Moreover, the combination of the spurionic invariances (14) and (15) implies that, if $m_0 = M_{\text{cr}}$ fulfills a certain condition defining the critical mass for a given r (and m_2), then $m_0 = -M_{\text{cr}}$ satisfies the same condition for $-r$ (and the same m_2). The critical mass counterterm is thus an odd function of r :

$$M_{\text{cr}}(-r) = -M_{\text{cr}}(r). \quad (19)$$

If there exists an interval $[s_1, s_2]$ of m_0 -values for which the condition defining $M_{\text{cr}}(r)$ is satisfied, the invariances (14) and (15) imply that the m_0 -values in the interval $[-s_2, -s_1]$ are solutions of the same condition for $-r$. In finite volume, analyticity in m_0 excludes the possible existence of such intervals of solutions. They may however show up, though with a width vanishing as $a \rightarrow 0$, if M_{cr} is defined through some infinite volume quantity, as for instance when it is determined by the vanishing of the charged pion mass and the ‘‘Aoki phase scenario’’ [8] is realized ⁴.

Once a definition of $M_{\text{cr}}(r)$ has been chosen for, say, $r > 0$, one must (and, as shown above, always can) take $M_{\text{cr}}(-r) = -M_{\text{cr}}(r)$. Otherwise one would unnecessarily spoil –by an artificial choice of the untwisted mass counterterm– the spurionic symmetry $\mathcal{R}_5 \times \tilde{\mathcal{P}} \times (r \rightarrow -r) \times (m_0 \rightarrow -m_0)$

⁴In this scenario the intervals of m_0 -values with zero charged pion mass are expected to have $\mathcal{O}(a^2)$ widths.

enjoyed by the lattice theory prior to renormalization. For these reasons the criticism to eq. (19) raised in ref. [9] is unjustified.

2. Mass non-degenerate flavours

For the case of $\omega = \pi/2$ (maximal twist), the fermionic action of an $SU_f(2)$ pair of mass non-degenerate quark is conveniently written [10] as

$$S_{\text{Fnd}}^{(\pi/2)}[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\psi}(x) \left[\gamma \cdot \tilde{\nabla} + \right. \quad (20) \\ \left. -i\gamma_5 \tau_1 W_{\text{cr}}(r) + m_q - \tau_3 \epsilon_q \right] \psi(x),$$

where to keep the mass term real and flavour diagonal I have used the matrix τ_3 to split the masses of the members of the doublet. Consequently the Wilson term has been twisted with the flavour matrix τ_1 . Note that m_q and ϵ_q are both real.

It has been shown [10] that the quark mass splitting ϵ_q is only multiplicatively renormalized and the continuum flavour chiral WTI’s can be implemented up to $\mathcal{O}(a)$, provided that ⁵

$$\hat{m}_q^{(-)} = \hat{m}_q - \hat{\epsilon}_q = Z_P^{-1} m_q - Z_S^{-1} \epsilon_q, \quad (21)$$

$$\hat{m}_q^{(+)} = \hat{m}_q + \hat{\epsilon}_q = Z_P^{-1} m_q + Z_S^{-1} \epsilon_q \quad (22)$$

are identified as the renormalized (current) masses of the quarks in the doublet.

Remarkably, the fermionic determinant ⁶,

$$\det[D_{\text{Fnd}}^{(\pi/2)}] \geq \det[Q_{\text{cr}}^2 + m_q^2 - \epsilon_q^2], \quad (23)$$

$$Q_{\text{cr}} \equiv \gamma_5 [\gamma \cdot \tilde{\nabla} + W_{\text{cr}}(r)] = Q_{\text{cr}}^{\dagger}. \quad (24)$$

is real and positive, as long as $\epsilon_q^2 < m_q^2$ [10]. This allows for unquenched Monte Carlo simulations, for instance by means of algorithms of the multi-boson or PHMC type based on some polynomial approximation of $[D_{\text{Fnd}}^{\dagger} D_{\text{Fnd}}]^{-1/2}$.

Alternatively, lattice formulations of QCD with N_f mass non-degenerate quarks of the (twisted) Wilson type can be obtained by taking the fermionic action of the form [11,12,13]:

$$S_F^{OS} = \sum_{f=1}^{N_f} \left\{ a^4 \sum_x \bar{q}_f(x) \right. \\ \left. [\gamma \cdot \tilde{\nabla} + e^{-i\gamma_5 \theta_f} W_{\text{cr}}(r_f) + m_f] q_f(x) \right\}, \quad (25)$$

⁵ Z_S is the renormalization constant of $\bar{\chi} \tau_a \chi$, $a = 1, 2, 3$.

⁶Here D_{Fnd} denotes the Dirac operator (a 2×2 matrix in flavour space) corresponding to the fermionic action (20).

with $\theta = \sum_f \theta_f = 0$ ⁷ and $W_{\text{cr}}(r_f)$ defined as in eq. (3). For all flavours f , the critical mass $M_{\text{cr}}(r_f)$ can be taken independent of θ_f and is given by the same dimensionless function $aM_{\text{cr}}(r)$ as in the case of (un)twisted mass-degenerate Wilson quarks [13,14]. The renormalized counterpart of the bare quark mass m_f reads [11,13,14]

$$\hat{m}_f = Z_m(r_f)m_f, \quad f = u, d, s, c, \dots \quad (26)$$

Choosing $\theta_f \neq 0$ (at least for u and d quarks) allows to get rid of spurious quark zero modes, while keeping the action diagonal in flavour space. However, even if $\theta = 0$, for generic values of the m_f 's the lattice fermionic determinant corresponding to the action (25) is complex [11,12,14]. Actions of the type (25) can thus be used in general only for valence quarks (see sect. 5). Particular cases with positive quark determinant exist: e.g. the $N_f = 4$ case with $m_u = m_d$, arbitrary m_s and m_c , $\theta_u = -\theta_d = \pi/2$ and $\theta_s = \theta_c = 0$ [14].

3. Continuum and chiral limits

In tmLQCD the relative magnitude of the lattice cutoff effects arising from the chiral violating action terms may change significantly as a function of the quark mass. This remark is particularly relevant in the regime where chiral symmetry is spontaneously broken. In this situation, in the continuum theory the chiral phase of the vacuum is driven by the phase of the quark mass term. The same must be true on the lattice, thus ideally the continuum limit should be taken –at non-zero \hat{m}_q – before letting $\hat{m}_q \rightarrow 0$.

3.1. O(a) improvement

From an analysis à la Symanzik [15] of the leading cutoff effects in tmLQCD (eq. (1)) and by exploiting the invariances (8) and⁸

$$\mathcal{R}_5^{sp} \equiv \mathcal{R}_5 \times (r \rightarrow -r) \times (m_q \rightarrow -m_q), \quad (27)$$

it follows [3] that the expectation values of multi-local, gauge invariant, multiplicatively renormalizable (m.r.) operators O satisfy

$$\langle O(x_1, \dots, x_n) \rangle_{(r, m_q)}^{(\omega)} + \langle O(x_1, \dots, x_n) \rangle_{(-r, m_q)}^{(\omega)} =$$

⁷The condition $\theta = 0$ is necessary for the corresponding continuum limit theory be QCD with vanishing “ θ -term”.

⁸The spurionic symmetry \mathcal{R}_5^{sp} holds for generic ω and follows directly from the symmetry (14) using (13) and (19).

$$= 2\zeta_O^O(r; \omega) \langle O(x_1, \dots, x_n) \rangle_{(m_q)}^{\text{cont}} + O(a^2). \quad (28)$$

$O(a)$ effects cancel⁹ in the average of results obtained with opposite values of r (Wilson average).

In case of maximal twist, $\omega = \pm\pi/2$, one can obtain $O(a)$ improved results even from one single simulation at a given r -value. In fact invariance of the action under $\mathcal{P} \times (r \rightarrow -r)$ ¹⁰ implies

$$\langle O(x_1, \dots) \rangle_{(-r, m_q)}^{(\pm\pi/2)} = \eta_O \langle O(x_1^P, \dots) \rangle_{(r, m_q)}^{(\pm\pi/2)}, \quad (29)$$

with η_O the formal parity of O , from which the second term in the l.h.s. of (28) can be obtained.

As eq. (28) holds for arbitrary space-time separations, all the quantities derived from m.r. lattice correlators are free from $O(a)$ cutoff effects, once the Wilson average, or –if $\omega = \pm\pi/2$ – the average over opposite values of all external three-momenta (see eq. (29)), has been taken.

The extension to mass non-degenerate quarks is straightforward, since the symmetries (8) and (with obvious modifications) (27) remain valid.

At variance with the familiar Symanzik's programme for $O(a)$ improvement [15], this method does not require the addition (and the determination) of action and operator counterterms.

The statement that a certain lattice quantity is $O(a)$ improved simply means that in the renormalized theory it approaches its continuum limit with a rate (asymptotically) quadratic in a . Moreover, as far as one is interested in continuum QCD with massive pions, the renormalized quark mass $\hat{m}_q \propto m_q$ must have a non-zero limit as $a \rightarrow 0$ ¹¹. Under this condition $O(a)$ improvement holds irrespectively of the value of \hat{m}_q . The contrary result of ref. [9] comes from the fact that there \hat{m}_q is allowed to be an $O(a)$, or $O(a^2)$, quantity. The question of the magnitude of residual cutoff effects, even if parametrically $O(a^2)$, when \hat{m}_q is numerically small is however important in practice (e.g. for extrapolations of results to $a = 0$).

Large cutoff effects can arise when working on a coarse lattice at small m_q -values, just because both the quark mass term proportional to m_q and the Wilson term (which has a different chiral orientation if $\omega \neq 0$) contribute to the breaking of

⁹The same is true also for $O(a^{2k+1})$ effects (k integer).

¹⁰This symmetry follows directly from eqs. (7) and (19).

¹¹If $a \rightarrow 0$ at $\hat{m}_q = 0$ and fixed physical volume, no spontaneous breaking of chiral symmetry occurs and chiral breaking cutoff effects are expected to be harmless [3].

chirality (and in particular to the chiral phase of the vacuum). In general, to avoid large lattice artifacts one should work with parameters such that $a\Lambda_{\text{QCD}}^2 \ll m_q$, but for $O(a)$ improved quantities it is conceivable that in several cases the relative magnitude of the dominant (as $a \rightarrow 0$) cutoff effects is just $O(a^2\Lambda_{\text{QCD}}^3/m_q)$. Scaling tests are thus important to assess the magnitude of cutoff effects on various observables as a function of \hat{m}_q .

3.2. ChPT analyses and phase diagram

Lattice chiral perturbation theory (ChPT) is an expansion in powers of the quark mass (or external momenta) and the lattice spacing and provides an explicit representation of (lattice estimates of) physical observables in the Goldstone boson sector, as well as of the chiral phase of the lattice vacuum, in terms of a few low energy constants to be determined [16]. Such a representation of observables, though usually limited to $O(m_\pi^4)$, can be very useful in extracting physical information from simulation data. When applied to tmLQCD [17], lattice ChPT involves both the quark mass \hat{m}_q and the (rescaled) twist angle ¹² $\hat{\omega} = \tan^{-1}[Z_{S^0}Z_P^{-1}\tan\omega]$, or equivalently

$$\hat{m}_1 \equiv \hat{m}_q \cos \hat{\omega}, \quad \hat{m}_2 \equiv \hat{m}_q \sin \hat{\omega}. \quad (30)$$

Here I shortly summarize the main results obtained to $O(m_\pi^4)$, including $O(a)$ [17] and $O(a^2)$ [18,19,20] artifacts, for tmLQCD in different regimes.

Regime $\hat{m}_q \gg O(a)$: The chiral phase of the vacuum is determined by the term $\propto \hat{m}_q$, up to negligible higher order corrections. $O(a)$ cutoff effects on pion masses and decay constants cancel in Wilson averages at generic $\hat{\omega}$ and are automatically absent at $\hat{\omega} = \omega = \pm\pi/2$. The pion mass splitting is $O(a^2)$, as expected. Lattice artifacts of order a^2/\hat{m}_q however arise from $O(a)$ terms in the chiral Lagrangian, e.g. in the pion masses [20].

Regime $\hat{m}_q \sim O(a)$: Once the $O(a)$ corrections to the untwisted quark mass (related to the intrinsic non-perturbative uncertainty on M_{cr}) are taken into account, the chiral phase of the vacuum is still determined by the term $\propto \hat{m}_q$ (up to small corrections). The concept of $O(a)$ improvement is of course no longer applicable.

Regime $\hat{m}_q \sim O(a^2)$: After taking into account the $O(a)$ corrections to the untwisted quark

mass, one finds that $O(a^2)$ and $O(\hat{m}_q)$ terms in the chiral Lagrangian compete with each other in determining the chiral phase of the vacuum. Depending on the sign of the coefficient, c_2 , of the term $\propto a^2$ in the chiral effective potential, one finds two possible scenarios ¹³, which extend to twisted mass $\hat{m}_2 \neq 0$ the ‘‘Aoki phase’’ ($c_2 > 0$) and ‘‘normal’’ ($c_2 < 0$) scenarios of ref. [8]. For $c_2 > 0$, the Aoki phase transitions are washed out into a crossover if \hat{m}_2 is non-zero. For $c_2 < 0$, the first order phase transition in the untwisted mass parameter \hat{m}_1 that occurs at $(\hat{m}_1, \hat{m}_2) = (0, 0)$ extends itself in the twisted mass plane, ending with two symmetrical second order points, $(\hat{m}_1, \hat{m}_2) = (0, \pm\hat{m}_{2c})$, where the neutral pion mass vanishes. Note that $\hat{m}_{2c} \sim |c_2|a^2$. The value of c_2 , which depends on (un)quenched, g_0^2 and many details of the lattice action, is closely related to pion mass splitting: $m_{\pi^0}^2 - m_{\pi^\pm}^2 \propto c_2 a^2$, with positive proportionality constant.

4. Numerical investigations

A first convincing numerical evidence in favour of $O(a)$ improvement via Wilson average and its remarkable simplicity in the case of tmLQCD with $\omega = \pm\pi/2$ was obtained in ref. [21]. There the scaling behaviour of the vector meson mass and the pseudoscalar decay constant was studied in quenched tmLQCD with two mass degenerate quarks for lattice spacings in the range $0.068 \div 0.123$ fm. All quantities were expressed in units of Sommer’s scale r_0 and, for each value of g_0^2 , m_0 was set to M_{cr} and m_2 was chosen so as to obtain $r_0 m_{\text{PS}} = 1.79$. The results were also compared with the corresponding ones for plain and clover-improved Wilson quarks.

An extension of this scaling study to lower quark masses (down to $r_0 m_{\text{PS}} \sim 0.6$) and few more lattice spacings is currently in progress [22] and should provide useful insights on the size of residual cutoff effects as a function of the quark mass. A comparative study of several hadronic observables for a wide range of quark masses, using tm and overlap fermions in the quenched approximation at $\beta = 5.85$, has also been presented [23]. Given the different chiral prop-

¹²Dependence on $\hat{\omega}$ disappears as $a \rightarrow 0$ at fixed \hat{m}_q .

¹³Higher order corrections are unlikely to alter the qualitative features of these scenarios [18,19].

erties of tm and overlap fermions, information on the continuum limit, coming from scaling tests, such as those of ref. [21,22], may be very beneficial for the interpretation of the results. The performance of linear solvers for quark propagators in the two different lattice formulations has also been studied at $\beta = 5.85$ for given volumes and pion masses [24].

A nice computation of the pion form factor, $F(Q^2)$, has been carried out in quenched tmLQCD at $\omega = \pi/2$, $\beta = 6.0$ and pion masses of about 660 MeV and 470 MeV [25]. The conserved (one-point split) isotriplet vector current V_μ^3 [1] was employed and the matrix element

$$\langle \pi^+(\mathbf{p}_f) | V_0^3 | \pi^+(\mathbf{p}_i) \rangle = F(Q^2) [E(\mathbf{p}_f) + E(\mathbf{p}_i)], \quad (31)$$

with $Q = p_f - p_i$, was $O(a)$ improved by averaging over opposite values of $\mathbf{p}_{i,f}$ [3]. Results agree with meson vector dominance and other $O(a)$ improved computations and cover a wide range of Q^2 -values.

The phase diagram of unquenched tmLQCD with two mass degenerate quarks (action (12)) in the plane (m_0, m_2) is currently under study as a function of g_0^2 and for different choices of the pure gauge action (plaquette and DBW2). Results with the plaquette YM action at $\beta = 5.2$ did reveal the presence of metastabilities in several observables (plaquette, pion mass, m_χ^{PCAC})¹⁴ for small values of m_2 and m_0 close to its critical value (the value where m_χ^{PCAC} should vanish) [5].

Long living metastable states (associated with values of m_χ^{PCAC} of different sign) were identified and the metastability in the plaquette was related to that in the untwisted condensate $\langle \bar{\chi}\chi \rangle$. These findings were interpreted as evidence for a segment of first order phase transition, i.e. the scenario arising from lattice ChPT in the regime $\hat{m}_q \sim O(a^2)$ for $c_2 < 0$. Metastabilities in unquenched simulations with Wilson-like quarks would thus be ultimately related to chiral violating cutoff effects and should be absent for $m_2 \neq 0$, if $c_2 > 0$, and for $|m_2| > m_{2c} \sim |c_2|a^2$, if $c_2 < 0$. Whenever $c_2 < 0$, the condition $|m_2| > m_{2c}$ can always be fulfilled by sufficiently decreasing a , but it would of course be desirable to have both c_2 small in modulus and a good a^2 -scaling behaviour already for $a \sim 0.10 \div 0.15$ fm. To what extent this can be achieved by simple modifications of the irrelevant terms of the lattice action is still an open question.

¹⁴See eq. (12) of ref. [5a] for the definition of m_χ^{PCAC} .

5. Weak matrix elements and tm quarks

Although tm Wilson quarks do not preserve full chiral symmetry, they can be used to compute many weak matrix elements with no or substantially reduced (with respect to standard Wilson fermions) operator mixings. To achieve this remarkable result, the key step is to choose the details of the UV regularization of the (typically four quark) operators on a case by case basis. More precisely, one can conveniently employ for the valence quarks in the operator of interest a flavour diagonal action of the form (25) and choose the value of the angles θ_f so as to maximally simplify operator renormalization. If the UV regularization chosen for the quarks in the operator (and the hadron interpolating fields) does not admit a positive defined fermionic determinant, in unquenched studies one can adopt a different regularization for the sea quarks, e.g. that provided by maximally twisted Wilson quarks (see eq. (20)). In this case, to make contact, as $a \rightarrow 0$, with continuum QCD¹⁵, the masses of the sea and valence quarks of the same flavour must be matched, for instance by imposing the equality of the renormalized current quark masses [13].

Within the quenched approximation, use of tm Wilson quarks and non-perturbative renormalization (in the SF scheme) has recently led to a precise computation of B_K , with no mixings and extrapolation of results to the continuum limit [26].

Moreover it has been remarked [14] that by using the clover improved version of the valence quark action (25), with $N_f = 4$, $\theta_u = -\theta_d = \pi/2$ and either $\theta_s = -\theta_c = \pi/2$ or $\theta_s = \theta_c = 0$, the renormalization of $K \rightarrow \pi$ matrix elements requires at most linearly divergent counterterms, which can be determined by enforcing parity.

A rather general strategy to make use of maximally twisted Wilson quarks for evaluating (un)quenched weak matrix elements with neither wrong chirality operator mixings nor $O(a)$ cutoff effects has been presented in ref. [13]. Besides using different (cleverly chosen) tm Wilson regularizations for sea and valence quarks, the approach is based on the remark that renormalizable lattice models with four sea quarks and a certain

¹⁵Here I am interested in the theory with massive quarks.

number of (possibly replicated) valence quarks, plus corresponding ghosts, yield, among others, operator matrix elements that coincide –in the continuum limit and provided the renormalized sea and valence quark masses are appropriately matched– with those of $N_f = 4$ Euclidean QCD. The method was illustrated in ref. [13] by discussing the evaluation of B_K as well as $K \rightarrow \pi\pi$ and $K \rightarrow \pi$ amplitudes. In ref. [27] it has been pointed out that it can be extended to static quarks and applied e.g. to the computation of B_B .

6. Conclusions

Lattice formulations of QCD with chirally twisted Wilson quarks provide a framework for non-perturbative numerical studies where spurious quark zero modes are absent and $O(a)$ cutoff effects, whenever present, can be removed either à la Symanzik or by a new simpler method. The magnitude of the residual $O(a^2)$ scaling violations in the small quark mass region is under investigation. In unquenched simulations a non-trivial phase diagram is also expected, and numerical studies of it are in progress. Finally, tmLQCD can be in various ways extended to mass non-degenerate quarks and adapted so as to make possible computations of matrix elements of the weak effective Hamiltonian with reduced or no mixings.

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REFERENCES

1. R. Frezzotti *et al.*, Nucl. Phys. **B** (Proc. Suppl.) **83** (2000) 941 and JHEP **0108** (2001) 058. For first tests, see: R. Frezzotti *et al.*, JHEP **0107** (2001) 048; M. Della Morte *et al.* [ALPHA coll.], JHEP **0110** (2001) 041.
2. R. Frezzotti, Nucl. Phys. **B** (Proc. Suppl.) **119** (2003) 140.
3. R. Frezzotti and G. C. Rossi, JHEP **0408** (2004) 007; Nucl. Phys. Proc. Suppl. **129** (2004) 880.
4. S. Aoki and A. Gocksch, Phys. Rev. Lett. **63** (1989) 1125; Phys. Lett. B **231** (1989) 449.
5. F. Farchioni *et al.*, hep-lat/0406039; hep-lat/0409098.
6. A.D. Kennedy, review talk at Lattice 2004 (to appear in these proceedings).
7. G. Immirzi and K. Yoshida, Nucl. Phys. **B210** (1982) 499.
8. S. Aoki, Phys. Rev. D **30** (1984) 2653; S. R. Sharpe and R. J. Singleton, Phys. Rev. D **58** (1998) 074501.
9. S. Aoki and O. Bar, hep-lat/0409006.
10. R. Frezzotti and G. C. Rossi, Nucl. Phys. Proc. Suppl. **128** (2004) 193.
11. K. Osterwalder and E. Seiler, Ann. of Phys. **110** (1978) 440.
12. W. Kerler, Phys. Rev. **D24** (1981) 1595; E. Seiler, I.O. Stamatescu, Phys. Rev. **D25** (1982) 2177 [Erratum-ibid. **D26** (1982) 534].
13. R. Frezzotti and G. C. Rossi, hep-lat/0407002.
14. C. Pena, S. Sint and A. Vladikas, hep-lat/0405028 and Nucl. Phys. **B** (Proc. Suppl.) **119** (2003) 368.
15. K. Symanzik, Nucl. Phys. **B226** (1983) 187 and 205; B. Sheikholeslami and R. Wohlert, Nucl. Phys. **B259** (1985) 572; G. Heatlie *et al.*, Nucl. Phys. **B352** (1991) 266; M. Lüscher *et al.*, Nucl. Phys. **B478** (1996) 365.
16. O. Bär, hep-lat/0409123 and references therein.
17. G. Münster and C. Schmidt, Europhys. Lett. **66** (2004) 652; G. Münster, C. Schmidt and E.E. Scholz, hep-lat/0402003 and hep-lat/0409066.
18. G. Münster, hep-lat/0407006.
19. S. Sharpe and J. Wu, hep-lat/0407025 and hep-lat/0407035.
20. L. Scorzato, hep-lat/0407023.
21. K. Jansen *et al.*, Phys.Lett. B **586** (2004) 432.
22. A. Shindler [for the XLF coll.], talk at Lattice 2004 (to appear in these proceedings).
23. W. Bietenholz *et al.*, hep-lat/0409109.
24. T. Chiarappa *et al.*, hep-lat/0409107.
25. A.M. Abdel-Rehim and R. Lewis, hep-lat/0408033.
26. P. Dimopoulos *et al.*, hep-lat/0409026.
27. M. Della Morte, hep-lat/0409012.