Spectrum of q-Deformed Schrödinger Equation

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Abstract

The energy spectrum of q-deformed Schrödinger equation is demonstrated. This spectrum includes an exponential factor with new quantum numbers–the q-exciting number and the scaling index. The pattern of quark and lepton masses is qualitatively explained by such a q-deformed spectrum in a composite model.

Quarks and leptons are considered as point-like particles which is correct at least down to $10^{-17}cm$. The regularities found in the quark and lepton mass spectrum and several other motivations suggest a possibility that at short distances much smaller than $10^{-17}cm$ quarks and leptons are composites of some basic entities. For an earlier consideration of a substructure, see e. g. Refs. [1, 2]. One of the common problem of composite models is to understand the mysterious mass spectrum of quarks and leptons. The striking features of the quark-lepton mass spectrum are that it is similar (the family structure and the generation structure) and has a large mass range (the mass ratios reach as large as 10^5 order) with exponential interval.

Because such substructures involve distances which are many orders of magnitude below ones of present physics, dynamics at very short distances may be radically different, and is likely to involve some entirely new principles. According to the present tests of quantum electrodynamics, quantum theories based on Heisenberg's commutation relation (Heisenberg's algebra) are correct at least down to 10^{-17} cm. For possible new quantum theories it is likely that a modification of Heisenberg's algebra must be at short distances much smaller than 10^{-17} cm. In searching for such a possibility at short distances consideration of the space structure is a useful guide.

Recently q-deformed quantum mechanics is proposed [3–9] in the framework of quantum group. Quantum groups are a generalization of symmetry groups which we have successfully used in physics. A general feature of space carrying a quantum group structure is that they are noncommutative and inherit a well-defined mathematical structure from quantum group symmetries. There is a possibility that noncommutativity of space might be a realistic picture of space at short distances. q-deformed quantum mechanics is based on the q-deformed Heisenberg algebra [4, 6] which is a generalization of Heisenberg's algebra. Starting from the q-deformed Heisenberg algebra a general dynamical equation of q-deformed quantum mechanics is obtained [9]. A general feature of this equation is that its energy spectrum shows an exponential q-structure [6–9]. In this letter we report that the pattern of the quark-lepton masses can be qualitatively explained by such a q-deformed spectrum in a composite model, for example, the rishon model [2]. The calculated quark and lepton masses agree with known data.

In the q-deformed phase space Refs. [4, 6] generalized the Heisenberg algebra to the following q-deformed Heisenberg algebra:

$$q^{1/2}XP - q^{-1/2}PX = i\Lambda^{-1}, \quad \Lambda X = qX\Lambda, \quad \Lambda P = q^{-1}P\Lambda, \tag{1}$$

where the position X and the momentum P are hermitian, Λ is unitary:

$$X^\dagger = X, \quad P^\dagger = P, \quad \Lambda^\dagger = \Lambda^{-1}. \tag{2}$$

In (1) the parameter q is real and $q > 1$, Λ is called scaling operator. The algebra (1) is based on the definition of the hermitian P. However, if X is assumed to be a hermitian operator in a Hilbert space, the usual quantization rule $P \to -i\partial_X$ does not yield a hermitian momentum operator. Ref. [6] showed that a hermitian momentum operator P is related to ∂_X and X in a nonlinear way by introducing a scaling operator Λ

$$\Lambda \equiv q^{1/2}[1 + (q-1)X\partial_X], \quad \bar{\partial}_X \equiv -q^{-1/2}\Lambda^{-1}\partial_X, \quad P \equiv -\frac{i}{2}(\partial_X - \bar{\partial}_X),$$

where $\bar{\partial}_X$ is the conjugate of ∂_X. Because the scaling operator Λ is introduced in the definition of the hermitian momentum operator, it closely relates to properties of dynamics and plays an essential role in q-deformed quantum mechanics. The nontrivial properties of Λ leads to that the algebra (1) has a richer structure than the Heisenberg algebra. In the case of q approaching to one the scaling operator Λ reduces to the unit operator, thus the algebra (1) reduces to the Heisenberg algebra.

The variables X, P of the q-deformed algebra (1) can also be expressed in terms of the variables of an undeformed algebra. There are three pairs of canonically conjugate variables [6]:

1. The variables $\hat{x}$ and $\hat{p}$ of the undeformed quantum mechanics which satisfy: $[\hat{x}, \hat{p}] = i$.

2. The variables $\tilde{x}$ and $\tilde{p}$ which are obtained by a canonical transformation of $\hat{x}$ and $\hat{p}$: $\tilde{p} = f(\hat{z})\hat{p}$, $\tilde{x} = \hat{x}f^{-1}(\hat{z})$ where

$$f^{-1}(\hat{z}) = \frac{[\hat{z} - \frac{1}{2}]}{\hat{z} - \frac{1}{2}}, \quad \hat{z} = -\frac{i}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}), \tag{3}$$

and $[A] = (q^A - q^{-A})/(q - q^{-1})$. The function defined by (3) satisfy $\hat{x}f(\hat{z}) = f(\hat{z} + 1)\hat{x}$, $\hat{p}f(\hat{z}) = f(\hat{z} - 1)\hat{p}$. The variables $\tilde{x}$ and $\tilde{p}$ also satisfy the same commutation relation as

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$\hat{x}$ and $\hat{p}$: $[\tilde{x}, \tilde{p}] = i$. Thus in the $\tilde{x}$ representation $\tilde{p} = -i\tilde{\partial}$, where $\tilde{\partial} = \partial/\partial\tilde{x}$, and all the machinery of quantum mechanics can be used for the $(\tilde{x}, \tilde{p})$ system.

3. The q-deformed variables X and P, where X, P and the scaling operator Λ are related to $\tilde{x}$ and $\tilde{p}$ in the following way:

$$X = \tilde{x}, \quad P = f^{-1}(\tilde{z})\tilde{p}, \quad \Lambda = q^{-\tilde{z}}. \tag{4}$$

In (4) $\tilde{z}$ and $f^{-1}(\tilde{z})$ are defined by the same equations (3) for $\hat{z}$ and $f^{-1}(\hat{z})$. It is easy to check that X, P and Λ in (4) satisfy (1) and (2).

In order to derive the dynamical equation of q-deformed quantum mechanics our starting point is to use the q-deformed variables to write down the Hamiltonian in analogy with the undeformed system, then using (4) to represent X, P and Λ by $\tilde{x}$ and $\tilde{p}$. From (1) and (2) it follows that

$$P^2 = -(q - q^{-1})^{-2} X^{-2} \left[q^2 \Lambda^{-2} - (q + q^{-1}) + q^{-2} \Lambda^2 \right]. \tag{5}$$

Using (4) and (5) we obtain the stationary dynamical equation of q-deformed quantum mechanics

$$\left\{ -\frac{1}{2\mu}(q - q^{-1})^{-2}\tilde{x}^{-2} \left[q(q^{-2\tilde{x}\tilde{\partial}} - 1) + q^{-1}(q^{2\tilde{x}\tilde{\partial}} - 1) \right] + V(\tilde{x}) \right\} \psi(\tilde{x}) = E_q \psi(\tilde{x}). \tag{6}$$

Eq. (6) is a non-linear equation, which is a q-generalization of the Schrödinger equation.

For the case of q closing to 1, let $q = e^f$, $0 < f \ll 1$. The perturbation expansion of (6) reads

$$\begin{aligned} H = \quad & -\frac{1}{2\mu}(2f + \frac{1}{3}f^3 + \cdots)^{-2}\tilde{x}^{-2} \big[4f^2\tilde{x}^2\tilde{\partial}^2 \\ & + \frac{1}{3}f^4(4\tilde{x}^4\tilde{\partial}^4 + 16\tilde{x}^3\tilde{\partial}^3 + 10\tilde{x}^2\tilde{\partial}^2) + \cdots \big] + V(\tilde{x}). \end{aligned} \tag{7}$$

To the lowest order of f, (6) reduces to the Schrödinger equation of the $(\tilde{x}, \tilde{p})$ system

$$\left[-\frac{1}{2\mu}\tilde{\partial}^2 + V(\tilde{x}) \right] \psi(\tilde{x}) = E\psi(\tilde{x}). \tag{8}$$

In (7) the next order correction of H shows a complex structure which amounts to some additional momentum dependent interaction. As an example, we consider the harmonic

oscillator potential $V(\tilde{x}) = \frac{1}{2}\mu\omega^2\tilde{x}^2$. The spectrum of the un-perturbation Hamiltonian $H_0 = -\frac{1}{2\mu}\tilde{\partial}^2 + \frac{1}{2}\mu\omega^2\tilde{x}^2$ is equal interval, $E_n^{(0)} = \omega(n+\frac{1}{2})$, $(n = 0, 1, 2, \cdots)$. The perturbation Hamiltonian H_I in (7) is

$$H_I = -\frac{f^2}{12\mu}(2\tilde{x}^2\tilde{\partial}^4 + 8\tilde{x}\tilde{\partial}^3 + 3\tilde{\partial}^2). \tag{9}$$

The perturbation shifts of the energy levels are $\Delta E_n^{(0)} = \int d\tilde{x}\psi_n^{(0)*}(\tilde{x})H_I\psi_n^{(0)}(\tilde{x})$, where

$$\psi_n^{(0)}(\tilde{x}) = 2^{-n/2}(n!)^{-1/2}(\mu\omega/\pi)^{1/4}exp(-\mu\omega\tilde{x}^2/2)H_n(\sqrt{\mu\omega}\tilde{x})$$

and H_n is the Hermite polynomials. The first few shifts are $\Delta E_0^{(0)} = -3f^2\omega/16$, $\Delta E_1^{(0)} = -5f^2\omega/12$, $\Delta E_2^{(0)} = -f^2\omega/2$, etc. Thus the intervals of the total spectrum $E_n = E_n^{(0)} + \Delta E_n^{(0)}$ are not equal.

For the non-perturbation properties of the q-deformed system, we consider a simple case, the "free" Hamiltonian $H_0 = \frac{1}{2\mu}P^2$. Suppose that the eigenvalue of H_0 is solved: $H_0|\epsilon_0\rangle = \epsilon_0|\epsilon_0\rangle$ with the normalization condition of $|\epsilon_0\rangle$, $\langle\epsilon_0'|\epsilon_0\rangle = \delta(\epsilon_0' - \epsilon_0)$. H_0 is semi-positive definite, i.e. $\epsilon_0 \geq 0$. The state $|\epsilon_0\rangle$ is a common eigenstate of H_0 and the momentum P, $P|\epsilon_0\rangle = \pm(2\mu\epsilon_0)^{1/2}|\epsilon_0\rangle$, here the plus and minus sign correspond, respectively, to the right and left moving modes. Now we consider the action of the scaling operator Λ on the state $|\epsilon_0\rangle$. From the algebra (1) we have $H_0(\Lambda^M|\epsilon_0\rangle) = \epsilon_0 q^{2M}(\Lambda^M|\epsilon_0\rangle)$. i. e. $\Lambda^M|\epsilon_0\rangle = |\epsilon_0 q^{2M}\rangle$ with the normalization condition $\langle\epsilon_0' q^{2M'}|\epsilon_0 q^{2M}\rangle = \delta_{M'M}\delta(\epsilon_0' - \epsilon_0)$. Thus in the general case the q-deformed spectrum of H_0 is $H_0|\epsilon_0 q^{2nM}\rangle = E_n|\epsilon_0 q^{2nM}\rangle$, $|\epsilon_0 q^{2nM}\rangle = (\Lambda^M)^n|\epsilon_0\rangle$, $E_n = \epsilon_0 q^{2nM}$, where $n = 0, 1, 2, \cdots$; $M = 0, 1, 2, \cdots$. The unde-formed energy ϵ_0 is determined by dynamics, the exponential factor q^{2nM} (the q-exciting structure) is determined by the algebra (1). The non-trivial properties of the scaling operator Λ leads to a richer structure of the algebra (1) than the Heisenberg algebra and, as a result, leads to the q-structure of the spectrum. This spectrum includes new quantum numbers, the q-exciting number n and the scaling index M. In the case of $q = 1$, which corresponds to the present-day physics, the q-exciting degree of freedom freezes, and the q-exciting spectrum E_n reduces to the undeformed one ϵ_0.

In the following in a composite scheme we use the q-deformed spectrum to qualitatively explain the mass pattern of quarks and leptons. The only strong gauge interaction that

we understand in any detail is QCD. But for the non-perturbation aspect of the strong coupling it is not clear in QCD how to calculate bound states of quarks to yield the hadron spectrum except for some simple cases which can be treated by lattice QCD. Thus it is helpful to treat the hadron spectrum by some phenomenological approaches which may guide us to the right direction. In analogy with calculations of the hadron mass spectrum we calculate bound states of substructure to yield the quark-lepton mass spectrum in a phenomenological approach. Suppose that the composite system is described by the Hamiltonian $H = H_0 - V_0$, where H_0 is the "free" Hamiltonian and $V_0 > 0$ is the binding energy. In order to reduce the number of phenomenological parameters we consider a simplified example, the rishon model [2] in which the most economical set of building blocks consists of two spin $J = 1/2$ rishons, a charged $T(Q = 1/3)$ and a neutral $V(Q = 0)$. Their antiparticles are $\bar{T}(Q = -1/3)$ and $\bar{V}(Q = 0)$. The first generation of composite fermions is u-quark (TTV, TVT, VTT), d-quark $(\bar{T}\bar{V}\bar{V}, \bar{V}\bar{T}\bar{V}, \bar{V}\bar{V}\bar{T})$, neutrino $\nu_e(VVV)$ and electron $e(\bar{T}\bar{T}\bar{T})$. The dynamics is supposed to be that the three states of quarks are degenerate. The q-deformed spectrum in the above simplified example is

$$H|\epsilon_0^{(i)}q^{2nM_i}\rangle = E_n^{(i)}|\epsilon_0^{(i)}q^{2nM_i}\rangle, \tag{10}$$

$$E_n^{(i)} = \epsilon_0^{(i)}q^{2nM_i} - V_0^{(i)}, \quad (n = 0, 1, 2, \cdots; M_i = 0, 1, 2, \cdots), \tag{11}$$

$$\epsilon_0^{(1)} = 2\mu_T + \mu_V, \quad \epsilon_0^{(2)} = \mu_T + 2\mu_V, \quad \epsilon_0^{(3)} = 3\mu_T, \quad \epsilon_0^{(4)} = 3\mu_V, \tag{12}$$

$$V_0^{(1)} = V_0^{(2)} = V_0^{(3)} = V_0, \quad V_0^{(4)} = V_0'. \tag{13}$$

The physical contents of the spectrum (10)-(13) are as follows. The index $i = 1, 2, 3, 4$ represents families, i. e. the scaling indices M_1, M_2 and M_3, M_4 correspond to, respectively, the quark families (u, c, t), (d, s, b) and the lepton families (e, μ, τ), $(\nu_e, \nu_\mu, \nu_\tau)$. The q-exciting quantum numbers $n = 0, 1, 2$ correspond to, respectively, the first, the second and the third generation $(u, d; \nu_e, e)$, $(c, s; \nu_\mu, \mu)$ and $(t, b; \nu_\tau, \tau)$ [1] . The undeformed spectrum

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[1]In composite models the next generation are simply considered as higher excitations of the first gen-

$\epsilon_0^{(i)}$ are flavor dependent, and μ_T and μ_V are, respectively, the ground state energies of the "free" T and V. The binding energy V_0' of the neutrino family is supposed to be different from the other ones.

Eq. (12) gives a simple regularity among m_u, m_d and m_e

$$2m_u - m_d = m_e. \tag{14}$$

which we were not previously aware of.

If we put in

$$q^{2M_1} = 128.98, \quad q^{2M_2} = 17.28, \quad q^{2M_3} = 15.90,$$
$$\mu_T = 2.35\, MeV, \quad \mu_V = 5.70\, MeV,$$
$$V_0 = 6.55\, MeV. \tag{15}$$

Eqs. (11)-(13) give (in MeV units)

$$
\begin{aligned}
1.\ m_u &= E_1^{(1)} = 4\ (1.5 - 5), \\
m_c &= E_2^{(1)} = 1340\ (1100 - 1400), \\
m_t &= E_3^{(1)} = 170 \times 10^3 \left(\begin{array}{c} (73.8 \pm 5.2) \times 10^3 \\ (170 \pm 7(+14)) \times 10^3 \end{array} \right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
2.\ m_d &= E_1^{(2)} = 7\ (3 - 9), \\
m_s &= E_2^{(2)} = 230\ (60 - 170), \\
m_b &= E_3^{(2)} = 4100\ (4100 - 4400).
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
3.\ m_e &= E_1^{(3)} = 0.5\ (0.51099907 \pm 0.00000015), \\
m_\mu &= E_2^{(3)} = 106\ (105.658389 \pm 0.000034), \\
m_\tau &= E_3^{(3)} = 1777 \left(1777.05\ {}^{+0.29}_{-0.26} \right).
\end{aligned}
\tag{18}
$$

eration. At the present stage one of the common open problems in composite models is that there is no principle to govern a choice of the value n of generations. The only way of fixing the maximum value of n is the experimental results from measurements at the Z peak. These measurements establish the number of light neutrino generations to be $n_\nu = 2.994 \pm 0.012$ (Standard Model fits to LEP data) and $n_\nu = 3.07 \pm 0.12$ (Direct measurement of invisible Z) [12].

And

$$m_u/m_d = 0.54 \ (0.20 - 0.70),$$

$$m_s/m_d = 32 \ (17 - 25),$$

$$\bar{m} = (m_u + m_d)/2 = 5.6 \ (2 - 6),$$

$$[m_s - (m_u + m_d)/2]/(m_d - m_u) = 68 \ (34 - 51). \tag{19}$$

In (16)-(19) the data in the brackets are cited from Ref. [10]. (In (16) for top quark the datum in the first line from direct observation of top events; the one in the second line from Standard Model electroweak fit). The calculated masses agree with known data [2] .

At present little about neutrino masses can be predicted because of lack of definite data. In Standard Model of particles neutrinos could be exactly massless, although this would violate naive quark-lepton symmetry. According to (11) massless neutrinos correspond to $\epsilon_0^{(4)} = V_0^{(4)}$, $q^{2M_4} = 1$. There are several hints for non-vanishing neutrino masses which can be inferred from the observations of the solar neutrinos [11], the atmospheric neutrinos [12] and the results of LSND [13]. These data can be understood in terms of neutrino oscillations which depend on the different neutrino mass-squared differences Δm_{ij}^2. The solution of the MSW type [14] to the solar neutrino puzzle yields the so-called small angle solution [11] $\Delta m_{ei}^2 \sim 4 \times 10^{-6} - 1.2 \times 10^{-5} \, eV^2$. Assuming $\nu_\mu - \nu_\tau$ oscillation the presently available atmospheric neutrino data yields [12] $\Delta m_{\mu\tau}^2 \sim 4 \times 10^{-4} - 5 \times 10^{-3} \, eV^2$. Finally, the LSND data suggests [13] that $\Delta m_{e\mu}^2 \sim 0.2 - 10 \, eV^2$. The Solar MSW small angle and atmospheric neutrino along [15, 16] indicate very small differences between the neutrino masses [3] . We may suggest a degenerate scheme where all three masses are large relative to their splitting and almost degenerate. There is no clear way to set a meaningful limit on m_{ν_e}. If the three neutrinos are the candidate for the hot dark matter, an estimation of the total mass of neutrinos is about 4.8 eV [17]. In (11)-(13) the estimations of $m_{\nu_e} \sim m_{\nu_\mu} \sim m_{\nu_\tau} \sim 1.6 \, eV$,

[2]The concept of quark mass is involved. The values of the quark masses depend on the energy scale where they are calculated. As in the quark model of hadrons, the free parameters in (11)-(13) are a phenomenological input of the theory. In (16) and (17) the values of m_u, m_t and m_b cited from Ref. [12] are used to fix the parameters. Thus the energy scale of the calculated quark masses in (16) and (17) is related to the energy scale of quark masses cited in Ref. [12].

[3]For the mass-squared differences Δm_{ei}^2 in the MSW small angle result, the type of neutrino ν_i is, depending on the specific version of the effects, a ν_μ, a ν_τ, a $\nu_\mu - \nu_\tau$ mixture, or perhaps a sterile neutrino ν_s.

$\Delta m_{e\mu}^2 \sim \Delta m_{\mu\tau}^2 \sim 10^{-4} \, eV^2$ correspond to inputs $3\mu_V - V_0' \sim 1.6 \, eV$, $q^{2M_4} - 1 \sim 10^{-10}$.

In (16)-(18) 9 observed masses are explained by a fit to 6 free parameters, which are a phenomenological input of the theory. If we find an effective way to solve the nonlinear q-deformed Schrödinger equation (6), the parameters μ_T, μ_V and V_0 are expected to be calculable.

If q-deformed quantum mechanics is a correct theory, its effects mainly manifest at short distances much smaller than $10^{-17} \, cm$. At such short distances if rishon dynamics is governed by a q-deformed gauge theory, we may expect a better explanation of the quark-lepton mass spectrum by bound state calculation in q-deformed gauge theory. Of course, this will be very difficult topics, perhaps much difficult than bound state treatment in QCD.

In summary, in this letter we show that the new degrees of freedom in the q-deformed spectrum emerge. The qualitative explanation of the mass spectrum of quarks and leptons by this spectrum is encouragement which may guide us to the right direction in understanding dynamics at very short distances.

Acknowledgements

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