

Formulation of chiral gauge theories*

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We present a formulation of chiral gauge theories, which admits more general spectra of Dirac operators and reveals considerably more possibilities for the structure of the chiral projections. Our two forms of correlation functions both also apply in the presence of zero modes and for any value of the index. The decomposition of the total set of pairs of bases into equivalence classes is carefully analyzed. Transformation properties are derived.

1. CHIRAL PROJECTIONS

Starting from the basic structure of previous approaches to chiral gauge theories [1,2] we have recently presented a generalization [3] in which the Dirac operator and the chiral projections have been considered as functions of a certain unitary and γ_5 -Hermitian operator. Here we avoid the restrictions introduced by referring to such an operator by determining the possible structures of the chiral projections for given Dirac operator D .

For operators satisfying $[D^\dagger, D] = 0$ and $D^\dagger = \gamma_5 D \gamma_5$ we have the spectral representation

$$D = \sum_j \hat{\lambda}_j (P_j^+ + P_j^-) + \sum_k (\lambda_k P_k^{(I)} + \lambda_k^* P_k^{(II)}) \quad (1)$$

with $\text{Im } \hat{\lambda}_j = 0$ and $\text{Im } \lambda_k > 0$ and where $\gamma_5 P_j^\pm = P_j^\pm \gamma_5 = \pm P_j^\pm$ and $\gamma_5 P_k^{(I)} = P_k^{(II)} \gamma_5$. Since $\text{Tr}(\gamma_5 \mathbf{1}) = \text{Tr}(\gamma_5 P_k^{(I)}) = \text{Tr}(\gamma_5 P_k^{(II)}) = 0$ we get for $N_j^\pm = \text{Tr } P_j^\pm$

$$\sum_j (N_j^+ - N_j^-) = 0. \quad (2)$$

Associating $j = 0$ to zero modes the index of D is given by $I = N_0^+ - N_0^-$.

In contrast to the Dirac operators considered previously those in (1) are no longer restricted to one real eigenvalue in addition to zero and also admit more general complex ones. They have nevertheless appropriate realizations which also allow numerical evaluation [4].

For the chiral projections P_- and \bar{P}_+ the fundamental relation

$$\bar{P}_+ D = D P_- \quad (3)$$

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is required. Then because of $[P_-, DD^\dagger] = [\bar{P}_+, DD^\dagger] = 0$ we obtain the decomposition

$$P_- = \sum_j P_j^X + \sum_k P_k^R, \quad \bar{P}_+ = \sum_j \bar{P}_j^X + \sum_k \bar{P}_k^R \quad (4)$$

in which the projections P_k^R and \bar{P}_k^R are given by

$$P_k^R = c_k P_k^{(I)} + (1 - c_k) P_k^{(II)} - \sqrt{c_k(1 - c_k)} \gamma_5 (e^{i\varphi_k} P_k^{(I)} + e^{-i\varphi_k} P_k^{(II)}), \quad (5)$$

$$\bar{P}_k^R = c_k P_k^{(I)} + (1 - c_k) P_k^{(II)} + \sqrt{c_k(1 - c_k)} \gamma_5 (e^{-i\bar{\varphi}_k} P_k^{(I)} + e^{i\bar{\varphi}_k} P_k^{(II)}), \quad (6)$$

where $0 \leq c_k \leq 1$, $e^{i(\varphi_k + \bar{\varphi}_k - 2\alpha_k)} = -1$, $e^{i\alpha_k} = \lambda_k / |\lambda_k|$ and

$$\text{Tr } P_k^R = \text{Tr } \bar{P}_k^R = \text{Tr } P_k^{(I)} = \text{Tr } P_k^{(II)} =: \tilde{N}_k. \quad (7)$$

For the other projections, with $\bar{N} - N = I$ for $\bar{N} = \text{Tr } \bar{P}_+$ and $N = \text{Tr } P_-$, we get

$$\bar{P}_0^X = P_0^+, \quad P_0^X = P_0^-, \quad (8)$$

and have for $j \neq 0$ the two possibilities

$$\bar{P}_j^X = P_j^X = P_j^+ \quad \text{or} \quad \bar{P}_j^X = P_j^X = P_j^-. \quad (9)$$

With these relations for the chiral projections we see that, introducing $\text{Tr } \mathbf{1} = 2d$, we have

$$\bar{N} = d, \quad N = d - I \quad \text{or} \quad \bar{N} = d + I, \quad N = d \quad (10)$$

for the two choices in (9), respectively, and that

$$N = N_0^- + L, \quad \bar{N} = N_0^+ + L, \quad L = \sum_{j \neq 0} N_j^\pm + \sum_k \tilde{N}_k \quad (11)$$

holds, where \pm refers to such two choices.

2. CORRELATION FUNCTIONS

Non-vanishing fermionic correlation functions are given by

$$\langle \psi_{\sigma_{r+1}} \cdots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \cdots \bar{\psi}_{\bar{\sigma}_N} \rangle_f = \quad (12)$$

$$\frac{1}{r!} \sum_{\bar{\sigma}_1 \cdots \bar{\sigma}_r} \sum_{\sigma_1, \dots, \sigma_r} \bar{\Upsilon}_{\bar{\sigma}_1 \cdots \bar{\sigma}_N}^* \Upsilon_{\sigma_1 \cdots \sigma_N} D_{\bar{\sigma}_1 \sigma_1} \cdots D_{\bar{\sigma}_r \sigma_r}$$

with the alternating multilinear forms

$$\Upsilon_{\sigma_1 \cdots \sigma_N} = \sum_{i_1, \dots, i_N=1}^N \epsilon_{i_1, \dots, i_N} u_{\sigma_1 i_1} \cdots u_{\sigma_N i_N}, \quad (13)$$

$$\bar{\Upsilon}_{\bar{\sigma}_1 \cdots \bar{\sigma}_N} = \sum_{j_1, \dots, j_N=1}^{\bar{N}} \epsilon_{j_1, \dots, j_N} \bar{u}_{\bar{\sigma}_1 j_1} \cdots \bar{u}_{\bar{\sigma}_N j_N}, \quad (14)$$

in which the bases $\bar{u}_{\bar{\sigma}j}$ and $u_{\sigma i}$ satisfy

$$P_- = uu^\dagger, \quad u^\dagger u = \mathbb{1}_w, \quad \bar{P}_+ = \bar{u}\bar{u}^\dagger, \quad \bar{u}^\dagger \bar{u} = \mathbb{1}_{\bar{w}}. \quad (15)$$

While P_- and \bar{P}_+ are invariant under unitary basis transformations $u^{(S)} = uS$, $\bar{u}^{(\bar{S})} = \bar{u}\bar{S}$, the forms $\Upsilon_{\sigma_1 \cdots \sigma_N}$ and $\bar{\Upsilon}_{\bar{\sigma}_1 \cdots \bar{\sigma}_N}$ get multiplied by $\det_w S$ and $\det_{\bar{w}} \bar{S}$, respectively. Therefore, in order that all general correlation functions remain invariant we have to impose the condition

$$\det_w S \cdot \det_{\bar{w}} \bar{S}^\dagger = 1. \quad (16)$$

Without this condition all bases related to a chiral projection are connected by unitary transformations. With it the total set of pairs of bases u and \bar{u} decomposes into equivalence classes of which one is to be chosen to describe physics. Different equivalence classes are related by pairs of basis transformations with

$$\det_w S \cdot \det_{\bar{w}} \bar{S}^\dagger = e^{i\Theta}, \quad \Theta \neq 0. \quad (17)$$

The phase factor $e^{i\Theta}$ determines how the results of the respective formulations of the theory differ.

The relations obtained for the chiral projections imply ones for the bases, too. Thus from

$$\bar{P}_k^R = |\lambda_k|^{-2} D P_k^R D^\dagger \quad (18)$$

putting $P_k^R = \sum_{l=1}^{\bar{N}_k} u_l^{[k]} u_l^{[k]\dagger}$ we get

$$\bar{u}_l^{[k]} = e^{-i\Theta_k} |\lambda_k|^{-1} D u_l^{[k]} \quad (19)$$

with phases Θ_k so that $\bar{P}_k^R = \sum_{l=1}^{\bar{N}_k} \bar{u}_l^{[k]} \bar{u}_l^{[k]\dagger}$. For $P_j^\pm = \sum_{l=1}^{N_j^\pm} u_l^{\pm[j]} u_l^{\pm[j]\dagger} = \sum_{l=1}^{N_j^\pm} \bar{u}_l^{\pm[j]} \bar{u}_l^{\pm[j]\dagger}$ where $j \neq 0$ we have with phases Θ_j^\pm

$$\bar{u}_l^{\pm[j]} = e^{-i\Theta_j^\pm} |\hat{\lambda}_j|^{-1} D u_l^{\pm[j]}. \quad (20)$$

From (19) and (20) it becomes obvious that the $L \times L$ submatrix \check{M} of $\bar{u}^\dagger D u$, which occurs according to (11), has the eigenvalues

$$e^{i\Theta_k} |\lambda_k|, \quad e^{i\Theta_j^\pm} |\hat{\lambda}_j|, \quad (21)$$

with multiplicities \check{N}_k and N_j^\pm , respectively. Using \check{M} and introducing $P_0^- = \sum_{l=L+1}^N u_l u_l^\dagger$ and $P_0^+ = \sum_{l=L+1}^{\bar{N}} \bar{u}_l \bar{u}_l^\dagger$ for the zero mode part we find for the correlation functions the form

$$\langle \psi_{\sigma_{r+1}} \cdots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \cdots \bar{\psi}_{\bar{\sigma}_N} \rangle_f = \quad (22)$$

$$\sum_{\sigma'_{r+1}, \dots, \sigma'_N} \epsilon_{\sigma'_{r+1}, \dots, \sigma'_N} \sum_{\bar{\sigma}'_{r+1}, \dots, \bar{\sigma}'_{\bar{N}}} \epsilon_{\bar{\sigma}'_{r+1}, \dots, \bar{\sigma}'_{\bar{N}}} \frac{1}{(L-r)!}$$

$$\mathcal{G}_{\sigma'_{r+1} \bar{\sigma}'_{r+1}} \cdots \mathcal{G}_{\sigma'_L \bar{\sigma}'_L} e^{-i\theta_z^-} u_{\sigma_{L+1}, L+1} \cdots u_{\sigma_N N}$$

$$e^{i\theta_z^+} \bar{u}_{L+1, \bar{\sigma}_{L+1}}^\dagger \cdots \bar{u}_{\bar{N}, \bar{\sigma}_{\bar{N}}}^\dagger \det_L \check{M},$$

with $\mathcal{G} = \check{P}_- \check{D}^{-1} \check{P}_+$, where \check{D} , \check{P}_- , \check{P}_+ are the restrictions of the operators D , P_- , P_+ to the subspace on which $\mathbb{1} - P_0^+ - P_0^-$ projects. With θ_z^+ and θ_z^- being related to the zero-mode part, the equivalence class of pairs of bases is characterized by the value of

$$\sum_k N_k \Theta_k + \sum_{j \neq 0} N_j^\pm \Theta_j^\pm + \theta_z^+ - \theta_z^-. \quad (23)$$

3. GAUGE TRANSFORMATIONS

Conditions (15) and (16) determine the equivalence class of pairs of bases uS , $\bar{u}\bar{S}$. Gauge transformations $P'_- = \mathcal{T} P_- \mathcal{T}^\dagger$, $\bar{P}'_+ = \mathcal{T} \bar{P}_+ \mathcal{T}^\dagger$ for $[\mathcal{T}, P_-] \neq 0$, $[\mathcal{T}, \bar{P}_+] \neq 0$ imply that the transformed equivalence class is given by

$$u' S' = \mathcal{T} u S S', \quad \bar{u}' \bar{S}' = \mathcal{T} \bar{u} \bar{S} \bar{S}', \quad (24)$$

where u' , \bar{u}' , S' , \bar{S}' satisfy the transformed conditions (15) and (16), and where the unitary transformations $\mathcal{S}(\mathcal{T}, \mathcal{U})$ and $\bar{\mathcal{S}}(\mathcal{T}, \mathcal{U})$ with

$\det_{\mathbb{W}}\mathcal{S}(\mathbb{1},\mathcal{U})(\det_{\bar{\mathbb{W}}}\bar{\mathcal{S}}(\mathbb{1},\mathcal{U}))^* = 1$ are introduced for full generality. Insertion of (24) into (12) gives for the correlation functions

$$\langle \psi'_{\sigma'_1} \dots \psi'_{\sigma'_R} \bar{\psi}'_{\bar{\sigma}'_1} \dots \bar{\psi}'_{\bar{\sigma}'_R} \rangle_{\mathbb{f}} = \quad (25)$$

$$e^{i\vartheta_{\mathcal{T}}} \sum_{\sigma_1, \dots, \sigma_R} \sum_{\bar{\sigma}_1, \dots, \bar{\sigma}_R} \mathcal{T}_{\sigma'_1 \sigma_1} \dots \mathcal{T}_{\sigma'_R \sigma_R}$$

$$\langle \psi_{\sigma_1} \dots \psi_{\sigma_R} \bar{\psi}_{\bar{\sigma}_1} \dots \bar{\psi}_{\bar{\sigma}_R} \rangle_{\mathbb{f}} \mathcal{T}_{\bar{\sigma}_1 \bar{\sigma}'_1}^\dagger \dots \mathcal{T}_{\bar{\sigma}_R \bar{\sigma}'_R}^\dagger.$$

In this relation the factor

$$e^{i\vartheta_{\mathcal{T}}} = \det_{\mathbb{W}}\mathcal{S} \cdot \det_{\bar{\mathbb{W}}}\bar{\mathcal{S}}^\dagger \quad (26)$$

for $\vartheta_{\mathcal{T}} \neq 0$ has just the form met in (17) for the transformations to inequivalent subsets of pairs of bases. Thus to prevent arbitrary switching to different equivalence classes the condition $\vartheta_{\mathcal{T}} = 0$ is to be imposed. That this is to be done follows [4], on the other hand, also from the covariance requirement for the current in Ref. [2].

In the special case $[\mathcal{T}, P_-] \neq 0$, $[\mathcal{T}, \bar{P}_+] = 0$ the equivalence class can be represented [4] by pairs $uS, \bar{u}_c \bar{S}_c$ where \bar{u}_c and \bar{S}_c are independent of the gauge field so that instead of (24) we have

$$u'S' = \mathcal{T}uS\mathcal{S}, \quad \bar{u}_c \bar{S}_c = \text{const.} \quad (27)$$

Because of $[\mathcal{T}, \bar{P}_+] = 0$ it is now possible to rewrite $\bar{u}_c = \mathcal{T}\bar{u}_c \bar{S}_{\mathcal{T}}$. With this and (27) we get again the form (25), however, with

$$e^{i\vartheta_{\mathcal{T}}} = \det_{\mathbb{W}}\mathcal{S} \cdot \det_{\bar{\mathbb{W}}}\hat{\mathcal{S}}_{\mathcal{T}}^\dagger. \quad (28)$$

Here $\det_{\mathbb{W}}\mathcal{S} = 1$ remains to be required to prevent arbitrary switching to different equivalence classes. For the factor $\det_{\bar{\mathbb{W}}}\hat{\mathcal{S}}_{\mathcal{T}}^\dagger$ with $\mathcal{T} = \exp(\mathcal{B})$ we obtain the constant result

$$\det_{\bar{\mathbb{W}}}\hat{\mathcal{S}}_{\mathcal{T}}^\dagger = \exp\left(\frac{1}{2}\text{Tr}\mathcal{B}\right). \quad (29)$$

It should be noted that in the continuum limit certain compensations of terms present on the lattice disappear so that one arrives just at the usual features of continuum perturbation theory [3,4].

4. CP TRANSFORMATIONS

For CP transformations of the chiral projections we have

$$P_-^{\text{CP}}(\mathcal{U}^{\text{CP}}) = \mathcal{W}\bar{P}_+^{\text{T}}(\mathcal{U})\mathcal{W}^\dagger, \quad (30)$$

$$\bar{P}_+^{\text{CP}}(\mathcal{U}^{\text{CP}}) = \mathcal{W}P_-^{\text{T}}(\mathcal{U})\mathcal{W}^\dagger, \quad (31)$$

with $\mathcal{W} = \mathcal{P}\gamma_4 C^\dagger$, $\mathcal{P}_{n'n} = \delta_{\tilde{n}'\tilde{n}}^4$, $U_{4n}^{\text{CP}} = U_{4\tilde{n}}^*$ and $U_{kn}^{\text{CP}} = U_{k,\tilde{n}-\hat{k}}^*$ for $k = 1, 2, 3$, where $\tilde{n} = (-\vec{n}, n_4)$.

Writing $P_-(\mathcal{U})$ and $\bar{P}_+(\mathcal{U})$ in the form

$$P_- = \frac{1}{2}(\mathbb{1} - \gamma_5 G), \quad \bar{P}_+ = \frac{1}{2}(\mathbb{1} + \bar{G}\gamma_5) \quad (32)$$

we get for $P_-^{\text{CP}}(\mathcal{U}^{\text{CP}})$ and $\bar{P}_+^{\text{CP}}(\mathcal{U}^{\text{CP}})$

$$P_-^{\text{CP}} = \frac{1}{2}(\mathbb{1} - \gamma_5 \bar{G}), \quad \bar{P}_+^{\text{CP}} = \frac{1}{2}(\mathbb{1} + G\gamma_5). \quad (33)$$

Obviously the transformed projections differ by an interchange of G and \bar{G} , in which context it is to be noted that generally $\bar{G} \neq G$ holds [4].

With the conditions (15) and (16) satisfied by u, \bar{u}, S, \bar{S} as well as by $u^{\text{CP}}, \bar{u}^{\text{CP}}, S^{\text{CP}}, \bar{S}^{\text{CP}}$, the equivalence class of pairs of bases transforms as

$$u^{\text{CP}} S^{\text{CP}} = \mathcal{W}\bar{u}^* \bar{S}^* S_\zeta, \quad \bar{u}^{\text{CP}} \bar{S}^{\text{CP}} = \mathcal{W}u^* S^* \bar{S}_\zeta \quad (34)$$

where the unitary transformations S_ζ and \bar{S}_ζ are introduced for full generality. Inserting this into (12) we get for the correlation functions

$$\langle \psi_{\sigma'_1}^{\text{CP}} \dots \psi_{\sigma'_R}^{\text{CP}} \bar{\psi}_{\bar{\sigma}'_1}^{\text{CP}} \dots \bar{\psi}_{\bar{\sigma}'_R}^{\text{CP}} \rangle_{\mathbb{f}}^{\text{CP}} = \quad (35)$$

$$e^{i\vartheta_{\text{CP}}} \sum_{\sigma_1, \dots, \sigma_R} \sum_{\bar{\sigma}_1, \dots, \bar{\sigma}_R} \mathcal{W}_{\bar{\sigma}_1 \bar{\sigma}'_1}^\dagger \dots \mathcal{W}_{\bar{\sigma}_R \bar{\sigma}'_R}^\dagger$$

$$\langle \psi_{\sigma_1} \dots \psi_{\sigma_R} \bar{\psi}_{\bar{\sigma}_1} \dots \bar{\psi}_{\bar{\sigma}_R} \rangle_{\mathbb{f}} \mathcal{W}_{\sigma'_1 \sigma_1} \dots \mathcal{W}_{\sigma'_R \sigma_R}.$$

Here the factor

$$e^{i\vartheta_{\text{CP}}} = \det_{\bar{\mathbb{W}}}\mathcal{S}_\zeta \cdot \det_{\mathbb{W}}\bar{\mathcal{S}}_\zeta^\dagger \quad (36)$$

is subject to the condition that repetition of the transformation must always lead back, which is satisfied by restricting S_ζ and \bar{S}_ζ to choices for which ϑ_{CP} is a universal constant. Then the resulting factor gets irrelevant in full correlation functions so that one may put $\vartheta_{\text{CP}} = 0$.

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