BRANE THEORY SOLITONS

Cargèse Lectures 1999

P.K. TOWNSEND
CENTRE FOR MATHEMATICAL SCIENCES,
UNIVERSITY OF CAMBRIDGE,
WILBERFORCE ROAD,
CAMBRIDGE, U.K.

Abstract. Field theories that describe *small* fluctuations of branes are limits of 'brane theories' that describe *large* fluctuations. In particular, supersymmetric sigma-models arise in this way. These lectures discuss the soliton solutions of the associated 'brane theories' and their relation to calibrations.

1. Preamble

The last five years or so have seen some exciting developments in high energy/gravitational physics, with branes as the common feature. In particular, branes have revolutionized our ideas about quantum field theory, both on the technical level, by giving us new and powerful methods that allow us to go beyond perturbation theory, and on the conceptual level, by providing us with a new insight into its nature: it now seems likely that all consistent quantum field theories can be viewed as effective descriptions of low-energy fluctuations of branes.

The small fluctuations of a single brane are governed by a free field theory. Going beyond small fluctuations, but still on a single brane, introduces interactions but these are of higher-derivative type, and hence associated with a characteristic length scale L. The 11-dimensional supermembrane provides a simple example, with L determined by the membrane tension. Interactions of conventional field theory type arise from *inter-brane* interactions. These will dominate if the branes are separated by distances much less than L but if L also sets the scale for the brane 'core', as will typically be the case, then the inter-brane dynamics cannot be separated from the unknown core dynamics. Exceptions to this state of affairs can arise only when there is a separate length scale L_c determined by the size of the core, with $L_c \ll L$, and this implies the existence of a small dimensionless constant $g = L_c/L \ll 1$. This scenario is realized by the D-branes of superstring theory, with $L = l_s$ the string length (set by the string tension) and $g^{p+1} = g_s$, for a Dp-brane, with g_s the string coupling constant (which must be small for superstring theory to be a valid approximation); in this case the inter-brane interactions between parallel D-branes separated by distances l with $g_s l_s \ll l \ll l_s$ are supersymmetric gauge theories. If, instead, $l \gg l_s$ then we cannot ignore interactions due to brane fluctuations but we can ignore inter-brane interactions. In this limit the dynamics of each brane is governed (at sufficiently low energy) by a Dirac-Born-Infeld (DBI) action. At intermediate length scales we have interactions of both types. This regime is the one that we are going to study in these lectures, although not for D-branes. We want to include the inter-brane interactions that can lead to interacting field theories, but we want to go beyond the field theory approximation by including the interactions due to large brane fluctuations. We shall call this 'brane theory'.

One might expect some general features of field theory to remain valid in brane theory, and others not. Of importance to these lectures is the fact that many supersymmetric field theories admit supersymmetric soliton solutions saturating a Bogomolnyi-type bound. As these bounds are usually a consequence of the supersymmetry algebra one would expect them to hold beyond the field theory approximation, but it is not immediately clear how, or whether, the bound continues to be saturated because the brane theory equations are different. In the D-brane case, for example, the equations are of non-abelian DBI type. They reduce to standard gauge theory equations in the field theory limit, with their standard gauge theory soliton solutions, but to determine whether these solitons continue solve the brane theory equations and, if so, whether they continue to saturate the energy bound implied by supersymmetry requires a precise knowledge of the non-abelian DBI equations. Here we confront a general difficulty: to get interactions of field theory type we need more than one brane, but the inter-brane interactions are known precisely only in the field theory limit.

There is one way to avoid this dilemma. We can fix our attention on one brane and replace the others with which it interacts by the supergravity background that they induce at 'our' brane. This way we take the inter-brane interactions into account in an approximation that is exact in the limit of large brane charges for the 'other' branes, which are effectively macroscopic and can be replaced by the supergravity background they induce. What we now have is a single brane in a brane background. As applied to D-branes, this method can be used to recover many of the finite-energy soliton solutions of D=4 SYM theory but as soliton solutions on a *sin-*

gle brane, with an *abelian* gauge group [1]. In this application, the 'other' branes are parallel D3-branes. Another application of this idea, and one to be reviewed in lecture 2, involves M2-branes in a two-centre M-monopole (i.e. M-theory Kaluza-Klein monopole) background, which allows a brane realization [2] of the sigma model 'lump' soliton of hyper-Kähler sigma-models. Of course, other backgrounds can yield brany generalizations of sigma-models with other target spaces, and other branes can yield models in other dimensions. A general feature is that the relativitistic aspects of the field theory are extended to a bulk-space relativity. This is explained in the first lecture, where it is presented as a consequence of implementing the principle of 'field-space democracy' [3].

It is also a general feature that sigma-model solitons survive as solutions of the brane equations and continue to minimize the energy in their charge sector. Moreover, these solitons now acquire a new geometrical interpretation, as minimal surfaces in the simple cases discussed in these lectures. At this point we can see that brane theory goes beyond field theory because there are many types of minimal surface and not all of them have an interpretation as field theory solitons. Derrick's theorem states that static minimal energy solutions of (conventional) scalar field theories in p space dimension cannot exist for $p > 2$, but Derrick's theorem no longer applies once we have made the transition from field theory to brane theory.

When discussing solitons that minimise the energy it is natural to start from the Hamiltonian rather than the Lagrangian, and this will be the strategy adopted here, following [4]. Only the standard Dirac-type p-brane action, and its Hamiltonian, will be needed in these lectures, apart from the coupling to a background (p+1)-form gauge potential A, which will play a minor role because only backgrounds with vanishing field strength $F = dA$ will arise. Neither will we need terms involving worldvolume gauge fields because no attempt will be made here to review the brane theory status of *gauge theory* solitons. A further restriction will be to *static* solitons. A general framework for those cases that remain is provided by the theory of calibrations [5]. As we shall see in the third lecture, sigma model solitons on the M2-brane fit into this framework as examples of Kähler calibrations [6], but for other branes there are solitons with no field theory analogue that arise from more complicated types of calibration [7, 8]. The simplest example, albeit one with infinite energy, is the Special Lagrangian 3-surface in $\mathbb{E}^6$. Its realization by intersecting M5-branes [9, 10, 11] will be reviewed in the fourth, and last, lecture. Hopefully, the detailed treatment of these few cases of 'brane theory solitons' will compensate for the restricted focus.

One other restriction is implicit in the above discussion. We have taken the term 'field theory' to exclude gravity (and hence supergravity). It has long been appreciated that gravity is a rather special kind of 'field theory', and branes have provided us with a new reason for believing this. Gravitons (and superpartners) propagate in the 'bulk' while 'matter' propagates on branes. There are reasons for this that go beyond the simple statement that it is 'difficult' to confine gravity to a brane but these reasons need to be reassessed in light of the demonstration [12] that it is, nevertheless, possible. The epilogue that concludes these lectures deals with some of these issues.

2. Lecture 1: Field theory vs Brane Theory

The n scalar fields of a non-linear sigma-model define a map from a (p+1)-dimensional Minkowski spacetime W with metric η (diagonal with entries $(-1,1,\ldots,1)$ in cartesian coordinates) to an n-dimensional Riemannian target space M with metric G. Let ξ^μ, $(\mu=0,1,\ldots,p)$ be *cartesian* coordinates for W, and let X^i $(i=1,\ldots,n)$ be coordinates for M, so that $X^i(\xi)$ are the scalar fields. The Lagrangian density of the massless sigma model is then

$$\mathcal{L}_\sigma = \frac{1}{2}\eta^{\mu\nu}\partial_\mu X^i \partial_\nu X^j G_{ij}(X)\,. \tag{1}$$

The corresponding Hamiltonian density is

$$\mathcal{H}_\sigma = \frac{1}{2}G^{ij}P_iP_j + \frac{1}{2}\boldsymbol{\nabla} X^i\cdot\boldsymbol{\nabla} X^j G_{ij}\,, \tag{2}$$

where the variables $P_i(\xi)$ are the momenta canonically conjugate to the fields $X^i(\xi)$. Although this field theory is 'relativistic' as a field theory on W, there is another sense in which it is *not* relativistic. Consider the special case of $p=0$, and set $P_i = p_i/\sqrt{\mu}$; then $\mathcal{H}_\sigma = H_{NR}$, where

$$H_{NR} = \frac{1}{2\mu}G^{ij}p_ip_j\,. \tag{3}$$

This is the Hamiltonian of a *non-relativistic* particle of mass μ. This may be contrasted with the Hamiltonian

$$H = \sqrt{G^{ij}p_ip_j + \mu^2} \tag{4}$$

for a *relativistic* particle. Is there an analogous '*relativistic*' Hamiltonian density $\mathcal{H}$ for a scalar field theory?

There is, and one way to find it is by implementation of 'field-space democracy' [3] (a principle invoked for similar reasons in [13]). We begin with a Lorentzian spacetime $\mathcal{M}$ of dimension $D=p+n+1$ with coordinates (ξ^μ, X^i). The sigma model fields $X^i(\xi)$ now define a (p+1)-dimensional surface W in $\mathcal{M}$. This (p+1)-surface can also be specified parametrically by giving D coordinates X^m $(m=0,1,\ldots,p+n)$, as functions of $(p+1)$

parameters ξ^μ, such that $X^\mu = \xi^\mu$ for some particular parametrization. The equations for the (p+1)-surface W now take the form $X^m = X^m(\xi)$. We will assume that the induced metric on W is Lorentzian, so that W is a (p+1)-dimensional worldvolume swept out in time by a p-dimensional 'worldspace' w. The evolution of w is governed by an action $S[X]$ that must be reparametrization invariant in order to allow, at least locally, the 'physical' gauge choice

$$X^\mu(\xi) = \xi^\mu \,. \tag{5}$$

Let $\mathcal{G}$ be the Lorentzian metric on $\mathcal{M}$ and let g be the metric it induces on W. The reparametrization invariant action with a Lagrangian density of lowest dimension is

$$S = -T \int d^{p+1}\xi \, \sqrt{-\det g} \,, \tag{6}$$

which is the Dirac-Nambu-Goto action for a p-brane of tension T.

To make contact with the sigma model we set $T = 1$ and take the metric $\mathcal{G}$ on $\mathcal{M}$ to be of the form

$$\mathcal{G}_{mn} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ij} \end{pmatrix} . \tag{7}$$

The physical gauge metric that this induces on W is

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu X^i \partial_\nu X^j G_{ij} \,. \tag{8}$$

Choosing local *cartesian* coordinates for W, we then have

$$T^{-1}\mathcal{L} = -\det g = 1 + \frac{1}{2}\eta^{\mu\nu}\partial_\mu X^i \partial_\nu X^j G_{ij} + \mathcal{O}\left((\partial X)^4\right) . \tag{9}$$

Apart from the constant term, and the higher-derivative corrections, this is the sigma-model Lagrangian density. The low-energy 'non-relativistic' dynamics of the brane is therefore governed by the sigma-model. To complete the picture we now need to determine the p-brane Hamiltonian and show that it provides the required generalization of the relativistic particle hamiltonian.

We could find the physical-gauge Hamiltonian by performing a Legendre transformation on the gauge-fixed Lagrangian. Instead, we will first proceed to the Hamiltonian form of the gauge-invariant action. This has the advantage of maintaining manifest invariance under any isometries of $\mathcal{G}$, until a gauge choice is made. We will need to make a worldvolume space/time split, so we write

$$\xi^\mu = (t, \sigma^a) \,, \tag{10}$$

where σ^a $(a = 1, \ldots, p)$ are coordinates for the p-dimensional worldspace w. Let us write the induced metric on W as

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{ta} \\ g_{tb} & m_{ab} \end{pmatrix} , \tag{11}$$

so that m is the metric induced on w, with components

$$m_{ab} = \partial_a X^m \partial_b X^n \mathcal{G}_{mn} \,. \tag{12}$$

One can now use standard methods to obtain the Hamiltonian form of the action in which the independent variables are the scalar fields $X^m(\xi)$ and their canonically conjugate momenta $P_m(\xi)$. The result is [14]

$$S = \int dt \int d^p\sigma \, \left[\dot{X}^m P_m - s^i \mathcal{H}_i - \ell \mathcal{H}_t\right] , \tag{13}$$

where s^a and ℓ are Lagrange multipliers for the constraints $\mathcal{H}_a = 0$ and $\mathcal{H}_t = 0$, with

$$\mathcal{H}_a = \partial_a X^m P_m \,, \qquad \mathcal{H}_t = \frac{1}{2}\left(\mathcal{G}^{mn} P_m P_n + T^2 \det m\right) . \tag{14}$$

This form of the action is to be expected from the general covariance of the initial action (6). The Lagrange multipliers are analogous to the 'shift' and 'lapse' functions of General Relativity. The difference is that the geometry here is *extrinsic* whereas that of General Relativity is *intrinsic*.

It is simple to verify that the result given above is correct. Elimination of P_m in (13) by its Euler-Lagrange equation yields

$$S = \int dt \int d^p\sigma \, \left[\frac{1}{2\ell}\left(g_{tt} - 2s^a g_{ta} + s^a s^b m_{ab}\right) - \frac{1}{2}T^2 \ell \det m\right] . \tag{15}$$

We now eliminate s^a by its Euler-Lagrange equation, set

$$\ell = v / \det m \,, \tag{16}$$

and use the identity

$$\det g \equiv \det m \left(g_{tt} - m^{ab} g_{ta} g_{tb}\right) , \tag{17}$$

to get

$$S = \int dt \int d^p\sigma \, \left[\frac{1}{2v}\det g - \frac{1}{2}T^2 v\right] . \tag{18}$$

This is a well-known alternative form of the p-brane action. Provided $T \neq 0$, which we assume here, we can eliminate v by its algebraic Euler-Lagrange equation to recover (6).

To find the physical gauge Hamiltonian we have only to substitute the 'physical' gauge choice (5) into the constraints and then solve them for the momenta P_μ. However, it is instructive to proceed sequentially, first fixing only the time parameterization by the gauge choice $X^0(\xi) = t$. If we rename P_0 as $-\mathcal{H}$, and define

$$X^I = (X^a, X^i)\,, \qquad P_I = (P_a, P_i)\,, \tag{19}$$

then we now have

$$X^m = (t, X^I)\,, \qquad P_m = (-\mathcal{H}, P_I)\,. \tag{20}$$

It will also prove convenient to write the spacetime metric $\mathcal{G}$ as

$$\mathcal{G}_{mn} = \begin{pmatrix} G_{00} & G_{0I} \\ G_{0J} & M_{IJ} \end{pmatrix}\,. \tag{21}$$

Note that since $\partial_a X^0 = 0$ the metric m is now

$$m_{ab} = \partial_a X^I \partial_a X^J M_{IJ}\,. \tag{22}$$

The Hamiltonian constraint $\mathcal{H}_t = 0$ can now be solved to yield

$$\mathcal{H} = \mathcal{N}^I P_I \pm \mathcal{N}\sqrt{M^{IJ} P_I P_J + T^2 \det m}\,, \tag{23}$$

where

$$\mathcal{N}^I = -\frac{\mathcal{G}^{0I}}{\mathcal{G}^{00}}\,, \qquad \mathcal{N} = \frac{1}{\sqrt{-\mathcal{G}^{00}}}\,, \tag{24}$$

and M^{IJ} is the inverse of the space metric M_{IJ}. The action (13) now becomes

$$S = \int dt \int d^p\sigma\, \left[\dot{X}^I P_I - \mathcal{H}(X,P) - s^a \mathcal{H}_a\right]\,, \tag{25}$$

which is that of a $(p+1)$-dimensional field theory with Hamiltonian density $\mathcal{H}(X,P)$. The constraint imposed by s^a is *linear* in momenta and can therefore be viewed as the generator of a gauge invariance.

To fix this gauge invariance we set $X^a(\xi) = \sigma^a$. The constraint $\mathcal{H}_a = 0$ can then be solved for P_a,

$$P_a = -\partial_a X^i P_i\,, \tag{26}$$

and the action then takes the canonical form

$$S = \int dt \int d^p\sigma\, \left[\dot{X}^i P_i - \mathcal{H}\right]\,, \tag{27}$$

where the Hamiltonian density is now a function only of the physical phase-space variables (X^i, P_i).

For a metric on $\mathcal{M}$ of the form (7) we have $\mathcal{N}^I = 0$, $\mathcal{N} = 1$ and

$$M_{IJ} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix} . \tag{28}$$

The physical-gauge metric on w is therefore

$$m_{ab} = \delta_{ab} + \partial_a X^i \partial_b X^j G_{ij} \, . \tag{29}$$

The Hamiltonian density for this case is

$$\mathcal{H} = \sqrt{\left(G^{ij} + \boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j\right) P_i P_j + T^2 \det\left(\mathbb{I} + \boldsymbol{\nabla} X^i \boldsymbol{\nabla} X^j G_{ij}\right)} \, . \tag{30}$$

For $p = 0$ the $\boldsymbol{\nabla} X$ terms are absent and $T = \mu$, a mass parameter. The Hamiltonian density then reduces to the Hamiltonian (4) for a relativistic particle; we have thus found the sought $p > 0$ generalization of this Hamiltonian. If we now set $T = 1$ and write (30) as a double expansion in powers of P and $\boldsymbol{\nabla} X$, we find that

$$\mathcal{H} = 1 + \frac{1}{2}\left[G^{ij} P_i P_j + \boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j G_{ij}\right] + \dots \tag{31}$$

The leading term is the p-surface tension energy of the brane. The next term is just the sigma model hamiltonian. The remaining terms, indicated by the dots, are ‘relativistic’ corrections; these can be ignored if (i) all speeds are much less than light, *and* (ii) all fields are slowly varying. To this we should add that the validity of the Dirac-Nambu-Goto action from which we began requires all accelerations to be small.

Although the non-zero vacuum energy is expected, it is natural to define the energy on any given worldspace as

$$\mathcal{E} = \mathcal{H} - T \, , \tag{32}$$

because this vanishes in the vacuum. Although no mention has been made of supersymmetry, it is nevertheless the case that the analogous analysis for a super-p-brane action yields a supersymmetric worldvolume theory for which the energy density must vanish in the vacuum. In fact, it is $\mathcal{E}$, rather than $\mathcal{H}$, that plays the role of the Hamiltonian density in the worldvolume supersymmetry current algebra [15], and this is what allows the brane vacuum to preserve half of the supersymmetry of the spacetime vacuum. This can also be understood, from the spacetime perspective, as due to a p-form charge in the spacetime supersymmetry algebra [16]. For these reasons, we will usually focus on the worldspace energy density $\mathcal{E}$ given in the spacetimes of interest here, and for $T=1$, by the formula

$$(\mathcal{E}+1)^2 = \left(G^{ij} + \nabla X^i \cdot \nabla X^j\right) P_i P_j + T^2 \det\left(\mathbb{I} + \nabla X^i \nabla X^j G_{ij}\right) . \quad (33)$$

3. Lecture 2: Sigma-model solitons on branes

Let us begin with the (2+1) dimensional sigma-model Hamiltonian

$$H_\sigma = \frac{1}{2}\int d^2\sigma \left\{|P|^2 + |\nabla X|^2\right\}, \quad (34)$$

where the norm $|.|$ is defined by contraction with the target space metric and, where applicable, with the Euclidean 2-space metric. We use here standard vector calculus notation for differential operators on $\mathbb{E}^2$. For example, in cartesian coordinates we have

$$\nabla = (\partial_1, \partial_2), \qquad \star\nabla = (\partial_2, -\partial_1) . \quad (35)$$

We will not consider models with fermions, such as supersymmetric models. However, all the models we will consider are supersymmetrizable, so that supersymmetry will be implicit in much of the discussion and it will pay to keep in mind some of its implications. The simplest, N=1, (2+1)-dimensional supersymmetric sigma model has one real $Sl(2;\mathbb{R})$ spinor charge. If the target space has a metric of reduced holonomy then there may be additional supersymmetries. Specifically, if the target space is Kähler then there will be two spinor charges [17] ($N=2$ supersymmetry) and if it is hyper-Kähler there will be four spinor charges [18] (N=4 supersymmetry). A summary of what these Kähler and hyper-Kähler conditions mean now follows.

If the target space M is almost-complex then it will admit an almost complex structure, which is a (1,1) tensor I such that $I^2 = -\mathbb{I}$, where $\mathbb{I}$ is the identity matrix. Given an almost-complex structure I we may define the associated Nijenhuis tensor

$$N_{ij}{}^k(I) = 4\left(\partial_\ell I_{[i}{}^k I_{j]}{}^\ell + \partial_{[i} I_{j]}{}^\ell I_\ell{}^k\right) . \quad (36)$$

If $N(I)$ vanishes then I is a complex structure and M is a complex manifold. A metric G on M satisfying

$$I_{(i}{}^j G_{j)k} = 0 \quad (37)$$

is Hermitian with respect to I. For a Hermitian metric the tensor I_{ij} is antisymmetric and hence defines a 2-form

$$\Omega = \frac{1}{2} I_{ij}\, dX^i \wedge dX^j . \quad (38)$$

The metric is Kähler if this 2-form is closed, $d\Omega = 0$, and Ω is then called the Kähler 2-form (associated to the complex structure I). For a complex manifold, with vanishing Nijenhuis tensor, this condition is equivalent to the apparently weaker condition that I be covariantly constant (with respect to the usual affine metric connection). A hyper-Kähler manifold is one with a metric that is Kähler with respect to three independent complex structures I, J, K, obeying the algebra of the quaternions ($IJ = K$ and cyclic).

We begin our study of solitons by seeking minimal energy configurations of a Kähler sigma model. The Hamiltonian (34) can be rewritten as

$$H_\sigma = \frac{1}{4}\int d^2\sigma \,\{2|P|^2 + |\boldsymbol{\nabla} X \mp \star\boldsymbol{\nabla} X I|^2\} \mp L. \tag{39}$$

where L is the topological 'lump' charge

$$L = \int_w \Omega\,. \tag{40}$$

The integrand is the the Kähler 2-form Ω, which is integrated over the 2-surface w into which the Euclidean 2-space is mapped by the sigma models map. To check the equivalence of (39) to the original form (34) it suffices to note that L cancels against the cross term from

$$G_{ij}\left(\boldsymbol{\nabla} X^i \mp \star\boldsymbol{\nabla} X^k I_k{}^i\right)\left(\boldsymbol{\nabla} X^j \mp \star\boldsymbol{\nabla} X^l I_l{}^j\right)\,, \tag{41}$$

while the identity $I_{(k}{}^i I_{\ell)}{}^j G_{ij} = G_{k\ell}$ ensures equality of the remaining two terms.

Since L is a topological invariant, the variation of the fields for fixed boundary conditions will not change its value, and since the other terms in H are non-negative we deduce the bound [19]

$$H_\sigma \geq |L|\,, \tag{42}$$

which is saturated by *static* solutions of the first order equations

$$\boldsymbol{\nabla} X^i = \pm\star\boldsymbol{\nabla} X^k I_k{}^i\,. \tag{43}$$

Locally we may choose complex coordinates Z^α on a chart of M for which I is diagonal with eigenvalues $\pm i$. We may also view $\mathbb{E}^2$ as the complex plane with complex coordinate $\zeta = \sigma_1 \pm \sigma_2$. The equations (43) then reduce to

$$\bar\partial Z^\alpha = 0\,, \tag{44}$$

where $\bar\partial \equiv \partial/\partial\bar\zeta$. That is, the functions $Z^\alpha(\zeta)$ are holomorphic functions. Globally this means that the solutions of (43) are holomorphic curves on M.

As a simple example, suppose that $M = \mathbb{C}$, with complex coordinate Z and flat metric $dZd\bar{Z}$. The complex sigma-model field is $Z(t,\zeta,\bar{\zeta})$, but as there are no interactions we can hardly expect to find solitons. All the same, it will prove instructive to consider how one might go about looking for them. As we have seen, static solutions of minimum energy correspond to holomorphic functions $Z(\zeta)$. For a localized energy density we require that $|Z| \to 0$ as $|\zeta| \to \infty$. This means that any non-zero $Z(\zeta)$ must have singularities, and the simplest choice is a point singularity at the origin. For this choice we have

$$Z(\zeta) = c/\zeta \tag{45}$$

for complex constant c, with $|c|$ determining the objects's 'size'. It would be misleading to call this object a 'soliton' because its energy is infinite. To see this we note that the Kahler 2-form on $\mathbb{C}$ is $\Omega = idZ \wedge d\bar{Z}$ so its pullback to the complex ζ-plane, when Z is holomorphic has magnitude $|Z'|^2$. The soliton energy is therefore

$$\begin{aligned} E &= \int d^2\sigma |Z'|^2 = |c|^2 \int d^2\sigma |\zeta|^{-4} && (46)\\ &= -\pi |c|^2 [r^{-2}]_0^\infty = [\pi R^2]_0^\infty \,, && (47) \end{aligned}$$

where r is distance from the origin in the ζ-plane, and R is distance from the origin in the Z-plane. The energy is infinite because it equals the infinite area of the target 2-space. In general, a finite energy soliton saturating the energy bound is possible only if the target space has a compact holomorphic 2-cycle. A holomorphic map $Z^\alpha(\zeta)$ then yields finite energy if it maps the ζ-plane to this 2-cycle, and the energy will be the area of the 2-cycle. Obviously, a flat target space, which yields a free field theory, does not have such 2-cycles.

A flat target space has trivial holonomy. In some respects, the simplest non-flat sigma models are those for which the holonomy group is the smallest non-trivial subgroup of $SO(n)$. If one also requires a Ricci flat metric (this being motivated by its ultimate interpretation as part of a background supergravity solution) then the simplest case is $n = 4$ with holonomy $SU(2) \subset SO(4)$. Such 4-manifolds are hyper-Kähler. In this case there is a triplet $\mathbf{I}$ of complex structures. For any unit 3-vector $\mathbf{n}$ the tensor $I = \mathbf{n} \cdot \mathbf{I}$ is also a complex structure, which we can identify as the one of the above discussion. Similarly, $\Omega = \mathbf{n} \cdot \mathbf{\Omega}$, where $\mathbf{\Omega}$ is the triplet of Kähler 2-forms. An important class of hyper-Kähler 4-manifolds are those admitting a tri-holomorphic Killing vector field; that is, a Killing vector k field for which $\mathcal{L}_k \mathbf{\Omega}$ vanishes. All such manifolds are circle bundles over $\mathbb{E}^3$ [20]. We can choose coordinates such that

$$k = \partial/\partial\varphi \,, \tag{48}$$

where φ parametrizes the circle. The metric then takes the form

$$ds_4^2 = V^{-1}(d\varphi - \mathbf{A}\cdot d\mathbf{X})^2 + V d\mathbf{X}\cdot d\mathbf{X} \tag{49}$$

where $\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}V$. This implies that $V(\mathbf{X})$ is harmonic on $\mathbb{E}^3$, except at isolated poles. The metric (49) is complete provided that (i) the residues of V at its poles are equal and positive, and (ii) φ is an angular variable with period 4π times this common residue. Under these circumstances the poles of V are coordinate singularities of the metric, called its 'centres'. If we also take $V\to 1$ as $|\mathbf{X}|\to\infty$ then the metric is asymptotically flat. A simple example is the 2-centre metric with $\varphi\sim\varphi+2\pi$ and

$$V = 1 + \frac{1}{2}\left[\frac{1}{|\mathbf{X}+\mathbf{a}|} + \frac{1}{|\mathbf{X}-\mathbf{a}|}\right]. \tag{50}$$

In terms of the frame 1-forms

$$e^{\varphi} = V^{-\frac{1}{2}}\left(d\varphi - \mathbf{A}\cdot d\mathbf{X}\right), \qquad \mathbf{e} = V^{\frac{1}{2}} d\mathbf{X}, \tag{51}$$

the triplet of Kähler 2-forms is

$$\boldsymbol{\Omega} = e^{\phi}\mathbf{e} - \frac{1}{2}\mathbf{e}\times\mathbf{e}, \tag{52}$$

where the wedge product of forms is implicit here. In the two-centre case there is a preferred direction $\mathbf{n} = \mathbf{a}/|\mathbf{a}|$ and hence a preferred complex structure $\Omega = \mathbf{n}\cdot\boldsymbol{\Omega}$. The 2-centre metric is the simplest multi-centre metric, all of which admit finite energy lump solutions corresponding to holomorphic maps from $\mathbb{C}$ to homology 2-cycles. In the 2-centre case there is just one such 2-cycle. This is the 2-sphere with poles at the centres, where k vanishes, and orbits of k as its lines of latitude. The lump solution can be found from the ansatz $\mathbf{X} = X\mathbf{n}$, which leaves leaves $\varphi(\boldsymbol{\sigma})$ and $X(\boldsymbol{\sigma})$ as the two 'active' coordinates. When restricted to this subspace, the Kähler 2-form is $\Omega = d\varphi\wedge dX$, and hence $|L| = 4\pi|\mathbf{a}|$.

We now wish to generalize these considerations from field theory to brane theory. Our starting point will be the $p=2$ case of the formula (33) for the physical-gauge p-brane energy density $\mathcal{E}$. We expand the 2×2 determinant to obtain

$$\begin{aligned}(\mathcal{E}+1)^2 \;=\;& 1 + |\boldsymbol{\nabla}X|^2 + (G^{ij} + \boldsymbol{\nabla}X^i\cdot\boldsymbol{\nabla}X^j)P_iP_j \\ &+ 2X^{ij}X^{kl}G_{ik}G_{jl}\,,\end{aligned} \tag{53}$$

where we have set

$$X^{ij} \equiv \frac{1}{2}\boldsymbol{\nabla}X^i\times\boldsymbol{\nabla}X^j\,. \tag{54}$$

Previously we were able to express the energy as a sum of a topological charge and a manifestly non-negative integral. This is the trick introduced by Bogomol'nyi for deriving energy bounds in field theory [21]. Its generalization to brane theory involves writing $(\mathcal{E}+1)^2$ as a sum of squares [5, 4, 2]. To simplify we will put the momentum to zero. For the case in hand we can then rewrite (53) as

$$\begin{aligned}(\mathcal{E}+1)^2 &= \left(1 \mp X^{ij} I_{ij}\right)^2 + \frac{1}{2}|\boldsymbol{\nabla} X \mp \star\boldsymbol{\nabla} X I|^2 \\ &\quad + \left(X^{ij} J_{ij}\right)^2 + \left(X^{ij} K_{ij}\right)^2 . \end{aligned} \tag{55}$$

To verify this one needs the identity

$$\delta_i{}^{(j}\delta_l{}^{k)} + I_i{}^{(j} I_l{}^{k)} + J_i{}^{(j} J_l{}^{k)} + K_i{}^{(j} K_l{}^{k)} \equiv G^{jk} G_{il} . \tag{56}$$

It now follows (for one choice of sign) that

$$\mathcal{E} \geq |X^{ij} I_{ij}| . \tag{57}$$

This bound is saturated by static solutions of the same first-order equations (43) as we found before because, for example, these imply that

$$\begin{aligned} X^{ij} J_{ij} &= \mp\frac{1}{2}\boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j (IJ)_{ij} \\ &= \mp\frac{1}{2}\boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j K_{ij} \equiv 0 , \end{aligned} \tag{58}$$

where we have used $IJ = K$ in the last line. Of course, the choice of complex structure I is arbitrary; we could take $I = \mathbf{n}\cdot\mathbf{I}$. Let

$$\mathbf{L} = \int_w \boldsymbol{\Omega} \tag{59}$$

and let $\bar{\mathbf{n}}$ be the direction that minimises $\mathbf{n}\cdot\mathbf{L}$. Then we deduce the bound

$$\mathcal{E} \geq |X^{ij}\Omega_{ij}| , \tag{60}$$

where $\Omega = \bar{\mathbf{n}}\cdot\boldsymbol{\Omega}$. Integration of (60) yields the bound

$$E \equiv \int d^2\sigma\, \mathcal{E} \geq |L| \tag{61}$$

on the total energy. Given that the bound (60) is saturated, the bound (61) will also be saturated *provided that the integrand of L does not change sign.* Recalling that $\Omega_{ij} \equiv I_{ij}$, we see that this condition is satisfied because (43) implies that

$$X^{ij}\Omega_{ij} = \pm|\boldsymbol{\nabla} X|^2 . \tag{62}$$

Thus, the bound (61) is saturated by static solutions of (43). We see that the additional non-quadratic terms in the membrane Hamiltonian make no difference to the final result.

Let us now reconsider the case in which $M = \mathbb{C}$. This corresponds to a membrane in a 5-dimensional Minkowski spacetime, which we can view as the product of a real time-line with $\mathbb{C}^2$; the $\mathbb{C}^2$ coordinates are (Z, ζ). Minimal energy membranes are static holomorphic curves in $\mathbb{C}^2$, which are specified by an equation of the form $f(Z, \zeta) = 0$ for some holomorphic function of Z and ζ. This equation has a solution of the form $Z = Z(\zeta)$ in which ζ parametrizes the membrane worldspace w and $Z(\zeta)$ can now be interpreted both as a worldvolume field and as the displacement of the membrane in the Z-plane at the coordinate ζ. If we want the field $Z(\zeta)$ to be single-valued on the ζ-plane, a condition that is normally required of a sigma model, we must choose f to be linear in Z. In contrast, we might expect any given value of Z to occur for several values of ζ; for instance, if we have k identical widely-separated solitons we expect each value of Z to occur at least k times. This will happen if f is a k'th order polynomial in ζ, but this suggests that the one soliton sector is described by a function f that is *linear* in ζ. The simplest soliton solution should therefore be found by choosing $f = \zeta Z - c$. Provided that $c \neq 0$, this yields the solution $Z(\zeta) = c/\zeta$ discussed above. In that discussion the limit $c \to 0$, which shrinks the 'soliton' to a point, would simply yield the sigma-model vacuum $Z \equiv 0$. In the membrane context, however, this limit yields the equation

$$\zeta Z = 0 \,, \tag{63}$$

which has two solutions: $Z = 0$ or $\zeta = 0$. The second solution makes no sense in the sigma model context but it does in the membrane context. Since this equation is symmetric under the interchange of Z and ζ we could equally well interpret Z as a worldspace coordinate and $\zeta(Z)$ as its displacement in the ζ-plane. Thus, the equation (63) describes two membranes intersecting at the point $Z = \zeta = 0$. Recalling that this is a limit of the equation

$$\zeta Z = c \tag{64}$$

we see that the 'soliton' solution $Z = c/\zeta$ describes the desingularized intersection of two membranes [22]. Either membrane can be viewed as an infinite-energy 'soliton' on the worldspace of the other one, the energy being infinite because the 'soliton' membrane has constant surface tension and infinite area. Of course, it is also possible to view the desingularized intersection as a *single* membrane in $\mathbb{E}^4$ with two asymptotic planes. This single membrane will have minimal energy if it is a minimal surface in $\mathbb{E}^4$. The study of sigma model solitons is therefore closely related to the study of minimal surfaces. We shall return to this theme in the next two lectures.

We now turn to the membrane version of the finite energy soliton of the hyper-Kähler sigma model. We start from the D=11 supermembrane in a D=11 supergravity M-monopole background. The supermembrane is a super version of the membrane already considered, and can be consistently formulated in any background that solves the D=11 supergravity field equations [23]. The M-monopole is a solution of D=11 supergravity [24] for which the only non-vanishing field is the 11-metric, which takes the form

$$ds_{11}^2 = ds^2(\mathbb{E}^{(1,6)}) + G_{ij}(X)dX^i dX^j \,, \tag{65}$$

where G is a hyper-Kähler 4-metric of the type considered above. We now place a probe membrane in this background and choose its vacuum to be a Minkowski 3-space in $\mathbb{E}^{(1,6)}$. Restricting attention to deformations of the supermembrane described by the worldvolume fields X^i, we find the induced worldvolume 3-metric to be exactly as in (8). As we have seen, this leads, in the field theory limit, to a sigma-model with target space metric G. In the case of the supermembrane this becomes an N=4 supersymmetric sigma model. As we have seen this model admits finite energy (and 1/2 supersymmetric) lump solutions corresponding to particular holomorphic curves. We have also seen that the *same* configurations minimise the brane theory energy, with the membrane worldspace as the holomorphic curve. For the choice of a 2-centre hyper-Kähler metric with V given by (50) we have the finite area homology 2-sphere previously described and the lump is a membrane wrapped on it. This appears as a soliton on a probe brane that intersects the lump brane. Non-singular intersections, which can be viewed as *single* membranes asymptotic to the vacuum membrane, are obtained as solutions of (43). This entire set up can be summarized by the array

$$\begin{array}{lcccccc}
MK: & & & \times & - & - & - \\
M2: & 1 & 2 & & & & - \\
M5: & & & 3 & 4 & &
\end{array}$$

where 'MK' indicates the (multi-centre) M-monopole background solution of D=11 supergravity; the cross represents the compact direction of this background. The second row is the probe supermembrane, or M2-brane, and the third row the soliton M2-brane; of course, in the case of a non-singular intersection, there is really only one M2-brane. This array is associated with the constraints

$$\Gamma_{3456}\epsilon = \epsilon \,, \qquad \Gamma_{012}\epsilon = \epsilon \,, \qquad \Gamma_{034}\epsilon = \epsilon \,, \tag{66}$$

where ϵ is a 32-component real D=11 spinor. I refer to my previous Cargèse lectures [25] for an explanation of these constraints, which will be needed in the following lecture. The fact that their solution space is is 4-dimensional implies that the configuration as a whole preserves 1/8 of the supersymmetry of the M-theory vacuum. As the hyper-Kähler sigma model vacuum preserves eight supersymmetries the lump soliton on the supermembrane preserves 1/2 of the supersymmetry of the brane theory vacuum.

To conclude this lecture, we will consider the IIA interpretation of the above sigma model lump. Because of the holomorphicity of k, reduction on its orbits preserves all supersymmetries of the original configuration. I will not prove this here, but it can be verified directly from the resulting IIA configuration, which (after a permutation of the columns) is represented by the array

$$\begin{array}{lccccccc} D6: & 1 & 2 & 3 & 4 & 5 & 6 & \\ D2: & 1 & 2 & & & & & \\ F1: & & & & & & & 7 \end{array}$$

We now have two parallel D6-branes, represented by the first row. The probe M2-brane has become a D2-brane parallel to the D6-branes and the 'soliton' M2-brane a IIA string stretched between the D6-branes. An intersection of the string with the D2-brane corresponds to a singular intersection of the two M2-branes. The deformation of the M2-branes to a non-singular lump on a single M2-brane now has a IIA interpretation as the splitting of the IIA string intersection with the D2-brane into two endpoints, yielding two separate IIA strings stretched between the D2-brane and each of the D6-branes.

4. Lecture 3: Solitons and Kähler Calibrations

We have been considering p-branes in D-dimensional spacetimes $\mathcal{M} = \mathbb{R}\times\mathcal{S}$ with metric

$$ds^2 = -(dx^0)^2 + M_{IJ}dX^I dX^j\,, \tag{67}$$

so that M is the Riemannian metric on the (D-1)-dimensional space $\mathcal{S}$. We shall assume that a time parametrization has been chosen so that $X^0(\xi) = t$. A static p-brane is then an immersed p-surface w in $\mathcal{S}$ specified by functions $X^I(\xi)$. The metric induced on w is the metric m of (22). Let Γ_I be the spatial Dirac matrices that anticommute with γ_0 and satisfy

$$\{\Gamma_I, \Gamma_J\} = 2M_{IJ}\,, \tag{68}$$

and let $\Gamma_{IJ\ldots}$ be antisymmetrized products of Dirac matrices (with 'strength one', so that $\Gamma_{12\ldots} = \Gamma_1\Gamma_2\cdots$ when M is diagonal). The matrix

$$\Gamma = \frac{1}{p!\sqrt{\det m}}\,\epsilon^{a_1\ldots a_p}\partial_{a_1}X^{I_1}\ldots\partial_{a_p}X^{I_p}\gamma_0\Gamma_{I_1\ldots I_p} \tag{69}$$

will play an important role in what follows. It has the property that

$$\Gamma^2 = (-1)^{(p-2)(p-5)/2} \,. \tag{70}$$

To verify this one notes first that, as for any product of Dirac matrices,

$$\Gamma^2 = \sum_k \frac{1}{k!} C^{I_1 \dots I_k} \Gamma_{I_1 \dots I_k} \equiv \sum_k C^{(k)} \cdot \Gamma_{(k)} \,, \tag{71}$$

for some coefficient functions $C^{(k)}$; one then observes that $C^{(k)}$ must vanish for $k \neq 0$ because no antisymmetric tensor can be constructed from the $(p-k)$ factors of the induced metric arising from the 'contractions' of Dirac matrices that must be made to get the k'th term. Evaluation of the zeroth term then yields the result. We will restrict ourselves to the cases $p = 2, 5$, for which

$$\Gamma^2 \equiv 1 \,. \tag{72}$$

We will also take $D = 11$, so $\mathcal{S}$ is 10-dimensional. In this case the 2-brane and 5-brane have a natural interpretation as the M2-brane and M5-brane of M-theory (with the tensor gauge field set to zero in the latter case).

We may, and will, choose the 32×32 D=11 Dirac matrices to be real. Let $\epsilon(X)$ be a real (commuting) time-independent 32-component spinor field on $\mathcal{M}$, normalized so that

$$\epsilon^T \epsilon = 1 \,, \tag{73}$$

and let Φ be the p-form on $\mathcal{S}$ defined by

$$\Phi = \frac{1}{p!} (\bar\epsilon \Gamma_{I_1 \dots I_p} \epsilon) \, dX^{I_1} \wedge \dots \wedge dX^{I_p} \,, \tag{74}$$

where

$$\bar\epsilon \equiv \epsilon^T \gamma_0 \,. \tag{75}$$

We shall choose ϵ to be covariantly constant with respect to a metric spin connection. In this case Φ is a closed form,

$$d\Phi = 0 \,. \tag{76}$$

This p-form Φ induces a p-form ϕ on w, given by

$$\phi = vol \, (\epsilon^T \Gamma \epsilon) \,. \tag{77}$$

where

$$vol \equiv d\sigma^1 \wedge \dots \wedge d\sigma^p \sqrt{\det m} \tag{78}$$

is the volume p-form on w in the induced metric m. From the property (72) it follows that

$$\phi \leq vol\,. \tag{79}$$

A closed p-form Φ with this property is called a (p-form) *calibration* [5]. A p-surface in $\mathcal{S}$ for which this inequality is everywhere saturated is said to be a *calibrated* surface, calibrated by Φ.

The significance of calibrations resides in their connection to minimal p-surfaces [5]. Let w be a calibrated surface and let U be an open subset of w. Then, by hypothesis,

$$vol(U) = \int_U \Phi\,. \tag{80}$$

Now let V be any deformation of U in $\mathcal{S}$ such that $U - V = \partial D$ where D is some (p+1)-surface in $\mathcal{S}$. Then

$$\int_U \Phi = \int_V \Phi + \int_D d\Phi = \int_V \Phi\,. \tag{81}$$

where the second equality follows from the fact that Φ is a closed form. Because Φ is a calibration we have

$$\int_V \Phi \leq vol(V)\,. \tag{82}$$

Putting everything together we deduce that

$$vol(U) \leq vol(V)\,, \tag{83}$$

which shows that w is a minimal surface.

Given a p-surface w we may evaluate on any of its tangent p-planes the matrix Γ, and hence the p-form ϕ induced by the calibration p-form Φ. The p-surface will be calibrated by Φ if and only if there exists a covariantly constant normalized spinor ϵ such that

$$\Gamma\epsilon = \pm\epsilon \tag{84}$$

for all p-planes tangent to w. Because of the identity (72), this equation is automatically satisfied for any *given* tangent p-plane, the solutions spanning a 16-dimensional subspace of spinor space. The intersection of these spaces for all tangent p-planes is the solution space of the equation (84), which therefore has dimension ≤ 16. For a generic p-surface the dimension will vanish, so a generic p-surface is not calibrated by Φ, but special surfaces, which will necessarily be minimal, may be. It follows that minimal surfaces can be found by seeking solutions of (84). These minimal surfaces have the feature that they partially preserve the supersymmetry of the M-theory vacuum; this can be understood either as a consequence of the 'κ-symmetry' of super-brane actions [26, 7, 27] or directly from the space-time supersymmetry algebra [6], but the details of this connection between supersymmetry and calibrations will not be needed here.

Examples are provided by a p-brane, with p=2 or p=5, in D=11 space-times of the form (7), for which $\mathcal{S} = \mathbb{E}^p \times M$. In this case M_{IJ} takes the form (28) and the induced metric m in the physical gauge is given by (29). It then follows that

$$\begin{aligned} \det m &= 1 + \boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j G_{ij} \\ &+ \frac{1}{2}(\boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j)(\boldsymbol{\nabla} X^k \cdot \boldsymbol{\nabla} X^l)\,(G_{ij}G_{kl} - G_{ik}G_{jl}) \\ &+ \ \ldots + \det\left(\boldsymbol{\nabla} X^i \boldsymbol{\nabla} X^j G_{ij}\right) . \end{aligned} \tag{85}$$

We also have

$$\sqrt{\det m}\,\Gamma = \left(\sum_{k=0}^{p} \frac{(-1)^{k(k+1)/2}}{k!} \gamma^{a_1\ldots a_k}\partial_{a_1}X^{i_1}\cdots\partial_{a_k}X^{i_k}\Gamma_{i_1\ldots i_k}\right)\Gamma_* \tag{86}$$

where Γ_* is the constant matrix

$$\Gamma_* \equiv \gamma_0 \Gamma_{1\ldots p}\,. \tag{87}$$

We are now in a position to find calibrated p-surfaces from the calibration condition (84). Consider first the brane theory vacuum; in this case the calibration condition reduces (for one choice of sign) to

$$\Gamma_* \epsilon = \epsilon\,. \tag{88}$$

Since $\Gamma_*^2 = 1$ and $\operatorname{tr}\Gamma_* = 0$, this condition reduces by half the space spanned by covariantly constant spinors on $\mathcal{S}$. The calibrated p-surface is a planar p-surface that fills the $\mathbb{E}^p$ factor of $\mathcal{S}$. It is calibrated by the p-form

$$\Phi = dx^1 \wedge \ldots \wedge dx^p\,. \tag{89}$$

Since every p-surface is locally planar the condition (88) must always be satisfied, but for non-planar p-surfaces it will not be sufficient. To determine the required additional conditions we can use (88) in (84) to reduce the latter to

$$\begin{aligned} \sqrt{\det m}\,\epsilon &= \Big(1 - \gamma^a\partial_a X^i\Gamma_i - \frac{1}{2}\gamma^{ab}\partial_a X^i\partial_b X^j\Gamma_{ij} \\ &\quad + \frac{1}{6}\gamma^{abc}\partial_a X^i\partial_b X^j\partial_c X^k\Gamma_{ijk} + \ldots\Big)\epsilon \end{aligned} \tag{90}$$

The simplest non-trivial way to solve this condition is to suppose that each power of ∂X cancels separately. The cancellation of the linear term requires

$$\gamma^a \partial_a X^i \Gamma_i \epsilon = 0\,. \tag{91}$$

Remarkably, this implies that all higher powers in ∂X cancel [5]. Here we shall verify this for $p = 2$ [2]. Iteration of (91) yields

$$-\gamma^{ab}\partial_a X^i \partial_b X^j \Gamma_{ij}\epsilon = \boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} X^j G_{ij}\epsilon\,. \tag{92}$$

Since ϵ is non-zero by hypothesis, the calibration condition is now reduced to the condition

$$\sqrt{\det(\mathbb{I} + \tilde m)} = 1 + \frac{1}{2}\mathrm{tr}\,\tilde m \tag{93}$$

where we have set

$$\tilde m_{ab} = \partial_a X^i \partial_b X^j G_{ij}\,. \tag{94}$$

This condition is equivalent to

$$\mathrm{tr}\,\tilde m^2 = \frac{1}{2}(\mathrm{tr}\,\tilde m)^2\,. \tag{95}$$

This is indeed a consequence of (91) and can be proved by iteration of (92) and use of the Dirac matrix identity

$$\Gamma_{IJ}\Gamma_{KL} = \Gamma_{IJKL} + 2M_{L[I}\Gamma_{J]K} - 2M_{K[I}\Gamma_{J]L} + 2M_{j[k}M_{L]I}\,. \tag{96}$$

We have just seen that we can find non-planar calibrated membranes in a 6-dimensional subspace $\mathbb{E}^2 \times M$ of the 10-dimensional space $\mathcal{S}$ by seeking fields $X^i(\sigma)$ for which (91) admits non-zero solutions for constant ϵ. We are now going to make contact with the results of the previous lecture by showing that solutions of the Bogomol'nyi-type equation (43) are precisely the required configurations. We will choose the top sign, for convenience, and rewrite this equation as

$$\partial_2 X^i = \partial_1 X^j I_j{}^i\,. \tag{97}$$

Subsitution of this into (91), followed by multiplication by γ^1, yields

$$\partial_1 X^i \left(\Gamma_i + \gamma^{12}\Gamma_j I_i{}^j\right)\epsilon = 0\,. \tag{98}$$

In coordinates for which I takes a standard skew-diagonal form, with two 2×2 blocks, this becomes

$$\begin{aligned}&\Big[e^1\left(\Gamma_1 + \gamma^{12}\Gamma_2\right) + e^2\left(\Gamma_2 - \gamma^{12}\Gamma_1\right)\\ &\quad + e^3\left(\Gamma_3 + \gamma^{12}\Gamma_4\right) + e^4\left(\Gamma_4 - \gamma^{12}\Gamma_3\right)\Big]\epsilon = 0\end{aligned} \tag{99}$$

where we have set $e^i = \partial_1 X^i$. This is equivalent to

$$\sum_{k=1}^{2} \left(\Gamma_{2k-1} e^{2k-1} + \Gamma_{2k} e^{2k}\right) \left(1 + \gamma^{12} \Gamma_{2k-1} \Gamma_{2k}\right) \epsilon = 0\,. \tag{100}$$

Thus, for each $k = 1, 2$, either e^{2k-1} and e^{2k} vanish, which is equivalent to requiring one complex field to be constant, or

$$\left(1 + \gamma^{12} \Gamma_{2k-1} \Gamma_{2k}\right) \epsilon = 0\,. \tag{101}$$

Each such condition reduces the space of solutions of (91) by 1/2. The generic solution of (43) has all four scalar fields 'active', and is hence 1/4 supersymmetric. However, the generic solution does not have finite energy. As we have seen, finite energy solutions correspond to membranes wrapped on finite area holomorphic 2-cycles. Consider the two-centre model discussed previously. The minimal energy membrane, wrapped on the one finite-area holomorphic 2-cycle, has $\mathbf{X} = X\mathbf{n}$. It is manifestly a configuration with two 'active' scalars, which can be replaced by the single complex scalar $Z = Xe^{i\phi}$. As a solution of the supermembrane equations, this finite-energy lump solution is therefore 1/2 supersymmetric [2].

We have now seen how sigma model lump solutions of the membrane equations provide examples of calibrated surfaces. The calibration 2-form for this special class of calibrated surfaces is called a *Kähler* calibration, for reasons that will now be explained. We begin by recalling that

$$\Phi = \frac{1}{2} (\bar\epsilon \Gamma_{IJ} \epsilon)\, dX^I dX^J\,. \tag{102}$$

In the physical gauge, this becomes

$$\Phi = \frac{1}{2} (\bar\epsilon \gamma_{ab} \epsilon)\, d\sigma^a d\sigma^b + (\bar\epsilon \gamma_a \Gamma_i \epsilon)\, d\sigma^a dX^i + (\bar\epsilon \Gamma_{ij} \epsilon)\, dX^i dX^j\,, \tag{103}$$

the wedge product being understood here and in what follows. As a result of the constraint (88), which is here equivalent to $\gamma_{012}\epsilon = \epsilon$, and the normalization (73) of ϵ, the first term equals $d\sigma^1 d\sigma^2$. Furthermore, when $X = X(\boldsymbol{\sigma})$ the term linear in dX is

$$(\bar\gamma_a \Gamma_I \epsilon)\, d\sigma^a d\sigma^b \partial_b X^i = d\sigma^1 d\sigma^2 (\epsilon^T \gamma^b \Gamma_i \gamma_{012} \epsilon)\, \partial_b X^i\,, \tag{104}$$

but this vanishes on using the constraints (88) and (91). We therefore drop the term linear in dX. In coordinates $(\varphi, \mathbf{X})$ for M we are then left with

$$\Phi = d\sigma^1 d\sigma^2 + (\bar\epsilon \Gamma_3 \Gamma_a \epsilon) e^\phi e^a + \frac{1}{2} (\bar\epsilon \Gamma_{ab} \epsilon) e^a e^b \tag{105}$$

where (e^φ, e^a) are the frame 1-forms for which the 4-metric takes the form

$$ds^2 = (e^\varphi)^2 + \sum_{a=1}^{3} (e^a)^2 . \tag{106}$$

Referring to the M-theory array of the previous lecture, we see that the 3-direction is the one with coordinate φ and that we should take $a = (3, 4, 5)$. The constraints (66 associated with that array imply

$$(\bar\epsilon\Gamma_{34}\epsilon) = (\bar\epsilon\Gamma_{56}\epsilon) = 1 , \tag{107}$$

and

$$(\bar\epsilon\Gamma_{35}\epsilon) = (\bar\epsilon\Gamma_{36}\epsilon) = (\bar\epsilon\Gamma_{45}\epsilon) = (\bar\epsilon\Gamma_{46}\epsilon) = 0 . \tag{108}$$

After relabelling $a = (4, 5, 6) \to (1, 2, 3)$ we then find that the surviving terms in Φ are

$$\Phi = d\sigma^1 d\sigma^2 + e^\phi e^1 - e^2 e^3 . \tag{109}$$

That is

$$\Phi = d\sigma^1 d\sigma^2 + \Omega \tag{110}$$

where $\Omega = \mathbf{\Omega} \cdot \mathbf{n}$ is the target space Kähler 2-form. Clearly, the calibration form Φ is a Kähler 2-form on the larger space $\mathbb{E}^2 \times M$; it is therefore called a Kähler calibration.

5. Lecture 4: Beyond Field Theory

So far, we have seen how field theory solitons are interpreted within brane theory. Now we are going to see how brane theory allows additional static 'solitons' for which there is no field theory analogue. It will be useful to begin by reviewing the *a priori* limitations due to Derrick's theorem. Suppose that we have an energy functional of the form

$$E[X] = \sum_k E_k[X] , \tag{111}$$

where the functionals E_k are p-dimensional integrals with integrands that are homogeneous of degree k in derivatives of a set of scalar fields $X(\sigma)$. To any given field configuration $\overline{X}(\sigma)$ corresponds a value $\overline{E}_k$ of E_k and hence an energy $\overline{E}$. Now $\overline{E}_k \to \lambda^{k-p}\overline{E}_k$ under a uniform scaling $\sigma \to \lambda\sigma$ of the coordinates σ, so a necessary condition for $\overline{X}(\sigma)$ to minimise E is that

$$\sum_k (k-p)\overline{E}_k = 0 , \tag{112}$$

since there otherwise exists a λ for which the field configuration $X(\sigma) = \overline{X}(\lambda\sigma)$ has lower energy. If E_k is non-negative for all k in the sum then for any finite sum there will be a p for which $(k-p)$ is always negative and (112) cannot be satisfied unless $E_k = 0$ for all $k \neq p$. For conventional scalar field theories one has $k = 0, 2$, and since $\overline{E}_2$ vanishes only in the vacuum there can be no static solitons for $p > 2$. This is Derrick's theorem. A corollary is that when $p = 2$ we must have $\overline{E}_0 = 0$, which can normally be satisfied only if the scalar potential vanishes.

For a brane theory the energy functional is non-polynomial in derivatives and Derrick's theorem no longer applies. Of course, solutions that are 'Derrick-forbidden' must involve a cancellation of terms of different scaling weight and cannot be solutions of *first-order* equations of Bogomol'nyi-type; the relevant equations are necessarily non-linear in derivatives. The simplest example is provided by a 3-brane in $\mathbb{E}^6$. The $\mathbb{E}^6$ coordinates are (X^a, X^i) with $a = (1,2,3)$ and $i = (4,5,6)$. In a physical gauge we have $X^a = \sigma^a$, where σ^a are the worldspace coordinates, and $X^i(\boldsymbol{\sigma})$ are the physical fields. Let us define the 3-vector

$$\mathbf{X} = (X^4, X^5, X^6) \tag{113}$$

so that, for example, $\boldsymbol{\nabla} \cdot \mathbf{X} \equiv \mathrm{tr}\,(\partial X)$. In the physical gauge,

$$(\mathcal{E}+1)^2 = \det\Big(\mathbb{I} + (\partial X)(\partial X)^T\Big) \tag{114}$$

Now, we use the identity[1]

$$\begin{aligned} \det\Big(\mathbb{I} + (\partial X)(\partial X)^T\Big) \;\; &\equiv \;\; [1 - \star\psi]^2 + (\boldsymbol{\nabla} \cdot \mathbf{X} - \det \partial X)^2 \\ &+ \;\; |\boldsymbol{\nabla} \times \mathbf{X}|^2 + \sum_{i=4}^{6} \Big(\boldsymbol{\nabla} X^i \cdot \boldsymbol{\nabla} \times \mathbf{X}\Big)^2 \end{aligned} \tag{115}$$

where $\star\psi$ is the worldspace dual of the closed worldspace 3-form

$$\psi = \frac{1}{2} d\boldsymbol{\sigma} \cdot d\mathbf{X} \times d\mathbf{X}\,, \tag{116}$$

the wedge product of forms being implicit. Given that $\star\psi$ is negative[2] we may deduce the bound

$$\mathcal{E} \geq |\psi| \tag{117}$$

[1]This is the 3×3 case of an identity given by Harvey and Lawson for the $n \times n$ case [5]. I thank Jerome Gauntlett for pointing this out and for helping to transcribe the 3×3 result to the notation used here.

[2]This assumption is necessary because, in contrast to the analogous identity for Kähler calibrations we are *not* free to adjust the signs in the identity (115). It is possible to find configurations for which $\star\psi$ is positive, and even such that $(1 - \star\psi)$ is negative, but the simplest examples are such that $|\mathbf{X}|$ does not vanish as $|\boldsymbol{\sigma}| \to \infty$. Presumably, this condition guarantees that $\star\psi \leq 0$.

with equality when

$$\boldsymbol{\nabla} \times \mathbf{X} = \mathbf{0}\,, \qquad \boldsymbol{\nabla} \cdot \mathbf{X} = \det \partial X\,. \tag{118}$$

These equations (118) describe a *special Lagrangian* (SLAG) 3-surface in $\mathbb{E}^6$. Note that these conditions combined with (115) imply that

$$\sqrt{\det m} = 1 - \star\psi\,. \tag{119}$$

The curl-free condition is equivalent to

$$d\boldsymbol{\sigma} \cdot d\mathbf{X} = 0\,. \tag{120}$$

The left hand side is a symplectic 2-form on $\mathbb{E}^6$ (the wedge product of forms again being implicit). A lagrangian submanifold is a 3-surface on which this form vanishes. Since $\mathbf{X}$ is curl free we have, locally,

$$\mathbf{X} = \boldsymbol{\nabla} S \tag{121}$$

for some scalar function $S(\boldsymbol{\sigma})$ of the three worldspace coordinates. Any such function provides a local description of a Lagrangian 3-surface. The additional 'special' condition is needed for it to be minimal. In terms of S, this condition is

$$\nabla^2 S = \det \mathrm{Hess} S \tag{122}$$

where the *Hessian* of S is the matrix of second partial derivatives of S.

We are now going to see how these equations can be understood via the theory of calibrations. For the Kähler calibrations considered previously, the calibration condition (90) was satisfied order by order in an expansion in powers of ∂X. This was to be expected from the fact that the 'BPS' condition was homogeneous in derivatives. Now we should expect to satisfy (90) by a cancellation between different powers of ∂X. Special Lagrangian 3-surfaces in $\mathbb{E}^6$ have an M-theory interpretation in terms of three M5-branes intersecting according to the array [10]

$$\begin{array}{lcccccccccccc} M5: & 1 & 2 & 3 & - & - & - & | & 7 & 8 & - & - & - \\ M5: & - & - & 3 & 4 & 5 & - & | & 7 & 8 & - & - & - \\ M5: & - & 2 & - & 4 & - & 6 & | & 7 & 8 & - & - & - \end{array}$$

Omitting the two common worldspace directions, and the last two transverse directions, neither of which plays a role, we have effectively three 3-branes in $\mathbb{E}^6$. We can read off from the array the conditions imposed on the spinor ϵ by these three branes, up to a choice of signs. For example

$$\Gamma_{012378}\epsilon = \epsilon\,, \qquad \Gamma_{034578}\epsilon = -\epsilon\,, \qquad \Gamma_{024678}\epsilon = \epsilon\,. \tag{123}$$

Each product of Dirac matrices on the left hand side of these equations has eigenvalues ± 1, and the corresponding constraint projects out one of these eigenspaces according to the sign chosen; the signs here have been chosen for convenience. Note that these constraints imply

$$\Gamma^{1245}\epsilon = \epsilon\,, \qquad \Gamma^{1346}\epsilon = \epsilon\,, \qquad \Gamma^{2356}\epsilon = \epsilon\,. \tag{124}$$

The above discussion assumes that the only constraints are those associated with the three tangent planes indicated in the array. This is obviously the case if the configuration represented by the array is a singular orthogonal intersection of three planar M5-branes, but it may be possible to smooth the intersection in such a way that no further constraints arise, in which case the whole configuration can be interpreted as a single M5-brane asymptotic to the three M5-branes of the array. Our aim is to find the equations that govern such smooth intersections. We may choose the first of the asymptotic planar M5-branes as the M5-brane vacuum, interpreting the rest as a 'solitonic' deformation about this vacuum. Note that the first constraint is then the vacuum constraint $\Gamma_*\epsilon = \epsilon$. Imposing this condition, and taking

$$\Gamma_a \to \gamma_a\,, \qquad (a = 1, 2, 3) \tag{125}$$

to accord with our earlier notation, we again arrive at (90), but we will no longer assume that the terms linear and cubic in ∂X must vanish separately. Instead we allow for the possibility that that they may conspire to cancel; noting that

$$\frac{1}{6}\gamma^{abc}\partial_a X^i \partial_b X^j \partial_c X^k \Gamma_{ijk} = \gamma^{123}\Gamma_{456}\det(\partial_a X^i) \tag{126}$$

and that $\gamma^{23}\Gamma_{56}\epsilon = \epsilon$, this cancellation requires

$$\gamma^a \partial_a X^i \Gamma_i \epsilon = \det(\partial X)\,\gamma^1\Gamma_4\epsilon\,. \tag{127}$$

We now observe that (124) implies

$$\gamma^1\Gamma_4\epsilon = \gamma^2\Gamma_5\epsilon = \gamma^3\Gamma_6\epsilon \tag{128}$$

and

$$\gamma^3\Gamma_5\epsilon = -\gamma^2\Gamma_6\epsilon\,, \qquad \gamma^1\Gamma_6\epsilon = -\gamma^3\Gamma_4\epsilon\,, \qquad \gamma^2\Gamma_4\epsilon = -\gamma^1\Gamma_5\epsilon\,. \tag{129}$$

These constraints imply, in turn, that

$$\gamma^a \partial_a X^i \Gamma_i\,\epsilon = \Big[(\boldsymbol{\nabla}\cdot\mathbf{X})\gamma^1\Gamma_4 + (\boldsymbol{\nabla}\times\mathbf{X})\cdot\mathbf{G}\Big]\,\epsilon\,, \tag{130}$$

where we have set

$$\mathbf{G} = (\gamma^2\Gamma_6, \gamma^3\Gamma_4, \gamma^1\Gamma_5). \tag{131}$$

Putting all this together we see that (127) is satisfied if and only if $\mathbf{X}(\boldsymbol{\sigma})$ satisfies (118). Thus the special Lagrangian equations are *necessary* for a smooth calibrated intersection. We next show that they are also *sufficient*.

Using (119) the calibration condition becomes

$$[1 - \star\psi]\,\epsilon = \left(1 - \frac{1}{2}\gamma^{ab}\partial_a X^i \partial_b X^j \Gamma_{ij}\right)\epsilon = , \tag{132}$$

since the terms linear and cubic in ∂X on the right hand side have cancelled. We will see that this condition is identically satisfied, without any further conditions imposed on ϵ. Firstly, interation of (127), and further use of (124), yields

$$\begin{aligned} \left(\gamma^a\partial_a X^i\Gamma_i\right)^2\epsilon &= -(\det\partial X)^2\epsilon + 2\left[(\boldsymbol{\nabla}\times\mathbf{X})_3\Gamma_{45} - (\boldsymbol{\nabla}\times\mathbf{X})_2\Gamma_{46}\right]\epsilon \\ &= (\mathrm{tr}\,\partial X)^2\epsilon\,, \end{aligned} \tag{133}$$

where (118) has been used to arrive at the second line. Multiplying out the Dirac matrices on the right hand side, and using (118) again, we find that

$$\gamma^{ab}\partial_a X^i \partial_b X^j \Gamma_{ij} = (\mathrm{tr}\,\partial X)^2 - \mathrm{tr}\,(\partial X)^2\,. \tag{134}$$

The calibration condition (132) is thus equivalent to

$$\star\psi = \frac{1}{2}\mathrm{tr}\,(\partial X)^2 - \frac{1}{2}(\mathrm{tr}\,\partial X)^2 \tag{135}$$

Howe but this is identically satisfied as a consequence of the special Lagrangian conditions (118).

Finally, we turn to the relation between the 3-form ψ of (116) and the calibration 3-form Φ. Recall that

$$\Phi = \frac{1}{6}(\bar\epsilon\Gamma_{IJK}\epsilon)\,dX^I dX^J dX^K \tag{136}$$

for the case at hand. On going to the physical gauge we can expand the right hand side in powers of $\partial_a X^i$. Because the linear and cubic terms cancel on the calibrated surface we may drop these terms. What is left is the zeroth term and the quadratic term, and these are

$$\Phi = \frac{1}{6}d\boldsymbol{\sigma}\cdot d\boldsymbol{\sigma}\times d\boldsymbol{\sigma} - \psi\,. \tag{137}$$

We can rewrite this as

$$\Phi = \mathcal{R}\mathrm{e}\left[d\mathbf{Z}\cdot d\mathbf{Z}\times d\mathbf{Z}\right]\,, \tag{138}$$

where $\mathbf{Z} = \boldsymbol{\sigma} + i\mathbf{X}$ is a set of complex coordinates for $\mathbb{C}^3$. This illustrates a general feature of special Lagrangian calibrations. A special Lagrangian p-surface in $\mathbb{C}^p$ is calibrated by the real part of a holomorphic p-form. For $p = 2$ this is the real part of a holomorphic 2-form, which we can identify as the Kähler 2-form Ω of the previous lecture.

Of course, to find finite energy SLAG solitons of the type disussed we would need to choose a background with a holomorphic 3-cycle of finite 3-volume and admitting a covariantly constant holomorphic 3-form, but this I leave until such time as I have understood it better. In the meantime, the reader is invited to consult [28] for some interesting applications of SLAG calibrations. I should not leave the impression that Kähler and SLAG calibrations are the only cases. There are also some 'exceptional' calibrations. These also have a realization in terms of intersecting M5-branes and I refer to [29] for a recent review of some applications.

6. Epilogue: the brane world

These lectures have argued that field theory can be understood as a limit of a more encompassing 'brane theory'. Brane theory identifies certain field theory solitons with minimal surfaces (in spaces of reduced holonomy) associated to simple calibrations, but it goes beyond field theory in allowing other types of minimal surface, associated with more complicated calibrations. Only scalar field theories were considered here but a similar case can be made for gauge theories via D-branes and generalizations of calibration theory to include worldvolume gauge fields [30, 31]. Gravitational field theories, on the other hand, do not have an analogous brane theory interpretation because gravitons (and superpartners) propagate in the 'bulk' and not on branes. This explains the universality of gravitational interactions[3]; while there can be many branes, and many types of brane, there is only one 'bulk'.

Thus, 'brane theory' is naturally non-gravitational. Of course, there could still be an effective gravity at sufficiently low energy if the bulk is compact. However, in this case we are dealing with Kaluza-Klein theory rather than brane theory. One cannot really consider the lower-dimensional spacetime as a brane in this case because this brane is not localized in the extra dimension. This is a necessary feature in the quantum theory since the uncertainty principle guarantees complete delocalization in a compact space at zero momentum. Until recently it used to be thought that any decompactification of the bulk would cause a loss of localization of the

[3]Branes may still describe gravity via an equivalence to field theory, as in the M(atrix) model and the adS/CFT correspondence, but the variety of these equivalences corresponds to varieties of supergravity theories and not to varieties of graviton.

graviton on the lower-dimensional space, but Randall and Sundrum have shown that this need not be the case; in particular, gravity is localized on a horospherical boundary in anti-de Sitter space. This boundary is called a 'Brane World'. The brane interpretation is problematic, however, because gravity couples to the energy-momentum-stress tensor of matter, and 'brane matter' does not have a conventional stress tensor.

The symmetric stress tensor $T^{\mu\nu}$ for a Minkowski spacetime field theory is essentially the set of four Noether currents associated with translational invariance. One way to find the Noether currents is to note that the variation of the action under an infinitesimal but *non-uniform* translation with parameters α must be of the form

$$\delta_\alpha S = \int j \wedge d\alpha \,. \tag{139}$$

The left hand side vanishes (off-shell) when $d\alpha = 0$ but it must vanish on-shell even when $d\alpha \neq 0$, so the coefficient form j must be closed on-shell. Its dual vector density is therefore conserved, on shell, and can be identified as the Noether current. However, this prescription fails to define a translation Noether current for a brane theory because the action is not only invariant under uniform translations but also non-uniform translations, so the variation (139) vanishes identically.

This problem can be circumvented by fixing the worldvolume reparameterizations before applying the Noether prescription. In the physical gauge, $X^\mu = \sigma^\mu$, we have $g = \eta + \tilde{g}$, where

$$\tilde{g}_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j G_{ij}(X) \tag{140}$$

and X^i are the physical worldvolume fields. The Lagrangian is now

$$\mathcal{L} = -\sqrt{-\det(\eta + \tilde{g})} \,. \tag{141}$$

The stress tensor is given by [32]

$$\sqrt{-\det \eta}\, T^{\mu\nu} = \sqrt{-\det g}\, g^{\mu\nu} \tag{142}$$

and $\partial_\mu T^{\mu\nu} = 0$ in cartesian coordinates, on shell, because the X^μ field equation in physical gauge is

$$\partial_\mu \left(\sqrt{-\det g}\, g^{\mu\nu}\right) = 0 \tag{143}$$

This looks non-covariant but that is to be expected after gauge-fixing.

If this stress tensor is used to couple the brane to worldvolume gravity one finds the Lagrangian

$$\mathcal{L} = -\sqrt{-\det(g + h)} \,, \tag{144}$$

where $h \equiv \gamma - \eta$ is the deviation of the *independent* worldvolume metric γ from the Minkowski metric. Since $g = \eta + \tilde{g}$ we can write this as

$$\mathcal{L} = -\sqrt{-\det(\gamma + \tilde{g})}\,. \tag{145}$$

An expansion in powers of $\tilde{g}$ now yields the sum of a cosmological term and a more-or-less standard action for scalar fields X^i coupled to gravity via the metric γ. From this interpretation one can see that we have now *regained* reparameterization invariance. But we have paid a price: since one cannot fix a gauge twice, the interpretation as a gauge-fixed brane action has been lost. If one attempts to recover this interpretation by returning to (144) and taking g to be the induced metric *prior* to choice of the physical gauge then one has a coupling to gravity that is explicitly background dependent since it depends on the perturbation h and not the full metric γ.

In the case of an adS background, the problem can be phrased in a different way. The action is invariant under all isometries of the background, but from the point of view of the brane these are symmetries of a non-linearly realized conformal invariance [33, 34]. This observation applies, in particular, to a brane for which the worldvolume is a horosphere near the adS boundary. Such a brane provides a realization of the Randall-Sundrum mechanism by which gravity is induced on the brane [12], but the coupling to gravity on the brane will now break the non-linearly realized conformal invariance. On the other hand, this brane is supposed to be equivalent to a CFT on the adS boundary with a UV cut-off [35]. But a cut-off also breaks conformal symmetry. From this perspective it is no surprise that gravity on the brane breaks the non-linearly realized conformal symmetry of a brane action.

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