

**NEUTRINO MIXING ANGLES AND EIGENSTATES;
CP PROPERTIES AND MASS HIERARCHIES**

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Abstract: In the presence of independent generations of leptons, I show that the same type of ambiguity in the mass spectrum arises as was discussed in [1] for neutral kaons. It results from the freedom to add to their Majorana mass matrix, usually taken to be symmetric, an antisymmetric term which vanishes as soon as fermions belonging to different generations anticommute.

In the simple examples proposed, dealing with two generations, this procedure introduces an extra (mass) parameter ρ , which is shown to connect the (CP violating) mixing angle to the hierarchy of neutrino masses. We use this opportunity to investigate the relations between the two; in particular, large hierarchies are no longer preferentially attached to small mixing angles; this can be relevant for the “Large Mixing Angle” solution strongly advocated by recent experiments on neutrinos oscillations [2].

I discuss how the ρ parameter could be fixed, which appears, in the absence of a substructure for leptons, still more delicate than for kaons.

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1 Introduction

It was shown in [1] that the spectrum of the pair of charge-conjugate electrically neutral kaons ¹ can be ambiguous as soon as they are considered to be independent and commuting fields. The introduction, in the Lagrangian, of a formally vanishing “mass term” proportional to their commutator induces a degeneracy for the set of basis in which the kinetic and mass terms become simultaneously diagonalizable.

A $U(1)$ group was pointed out, the phase θ of which became, by the process mentioned above, the indirect CP -violating parameter of the neutral kaon system. Mass eigenstates can vary from self- (or anti-self-) conjugate states to states of definite “flavor number” (K^0 and $\overline{K^0}$) and the hierarchical pattern of masses becomes a function of θ .

The different possible behaviors of mass eigenstates by charge conjugation suggested in [1] a parallel with Majorana (self-, or anti-self-conjugate) and / or Weyl / Dirac neutrinos. The purpose of this letter is to show, with very simple example in the case of two generations, how, indeed, the same type of ambiguity arises for neutrinos, and how the departure from Majorana states is controlled by a phase which determines their hierarchy pattern and CP properties.

2 The mass matrix for neutrinos

2.1 The customary framework [3]

The Majorana mass matrix for neutrinos is usually taken to be symmetric.

Let us consider the example of a pure Majorana mass terms for two generations of neutrinos. We accordingly introduce the vector of left-handed neutrinos ²

$$n_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \end{pmatrix} \quad (1)$$

and the mass term (the superscript “ c ” means “charge conjugate”)

$$\mathcal{L}_0^m = -\frac{1}{2} \overline{(n_L)^c} M_0 n_L + h.c.; \quad (2)$$

M_0 is the 2×2 mass matrix

$$M_0 = \begin{pmatrix} m_1 & d \\ d & m_2 \end{pmatrix} \quad (3)$$

that we shall take real for the sake of simplicity. (2) violates, as usual, the conservation of the leptonic number.

That M_0 is taken to be symmetric results from the anticommutation relations between fermions [3]

$$\overline{(\nu_{eL})^c} \nu_{\mu L} - \overline{(\nu_{\mu L})^c} \nu_{eL} = 0. \quad (4)$$

Consequently, M_0 , can be diagonalised by a unitary matrix V^0 according to

$$V^{0T} M_0 V^0 = D^0 = \text{diag}(M_1^0, M_2^0), \quad (5)$$

¹and it was extended there, too, to Higgs-like doublets.

²The indices have been named here e and μ only to recall that they are *flavor* indices; they do not relate specifically to the electron or to the muon neutrinos.

where the superscript “ T ” means “*transposed*”; V^0 is given by:

$$V^0 = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad (6)$$

with

$$\tan(2\theta) = \frac{2d}{m_1 - m_2}. \quad (7)$$

\mathcal{L}_m^0 given in (2) rewrites

$$\mathcal{L}_m^0 = -\frac{1}{2} \overline{N^0} D^0 N^0 + h.c., \quad (8)$$

in terms of the mass eigenstates N^0 , which are Majorana neutrinos ($N_L^0 = (N_R^0)^c$), given by [3]

$$N^0 = V^{0\dagger} n_L + (V^{0\dagger} n_L)^c. \quad (9)$$

θ is the “mixing angle” for neutrinos [3].

The eigenmasses M_1 and M_2 are given, in terms of θ , by

$$\begin{aligned} M_1 &= \left| \frac{m_1 \cos^2 \theta - m_2 \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} \right|, \\ M_2 &= \left| \frac{m_2 \cos^2 \theta - m_1 \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} \right|; \end{aligned} \quad (10)$$

we have used the fact that both eigenvalues can always be made positive by eventually multiplying V^0 by $\text{diag}(1, i)$ or $\text{diag}(i, 1)$.

The spectrum (M_1 and M_2) is drawn on Fig. 1 as a function of θ , for $m_1 = 2$ and $m_2 = 4$. On Fig. 2a and 2b are drawn respectively the ratios M_1/M_2 and M_2/M_1 .

One of the eigenmasses vanishes for $d^2 = m_1 m_2$; the other is then given by $(m_1 + m_2)$.

The case of maximal mixing ($|\theta| = \pi/4$) corresponds either to $d \rightarrow \infty$ or to $m_2 = m_1 = m, \forall d$; in the first case, the eigenmasses are $|\pm d| \rightarrow \infty$ in the last case they are $|m \pm d|^3$.

³The case $m_1 = m_2 = m$ is special since, then, as can be seen directly, the mixing angle θ is fixed to its maximal value $\pi/4$.

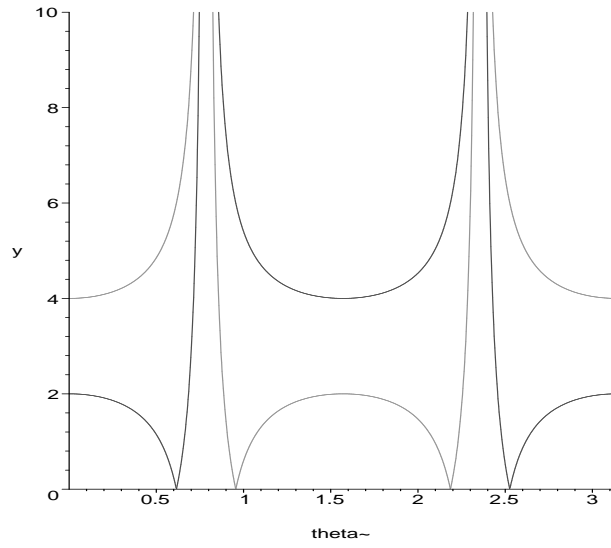


Fig. 1: M_1 and M_2 as functions of θ , for $m_1 = 2$, $m_2 = 4$ and $\rho = 0$

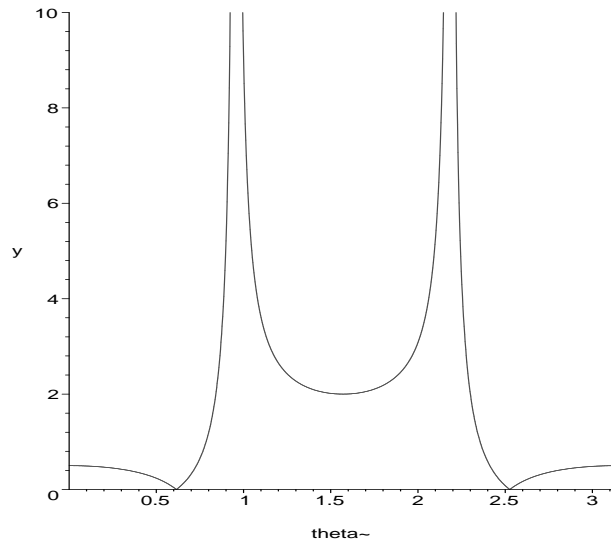


Fig. 2a: M_1/M_2 as function of θ for $m_1 = 2$, $m_2 = 4$ and $\rho = 0$

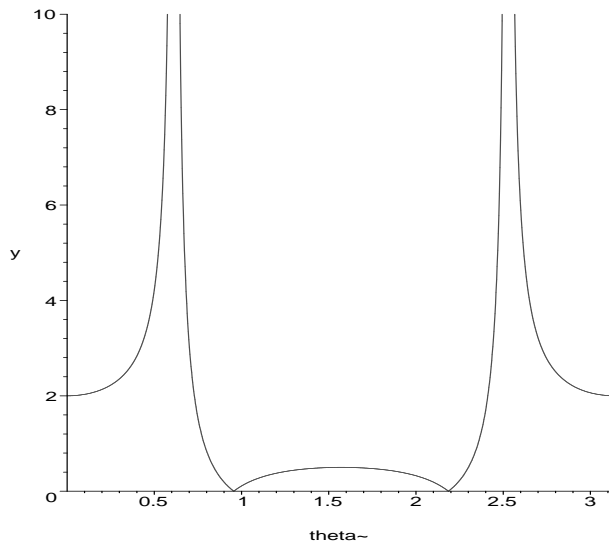


Fig. 2b: M_2/M_1 as function of θ for $m_1 = 2$, $m_2 = 4$ and $\rho = 0$

2.2 Independent generations of fermions

Taking for granted that, the two generations of fermions being supposed to be independent, the corresponding fields anticommute, the relation (4) holds. Instead of using, like in subsection 2.1 this property to take *a priori* the Majorana mass matrix to be symmetric, we shall use below the freedom that it yields to add to the latter a vanishing antisymmetric term.

Accordingly, the two mass matrices given respectively by M_0 in (3) and by M in (11) below

$$M = \begin{pmatrix} m_1 & d + \rho \\ d - \rho & m_2 \end{pmatrix}, \quad (11)$$

in which, for simplification, we also take ρ to be real, only differ by what corresponds to a vanishing mass term in the Lagrangian ⁴

$$-\rho/2 \left(\overline{(\nu_{1L})^c} \nu_{2L} - \overline{(\nu_{2L})^c} \nu_{1L} \right). \quad (12)$$

By this procedure, the mass Lagrangian goes from \mathcal{L}_m^0 given by (2) to \mathcal{L}_m given below

$$\mathcal{L}_m = -\frac{1}{2} \overline{(n_L)^c} M n_L + h.c.. \quad (13)$$

2.3 Diagonalization

2.3.1 Bi-unitary transformation

The mass matrix M in (11), which is no longer symmetric (neither hermitian), cannot be diagonalised any more by a single unitary transformation. It is however mandatory that the diagonalization be

⁴It also corresponds to the simplest type of “horizontal interaction”, and minimal in the sense that it vanishes.

performed with unitary matrices in order that the kinetic terms stay diagonal ⁵. One then accordingly proceeds with a bi-unitary transformation [4].

There exist two unitary matrices U and V such that

$$U^\dagger M V = D = \text{diag}(M_1, M_2). \quad (14)$$

U and V diagonalize respectively the hermitian matrices $M M^\dagger$ and $M^\dagger M$.

\mathcal{L}_m rewrites

$$\mathcal{L}_m = -\frac{1}{2} \overline{(U^T n_L)^c} D (V^\dagger n_L) + h.c.. \quad (15)$$

This leads to the two changes of basis below, which yield left-handed and right-handed mass eigenstates, according to

$$\begin{aligned} N_L &= V^\dagger n_L, \\ N_R &= (U^T n_L)^c = (U^T)^\dagger (n_L)^c = U^* (n_L)^c. \end{aligned} \quad (16)$$

The kinetic terms stay diagonal (see footnote 5) since

$$\frac{i}{2} (\overline{N_L} \gamma^\mu \partial_\mu N_L + \overline{N_R} \gamma^\mu \partial_\mu N_R) = i \overline{n_L} \gamma^\mu \partial_\mu n_L. \quad (17)$$

The diagonalised mass matrix no longer connects Majorana neutrinos but the Weyl spinors N_R and N_L ($N_L \neq (N_R)^c$).

The mixing angle for the left-handed neutrinos is given by the matrix V and has *a priori* no relation with θ defined by V^0 in (6).

The mass eigenvalues are no longer the solutions of the characteristic equation of M in (11) but the “square roots” of the eigenvalues of $M M^\dagger$. They can be very different from the eigenvalues of M_0 in (3).

Note that, unlike in (9), the Dirac spinor $N_L + N_R$ (see (16)) cannot be taken as the set of mass eigenstates since

$$(\overline{N_L} + \overline{N_R}) D (N_L + N_R) = \overline{(n_L)^c} M n_L + \overline{n_L} V D U^\dagger (n_L)^c, \quad (18)$$

the last contribution of which is unsuitable.

2.3.2 A short comment on oscillations

In the (N_L, N_R) basis, the Lagrangian (kinetic + mass terms) writes (recall that N_L and N_R are themselves 2-vectors)

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} \overline{N_L} & \overline{N_R} \end{pmatrix} \begin{pmatrix} \not{\partial} & & -M_1 & \\ & \not{\partial} & & -M_2 \\ -M_1 & & \not{\partial} & \\ & -M_2 & & \not{\partial} \end{pmatrix} \begin{pmatrix} N_L \\ N_R \end{pmatrix}; \quad (19)$$

while the kinetic terms are diagonal, the mass terms, which can only connect “left” to “right” Weyl fermions are now placed on the antidiagonal blocks.

⁵One recalls that the kinetic terms satisfy $i \overline{(\nu_L)^c} \gamma^\mu \partial_\mu (\nu_L)^c = i \overline{\nu_L} \gamma^\mu \partial_\mu \nu_L$.

One cannot use any more the standard form of the Dirac propagators for fermions; however, suppose that a N_{1L} is created with energy momentum p ; its propagator, expressed by the series depicted in Fig. 3, writes

$$\frac{i}{\not{p}} + \frac{i}{\not{p}} i M_1 \frac{i}{\not{p}} i M_1 \frac{i}{\not{p}} + \dots = \frac{i \not{p}}{p^2 - M_1^2} \quad (20)$$

showing as expected a pole at $p^2 = M_1^2$. The left-handed Weyl fermion N_{1L} does propagate like a particle of mass M_1 . A similar remark applies to N_{2L} .

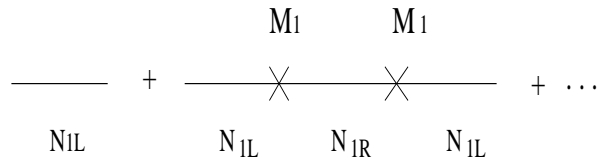


Fig. 3: propagating the Weyl fermion N_{1L} .

The phenomenon of oscillations proceeds as in the standard case: if a neutrino of a definite flavor is created at a space-time point x_0 , its two Weyl components, obtained through the simple inversion of (16), propagate with different frequencies and wavelengths corresponding respectively to the eigenmasses M_1 and M_2 ; after a certain time and distance of propagation, at point x , a left-handed neutrino, for example, if detected, will not have the same proportion of N_{1L} and N_{2L} as it had at point x_0 .

3 Examples

We shall successively treat three examples of increasing complexity corresponding to a mass matrix (11) with $\rho \neq 0$:

- the case $m_2 = m_1, d \neq 0$;
- the case $m_2 \neq m_1, d = 0$;
- the more general case $m_2 \neq m_1, d \neq 0$.

3.1 The case $m_2 = m_1, d \neq 0$

This very simple case, easily solvable, nevertheless exhibits most of the properties that we want to emphasize: departure from the status of Majorana neutrino and indirect CP violation, dependence of the spectrum on the mixing angle and, in this precise case, one-to-one relationship between the latter and the hierarchy of masses.

The mass matrix is

$$\mathcal{M} = \begin{pmatrix} m & d + \rho \\ d - \rho & m \end{pmatrix}, \quad (21)$$

and we choose the “diagonalizing” unitary matrices U and V to span, for conveniency, the two sets described by

$$A(\varphi) = \begin{pmatrix} c_\varphi & -s_\varphi \\ s_\varphi & c_\varphi \end{pmatrix}, \quad B(\omega) = \begin{pmatrix} -c_\omega & s_\omega \\ s_\omega & c_\omega \end{pmatrix}, \quad (22)$$

respectively with determinants $+1$ and -1 .

φ is the mixing angle for leptons [3]

3.1.1 Case $d > m$

The diagonalization equation

$$B^\dagger(\omega) \mathcal{M} A(\varphi) = \text{diag}(M_1, M_2) \quad (23)$$

is satisfied for

$$\begin{aligned} \omega &= \varphi + \pi/2, \\ \tan(2\varphi) &= -\frac{m}{\rho}; \end{aligned} \quad (24)$$

the mass eigenvalues are then

$$\begin{aligned} M_1 &= d + \frac{m}{\sin(2\varphi)} \\ M_2 &= d - \frac{m}{\sin(2\varphi)}. \end{aligned} \quad (25)$$

The condition $M_1 \geq 0$ requires $\sin(2\varphi) \leq -m/d$ or $\sin(2\varphi) \geq 0$. The condition $M_2 \geq 0$ requires $\sin(2\varphi) \geq m/d$ or $\sin(2\varphi) \leq 0$.

Inside the interval $-m/d \leq \sin(2\varphi) \leq m/d$, one among the two eigenmasses is negative; this problem is easily taken care of: for $-m/d \leq \sin(2\varphi) \leq 0$ one goes to a positive M_1 (M_2 is positive) by multiplying A and B by the matrix $\text{diag}(i, 1)$; for $0 \leq \sin(2\varphi) \leq m/d$, one goes to a positive M_2 (M_1 is positive) by multiplying A and B by the matrix $\text{diag}(1, i)$.

The spectrum is drawn on Fig. 4 for $m = 2$ and $d = 4$. Its periodicity with respect to φ has been used to draw the picture for $\varphi \in [0, \pi/2]$ rather than $\varphi \in [-\pi/4, \pi/4]$. It has been divided into three zones:

- in the central one, $\varphi \in [(1/2) \arcsin(m/d), \pi/2 - (1/2) \arcsin(m/d)]$, $M_1 + M_2 = 2d$ and $|M_2 - M_1| = |2m/\sin(2\varphi)|$. The extrema for the masses (one maximum and one minimum) are respectively $d + m$ and $d - m$ and correspond to $\rho = 0, \sin(2\varphi) = 1$; the mass matrix \mathcal{M} is then symmetric and the mass eigenstates can also be taken as Majorana neutrinos (see section 4);
- in the two lateral zones, $|M_2 - M_1| = 2d$ and $M_1 + M_2 = 2m/\sin(2\varphi)$;
- the mass of one of the neutrinos vanishes at the boundaries between the two zones above, which corresponds to $\rho^2 = d^2 - m^2$ or $\sin(2\varphi) = m/d$. The other mass is then equal to $2d$.

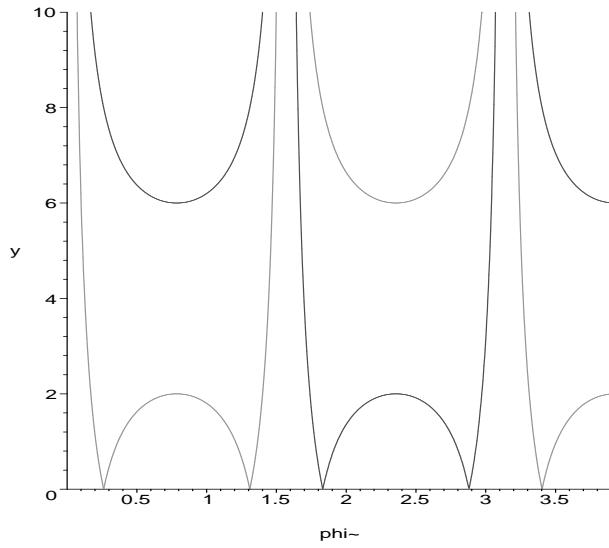


Fig. 4: M_1 and M_2 as functions of φ , case $d = 4 > m_1 = m_2 = 2$

The singularities, where one of the masses goes to ∞ , correspond to $\rho \rightarrow \infty$.

On Figs. 5a, 5b and 6 are plotted respectively the ratio M_1/M_2 , M_2/M_1 and $|M_2 - M_1|/(M_2 + M_1)$ for $m = 2$ and $d = 4$.

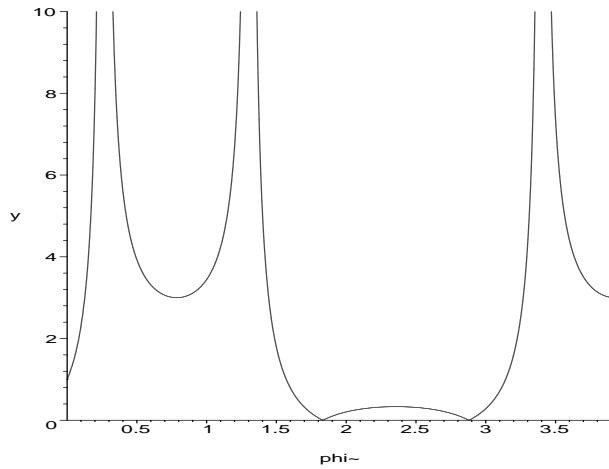


Fig. 5a: M_1/M_2 as a function of φ , case $d = 4 > m_1 = m_2 = 2$

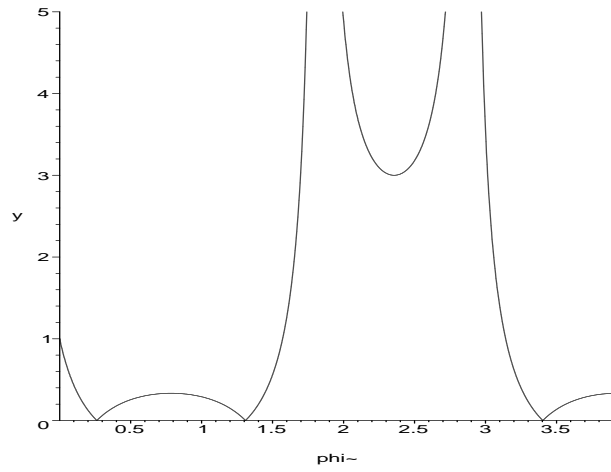


Fig. 5b: M_2/M_1 as a function of φ , case $d = 4 > m_1 = m_2 = 2$

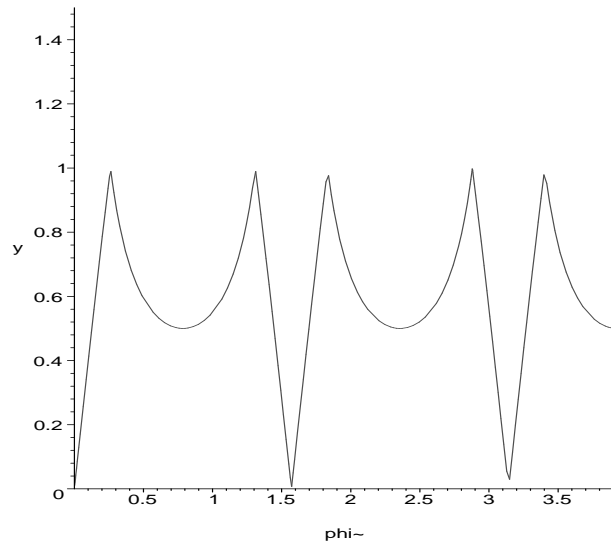


Fig. 6: $|M_2 - M_1|/(M_2 + M_1)$ as a function of φ , case $d = 4 > m_1 = m_2 = 2$

3.1.2 Case $d < m$

The spectrum is plotted on Fig. 7 for $m = 4$ and $d = 2$.

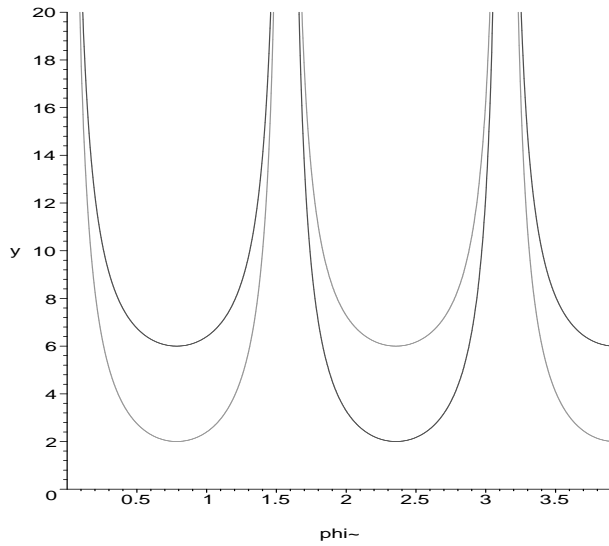


Fig. 7: M_1 and M_2 as functions of φ , case $d = 2 < m_1 = m_2 = 4$

One has $M_1 + M_2 = |2m/\sin(2\varphi)|$ and $|M_2 - M_1| = 2d$: while the mass splitting is constant, the sum of the masses reaches its minimum $2m$ for $\sin(2\varphi) = 1$, $\rho = 0$, in which case the mass matrix \mathcal{M} is symmetric and the mass eigenstates can also be taken as Majorana neutrinos.

None of the masses ever vanishes in this case, except when m_1 or m_2 vanishes. The singularities where one of the masses goes to ∞ correspond to $\rho \rightarrow \infty$.

The ratio M_1/M_2 is plotted on Fig. 8.

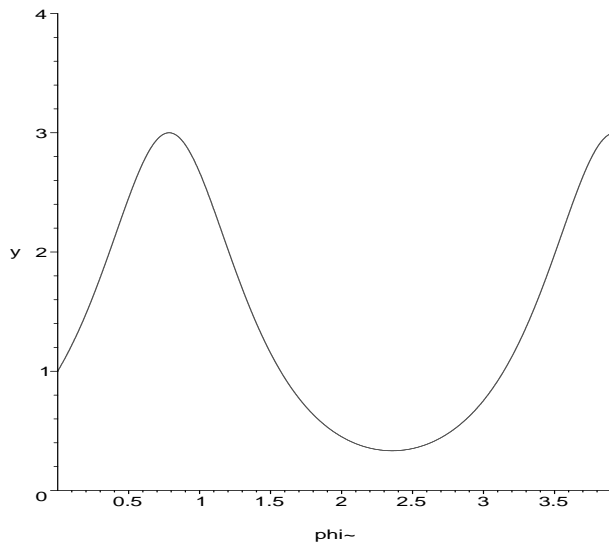


Fig. 8: M_1/M_2 as a function of φ , case $d = 2 < m_1 = m_2 = 4$

Note the continuity of this curve through the singularities of Fig. 7.

3.2 The case $m_2 \neq m_1, d = 0$

The corresponding mass matrix

$$\mathfrak{M} = \begin{pmatrix} m_1 & \rho \\ -\rho & m_2 \end{pmatrix} \quad (26)$$

is diagonalised through the relation

$$B(\varphi) \mathfrak{M} A(\varphi) = \text{diag}(M_1, M_2), \quad (27)$$

with

$$\tan(2\varphi) = \frac{2\rho}{m_1 + m_2}. \quad (28)$$

The two eigenmasses (which are easily made, as above, to be both positive) are given by

$$\begin{aligned} M_1 &= \left| \frac{m_1 \cos^2 \varphi + m_2 \sin^2 \varphi}{\cos^2 \varphi - \sin^2 \varphi} \right|, \\ M_2 &= \left| \frac{m_1 \sin^2 \varphi + m_2 \cos^2 \varphi}{\cos^2 \varphi - \sin^2 \varphi} \right|, \end{aligned} \quad (29)$$

such that

$$M_1 + M_2 = \left| \frac{m_1 + m_2}{\cos^2 \varphi - \sin^2 \varphi} \right|, \quad |M_2 - M_1| = |m_2 - m_1|. \quad (30)$$

None of them ever vanishes since the determinant of \mathfrak{M} cannot.

They are plotted on Fig. 9 for $m_1 = 2$ and $m_2 = 4$.

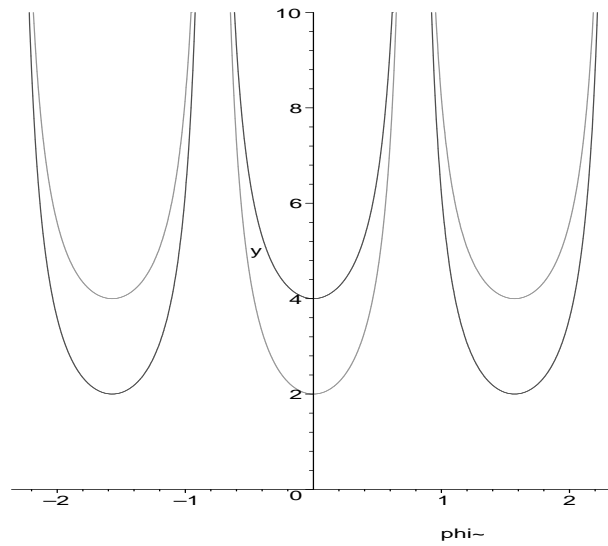


Fig. 9: M_1 and M_2 as functions of φ , case $m_1 = 2, m_2 = 4$ and $d = 0$

The singularities where one of the masses goes to ∞ always correspond to $\rho \rightarrow \infty$.

The ratio M_2/M_1 is plotted on Fig. 10. Note, like for Fig. 8, the continuity of this curve through the singularities of Fig. 9.

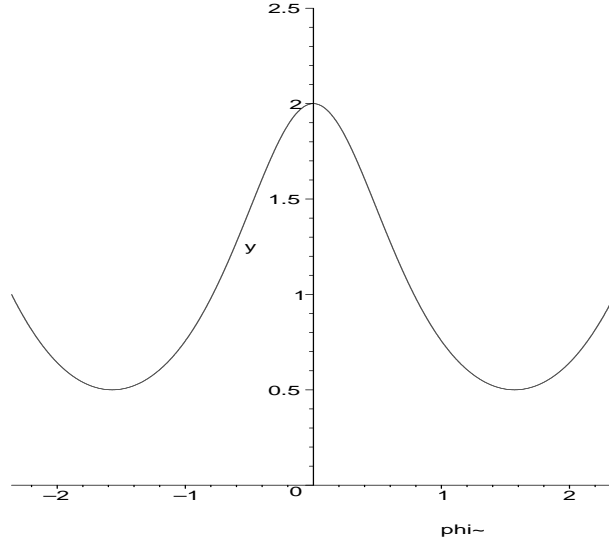


Fig. 10: M_1/M_2 as a function of φ , case $m_1 = 2$, $m_2 = 4$ and $d = 0$

3.3 The more general case $m_2 \neq m_1$, $d \neq 0$

The mass matrix (11) can be diagonalised by the bi-unitary transformation

$$B^\dagger(\omega)MA(\varphi) = \text{diag}(M_1, M_2), \quad (31)$$

such that

$$\begin{aligned} \tan(\omega + \varphi) &= \frac{2\rho}{m_1 + m_2}; \\ \tan(\omega - \varphi) &= \frac{2d}{m_2 - m_1}. \end{aligned} \quad (32)$$

The eigenmasses are ⁶

$$\begin{aligned} M_1 &= \left| \frac{m_1 \cos \varphi \cos \omega + m_2 \sin \varphi \sin \omega}{1 - \cos^2 \varphi - \cos^2 \omega} \right|, \\ M_2 &= \left| \frac{m_1 \sin \varphi \sin \omega + m_2 \cos \varphi \cos \omega}{1 - \cos^2 \varphi - \cos^2 \omega} \right|. \end{aligned} \quad (33)$$

On Fig. 11 is drawn the spectrum, M_1 and M_2 as functions of φ , for $m_1 = 2$, $m_2 = 4$ and $d = 1$, corresponding to $\tan(\omega - \varphi) = 1$ or $\omega = \varphi + \pi/4$.

⁶If one of the eigenmass turns out to be negative, one multiplies A and B by the suitable diagonal matrix $\text{diag}(1, i)$ or $\text{diag}(i, 1)$.

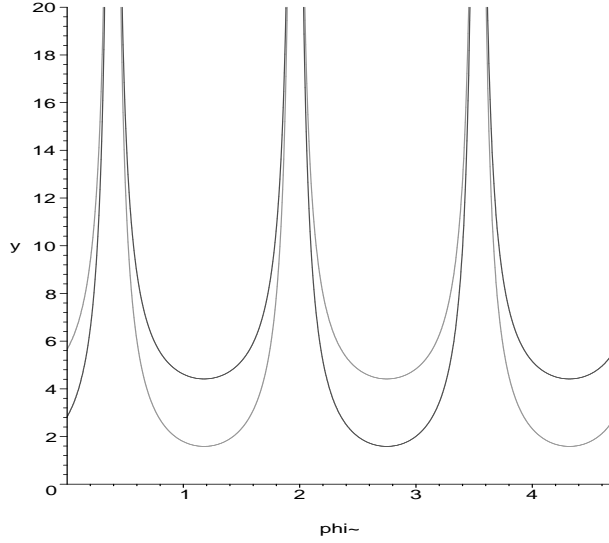


Fig. 11: M_1 and M_2 as functions of φ , for $m_1 = 2$, $m_2 = 4$ and $\omega = \varphi + \pi/4$

On Fig. 12 is drawn the spectrum for $m_1 = 2$, $m_2 = 4$ and $\omega = \varphi + 7\pi/15$.

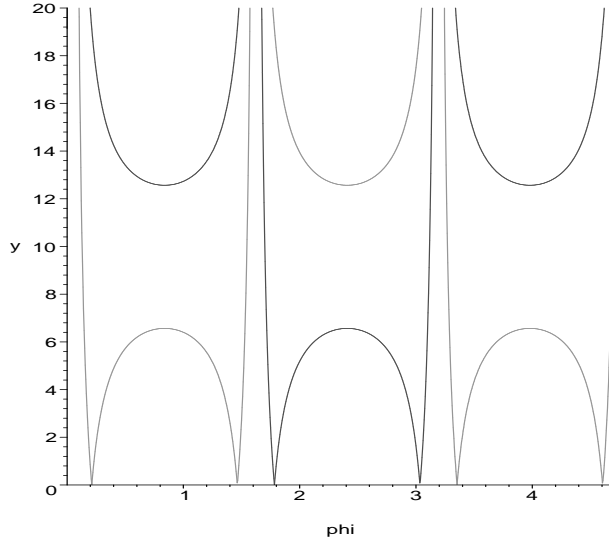


Fig. 12: M_1 and M_2 as functions of φ , for $m_1 = 2$, $m_2 = 4$ and $\omega = \varphi + 7\pi/15$

The two sets of curves above correspond to fixing the value of d , which is equivalent, by (32), to fixing the value of $\omega - \varphi$, and to varying ρ .

The same types of spectra as in previous sections appear, only shifted along the φ axis. For given m_1 , m_2 and d (corresponding to a “standard” mass matrix), the hierarchy of masses depend now on ρ . On Fig. 13 is plotted the ration M_2/M_1 as a function of φ for the two sets of parameters corresponding to the figures 10 and 11.

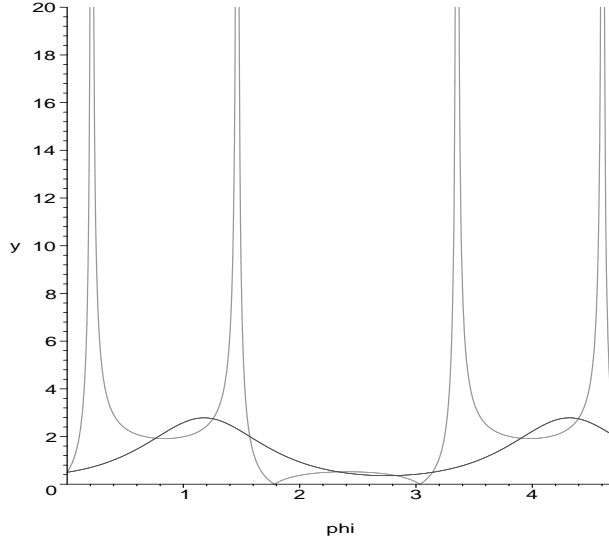


Fig. 13: M_2/M_1 as functions of φ , for the two sets of parameters corresponding to Fig. 11 and Fig. 12

On all examples of this section, the hierarchy of masses can become very large ($\rightarrow \infty$) when the determinant of the mass matrix vanishes. Tuning ρ can consequently also have the consequence that two among the four Weyl eigenstates decouple from the theory at low enough energy. That they are not only right-handed but form a left-right set distinguishes this mechanism from the usual see-saw.

4 Weyl or Majorana?

Turning on the antisymmetric “ ρ ” term in \mathcal{M} swaps the mass eigenstates from Majorana to Weyl.

In the case of a symmetric mass matrix ($\rho = 0$), the two points of view are, as shown below, equivalent. Let us deal with the simple case of subsection 3.1 corresponding to $m_2 = m_1 = m$.

The customary diagonalization method leading to Majorana mass eigenstates uses the relation

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} m & d \\ d & m \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} m+d & 0 \\ 0 & d-m \end{pmatrix}, \quad (34)$$

which corresponds, in (6), to

$$V^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (35)$$

The diagonal matrix $\text{diag}(1, i)$ in (35) ensures the positivity of both eigenmasses.

The diagonalization method advocated for in this work, which leads to left- and right- Weyl mass eigenstates, uses instead the relation (we consider the case $d > m$, such that $d - m$ is a positive eigenmass)

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} m & d \\ d & m \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} m+d & 0 \\ 0 & d-m \end{pmatrix}, \quad (36)$$

and corresponds, in (14), to

$$V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (37)$$

The Majorana mass eigenstates obtained from the customary diagonalization are given by

$$\begin{aligned} N_1^{Maj} &= \frac{1}{\sqrt{2}} ((n_{1L} + n_{2L}) + (n_{1L} + n_{2L})^c), \\ N_2^{Maj} &= \frac{1}{\sqrt{2}} ((i n_{1L} - i n_{2L}) + (i n_{1L} - i n_{2L})^c), \end{aligned} \quad (38)$$

while the Weyl mass eigenstates obtained in subsection 3.1 are

$$\begin{aligned} N_{1L}^{Weyl} &= \frac{1}{\sqrt{2}} (n_{1L} + n_{2L}) = (N_1^{Maj})_L, \\ N_{2L}^{Weyl} &= \frac{1}{\sqrt{2}} (-n_{1L} + n_{2L}) = (i N_2^{Maj})_L, \\ N_{1R}^{Weyl} &= \frac{1}{\sqrt{2}} (n_{1L} + n_{2L})^c = (N_1^{Maj})_R, \\ N_{2R}^{Weyl} &= \frac{1}{\sqrt{2}} (n_{1L} - n_{2L})^c = (-i N_2^{Maj})_R. \end{aligned} \quad (39)$$

Since the latter satisfy $N_{1R}^{Dir} = (N_{1L}^{Dir})^c$ and $N_{2R}^{Dir} = -(N_{2L}^{Dir})^c$, the two formulations (Weyl and Majorana) become equivalent for $\rho = 0$.

5 Mixing angles

One of the results of this work is that, unlike usually considered, small mixing angles are not *a priori* attached to large hierarchies of masses, neither, as a consequence, large mixing angles to a near degeneracy or “inverted” hierarchies. Let us be more explicit.

In the customary framework of a symmetric mass matrix (see Fig. 2b for example), the hierarchy M_1/M_2 starts by increasing (up to ∞) with the mixing angle, then decreases down to 1, gets inverted, goes down to 0, to finally (still inverted) increase again up to $1/2$.

Consider next the example of Fig. 5a. The mass hierarchy M_1/M_2 is minimal (close to 1) for φ small, increases to infinity for $\sin(2\varphi) = m/d$ and then decreases when φ increases up to the “maximal mixing” value $\varphi = \pi/4$. After this, it increases again to ∞ , goes down to 1, gets inverted down to 0 to finally slightly increase again. Note that the hierarchy goes to 1 when the two masses become very large, for $\varphi \rightarrow 0$ or $\pi/2$.

The next example is that of Fig. 8, where a large mixing angle ($\pi/4$) is associated with the largest hierarchy $M_1/M_2 = 3$.

Accordingly, the “Large Mixing Angle” solution for leptons [2] can also go along with large hierarchies among neutrino masses.

In the last example, Fig. 10, in agreement with the common prejudice, the largest hierarchies (2 or $1/2$ in this case) occur for a small mixing angle $\varphi = k\pi/2$, and the smallest hierarchy (1), occurs for maximal mixing $\varphi = \pi/4 + k\pi/2$.

In the more general case of subsection 3.3, the variety of the shifts than the spectrum can undergo along the mixing angle axis can produce *a priori* all kinds of variations of the mass hierarchy as a function of the mixing angle.

The relation between the two appears consequently as dependent of the structure of the mass matrix, and, specially, of the presence or not of the antisymmetric “ ρ ” term.

6 Symmetries

In general (when $\rho \neq 0$), CP is indirectly violated; the “left” and “right” mass eigenstates (Weyl spinors) are no longer CP eigenstates, and can be written as linear combinations of the Majorana (CP) eigenstates which correspond to $\rho = 0$, projected on a given helicity. Apart from the spin degrees of freedom, the similarity with the neutral kaon system clearly appears.

Like in [1], a $U(1)$ group of transformation can be associated with CP . The corresponding phase is the angle θ defined in (6). Though the eigenmasses depend on the value of the antidiagonal symmetric d term and thus on θ , whatever be the latter, the eigenvectors of the mass matrix M_0 given in (3), where no antidiagonal antisymmetric ρ term is present, are always Majorana neutrinos, independent of θ ; hence they are also CP eigenstates. Any choice for θ breaks this $U(1)$ symmetry but, since the eigenstates stay the same CP eigenstates, it does not break CP invariance; it only selects a specific mass pattern.

θ becomes the “ CP ” violating phase only when the ρ term is turned on, whatever its value (even when it goes to 0); the breaking of $U(1)$ and CP are consequently connected.

Whether this breaking is spontaneous or explicit is an ambiguous matter. For a non-vanishing commutator, $U(1)$ and CP are simultaneously explicitly broken when ρ is turned on. However, for independent generations, it amounts to introducing a vanishing perturbation; one is inclined in this case to consider that both symmetries become spontaneously broken ⁷.

The problem is twofold since one needs not only to fix ρ , but also to know whether fermions belonging to different generations truly anticommute.

What fixes ρ ? The example of the neutral kaon system taught us [1] that, if they are taken as composite and if their constituents undergo another interaction which is misaligned with flavor, the relevant commutator is expected not to vanish; the other interactions are, for the kaon system, electroweak interactions ⁸.

In the present case, the situation is more difficult, since one is reluctant to introduce for leptons another level of constituents.

A reasonable attitude is to consider that an ambiguity in the mass spectrum cannot exist and to interpret the paradox found above as a theoretical signal that nature has probably not made generations truly independent and anticommuting.

To go further in this direction, a conventional idea is to introduce additional, eventually gauged, horizontal interactions. They are likely to set up a new fundamental length scale, and it may be not unrealistic to think that it could be connected to ρ ⁹.

Then, the question arises of how the anticommutator of two fermions belonging to different generations can be determined. In terms of which fields? Is the commutator to be considered itself as a new (composite) field eventually coupled, as we did, to fermions? Can it be considered as independent or should constraints be introduced?

The situation is evidently far from clear and shedding light on it is evidently beyond the scope of this work.

We also mentioned in [1], in the case of composite Higgs-like doublets, that, for one of them, which includes the scalar flavor singlet and its three pseudoscalar partners, the ambiguity remains and its

⁷Different “domains” corresponding to different values of θ can exist in disconnected space-time regions; the $U(1)$ and CP symmetries can be expected to be restored only by averaging over the whole sets of domains.

⁸The term in the Lagrangian proportional to the commutator of K^0 and \bar{K}^0 was connected to the (integral of the) number of d quarks minus the number of s quarks, which is not conserved by electroweak interactions and “penguin”-type diagrams. This, however, did not tell us what would fix the value of ρ .

⁹ But are they really necessary since, when neutrinos are massive, flavor-changing (like $\mu - e$) transitions are induced through electroweak loops?

mass stays undetermined; this doublet commutes indeed with all other doublets of the same type. The eventual occurrence of the same phenomenon in the fermionic case, and the presence of a robust ambiguity, in the case of truly independent and anticommuting generations of fermions, should consequently not be overlooked.

7 Conclusion

Following a similar line of argumentation as we did for the neutral kaon system, we have shown that the existence of truly independent, anticommuting, generations of fermions leads to an ambiguity in the mass spectrum of neutrinos.

The introduction of the parameter ρ with the dimension of $[mass]$, closely related to the replication of families, sets a strong correlation between mixing angles, CP violation, and the hierarchy of masses. The presence of ρ brings a new point of view on the presence of large mixing angles for leptons [2] which are no longer preferentially associated with the smallest hierarchies [5].

If more generations are present, several arbitrary ρ -like parameters can be introduced; the variation of any among them will modify the mass pattern of neutrinos.

The main problem is that, for independent generations, ρ cannot play any role since it corresponds to a vanishing interaction term in the Lagrangian.

A first possible attitude would be to prove that introducing such terms is forbidden in field theory.

Otherwise, breaking this indetermination most likely requires that the generations be not independent, hence not anticommuting; the simplest assumption is, in analogy with the neutral kaon system, that subconstituents of fermions undergo a new type of interaction which is misaligned with electroweak interactions. However, introducing a new level of constituents for leptons is unwelcome, unless several other converging compelling constraints urge to do it.

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