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Locality bound for effective four-dimensional action of domain-wall fermion

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Abstract

We discuss locality in the domain-wall QCD through the effective four-dimensional Dirac operator which is defined by the transfer matrix of the five-dimensional Wilson fermion. We first derive an integral representation for the effective operator, using the inverse five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition in the fifth direction. Exponential bounds are obtained from it for gauge fields with small lattice field strength.

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1 Introduction

Locality properties of Neuberger's lattice Dirac operator [1], which is derived from the overlap formalism [2] and satisfies the Ginsparg-Wilson relation [3], has been examined by Hernándes, Jansen and Lüscher [4]. For a certain class of gauge fields with small lattice field strength, exponential bounds have been proved rigorously on the kernels of the Dirac operator and its differentiations with respect to the gauge field. These properties assure that the index theorem holds true on the lattice [5, 2]. The index theorem implies the topological properties of chiral anomalies. It plays the crucial role in the recent construction of lattice chiral gauge theories [6, 7, 8, 9, 10, 11]. Numerical studies of the locality of Neuberger's lattice Dirac operator are also found in [4, 12].

The purpose of this paper is to argue the locality of the low energy effective action of the domain-wall fermion [13, 14, 15] [16, 17] [18] [19] [20, 21, 22] [23, 24] [25]. It has been known that the partition function of the domain-wall fermion, in the anti-periodic subtraction scheme [18], reduces to a single determinant of an effective four-dimensional Dirac operator [26],

$$D_{\rm eff}^{(N)} = \frac{1}{2a}\left(1+\gamma_5 \tanh \frac{N}{2} a_5 \widetilde{H}\right). \tag{1.1}$$

Here N and a_5 denote the lattice size and the lattice spacing of the fifth dimension, respectively. $\widetilde{H}$ is defined through the transfer matrix of the five-dimensional Wilson fermion

$$T = e^{-a_5 \tilde{H}} = \left(\begin{array}{cc} \frac{1}{B} & -\frac{1}{B} C \\ -C^\dagger \frac{1}{B} & B + C^\dagger \frac{1}{B} C \end{array} \right), \tag{1.2}$$

where [1]

$$\begin{aligned} C &= a_5\, \sigma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right), && (1.3)\\ B &= 1 + a_5\left(-\frac{a}{2}\nabla_\mu \nabla_\mu^* - \frac{m_0}{a}\right). && (1.4)\end{aligned}$$

The limit $N\to\infty$ is defined well as long as $\widetilde{H}^2 > 0$. The effective Dirac operator Eq. (1.1) then reduces to Neuberger's lattice Dirac operator using $\widetilde{H}$,

$$D_{\rm eff} = \frac{1}{2a}\left(1+\gamma_5 \frac{\widetilde{H}}{\sqrt{\widetilde{H}^2}}\right), \tag{1.5}$$

[1] In this expression, the positivity of B is required for the transfer matrix to be defined consistently. It is assured when $0 < \frac{a_5}{a} m_0 < 1$. It is also assumed that N is even.

and turns out to satisfy the Ginsparg-Wilson relation.

Moreover, the propagator of the light fermion field which is introduced by Furman and Shamir [15],

$$q(x) = \psi_L(x,1) + \psi_R(x,N), \qquad \bar{q}(x) = \bar{\psi}_L(x,1) + \bar{\psi}_R(x,N), \tag{1.6}$$

can be expressed in terms of the effective Dirac operator [27]:

$$\langle q(x)\bar{q}(y)\rangle = \frac{a_5}{a^4}\left(\frac{1}{a}D_{\mathrm{eff}}^{(N)^{-1}} - \delta(x,y)\right). \tag{1.7}$$

The anomalous term in the axial Ward-Takahashi identity

$$X^{(N)}(x) = \frac{2}{a_5}\left\{\bar{\psi}_L(x,\frac{N}{2}+1)\psi_R(x,\frac{N}{2}) - \bar{\psi}_R(x,\frac{N}{2})\psi_L(x,\frac{N}{2}+1)\right\} \tag{1.8}$$

can also be expressed with it:

$$\begin{aligned} a^4\left\langle X^{(N)}(x)\right\rangle &= \mathrm{tr}\gamma_5\, 2\left(1 - aD_{\mathrm{eff}}^{(N)}\right)(x,x) \\ &\quad -\mathrm{tr}\left(\frac{1}{a}D_{\mathrm{eff}}^{(N)^{-1}}\gamma_5\frac{1}{\cosh^2\frac{N}{2}a_5\widetilde{H}}\right)(x,x). \end{aligned} \tag{1.9}$$

In the limit $N \to \infty$, this reduces to the chiral anomaly associated with the exact chiral symmetry [28, 29, 30, 31, 32, 33],

$$a^4\langle X(x)\rangle = \mathrm{tr}\gamma_5\, 2\left(1 - aD_{\mathrm{eff}}\right)(x,x). \tag{1.10}$$

It would have the topological properties, if the effective Dirac operator is local and depends smoothly on the gauge fields.

In view of this direct relation,[2] it seems reasonable to argue locality in the domain-wall fermion approach through the locality properties of the effective Dirac operators Eq. (1.1) and Eq. (1.5). It is expected that a similar exponential bound could be established under certain conditions, like the result obtained by Hernández, Jansen and Lüscher [4]. In our case,

[2] In Eq. (1.7) the propagator of the boundary variables turns out to be chiral invariant in the limit $N \to \infty$ due to the negative contact term. In this respect, it is important to note the contribution of massive modes (including the Pauli-Villars modes), which takes account of the chiral anomaly in Eqs. (1.8) and (1.9) and fills the gap to the Ginsparg-Wilson fermion [27]. Following [28, 34, 35], we may also introduce an "heavy" auxiliary field $a^4\sum_x \bar{\chi}(x)\chi(x)$ and redefine the boundary variables as $q'(x) = q(x) + \chi(x)$, $\bar{q}'(x) = \bar{q}(x) + \bar{\chi}(x)$, so that the contact term in the propagator is removed. It is also pointed out in [36] that a certain modification of the action of the domain-wall fermion leads directly to the propagotor of the boundary variables which satisfies the Ginsparg-Wilson relation.

though, the hermitian Wilson-Dirac operator should be replaced with $\widetilde{H}$. Then the use of the Legendre polynomials [4] does not lead to the expansion in terms of operators with finite ranges.

For our purpose, we first derive an integral representation for the effective Dirac operator. The inverse square root in Neuberger's lattice Dirac operator can be written by the integral:

$$\frac{1}{2}\left(1+\gamma_5\frac{H}{\sqrt{H^2}}\right)=\frac{1}{2}+\gamma_5\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{1}{i\gamma_5 p+\left(D_{\mathrm{w}}-\frac{m_0}{a}\right)}\gamma_5. \tag{1.11}$$

Corresponding to this, we can show that the effective Dirac operator admits the following representation [27]:

$$\begin{aligned} aD_{\mathrm{eff}}^{(N)} &= 1-P_R\left\{a_5\overline{D}_{5\mathrm{w}}\right\}^{-1}_{NN}P_L-P_L\left\{a_5\overline{D}_{5\mathrm{w}}\right\}^{-1}_{11}P_R \\ &\quad -P_R\left\{a_5\overline{D}_{5\mathrm{w}}\right\}^{-1}_{N1}P_R-P_L\left\{a_5\overline{D}_{5\mathrm{w}}\right\}^{-1}_{1N}P_L, \end{aligned} \tag{1.12}$$

where $\overline{D}_{5\mathrm{w}}$ is the five-dimensional Wilson-Dirac operator *with the anti-periodic boundary condition* in the fifth-dimension. Its inverse may be expressed as

$$\left\{a_5\overline{D}_{5\mathrm{w}}\right\}^{-1}_{st}=\frac{1}{N}\sum_p\frac{e^{ip(s-t)}}{i\gamma_5\sin p+1-\cos p+a_5\left(D_{\mathrm{w}}-\frac{m_0}{a}\right)}. \tag{1.13}$$

The summation is taken over the discrete momenta $p=\frac{2\pi}{N}(k-\frac{1}{2})$ $(k=1,2,\cdots,N)$ and in the limit $N\to\infty$ it reduces to the continuous integral.

From this representation, it is rather clear that the effective Dirac operator can be defined consistently if the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition is not singular and invertible for all N. In this respect, we should note that the lower bound on the square of the five-dimensional Wilson-Dirac operator is related closely to that on the square of the four-dimensional Wilson-Dirac operator [4, 37], because the gauge field is four-dimensional. In fact, the same lower bound can be set for the class of gauge fields with small lattice field strength.

Given the positive lower bound on the square of the five-dimensional Wilson-Dirac operator, it is possible to formulate a series expansion in terms of the five-dimensional Wilson-Dirac operator, using the generating function of the Chebycheff polynomials [38]. The exponential bounds on the effective Dirac operator and its differentiations can be established from it.

We may also discuss the Ginsparg-Wilson relation of the effective Dirac operator through this integral representation. We will see that this reduces to the question concerning the property of the five-dimensional Dirac operator under the chiral transformation introduced by Furman and Shamir [15].

Another interesting aspect of the effective Dirac operator is its behavior in case with the singular gauge configuration for which isolated eigenvalues of the hermitian Wilson-Dirac operator collapse to zero. In [4], it has been proved rigorously that Neuberger's Dirac operator in terms of the hermitian Wilson-Dirac operator remains local even with such singular gauge configurations. We will argue that it is also true for the effective Dirac operator Eq. (1.5).

The approach to the chiral symmetry limit from a finite N is the most important issue for the practical implementation of exact chiral symmetry using the domain-wall fermion [39, 40] [19, 22, 24] [41, 42, 43]. In this respect, our result of the locality and exact chiral symmetry of the effective four-dimensional action is restricted for the class of gauge fields with small lattice field strength. It is a nonperturbative result and it gives a sufficient condition for that the exact chiral symmetry based on the Ginsparg-Wilson relation can be implemented using the domain-wall fermion. But it does not assure that it would work practically in the numerical simulations using the standard Wilson's gauge action. Our result, however, presents an explicit method to connect the locality and chiral symmetry properties of the domain-wall fermion to the spectrum of the four-dimensional Wilson-Dirac operator. We hope that such a method would be useful in order to study the above practical issue.

This paper is organized as follows. In section 2, we briefly review the domain-wall fermion in order to fix our notation. In section 3, we describe how to derive the integral representation for the effective Dirac operator. We also discuss how the Ginsparg-Wilson relation for the effective Dirac operator follows in this integral representation. In section 4, we discuss the positivity of the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition. With this result, we consider exponential bounds for the effective Dirac operator and its differentiations in section 5. We also discuss the locality in the case with the singular gauge configurations. In section 6, we summarize our result and give some discussions concerning the issue of the approach to the chiral symmetry limit.

2 Domain-wall fermion in the anti-periodic subtraction scheme

In this section, we review the domain-wall fermion [13, 14, 15] and fix our notation. The domain-wall fermion is defined by the five-dimensional Wilson-Dirac fermion with the Dirichlet boundary condition.

$$S_{\rm DW} = \sum_{t=1}^{N} a^4 \sum_x \bar{\psi}(x,t) D_{5\rm w} \psi(x,t), \tag{2.1}$$

$$D_{5\rm w} = \gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right)\delta_{st} + P_L M_{st} + P_R M_{st}^\dagger. \tag{2.2}$$

We assume the lattice size of the fifth dimension N is even. For $N=6$, the mass matrix reads

$$M_{st} = \frac{1}{a_5}\begin{pmatrix} B & -1 & 0 & 0 & 0 & 0 \\ 0 & B & -1 & 0 & 0 & 0 \\ 0 & 0 & B & -1 & 0 & 0 \\ 0 & 0 & 0 & B & -1 & 0 \\ 0 & 0 & 0 & 0 & B & -1 \\ 0 & 0 & 0 & 0 & 0 & B \end{pmatrix}, \tag{2.3}$$

where B is defined by

$$B = 1 + a_5\left(-\frac{a}{2}\nabla_\mu \nabla_\mu^* - \frac{m_0}{a}\right). \tag{2.4}$$

The chiral transformation is introduced as vector-like one so that the symmetry breaking is minimized [15]:

$$\delta\psi(x,t) = -\psi(x,t) \qquad t \le \frac{N}{2}, \tag{2.5}$$

$$\delta\psi(x,t) = +\psi(x,t) \qquad t \ge \frac{N}{2}+1. \tag{2.6}$$

Accordingly, the anomalous term is given by

$$X^{(N)}(x) = \frac{2}{a_5}\left\{\bar{\psi}_L(x,\frac{N}{2}+1)\psi_R(x,\frac{N}{2}) - \bar{\psi}_R(x,\frac{N}{2})\psi_L(x,\frac{N}{2}+1)\right\} \tag{2.7}$$

The partition function of the domain-wall fermion may be defined with the subtraction of the Pauli-Villars fields, which is subject to the anti-periodic boundary condition in the fifth dimension. [3]

$$Z_{\mathrm{DW}} = \frac{\det D_{5\mathrm{w}}}{\det \overline{D}_{5\mathrm{w}}}, \tag{2.9}$$

where

$$\overline{D}_{5\mathrm{w}} = \gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right)\delta_{st} + P_L \overline{M}_{st} + P_R \overline{M}^\dagger_{st}, \tag{2.10}$$

$$\overline{M}_{st} = \frac{1}{a_5}\begin{pmatrix} B & -1 & 0 & 0 & 0 & 0 \\ 0 & B & -1 & 0 & 0 & 0 \\ 0 & 0 & B & -1 & 0 & 0 \\ 0 & 0 & 0 & B & -1 & 0 \\ 0 & 0 & 0 & 0 & B & -1 \\ 1 & 0 & 0 & 0 & 0 & B \end{pmatrix}, \qquad (N=6) \tag{2.11}$$

For later convenience, we perform a chirally asymmetric parity transformation in the fifth dimension:

$$\psi(x,t) = \left(P_R + P_L P\right)_{ts} \psi'(x,s), \tag{2.12}$$
$$\bar\psi(x,t) = \bar\psi'(x,s)\left(P_R P + P_L\right)_{st}, \tag{2.13}$$

where

$$P_{st} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (N=6). \tag{2.14}$$

Accordingly, the five-dimensional Dirac operators are transformed as follows:

$$D'_{5\mathrm{w}} = \left(P_R P + P_L\right) D_{5\mathrm{w}} \left(P_R + P_L P\right) \tag{2.15}$$
$$= \gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right) P_{st} + M^H_{st}, \tag{2.16}$$
$$\overline{D}'_{5\mathrm{w}} = \left(P_R P + P_L\right) \bar D_{5\mathrm{w}} \left(P_R + P_L P\right) \tag{2.17}$$
$$= \gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right) P_{st} + \overline{M}^H_{st}, \tag{2.18}$$

[3]For this subtraction to work consistently, we should require the positivity of $\overline{D}_{5\mathrm{w}}$:

$$\overline{D}^\dagger_{5\mathrm{w}} \overline{D}_{5\mathrm{w}} > 0. \tag{2.8}$$

We will see later that this requirement also assures the locality and the Ginsparg-Wilson relation of the effective Dirac operator.

where, for $N = 6$,

$$
\begin{aligned}
M^{\mathrm{H}}_{st} &= M_{st} P = P M^{\dagger}_{st} \\
&= \frac{1}{a_5} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & B \\ 0 & 0 & 0 & -1 & B & 0 \\ 0 & 0 & -1 & B & 0 & 0 \\ 0 & -1 & B & 0 & 0 & 0 \\ -1 & B & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{2.19}
$$

$$
\overline{M}_{st} = \frac{1}{a_5} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & B \\ 0 & 0 & 0 & -1 & B & 0 \\ 0 & 0 & -1 & B & 0 & 0 \\ 0 & -1 & B & 0 & 0 & 0 \\ -1 & B & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\tag{2.20}
$$

In this basis, the chiral transformation adopted by Shamir and Furman [15] can be expressed as follows:

$$
\delta\psi'_s(x) = (\Gamma_5)_{st}\, \psi'_t(x), \tag{2.21}
$$

where Γ_5 is given (for $N = 6$) by

$$
(\Gamma_5)_{st} = \begin{pmatrix} -\gamma_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_5 \end{pmatrix} \qquad (N = 6). \tag{2.22}
$$

With this definition of the chiral transformation, $D'_{5\mathrm{w}}$ and $\overline{D}'_{5\mathrm{w}}$ satisfy the following identities, respectively,

$$
\left\{\Gamma_5 D'_{5\mathrm{w}} + D'_{5\mathrm{w}}\Gamma_5\right\}_{st} = \frac{2}{a_5}\,\gamma_5 \delta_{s\frac{N}{2}}\delta_{t\frac{N}{2}}, \tag{2.23}
$$

$$
\left\{\Gamma_5 \overline{D}'_{5\mathrm{w}} + \overline{D}'_{5\mathrm{w}}\Gamma_5\right\}_{st} = \frac{2}{a_5}\,\gamma_5 \delta_{s\frac{N}{2}}\delta_{t\frac{N}{2}} + \frac{2}{a_5}\,\gamma_5 \delta_{sN}\delta_{tN}. \tag{2.24}
$$

The chiral symmetry breaking occurs at $t=\frac{N}{2}$ in the five-dimensional Dirac operator for the domain-wall fermion. On the other hand, it occurs both at $t=\frac{N}{2}$ and at $t=N$ for the Pauli-Villars field, because of the anti-periodic boundary condition.

3 Effective four-dimensional Dirac operator

3.1 An integral representation of the effective four-dimensional Dirac operator

The functional determinant of the domain-wall fermion, in the anti-periodic subtraction scheme, reduces to a single determinant of a four-dimensional Dirac operator,

$$\frac{\det D_{5\mathrm{w}}}{\det \overline{D}_{5\mathrm{w}}} = \det aD_{\mathrm{eff}}^{(N)}. \tag{3.1}$$

In this section, we reproduce this result and derive an integral representation for the effective four-dimensional Dirac operator.

We may write the partition function as follows:

$$\frac{\det D_{5\mathrm{w}}}{\det \overline{D}_{5\mathrm{w}}} = \frac{\det D'_{5\mathrm{w}}}{\det \overline{D}'_{5\mathrm{w}}} = \det\left(D'_{5\mathrm{w}}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1}\right). \tag{3.2}$$

Then, we note a simple relation between two five-dimensional Wilson-Dirac operators:

$$D'_{5\mathrm{w}} = \overline{D}'_{5\mathrm{w}} - \frac{1}{a_5}\delta_{sN}\delta_{Nt}. \tag{3.3}$$

This relation implies that

$$D'_{5\mathrm{w}}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1} = \delta_{st} - \frac{1}{a_5}\delta_{sN}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1}_{1t}. \tag{3.4}$$

Since this matrix is lower triangle in the lattice indices of the fifth dimension, we can easily see that its determinant reduces to a single four-dimensional determinant:

$$\det\left(D'_{5\mathrm{w}}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1}\right) = \det\left(1-\frac{1}{a_5}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1}_{NN}\right). \tag{3.5}$$

From this result, we may set

$$\begin{aligned} aD_{\mathrm{eff}}^{(N)} &= 1-\frac{1}{a_5}\left\{\overline{D}'_{5\mathrm{w}}\right\}^{-1}_{NN} && (3.6)\\ &= 1-\frac{1}{a_5}\Big(P_R\left\{\overline{D}_{5\mathrm{w}}\right\}^{-1}_{NN}P_L + P_L\left\{\overline{D}_{5\mathrm{w}}\right\}^{-1}_{11}P_R \\ &\qquad + P_R\left\{\overline{D}_{5\mathrm{w}}\right\}^{-1}_{N1}P_R + P_L\left\{\overline{D}_{5\mathrm{w}}\right\}^{-1}_{1N}P_L\Big). && (3.7)\end{aligned}$$

Thus the effective four-dimensional Dirac operator can be expressed in terms of the inverse of the five-dimensional Wilson-Dirac operator *with the anti-periodic boundary condition.* Since the gauge field is four-dimensional, the inverse of this five-dimensional Wilson-Dirac operator may be expressed as follows:

$$\left\{a_5 \overline{D}_{5\mathrm{w}}\right\}^{-1}_{st} = \frac{1}{N}\sum_{p} e^{ip(s-t)} \left\{ i\gamma_5 \sin p + 1 - \cos p + a_5 \left(D_{\mathrm{w}} - \frac{m_0}{a}\right)\right\}^{-1}, \tag{3.8}$$

where the summation is taken over the discrete momenta $p = \frac{2\pi}{N}(k - \frac{1}{2})$ $(k = 1, 2, \cdots, N)$ and D_{w} is the four-dimensional Wilson-Dirac operator

$$D_{\mathrm{w}} = \sum_{\mu} \left\{ \gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla^*_\mu\right) - \frac{a}{2}\nabla_\mu \nabla^*_\mu \right\}. \tag{3.9}$$

Then the effective Dirac operator may be expressed as follows:

$$\begin{aligned} aD^{(N)}_{\mathrm{eff}} \; = \; & 1 - P_R \frac{1}{N}\sum_p \frac{1}{i\gamma_5 \sin p + 1 - \cos p + a_5\left(D_{\mathrm{w}} - \frac{m_0}{a}\right)} P_L \\ & - P_L \frac{1}{N}\sum_p \frac{1}{i\gamma_5 \sin p + 1 - \cos p + a_5\left(D_{\mathrm{w}} - \frac{m_0}{a}\right)} P_R \\ & + P_R \frac{1}{N}\sum_p \frac{e^{-ip}}{i\gamma_5 \sin p + 1 - \cos p + a_5\left(D_{\mathrm{w}} - \frac{m_0}{a}\right)} P_R \\ & + P_L \frac{1}{N}\sum_p \frac{e^{+ip}}{i\gamma_5 \sin p + 1 - \cos p + a_5\left(D_{\mathrm{w}} - \frac{m_0}{a}\right)} P_L . \end{aligned} \tag{3.10}$$

In the limit $N \to \infty$, the summation over the discrete momentum reduces to the continuous integral:

$$\frac{1}{N} \sum_{p = \frac{2\pi}{N}(k - \frac{1}{2})} \Longrightarrow \int_{-\pi}^{\pi} \frac{dp}{2\pi}. \tag{3.11}$$

Note that, since we do not use the transfer matrix in this derivation, this expression could hold true even if B is not positive definite and the transfer matrix is not defined consistently. m_0 can be chosen as any value within $m_0 \in [0, 2]$ (when $a_5 = a$), as long as the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition in the fifth dimension is not singular and invertible.

3.2 The Ginsparg-Wilson relation

Next we discuss the Ginsparg-Wilson relation for the effective Dirac operator in the integral representation. As we have seen in the previous subsection, the effective Dirac operator, $D_{\rm eff}$, is defined by

$$a D_{\rm eff} = 1 - \frac{1}{a_5} \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{NN} \qquad (N = \infty). \tag{3.12}$$

If it would satisfies the Ginsparg-Wilson relation

$$\gamma_5 D_{\rm eff} + D_{\rm eff} \gamma_5 = 2a D_{\rm eff} \gamma_5 D_{\rm eff}, \tag{3.13}$$

then the following identity must hold true in the limit of $N \to \infty$:

$$\gamma_5 \left\{ \bar{D}'_{5{\rm w}} \right\}^{-1}_{NN} + \left\{ \bar{D}'_{5{\rm w}} \right\}^{-1}_{NN} \gamma_5 = \frac{2}{a_5} \left\{ \bar{D}'_{5{\rm w}} \right\}^{-1}_{NN} \gamma_5 \left\{ \bar{D}'_{5{\rm w}} \right\}^{-1}_{NN} \quad (N = \infty). \tag{3.14}$$

We may compare this identity with Eq. (2.24) which express the chiral property of $\bar{D}'_{5{\rm w}}$ under the chiral transformation introduced by Furman and Shamir Eq. (2.21). The latter we may write

$$\begin{aligned} \left\{ \Gamma_5 \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1} + \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1} \Gamma_5 \right\}_{st} \;=\; & \frac{2}{a_5} \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{s\frac{N}{2}} \gamma_5 \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{\frac{N}{2}t} \\ & + \frac{2}{a_5} \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{sN} \gamma_5 \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{Nt} . \end{aligned} \tag{3.15}$$

Setting $s = t = N$, we obtain

$$\begin{aligned} \gamma_5 \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{NN} + \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{NN} \gamma_5 \;=\; & \frac{2}{a_5} \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{N\frac{N}{2}} \gamma_5 \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{\frac{N}{2}N} \\ & + \frac{2}{a_5} \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{NN} \gamma_5 \left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{NN} . \end{aligned} \tag{3.16}$$

Then we can see that Eq. (3.14) is equivalent to the following condition in the limit of $N \to \infty$:

$$\left\{ \overline{D}'_{5{\rm w}} \right\}^{-1}_{N\frac{N}{2}} = 0 \qquad (N \to \infty). \tag{3.17}$$

As we will see below, this condition is fulfilled as long as the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition is not singular and invertible. Then we obtain Eq. (3.14) and Eq. (3.13), the Ginsparg-Wilson relation for the effective Dirac operator.

4 Positivity of the square of the five-dimensional Wilson-Dirac operator with anti-periodic boundary condition

From Eqs. (3.10) and its $N \to \infty$ limit, we see that for the effective four-dimensional Dirac operator to be defined consistently, it is required that the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition should be non-singular and invertible. In this section, we examine the positivity of the five-dimensional Wilson-Dirac operator square.

To examine this requirement, we evaluate the square of the five-dimensional Wilson-Dirac operator. Setting $a_5 = a$ for simplicity, we have

$$\begin{aligned} &\{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - m_0)\}^\dagger \{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - m_0)\} \\ &= 4\sin^2(p/2)\left(1 - m_0 - \frac{a^2}{2}\nabla_\mu \nabla_\mu^*\right) + (aD_{\mathrm{w}} - m_0)^\dagger (aD_{\mathrm{w}} - m_0). \end{aligned} \tag{4.1}$$

For $m_0 = 1$, the first term is positive semi-definite and then the positivity of $a^2 \overline{D}_{5\mathrm{w}}^\dagger \overline{D}_{5\mathrm{w}}$ is entirely determined by the positivity of the four-dimensional Wilson-Dirac operator square, $(aD_{\mathrm{w}} - 1)^\dagger (aD_{\mathrm{w}} - 1)$. According to the result of [4], if the plaquette variables $U(p)$ are uniformly bounded as

$$\| 1 - U(p) \| < \epsilon, \tag{4.2}$$

we obtain

$$\begin{aligned} &\| \{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - 1)\}^\dagger \{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - 1)\} \| \\ &= \| 4\sin^2(p/2)\left(-\frac{a^2}{2}\nabla_\mu \nabla_\mu^*\right) + (aD_{\mathrm{w}} - 1)^\dagger (aD_{\mathrm{w}} - 1) \| \\ &\geq 1 - 30\epsilon. \end{aligned} \tag{4.3}$$

For the generic value of $m_0 \in [0, 2]$, we also obtain [44, 45, 37]

$$\begin{aligned} &\| \{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - m_0)\}^\dagger \{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - m_0)\} \| \\ &\geq \left\{(1 - 30\epsilon)^{\frac{1}{2}} - |1 - m_0|\right\}^2 \qquad \text{if} \quad 1 - 30\epsilon > |1 - m_0|^2. \end{aligned} \tag{4.4}$$

Recently, it has been shown by Neuberger [37] that the constant 30 in the above bounds can be improved to $6(2 + \sqrt{2})$.

From these considerations, we may assume the positive lower and upper bounds of the square of the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition as

$$0 < \tilde{\alpha} \leq \left\{4\sin^2(p/2)\, B + (aD_{\mathrm{w}} - m_0)^\dagger (aD_{\mathrm{w}} - m_0)\right\} \leq \tilde{\beta}, \tag{4.5}$$

under the following condition,

$$\parallel 1 - U(p) \parallel < \epsilon, \qquad \epsilon < \frac{1}{6(2+\sqrt{2})}\left(1 - |1-m_0|^2\right). \tag{4.6}$$

5 Expansion with Chebycheff polynomials and an exponential bound

Given the bounds on the square of the five-dimensional Wilson- Dirac operator with the anti-periodic boundary condition, we will derive in this section exponential bounds on the effective four-dimensional Dirac operator and its differentiations with respect to the gauge field. We will also obtain an exponential bound on the inverse of the five-dimensional Wilson-Dirac operator which is needed to prove Eq. (3.17) and the Ginsparg-Wilson relation.

5.1 Locality bounds

From Eqs. (3.10) and its $N \to \infty$ limit, we see that the locality property of the effective Dirac operator of the domain-wall fermion is determined by the locality properties of the following operators in the integral representation:

$$I^{(N)} = \frac{1}{N} \sum_p \left\{1, e^{+ip}, e^{-ip}\right\} \frac{1}{i\gamma_5 \sin p + 1 - \cos p + (aD_{\rm w} - m_0)} \tag{5.1}$$

and

$$I = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \left\{1, e^{+ip}, e^{-ip}\right\} \frac{1}{i\gamma_5 \sin p + 1 - \cos p + (aD_{\rm w} - m_0)}. \tag{5.2}$$

The integrand can be written as

$$\begin{aligned} & \frac{1}{i\gamma_5 \sin p + 1 - \cos p + (aD_{\rm w} - m_0)} \left\{1, e^{+ip}, e^{-ip}\right\} \\ &= \frac{1}{4\sin^2(p/2)\, B + (aD_{\rm w} - m_0)^\dagger (aD_{\rm w} - m_0)} \times \\ & \qquad \left(1 - P_R e^{ip} - P_L e^{-ip} + (aD_{\rm w} - m_0)^\dagger\right) \left\{1, e^{+ip}, e^{-ip}\right\}. \end{aligned} \tag{5.3}$$

From this expression it is clear that the operators in the numerator are local and bounded. Then we may omit these operators in the following considerations.

We can obtain an expansion of the integrand using the generating function of the Chebycheff polynomials [38]

$$\frac{1}{1-2tz+t^2} = \sum_{k=0}^{\infty} t^k U_k(z), \qquad \|U_k(z)\| \le U_k(1) = k. \tag{5.4}$$

Following [4], we set

$$t = e^{-\tilde{\theta}}, \qquad \cosh\theta = \frac{\tilde{\beta}+\tilde{\alpha}}{\tilde{\beta}-\tilde{\alpha}}, \tag{5.5}$$

and

$$z = \frac{\tilde{\beta}+\tilde{\alpha} - 2\left\{4\sin^2(p/2)\,B + (aD_{\rm w}-m_0)^\dagger(aD_{\rm w}-m_0)\right\}}{\tilde{\beta}-\tilde{\alpha}}. \tag{5.6}$$

Then we obtain

$$\frac{1}{4\sin^2(p/2)\,B + (aD_{\rm w}-m_0)^\dagger(aD_{\rm w}-m_0)} = \frac{4t}{\tilde{\beta}-\tilde{\alpha}} \sum_{k=0} t^k U_k(z), \tag{5.7}$$

This defines an expansion in terms of the square of the Wilson-Dirac operator and B with only nearest-neighbor and next-to-nearest-neighbor couplings. In order to contribute to the kernel of the operator Eq. (5.7) between two lattice sites x and y of the lattice distance $d(x,y) = |x-y|$, the order of the polynomials $U_k(x)$ in the expansion must be greater than $\frac{d(x,y)}{2a}$:

$$k \ge \frac{d(x,y)}{2a} \tag{5.8}$$

Then for the given distance $d(x,y)$, the series expansion Eq. (5.7) can be arranged as follows:

$$\frac{4t}{\tilde{\beta}-\tilde{\alpha}} \exp\left\{-\frac{\tilde{\theta}}{2a} d(x,y)\right\} \cdot \sum_{k=0} t^k U_{k+d/2a}(z). \tag{5.9}$$

Noting the bound on the polynomials, $\|U_k(z)\| \le k$, we obtain

$$\left\| \frac{1}{4\sin^2(p/2)\,B + (aD_{\rm w}-m_0)^\dagger(aD_{\rm w}-m_0)}(x,y) \right\| \le \frac{4t}{\tilde{\beta}-\tilde{\alpha}} \exp\left\{-\frac{\tilde{\theta}}{2a} d(x,y)\right\} \cdot \sum_{k=0} t^k \|U_{k+d/2a}(z)\|$$

$$\leq \frac{4t}{\tilde{\beta}-\tilde{\alpha}}\left(\frac{1}{1-t}\frac{d(x,y)}{2a}+\frac{t}{(1-t)^2}\right)\exp\left\{-\frac{\tilde{\theta}}{2a}d(x,y)\right\}. \tag{5.10}$$

Since the summation over the momentum in $I^{(N)}$ (the integration in I) is normalized to unity, the above bounds implies the exponential bound for the integrals, $I^{(N)}$ and I.

As for the differentiations of the effective Dirac operator, we can also derive the exponential bounds, following [4]. We consider the differentiations of the Chebycheff expansion, Eq. (5.7). We first introduce an integral representation for the Chebycheff polynomials:

$$U_k(z) = \oint \frac{d\omega}{2\pi}\omega^{-k-1}\frac{1}{\omega^2-2\omega z+1}, \tag{5.11}$$

where the integration is defined along a circle in the complex plane centered at the origin. The radius r of the circle should be strictly less than 1 to avoid the singularities of the integrand. The denominator of the integrand can be factorized according to

$$\omega^2-2\omega z+1=(\omega-u^\dagger)(\omega-u), \qquad u=z+i(1-r)^{1/2}. \tag{5.12}$$

Since, $u^\dagger u=1$, it is clear that

$$\parallel \left(\omega^2-2\omega z+1\right)^{-1} \parallel \leq (1-r)^{-2}. \tag{5.13}$$

If we denote the differentiation of z with respect to the gauge fields as $\dot{z}$, then we obtain

$$\parallel \dot{U}_k(z) \parallel \leq 2 \parallel \dot{z} \parallel r^{-k}(1-r)^{-4}. \tag{5.14}$$

We may now adjust the radius r so that the factor $r^{-k}(1-r)^{-4}$ is minimized. We obtain

$$\parallel \dot{U}_k(z) \parallel \leq \text{constant} \;\parallel \dot{z} \parallel (1+k)^4. \tag{5.15}$$

With these bounds, we can see that the differentiated series Eq. (5.7) is also exponentially convergent with the same exponent as the original series. By similar estimations, we can see that this is also true for higher-order differentiations (each differentiation give rise to an additional factor of $(1+k)^2$ in the bound on the Chebycheff polynomials).

5.2 Exponential bounds in the fifth-direction and the Ginsparg-Wilson relation

We next consider the exponential bound on the inverse of the five-dimensional Wilson-Dirac operator which is necessary to prove Eq. (3.17) and the Ginsparg-Wilson relation. From the above derivation of the exponential bounds for the summation and integral Eqs. (5.1) and (5.2), we can see that the same bound holds true for the inverse of the five-dimensional Wilson-Dirac operator, $\overline{D}_{5\mathrm{w}}$, itself:

$$\left\{a\overline{D}_{5\mathrm{w}}\right\}^{-1}_{st} = \frac{1}{N}\sum_{p} \frac{e^{+ip(s-t)}}{i\gamma_5 \sin p + 1 - \cos p + (aD_{\mathrm{w}} - m_0)} \tag{5.16}$$

In fact, we can obtain

$$\left\| \left\{ a^2 \overline{D}^{\dagger}_{5\mathrm{w}} \overline{D}_{5\mathrm{w}} \right\}^{-1} (x,s;y,t) \right\| \leq C \, \exp\left\{ -\frac{\tilde{\theta}}{2a} d_5(x,s;y,t) \right\}, \tag{5.17}$$

where $d_5(x,s;y,t) = |x-y| + \min(|s-t|, N-|s-t|)$ and

$$C = \frac{4t}{\tilde{\beta}-\tilde{\alpha}} \left(\frac{1}{1-t} \frac{d_5(x,s;y,t)}{2a} + \frac{t}{(1-t)^2} \right). \tag{5.18}$$

From this bound, it follows immediately that

$$\lim_{N\to\infty} \left\{ \overline{D}'_{5\mathrm{w}} \right\}^{-1}_{N \frac{N}{2}} = 0. \tag{5.19}$$

This completes the proof of the Ginsparg-Wilson relation under the condition on the plaquette variables Eq. (4.6). Note again that this proof does not refer to the transfer matrix and it applies for any value of $m_0 \in [0,2]$.

5.3 Singular case

In this subsection, we examine locality of the effective Dirac operator Eq. (1.5) with the singular gauge configuration for which isolated eigenvalues of the hermitian Wilson-Dirac operator collapse to zero:

$$H\phi_0(x) = \lambda\phi_0(x), \qquad \lambda \simeq 0, \tag{5.20}$$

where

$$H = \gamma_5 (D_{\mathrm{w}} - \frac{m_0}{a}). \tag{5.21}$$

We will argue that the effective Dirac operator remains local even with such singular gauge configurations.

For this purpose, however, the integral representation and the Chebycheff expansion in terms of the five-dimensional Wilson-Dirac operator, considered so far, does not seem to be useful. When the isolated near-zero mode occurs in the four-dimensional hermitian Wilson-Dirac operator, it is associated with many modes with small fifth momenta in the spectrum of the five-dimensional Wilson-Dirac operator (with the anti-periodic boundary condition). This means that the continuum spectrum would collapse to zero in the five-dimensional Wilson-Dirac operator in the limit $N \to \infty$. Then the separation of the effect of the near-zero mode does not seem easy in this representation (cf. [4]). Therefore, in this section, we use the formula for the effective action in terms of the transfer matrix and $\widetilde{H}$.

In [4], it has been proved rigorously that the contribution of the near-zero-mode to Neuberger's Dirac operator,

$$\left.\frac{H}{\sqrt{H^2}}\right|_{\text{near}-\text{zero}}, \tag{5.22}$$

remains local. This result can be understood from the localization properties of the eigenvectors of the near-zero modes. In fact, it is well localized with exponentially decaying tails [4, 39].

Since $\widetilde{H}$ is related to H by the formula

$$e^{-a_5 \tilde{H}} + e^{+a_5 \tilde{H}} - 2 = a_5^2 H \frac{1}{B} H, \tag{5.23}$$

the exact zero mode of $\widetilde{H}$ is the exact zero mode of H [2, 39, 41, 34]. Therefore, the contribution of the zero mode of $\widetilde{H}$ to the effective Dirac operator is identical to the contribution of the zero mode of H to Neuberger's Dirac operator,

$$\left.\frac{\widetilde{H}}{\sqrt{\widetilde{H}^2}}\right|_{\text{zero}} = \left.\frac{H}{\sqrt{H^2}}\right|_{\text{zero}} = \lim_{\lambda \to 0} \text{sign}(\lambda)\, \phi_0(x)\phi_0^\dagger(y), \tag{5.24}$$

and it remains local, according to the result of [4]. It is expected that this localization properties persist also for the contribution of the near-zero modes of $\widetilde{H}$.

It is desirable to make the above argument rigorous. We will leave this issue for future study.

6 Discussion

We have argued locality in the domain-wall fermion approach through its effective four-dimensional Dirac operator. As expected, all the properties proved rigorously for Neuberger's Dirac operator holds true for the effective Dirac operator. In particular, we have shown explicitly that the locality properties of the domain-wall fermion depends crucially on the spectrum of the four-dimensional Wilson-Dirac operator, which is closely related to that of the five-dimensional Wilson-Dirac operator (with the anti-periodic boundary condition). Then we can see that the bound for the plaquette variables leads to the locality bound for the effective Dirac operator. We have also shown that the effective Dirac operator satisfies the Ginsparg-Wilson relation with the same bound for plaquette variables.

The approach to the chiral symmetry limit from a finite N is the most important issue for the practical implementation of exact chiral symmetry using the domain-wall fermion [19, 22, 24] [41, 42, 43]. In order to examine the effect of the finite N, the explicit breaking term in the axial Ward-Takahashi identity has been measured, among other physical quantities. This breaking term can be written by the correlation function between the middle and the boundary of the fifth dimension [27]:

$$\left\{a_5 D'_{5\mathrm{w}}\right\}^{-1}_{\frac{N}{2},N} = \frac{1}{2\cosh \frac{N}{2} a_5 \widetilde{H}} \times \left(\frac{1}{a} D_{\mathrm{eff}}^{(N)^{-1}}\right)(x,y). \tag{6.1}$$

From the point of view of the effective four-dimensional action, this effect may be examined through the breaking term in the Ginsparg-Wilson relation.[4] As we have seen in the section 5.2, it is given by the similar correlation function defined through the five-dimensional Wilson-Dirac operator with the anti-periodic boundary condition:

$$\begin{aligned} \left\{a_5 \overline{D}'_{5\mathrm{w}}\right\}^{-1}_{\frac{N}{2},N} &= \frac{1}{2\cosh \frac{N}{2} a_5 \widetilde{H}} \\ &= \frac{1}{N} \sum_p \frac{\left(e^{-ip}\right)^{\frac{N}{2}}}{i\gamma_5 \sin p + 1 - \cos p + a_5 \left(D_{\mathrm{w}} - \frac{m_0}{a}\right)}. \end{aligned} \tag{6.2}$$

It is the smallest eigenvalue of square of the four-dimensional Wilson-Dirac operator which determines the behavior of the breaking term in the limit $N \to \infty$. When the isolated near-zero mode occurs in the four-dimensional

[4] Numerical study of the Ginsparg-Wilson relation of the effective Dirac operator is found in [36].

hermitian Wilson-Dirac operator, it is associated with many modes with small fifth momenta in the spectrum of the five-dimensional Wilson-Dirac operator (with the anti-periodic boundary condition). This means that the continuum spectrum tends to collapse to zero and the lower bound for this continuum spectrum determines the rate of the exponential decay in the limit $N \to \infty$. Therefore, it would be important to examine the behavior of the near-zero modes as done in [4], also in the context of the domain-wall fermion, as suggested in [22]. It is also desirable to clarify the nature of the distribution of small eigenvalues of the four-dimensional Wilson-Dirac operator with a negative mass, for the gauge field configurations used in the current simulations [40].

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