CHIRAL FORMS AND THEIR DEFORMATIONS

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ABSTRACT

We systematically study deformations of chiral forms with applications to string theory in mind. To first order in the coupling constant, this problem can be translated into the calculation of the local BRST cohomological group at ghost number zero. We completely solve this cohomology and present detailed proofs of results announced in a previous letter. In particular, we show that there is no room for non-abelian, local, deformations of a pure system of chiral p-forms.

1   Introduction

A chiral p-form A is defined by the equation

$$F = {}^*F. \tag{1.1}$$

where $F \equiv dA$ is the corresponding fieldstrength. From this, it is clear that the dimension, d, of space-time is given by $d = 2(p + 1)$. Furthermore, p should be even in the Minkowski case and odd in the Euclidean case since only in those cases is the square of the Hodge $*$-operator equal to the identity. Throughout this paper we maintain a Minkowski signature. Chiral p-forms naturally appear in string or M-theory. Chiral bosons are essential in the worldsheet formulation of the heterotic string and correspond to $p = 0$. Chiral two-forms, which, as we will explain further, constitute the main motivation for the present study, are central in the description of the M5-brane. Finally, chiral four-forms appear in type IIB string theory where they signal the presence of D3-branes.

The strongest motivation for studying deformations of chiral forms arises from the study of coinciding M5-branes. The solitonic objects in M theory (viewed here as eleven dimensional supergravity) are M2- and M5-branes. These soliton solutions break half of the supersymmetries, reducing them from 32 to 16. Their effective worldbrane actions contain therefore 16 Goldstinos, which correspond to 8 propagating fermionic degrees of freedom. This should be matched by 8 bosonic degrees of freedom. Obvious candidates for the bosonic degrees of freedom of a p-brane living in d dimensions are the $d - p - 1$ transversal positions of the brane. For the M2 brane ($p = 2$ and $d = 11$), this saturates the number of bosonic degrees of freedom. For the M5-brane, however, one needs three additional bosonic degrees of freedom. The little group of the worldvolume theory is $Spin(6) = SU(2) \times SU(2)$, which means we need a (3,1) representation of this. This is precisely a chiral two-form in six dimensions.

In the low energy limit where bulk gravity decouples, a single M5-brane is described by a six dimensional $N = (2, 0)$ superconformal field theory [1], [2]. Its field content consists of five scalar fields and a single chiral two-form [1]. A Lorentz non-covariant action was constructed in [3], [4] and [5]. A covariant action was obtained in [6] and [7]. The covariant action contains appropriate extra auxiliary fields and gauge symmetries. Partial gauge fixing of the covariant action yields the non-covariant action.

Once n M5-branes coincide, the situation changes. This can be seen by compactifying one direction on a circle. For small radius, the resulting theory is weakly coupled type IIA string theory. When the M5-branes are transversal to the circle, they appear

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[1]Throughout this paper we ignore the fermionic degrees of freedom which does not change any of our conclusions.

in the type IIA theory as n coinciding NS5-branes. Not much is explicitly known about this system. However, when the M5-branes are longitudinal to the circle, they emerge as n coinciding D4-branes. The effective action for such a system is a $U(n)$ non-Abelian Born-Infeld action [8]. Its leading and next to leading terms are well understood but discussion about the subleading terms remains [9], [10]. Ignoring higher derivative terms and focussing on the leading term, one gets that the dynamics of the D4 system is governed by 5 scalar fields in the adjoint representation of $U(n)$ coupled to a 5-dimensional $U(n)$ gauge theory. Going back to the supergravity description, this observation suggests the existence of a non-abelian extension of chiral 2-forms.

Genuine non-abelian extensions of non-chiral p-forms, for $p \geq 2$ have not yet been constructed. Viewing a 2-form as a connection over loopspace, one can show that no straightforward non-abelian extension exists [11] (see also [12]). Dropping geometric prejudices, all local deformations continously connected to the free action were constructed in [13]. Though both known and novel deformations were discovered, none of them had the required property that the p-form gauge algebra becomes truly non-abelian.

Turning back to chiral 2-forms, one finds that M-theoretical considerations indicate that n coinciding M5-branes constitute a highly unusual physical system. Indeed, the supergravity description of n M5-branes predicts that both the entropy [14] and the two-point function for the stress-energy tensor [15] scale as n^3 in the large n limit. Anomaly considerations lead to a similar behaviour [16], [17]. So this suggests that a non-abelian extension of chiral two-forms falls outside the scope of finite dimensional semi-simple Lie groups as none of those have a dimension growing as fast as n^3 (where n would be the dimension of the Cartan sub-algebra). It has been argued that "gerbes" could provide the appropriate mathematical framework [18, 19].

In [20], we announced the result that no local field theory is able to describe a system of coinciding M5-branes. This result was obtained by showing that local deformations of the action cannot modify the abelian nature of the algebra of the 2-form gauge symmetries. It holds under the assumption that the deformed action is continuous in the coupling constant (i.e., possible non-perturbative "miracles" are not investigated) and reduces, in the limit of vanishing coupling constant, to the action describing free chiral 2-forms. In particular, no assumption was made on the polynomial order (cubic, quartic ...) of the interaction terms.

In the present paper we present detailed proofs of that assertion. The techniques used in this paper can be applied in a straightforward fashion to prove the results in [21] as well. There, deformations of chiral four-forms in ten dimensions were analyzed with as conclusion that the only consistent deformation was the type IIB coupling of the chiral four-form to the NS-NS and the R-R two-forms familiar from

IIB supergravity [22, 23].

The outline of this paper is as follows. In the next section, we review how the problem of consistent couplings can be reformulated as a cohomological problem [24, 25]. We then recall the non-covariant formalism for chiral 2-forms, and their BRST formulation (sections 3, 4 and 5). In particular, we point out that the BRST differential s naturally splits as the sum $s = \delta + \gamma$ of simpler building blocks. After a brief section in which we recall the so-called "algebraic Poincaré lemma", which provides an important tool for our investigations, we turn to the calculation of the BRST cohomology. First we compute the cohomology of γ (section 7). Next, we compute the cohomology of γ modulo d, where d is the spacetime exterior derivative (sections 8 and 9). In section 10, we compute the same cohomologies for the other piece involved in s, namely δ. In section 11, we put together the calculations of the previous sections to derive the announced result that the gauge symmetries for a set of free chiral 2-forms are rigid and cannot be deformed continuously in the local field theoretical context. Our paper ends with a short, concluding section.

2 Constructing consistent couplings as a deformation problem

The theoretical problem of determining consistent interactions for a given gauge invariant system has a long history. It has been formulated in general terms in [26] (see also [27]).

The equations for the consistent interactions are rather intricate because they are non linear and involve simultaneously not only the deformed action, but also the deformed structure functions of the deformed gauge algebra, as well as the deformed reducibility coefficients if the gauge transformations are reducible. The problem is further complicated by the fact that one has to factor out the "trivial" interactions that are simply induced by a change of variables.

As we now review, one can reformulate the problem as a cohomological problem [24]. This approach systematizes the recursive construction of the consistent interactions and, furthermore, enables one to use the powerful tools of homological algebra.

Starting with a "free" action $\overset{(0)}{S_0}[\varphi^i]$ with "free" gauge symmetries

$$\delta_\varepsilon \varphi^i = \overset{(0)i}{R_\alpha} \varepsilon^\alpha, \tag{2.1}$$

leading to the Noether identities

$$\frac{\delta \overset{(0)}{S}}{\delta \varphi^i} \overset{(0)i}{R_\alpha} = 0 , \tag{2.2}$$

we introduce a coupling constant g and modify $\overset{(0)}{S_0}$,

$$\overset{(0)}{S_0} \longrightarrow S_0 = \overset{(0)}{S_0} + g \overset{(1)}{S_0} + g^2 \overset{(2)}{S_0} + ... \qquad (2.3)$$

We consider only consistent deformations, meaning that the deformed action should be gauge invariant as well. In the generic case this requires a deformation of the gauge transformation rules,

$$\overset{(0)i}{R_\alpha} \longrightarrow R_\alpha^i = \overset{(0)i}{R_\alpha} + g \overset{(1)i}{R_\alpha} + g^2 \overset{(2)i}{R_\alpha} + ... \qquad (2.4)$$

Consistency is then translated into the requirement that the Noether identities should hold to all orders

$$\frac{\delta S}{\delta \varphi^i} R_\alpha^i = 0, \qquad (2.5)$$

where,

$$\delta_\varepsilon \varphi^i = R_\alpha^i \varepsilon^\alpha. \qquad (2.6)$$

Expanding Eq. (2.5) order by order in the coupling constant gives consistency condition of increasing complexity.

For reducible theories, which is the case relevant to chiral 2-forms, there is an additional constraint. The gauge transformations of the free theory are not independent,

$$\overset{(0)i}{R_\alpha} \overset{(0)\alpha}{Z_A} = 0 \qquad (2.7)$$

(possibly on-shell). One must then also impose that the gauge transformations remain reducible, possibly in a deformed way. This yields additional conditions on the coefficients R's in Eq. (2.4).

The deformations of an action fall into three classes. In the first one, gauge invariant terms are added to the original lagrangian and therefore no modification of the gauge transformations is required. Examples of this are functionals of the field strength and its derivatives, as well as Chern-Simons-like terms [28]. In the second class, both the action and the transformation rules are modified. However, the terms added to the transformation rules are invariant under the original gauge transformations. As a consequence, the gauge algebra is not modified to first order in the coupling constant. An example of this is the Freedman-Townsend model [29] for two-forms in four dimensions. Finally, in the last class, the additional terms in the deformed transformation rules are not gauge invariant. Therefore the gauge algebra itself gets modified as well. The best known example of this is the deformation of an abelian Yang-Mills theory to a non-abelian theory.

The key to translating the problem of consistent interactions into a cohomological problem is the antifield formalism [30, 31, 32] (for reviews, see [33, 34]). Let us assume

that we solved the master equation for the undeformed theory. Its solution is denoted by $\overset{(0)}{S}$, which satisfies $(\overset{(0)}{S}, \overset{(0)}{S}) = 0$. The existence of a consistent deformation of the original gauge invariant action implies the existence of a deformation of $\overset{(0)}{S}$, which we denote by S,

$$\overset{(0)}{S} \longrightarrow S = \overset{(0)}{S} + g\,\overset{(1)}{S} + g^2\,\overset{(2)}{S} + ... \tag{2.8}$$

Expanding the master equation for S, $(S, S) = 0$, order by order in the coupling constant yields various consistency relations,

$$(\overset{(0)}{S}, \overset{(0)}{S}) = 0 \tag{2.9}$$

$$(\overset{(0)}{S}, \overset{(1)}{S}) = 0 \tag{2.10}$$

$$2(\overset{(0)}{S}, \overset{(2)}{S}) + (\overset{(1)}{S}, \overset{(1)}{S}) = 0 \tag{2.11}$$

$$\vdots \quad .$$

The first equation is satisfied by assumption. As $(\overset{(0)}{S}, (\overset{(0)}{S}, \cdot)) = 0$, the second equation implies that $\overset{(1)}{S}$ is a cocycle for the free differential $\overset{(0)}{s} \equiv (\overset{(0)}{S}, \cdot)$. If $\overset{(1)}{S}$ is a coboundary, $\overset{(1)}{S} = (\overset{(1)}{T}, \overset{(0)}{S})$, one can show that this corresponds to a trivial deformation (i.e. a deformation which amounts to a simple redefinition of the fields).

In practice, we consider deformations which are local in spacetime, i.e., we impose that $\overset{(1)}{S}, \overset{(2)}{S}, ...$ be *local* functionals. Reformulating the equations in terms of the Lagrange densities takes care of this problem. E.g., rewriting equation (2.10) as

$$\overset{(0)}{s}\overset{(1)}{S} = \overset{(0)}{s}\left(\int \overset{(1)}{\mathcal{S}}\right) = 0 \Leftrightarrow \int (\overset{(0)}{s}\overset{(1)}{\mathcal{S}}) = 0, \tag{2.12}$$

we obtain the following condition on the Lagrange density $\overset{(1)}{\mathcal{S}}$,

$$\overset{(0)}{s}\overset{(1)}{\mathcal{S}} + d\mathcal{M} = 0, \tag{2.13}$$

where $\mathcal{M}$ is a local form of degree $n - 1$, where n is the dimensionality of space-time and d is the spacetime exterior derivative[2]. Again one can show that BRST-exact terms modulo d are trivial solutions of (2.13) and corresponds to trivial deformations. In the local context, the proper cohomology to evaluate is thus $H^{0,n}(\overset{(0)}{s}|\,d)$ where the first and second superscripts denote the ghost number and form degree, respectively.

Note that when all the representatives of $H^{0,n}(\overset{(0)}{s}|\,d)$ can be taken not to depend on the antifields, one may take the first-order deformations $\overset{(1)}{\mathcal{S}}$ to be antifield-independent. In this case Eq. (2.10) reduces to $(\overset{(0)}{S}, \overset{(2)}{S}) = 0$ and implies that the deformation at order g^2 defines also an element of $H^{0,n}(\overset{(0)}{s}|\,d)$. One can thus take $\overset{(2)}{S}$

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[2]Throughout this paper we ignore boundary contributions

not to depend on the antifields either. Proceeding in this manner order by order in the coupling constant, we conclude that the additional terms in S are all independent of the antifields. Since the antifield-dependent terms in the deformation of the master equation are related to the deformations of the gauge transformations, this means that there is no deformation of the gauge transformations. Summarizing, if there is no non-trivial dependence on the antifields in $H^{0,n}(\overset{(0)}{s}| \; d) = 0$, the only possible consistent interactions are of the first class and do not modify the gauge symmetry. This is the situation met for a system of chiral 2-forms, as we now pass to discuss.

3 System of free chiral 2-forms in 6 dimensions

The non-covariant action for a system of N free chiral 2-forms is [35],

$$S_0[A_{ij}^A] = \sum_A \int dt d^5x B^{Aij}(\dot{A}_{ij}^A - B_{ij}^A), \quad (A = 1, \ldots, N), \tag{3.1}$$

where

$$B^{Aij} = \frac{1}{6}\epsilon^{ijklm}F_{klm}^A = \frac{1}{2}\epsilon^{ijklm}\partial_k A_{lm}^A. \tag{3.2}$$

The integer N can be any function of the number n of coincident M5-branes (e.g., $N \sim n^3$). The action (3.1) differs from the one in [3]-[5] where a space-like dimension was singled out. Here we take time as the distinguished direction; from the point of view of the PST formulation [6, 7], the two approaches simply differ in the gauge fixing. We work in Minkowski spacetime. This implies, in particular, that the topology of the spatial sections R^5 is trivial. Most of our considerations would go unchanged in a curved background of the product form $R \times \Sigma$ provided the De Rham cohomology groups $H^2_{DeRham}(\Sigma)$ and $H^1_{DeRham}(\Sigma)$ of the spatial sections Σ vanish. [If $H^2_{DeRham}(\Sigma)$ is non-trivial, there are additional gauge symmetries besides (3.3) below, given by time-dependent spatially closed 2-forms; similarly, if $H^1_{DeRham}(\Sigma)$ is non-trivial, there are additional reducibility identities besides (3.4) below. One would thus need additional ghosts and ghosts of ghosts. These, however, would not change the discussion of *local* Lagrangians because they would be global in space (and local in t).]

The action S_0 is invariant under the following gauge transformations

$$\delta_\Lambda A_{ij}^A = \partial_i \Lambda_j^A - \partial_j \Lambda_i^A, \tag{3.3}$$

because B^{Aij} is gauge-invariant and identically transverse $(\partial_i B^{Aij} \equiv 0)$ [3]. As $\delta A_{ij}^A = 0$ for

$$\Lambda_i^A = \partial_i \varepsilon^A, \tag{3.4}$$

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[3]Since A_{0i}^A does not occur in the action – even if one replaces $\partial_0 A_{ij}^A$ by $\partial_0 A_{ij}^A - \partial_i A_{0j}^A - \partial_j A_{i0}^A$ (it drops out because B^{Aij} is transverse) –, the action is of course invariant under arbitrary shifts of A_{0i}^A.

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this set of gauge transformations is reducible. This exhausts completely the redundancy in Λ_i^A since $H^1_{DeRham}(R^5) = 0$.

The equations of motion obtained from $S_0[A_{ij}^A]$ by varying A_{ij}^A are

$$\epsilon^{ijklm}\partial_k \dot{A}_{lm}^A - 2\partial_k F^{Aijk} = 0 \Leftrightarrow \epsilon^{ijklm}\partial_k(\dot{A}_{lm}^A - B_{lm}^A) = 0. \qquad (3.5)$$

Using $H^2_{DeRham}(R^6) = 0$, one finds that the general solution of (3.5) is

$$\dot{A}_{ij}^A - B_{ij}^A = \partial_i\Lambda_j^A - \partial_j\Lambda_i^A. \qquad (3.6)$$

The ambiguity in the solutions of the equations of motion is thus completely accounted for by the gauge freedom (3.3). Hence the set of gauge transformations is complete.

We can view Λ_i^A as A_{0i}^A, so the equation (3.6) can be read as the self-duality equation

$$F_{0ij}^A - *F_{0ij}^A = 0, \qquad (3.7)$$

where $F_{0ij}^A = \dot{A}_{ij}^A + \partial_i A_{j0}^A + \partial_j A_{0i}^A$. Alternatively, one may use the gauge freedom to set $\Lambda_i^A = 0$, which yields the self-duality condition in the temporal gauge.

4 Fields - Antifields - Solution of the master equation

The solution of the master equation is easy to construct in this case because the gauge transformations are abelian. We refer to [30, 31, 32, 33] for the general construction.

The fields in presence here are

$$\{\Phi^M\} = \{A_{ij}^A, C_i^A, \eta^A\}. \qquad (4.1)$$

The ghosts C_i^A corresponds to the gauge parameters Λ_i^A, and the ghosts of ghosts η^A corresponds to ϵ^A.

Now, to each field Φ^M we associate an antifield Φ_M^*. The set of antifields is then

$$\{\Phi_M^*\} = \{A^{*Aij}, C^{*Ai}, \eta^{*A}\}. \qquad (4.2)$$

The fields and antifields have the respective parities

$$\epsilon(A_{ij}^A) = \epsilon(\eta^A) = \epsilon(C^{*Ai}) = 0 \qquad (4.3)$$

$$\epsilon(C_i^A) = \epsilon(A^{*Aij}) = \epsilon(\eta^{*A}) = 1. \qquad (4.4)$$

The antibracket is defined as

$$(X, Y) = \int d^n x \left(\frac{\delta^R X}{\delta \Phi^M(x)} \frac{\delta^L Y}{\delta \Phi_M^*(x)} - \frac{\delta^R X}{\delta \Phi_M^*(x)} \frac{\delta^L Y}{\delta \Phi^M(x)} \right) \qquad (4.5)$$

where $\delta^R/\delta Z(x)$ and $\delta^L/\delta Z(x)$ denote functional right- and left-derivatives.

Because the set of gauge transformations is complete and defines a closed algebra, the (minimal, proper) solution of the master equation $(S, S) = 0$ takes the general form

$$S = S_0 + \sum_M \int (-)^{\epsilon(M)} \Phi_M^* s\Phi^M, \tag{4.6}$$

where $\epsilon(M)$ is the Grassmann parity of Φ^M. More explicitly, we have

$$S = S_0 + \sum_A \int dt d^5x (A^{*Aij} \partial_i C_j^A - C^{*Ai} \partial_i \eta^A) \tag{4.7}$$

The solution S of the master equation captures all the information about the gauge structure of the theory : the Noether identities, the closure of the gauge transformations and the higher order gauge identities are contained in the master equation. The existence of S reflects the consistency of the gauge transformations.

5   BRST operator

The BRST operator s is obtained by taking the antibracket with the proper solution S of the classical master equation,

$$s\,X = (S, X). \tag{5.1}$$

The BRST operator can be decomposed as

$$s = \delta + \gamma \tag{5.2}$$

where δ is the Koszul–Tate differential [33]. What distinguishes δ and γ is the antighost number $(antigh)$ defined through

$$antigh(A_{ij}^A) = antigh(C_i^A) = antigh(\eta^A) = 0, \tag{5.3}$$

$$antigh(A^{*Aij}) = 1, \quad antigh(C^{*Ai}) = 2, \quad antigh(\eta^{*A}) = 3. \tag{5.4}$$

The ghost number (gh) is related to the antighost number by

$$gh = puregh - antigh \tag{5.5}$$

where $puregh$ is defined through

$$puregh(A_{ij}^A) = 0, \quad puregh(C_i^A) = 1, \quad puregh(\eta^A) = 2, \tag{5.6}$$

$$puregh(A^{*Aij}) = puregh(C^{*Ai}) = puregh(\eta^{*A}) = 0. \tag{5.7}$$

The differential δ is characterized by $antigh(\delta) = -1$, i.e. it lowers the antighost number by one unit and acts on the fields and antifields according to

$$\delta A_{ij}^A \;\;=\;\; \delta C_i^A = \delta \eta^A = 0, \qquad\qquad (5.8)$$

$$\delta A^{*Aij} \;\;=\;\; 2\partial_k F^{Akij} - \epsilon^{ijklm}\partial_k \dot{A}_{lm}^A, \qquad\qquad (5.9)$$

$$\delta C^{*Ai} \;\;=\;\; \partial_j A^{*Aij}, \qquad\qquad (5.10)$$

$$\delta \eta^{*A} \;\;=\;\; \partial_i C^{*Ai}. \qquad\qquad (5.11)$$

The differential γ is characterized by $antigh(\gamma) = 0$ and acts as

$$\gamma A_{ij}^A \;\;=\;\; \partial_i C_j^A - \partial_j C_i^A, \qquad\qquad (5.12)$$

$$\gamma C_i^A \;\;=\;\; \partial_i \eta^A, \qquad\qquad (5.13)$$

$$\gamma \eta^A \;\;=\;\; 0, \qquad\qquad (5.14)$$

$$\gamma A^{*Aij} \;\;=\;\; \gamma C^{*Ai} = \gamma \eta^{*A} = 0. \qquad\qquad (5.15)$$

Furthermore we have,

$$sx^\mu = 0, \;\; s(dx^\mu) = 0. \qquad\qquad (5.16)$$

6   Local forms - Algebraic Poincaré lemma

A *local function* is a function of the fields, the ghosts, the antifields, and their derivatives up to some finite order k (which depends on the function),

$$f = f(\Phi, \partial_\mu \Phi, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_k} \Phi). \qquad\qquad (6.1)$$

A *local function* is thus a function over a finite dimensional vector space J^k called "jet space". A *local form* is an exterior polynomial in the dx^μ's with local functions as coefficients. The algebra of local forms will be denoted by $\mathcal{A}$. In practice, the local forms are polynomial in the ghosts and the antifields, as well as in the differentiated fields, so we shall from now on assume that the local forms under consideration are of this type. One can actually show that polynomiality in the ghosts, the antifields and their derivatives follows from polynomiality in the derivatives of the A_{ij} by an argument similar to the one used in [36] for 1-forms; and polynomiality in the derivatives is automatic in our perturbative approach where we work order by order in the coupling constant(s).

Note also that we exclude an explicit x-dependence of the local forms. One could allow for one without change in the conclusions. In fact, as we shall indicate below, allowing for an explicit x-dependence simplifies some of the proofs. We choose not to do so here since the interaction terms in the Lagrangian should not depend explicitly on the coordinates in the Poincaré-invariant context.

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The following theorem describes the cohomology of d in the algebra of local forms, in degree $q < n$.

Theorem 6.1 *The cohomology of d in the algebra of local forms of degree $q < n$ is given by*

$$\begin{aligned} H^0(d) &\simeq R, \\ H^q(d) &= \{Constant\ Forms\},\ 0 < q < n. \end{aligned}$$

Constant forms are by definition polynomials in the dx^μ's with constant coefficients. This theorem is called the algebraic Poincaré lemma (for $q < n$). There exist many proofs of this lemma in the literature. One of the earliest can be found in [37, 38].

Constant q-forms are trivial in degree $0 < q < n$ in the algebra of local forms with an explicit x-dependence; e.g., $dx^0 = df$, where f is the x^0-dependent function $f = x^0$. Thus, in this enlarged algebra, the cohomology of d is simpler and vanishes in degrees $0 < q < n$. This is the reason that the calculations are somewhat simpler when one allows for an expicit x-dependence.

We work in a formalism where the time direction is privileged. For this reason, it is useful to introduce the following notation : the l-th time derivative of a field Φ (including the ghosts and antifields) is denoted by $\Phi^{(l)}$ $(= \partial_0^l \Phi)$, and the spatial differential is denoted by $\tilde{d} = dx^i \partial_i$.

A *local spatial form* is an exterior polynomial in the spatial dx^k's with coefficients that are local functions. If we write the set of the generators of the jet space J^k as

$$\{\Phi^{(l_0)}, \partial_{i_1} \Phi^{(l_1)}, \ldots, \partial_{i_1} \ldots \partial_{i_k} \Phi^{(0)};\ l_j = 0, \ldots, k - j\}, \tag{6.2}$$

it is clear that

Theorem 6.2 *The cohomology of $\tilde{d}$ in the algebra of local spatial forms of degree $q < n - 1$ is given by*

$$\begin{aligned} H^0(\tilde{d}) &\simeq R, \\ H^q(\tilde{d}) &= \{Constant\ spatial\ forms\},\ 0 < q < n - 1. \end{aligned}$$

A similar decomposition of space and time derivatives occurs of course in the Hamiltonian formalism. A discussion of the problem of consistent deformations of a gauge invariant action has been carried out in the Hamiltonian context in [39, 40, 41].

7   Cohomology of γ

The following theorem completely gives $H(\gamma)$.

Theorem 7.1 *The cohomology of γ is given by,*

$$H(\gamma) = \mathcal{I} \otimes V. \tag{7.1}$$

Here, the algebra $\mathcal{I}$ is the algebra of the local forms with coefficients that depend only on the variables F_{ijk}^A, the antifields ϕ_M^*, and all their partial derivatives up to a finite order ("gauge-invariant" local forms). These variables are collectively denoted by χ. The algebra V is the polynomial algebra in the ghosts η^A of ghost number two and their time derivatives.

Proof: The generators of $\mathcal{A}$ can be grouped in three sets:

$$T = \{t^i\} = \{\partial_{\mu_1...\mu_k} F_{ijk}^A, \partial_{\mu_1...\mu_k} \phi_M^*, \eta^{A(l)}, dx^\mu\} \tag{7.2}$$

$$U = \{u^\alpha\} = \{\partial_{(i_1...i_k} A_{[i)2j]_1}^{A(l)}, \partial_{(i_1...i_{k-1}} C_{i_k)}^{A(l)}\} \tag{7.3}$$

$$V = \{v^\alpha\} = \{\partial_{i_1...i_k} \partial_{[i} C_{j]}^{A(l)}, \partial_{i_1...i_k} \eta^{A(l)}\} \tag{7.4}$$

$(k, l = 0, \cdots)$ where $[\ ]$ and $(\ )$ mean respectively antisymmetrization and symmetrization; the subscript indicates the order in which the operations are made.

The differential γ acts on these three sets in the following way

$$\gamma T = 0, \quad \gamma U = V, \quad \gamma V = 0. \tag{7.5}$$

The elements of U and V are in a one-to-one correspondence and are linearly independent with respect to each other, so they constitute a manifestly contractible part of the algebra and can thus be removed from the cohomology.

No element in the algebra of generated by T is trivial in the cohomology of γ, except 0. Indeed, let us assume the existence of a local form $F(t^i) \neq 0$ which is γ-exact, then

$$F(t^i) = \gamma G(t^i, u^\alpha, v^\alpha) = v^\alpha \frac{\partial^L G}{\partial u^\alpha}(t^i, u^\alpha, v^\alpha). \tag{7.6}$$

But this implies that

$$F(t^i) = F(t^i) \mid_{v^\alpha = 0} = 0, \tag{7.7}$$

as announced. $\square$

Note that contrary to what happens in the non-chiral case, the temporal derivatives of the ghosts η^A are non-trivial in cohomology. There is thus an infinite number of generators in ghost number two for $H(\gamma)$, namely, all the $\eta^{A(l)}$'s. In contrast, in the non-chiral case, one has $\partial_0 \eta^A = \gamma C_0^A$ and so $\partial_0 \eta^A$ (and all the subsequent derivatives) are γ-exact. In the chiral case, there is no C_0^A.

Let$\{\omega^I\}$ be a basis of the vector space V of polynomials in the variables η^A and all their time derivatives. Theorem 7.1 tells us that

$$\gamma\alpha = 0, \ \alpha \in \mathcal{A} \quad \Leftrightarrow \quad \alpha = \sum_{\mathcal{I}} \mathcal{P}_{\mathcal{I}}(\chi)\omega^{\mathcal{I}} + \gamma\beta. \tag{7.8}$$

Furthermore, because ω^I is a basis of V

$$\sum_{I} P_I(\chi)\omega^I = \gamma\beta \quad \Rightarrow \quad P_I(\chi) = 0. \tag{7.9}$$

It will be useful in the sequel to choose a special basis $\{\omega^I\}$. The vector space V of polynomials in the ghosts η^A and their time derivatives splits as the direct sum V^{2k} of vector spaces with definite pure ghost number $2k$. The space V^0 is one-dimensional and given by the constants. We may choose 1 as basis vector for V^0, so let us turn to the less trivial spaces V^{2k} with $k \neq 0$. These spaces are themselves the direct sums of finite dimensional vector spaces V_r^{2k} containing the polynomials with exactly r time derivatives of the η's (e.g., $\partial_0\eta^A \partial_{00}\eta^B$ is in V_3^4). The following lemma provides a basis of V^{2k} for $k \neq 0$:

Lemma 7.1 *Let V^{2k} be the vector space of polynomials in the variables $\eta^{A(l)}$ with fixed pure ghost number $2k \neq 0$. V^{2k} is the direct sum*

$$V^{2k} = V_0^{2k} \oplus V_1^{2k} \oplus \ldots, \tag{7.10}$$

where V_m^{2k} is the subspace of V^{2k} containing the polynomials with exactly m derivatives of η^A. One has $dim V_m^{2k} \leq dim V_{m+1}^{2k}$. There exist a basis of V_m^{2k}

$$\{\omega_{(m)}^{I_m} : I_m = 1, \ldots, q_m; \ m = 0, \ldots\}, \tag{7.11}$$

which fulfills

$$\omega_{(m)}^{I_m} = \partial_0\omega_{(m-1)}^{I_m} \quad (I_m = 1, \ldots, q_{m-1}). \tag{7.12}$$

In other words, the first q_{m-1} basis vectors of V_m^{2k} are directly constructed from the basis vectors of V_{m-1}^{2k} by taking their time derivative ∂_0.

Proof: We will prove the lemma by induction. For $m = 0$, take an arbitrary basis of V_0^{2k} (space of polynomials in the undifferentiated ghosts η^A of degree k). Assume now that a basis with the required properties exists up to order $m-1$. Let $\{\omega_{(m-1)}^I; \ I = 0, \ldots, q_{m-1}\}$ be a basis with those properties for V_{m-1}^{2k}. We want to prove that it is possible to construct a basis of V_m^{2k} where the first q_{m-1} basis vectors are the time derivatives of the basis vectors of V_{m-1}^{2k}. We only have to show that the $\partial_0\omega_{(m-1)}^I$ are linearly independent (because they

can always be completed to form a basis of V_m^{2k}). In other words, we must prove that

$$\sum_{I=1}^{q_{m-1}} \lambda_I \partial_0 \omega_{(m-1)}^I = \partial_0 \left(\sum_{I=1}^{q_{m-1}} \lambda_I \omega_{(m-1)}^I \right) = 0 \qquad (7.13)$$

implies $\lambda_I = 0$. But (7.13) is equivalent to

$$\sum_{I=1}^{q_{m-1}} \lambda_I \omega_{(m-1)}^I = K, \qquad (7.14)$$

where K is a constant (algebraic Poincaré lemma in form degree 0). K must be equal to zero because we are in pure ghost number $\neq 0$. By hypothesis, the $\omega_{(m-1)}^I$ are linearly independant, hence the λ_I must be all equal to zero, which ends the proof. $\square$

8 Cohomology of γ modulo d at positive antighost number

Let be a^p a local p-form of antighost number $k \neq 0$ fulfilling

$$\gamma a^p + db^{p-1} = 0. \qquad (8.1)$$

We want to show that if we add to a^p an adequate d-trivial term, the equation (8.1) reduces to $\gamma a^p = 0$.

From (8.1), using the algebraic Poincaré lemma and the fact that γ is nilpotent and anticommute with d, we can derive the descent equations

$$\gamma a^p \quad + \quad db^{p-1} = 0 \qquad (8.2)$$
$$\gamma b^{p-1} \quad + \quad dc^{p-2} = 0 \qquad (8.3)$$
$$\vdots$$
$$\gamma e^{q+1} \quad + \quad df^q = 0 \qquad (8.4)$$
$$\gamma f^q \quad = \quad 0, \qquad (8.5)$$

Indeed, the fact that the antighost number is strictly positive eliminates the constants. [E.g., from (8.1), one derives $d\gamma b^{p-1} = 0$ and thus $\gamma b^{p-1} + dc^{p-2} = $ constant, but the constant must vanish since it must have strictly positive antighost number.] We suppose $q < p$, since otherwise $\gamma a^p = 0$, which is the result we want to prove. The equation (8.5) tells us that f^q is a cocycle of γ. It must be non-trivial in $H^q(\gamma)$ because if $f^q = \gamma g^q$, then (8.4) becomes $\gamma(e^{q+1} - dg^q) = 0$. The redefinition $e'^{q+1} = e^{q+1} - dg^q$ does not affect the descent equation before (8.4), which means that the descent stops one step earlier, at $q - 1$.

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Using theorem 7.1, we deduce from (8.5) that

$$f^q = \sum_{m,I_m} [\tilde{P}^{(m)}_{I_m}(\chi) + dx^0 \tilde{Q}^{(m)}_{I_m}(\chi)]\omega^{I_m}_{(m)}, \tag{8.6}$$

where $\tilde{P}^{(m)}_{I_m}$ and $\tilde{Q}^{(m)}_{I_m}$ are local spatial forms of respective degree q and $q-1$. We take the basis elements $\omega^{I_m}_{(m)}$ to fulfill the conditions of lemma 7.1. Differentiating (8.6), we find

$$
\begin{aligned}
df^q &= \sum_{m,I_m} \{\tilde{d}\tilde{P}^{(m)}_{I_m}\omega^{I_m}_{(m)} + \gamma(\tilde{P}^{(m)}_{I_m}\hat{\omega}^{I_m}_{(m)}) \\
&\quad + dx^0[(\partial_0\tilde{P}^{(m)}_{I_m} - \tilde{d}\tilde{Q}^{(m)}_{I_m})\omega^{I_m}_{(m)} + \tilde{P}^{(m)}_{I_m}\partial_0\omega^{I_m}_{(m)}]\}.
\end{aligned}
\tag{8.7}
$$

The local function $\hat{\omega}^{I_m}_{(m)}$ is defined by $\tilde{d}\omega^{I_m}_{(m)} = \gamma\hat{\omega}^{I_m}_{(m)}$ (and exists thanks to equation (5.13)).

Now, we will show that the component $\tilde{P}^{(m)}_{I_m}$ can be eliminated from f^q by a trivial redefinition of f^q. In order to satisfy (8.4), the term independent of dx^0 and the coefficient of the term linear in dx^0 in (8.7) must separately be γ-exact. The second condition gives explicitly

$$\sum_{m,I_m} [(\partial_0\tilde{P}^{(m)}_{I_m} - \tilde{d}\tilde{Q}^{(m)}_{I_m})\omega^{I_m}_{(m)} + \tilde{P}^{(m)}_{I_m}\partial_0\omega^{I_m}_{(m)}] = \gamma\beta, \tag{8.8}$$

To analyze precisely this equation, we define a degree T by

$$T(\chi) = 0, \quad T(\eta^{A(m)}) = m. \tag{8.9}$$

In fact, T simply counts the number of time derivative of η^A. We can decompose (8.8) according to the degree T. Let p be the highest degree occuring in f^q. Then, the highest degree occuring in (8.8) is $p+1$ and we must have

$$\sum_{I=1}^{q_p} \tilde{P}^{(p)}_I \partial_0\omega^I_{(p)} = \gamma\beta_{p+1}. \tag{8.10}$$

From the proof of the lemma 7.1, we find that

$$\tilde{P}^{(p)}_I = 0 \quad (I = 1, \ldots, q_p) \tag{8.11}$$

because the $\partial_0\omega^I_{(p)}$ are linearly independent. In T-degree p, (8.8) gives then

$$
\begin{aligned}
\gamma\beta_p &= -\sum_{I=1}^{q_p} \tilde{d}\tilde{Q}^{(p)}_I \omega^I_{(p)} + \sum_{I=1}^{q_{p-1}} \tilde{P}^{(p-1)}_I \partial_0\omega^I_{(p-1)} \tag{8.12} \\
&= \sum_{I=1}^{q_{p-1}} (\tilde{P}^{(p-1)}_I - \tilde{d}\tilde{Q}^{(p)}_I)\omega^I_{(p)} - \sum_{I=q_{p-1}+1}^{q_p} \tilde{d}\tilde{Q}^{(p)}_I \omega^I_{(p)}, \tag{8.13}
\end{aligned}
$$

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where we have used the property (7.12) of the basis $\{\omega^I\}$. This implies that

$$\tilde{P}_I^{(p-1)} = \tilde{d}\tilde{Q}_I^{(p)} \quad (I = 1, \ldots, q_{p-1}) \tag{8.14}$$

Inserting this equation in (8.6), we find that $\tilde{P}_I^{(p-1)}$ can be removed from f^q by eliminating a trivial cocycle of γ modulo d and redefining $\tilde{Q}_I^{(p-1)}$. It only affects e^{q+1} by a d-exact term. Next, the equation (8.8) at T-degree $p-1$ shows that $\tilde{P}_I^{(p-2)}$ is also $\tilde{d}$-exact and can thus also be removed. Proceeding in the same way until the order 1 in T, we have proved that all the $\tilde{P}_I^{(m)}$ can be eliminated from f^q.

Looking back at (8.8) and taking into account that $\tilde{P}_{I_m}^{(m)}$ can be set equal to zero by the above argument, we find that

$$\tilde{d}\tilde{Q}_{I_m}^{(m)} = 0. \tag{8.15}$$

Now, we must use the invariant Poincaré lemma (invariant means in the algebra $\mathcal{I}$ of gauge-invariant forms) stating that

Theorem 8.1 *Let be $\tilde{P}(\chi)$ a local spatial form of degree $q < 5$, then*

$$\tilde{d}\tilde{P}(\chi) = 0 \Rightarrow \tilde{P}(\chi) = \tilde{R}(F^{A(l)}) + \tilde{d}\tilde{Q}(\chi), \tag{8.16}$$

where $\tilde{R}(F^{A(l)})$ is a polynomial in the curvature forms $F^A = \frac{1}{6}F_{ijk}^A dx^i dx^j dx^k$ and all their time derivatives (with coefficients that may involve dx^k, which takes care of the constant forms).

Proof: The set of the generators of the algebra $\mathcal{I}$ is

$$\{\chi\} = \{\partial_{i_1 \ldots i_k} F_{ijk}^{A(l)}, \partial_{i_1 \ldots i_k} \phi_M^{*(l)}, \eta^{A(l)}, dx^\mu\} \tag{8.17}$$

The 1-form dx^0 is not present in our problem since $\tilde{P}$ is a spatial local form (it only involves dx^k). Considering l and A as only one label (call it α) and forgetting about dx^0, the set (8.17) is the same as the corresponding set of generators of the algebra $\mathcal{I}(\equiv H(\gamma)$ in pureghost number 0) for a system of spatial two-forms $\{A_{ij}^\alpha \equiv A_{ij}^A, \partial_0 A_{ij}^A, \partial_{00} A_{ij}^A, \cdots\}$ in 5 dimensions. Consequently, we can simply use the results demonstrated in [42] for a system of p-forms in any dimension. $\square$

We assumed before that f^q is of degree $q < 6$, hence $\tilde{Q}_I$ is of degree < 5. Thus, (8.15) implies

$$\tilde{Q}_{I_m}^{(m)} = \tilde{d}\tilde{R}_{I_m}^{(m)}, \tag{8.18}$$

where $\tilde{R}_{I_m}^{(m)}$ is a spatial form which only depends on the variables χ. There is no exterior polynomial in the curvatures in $\tilde{Q}_{I_m}^{(m)}$ because $\tilde{Q}_{I_m}^{(m)}$ has strictly positive antighost

number. We can therefore conclude that f^q is trivial in $H^q(\gamma \mid d)$ and can be eliminated by redefining e^{q+1}. The true bottom is then one step higher. We can proceed in the same way until we arrive at $\gamma a'^p = 0$ with $a'^p = a^p + dg^{p-1}$. This can be translated into the following theorem

Theorem 8.2 *Let be a local form a of antighost number $\neq 0$ fulfilling $\gamma a + db = 0$. There exists a local form c such as $a' := a + dc$ satisfies $\gamma a' = 0$.*

9 Cohomology of γ modulo d at zero antighost number

Now, we want to study $H^{6,0}(\gamma \mid d)$ in pureghost number 0. Let be $a^{(6,0)} \in \mathcal{A}$ of form degree 6, of antighost and pureghost number 0, and fulfilling $\gamma a^{(6,0)} + da^{(5,1)} = 0$. If $a^{(5,1)}$ is trivial γ modulo d, this equation reduces to $\gamma a^{(6,0)} + db^{(5,0)} = 0$, which gives $a^{(6,0)} = f(\partial_{\mu_1 \ldots \mu_k} F_{ijk}^A) d^6x$ plus a term trivial in the cohomology of γ modulo d.

Otherwise, we can derive the non trivial descent equations

$$\gamma a^{(6,0)} \quad + \quad da^{(5,1)} = 0 \tag{9.1}$$

$$\gamma a^{(5,1)} \quad + \quad da^{(4,2)} = 0 \tag{9.2}$$

$$\vdots$$

$$\gamma a^{(7-g,g-1)} \quad + \quad da^{(6-g,g)} = 0 \tag{9.3}$$

$$\gamma a^{(6-g,g)} \quad = \quad 0, \tag{9.4}$$

because $pureghost(\gamma a^{(6-i,i)}) > 0$ eliminates the constants. If $a^{(6-g,g)}$ is trivial γ modulo d, the bottom is really one step higher.

Eq. (9.4) implies that

$$a^{(6-g,g)} = \sum_I (\tilde{P}_I^{6-g}(\chi) + dx^0 \tilde{Q}_I^{5-g}(\chi))\omega^I + \gamma b^{(6-g,g-1)}, \tag{9.5}$$

where $\tilde{P}_I^{6-g}$ and $\tilde{Q}_I^{5-g}$ are local spatial forms, the superscript giving the form degree. Because the pureghost number of η is two, $a^{(6-g,g)}$ is non trivial only for g even. So, three cases are of interest: $g = 0, 2, 4$.

The case $g = 0$ corresponds to $\gamma a^{(6,0)} = 0$ and has been already studied so let us assume $g > 0$. The equations (9.3) and (9.5) imply together

$$\sum_I (\partial_0 \tilde{P}_I^{6-g} - \tilde{d}\tilde{Q}_I^{5-g})\omega^I + \sum_I \tilde{P}_I^{6-g}\partial_0\omega^I = \gamma\beta. \tag{9.6}$$

Repeating the same analysis as for the equation (8.8), we arrive at the conclusion that $\tilde{P}_I^{6-g}$ is trivial in the invariant cohomology of $\tilde{d}$ (or vanishes) and can thus be removed from $a^{(6-g,g)}$ by the addition of trivial terms in the cohomology of γ modulo

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d and a redefinition of $\tilde{Q}_I^{5-g}$. The case $g = 6$ is then eliminated because in that case $\tilde{Q}_I^{5-g}$ is not present at all. Hence, there remains only two cases to examine: $g = 2$ and $g = 4$.

Once $\tilde{P}_I^{6-g}$ is removed, the equation (9.6) gives $\tilde{d}\tilde{Q}_I^{5-g} = 0$. Using the invariant Poincaré lemma, we find $\tilde{Q}_I^{5-g} = \tilde{R}_I^{5-g}(F^{A(l)}) + \tilde{d}\tilde{S}_I^{(4-g,g)}(\chi)$. Hence, the form of the bottom is

$$a^{(6-g,g)} = dx^0 \sum_I \tilde{R}_I^{5-g}(F^{A(l)})\omega^I + \gamma b^{(6-g,g-1)} + dc^{(5-g,g)}. \qquad (9.7)$$

But $F^{A(l)}$ is of form degree 3, thus if $g = 4$, $\tilde{R}_I^{5-g}$ must be a constant spatial 1-form. In that instance, the ω^I must be quadratic in the ghosts $\eta^{A(l)}$. The lift of such a bottom is obstructed (i.e., leads to no $a^{6,0}$) unless it is trivial (see [13]), so that the case $g = 4$ need not be considered. [In the algebra of x-dependent local forms, the argument is simpler: the bottom is always trivial and removable since it involves a constant 1-form, which is trivial.]

It only remains to examine the case $g = 2$. $\tilde{R}$ must then be a 3-form. One can take $\tilde{R}$ linear in $F^{A(l)}$. In that case, the lift gives Chern-Simons terms, which are linear combinations of $dx^0 F^{A(l)} A^{B(m)}$, with $A^{B(m)} = \frac{1}{2} A_{ij}^{B(m)} dx^i dx^j$. Or one can take $\tilde{R}$ to be a constant 3-form. The corresponding deformation is linear in the 2-form $A^{A(l)}$ with coefficients that are constant forms. This second possibility is not $SO(5)$ invariant and leads to equations of motion that are not Lorentz invariant. It will not be considered further.

Dropping the latter possibility, all these results can be summarized in the

Theorem 9.1 *The non trivial elements of $H_0^{6,0}(\gamma \mid d)$ are of two types: (i) those that descend trivially; they are of the form $f(\partial_{\mu_1 \ldots \mu_k} B_{ij}^A) d^6 x$; (ii) those that descend non trivially; they are linear combinations of the Chern-Simons terms $\partial_0^l B^{Aij} \partial_0^m A_{ij}^B d^6 x$.*

Note that the kinetic term in the free action is precisely of the Chern-Simons type (with $l = 0$ and $m = 1$).

10 Invariant cohomology of δ modulo $\tilde{d}$ in antighost number $2, 4, 6, \ldots$

To pursue the analysis, we need some results on the cohomology of the Koszul-Tate differential δ as well as on its mod-d and mod-$\tilde{d}$ cohomologies.

We can rewrite the action of the Koszul-Tate differential in the following way

$$\delta A_{ij}^{A(l)} = \delta C_i^{A(l)} = \delta \eta^{A(l)} = 0, \qquad (10.1)$$

$$\delta A^{*A(l)ij} = 2\partial_k F^{A(l)kij} - \epsilon^{ijklm} \partial_k A_{lm}^{A(l+1)}, \qquad (10.2)$$

$$\delta C^{*A(l)i} = \partial_j A^{*A(l)ij}, \tag{10.3}$$

$$\delta\eta^{*A(l)} = \partial_i C^{*A(l)i}. \tag{10.4}$$

If we regard A and l as only one label, these equations corresponds to an infinite number of coupled non-chiral 2-forms in 5 dimensions.

It is useful to introduce a degree N defined as

$$N(\Phi^*_M) = 1, \quad N(\Phi^M) = 0, \tag{10.5}$$

$$N(\partial_k) = 1, \quad N(\partial_0) = 0 \tag{10.6}$$

$$N(dx^\mu) = 0. \tag{10.7}$$

N counts the number of spatial derivatives as well as the antifields (with equal weight given to each). According to this degree, δ decomposes as $\delta_0 + \delta_1$. The differential δ_1 acts exactly in the same way as the Koszul-Tate differential for a system of free 2-forms in 5 dimensions.

We are now able to prove the

Theorem 10.1 $H_i(\delta) = 0$ *for* $i > 0$, *where* i *is the antighost number, i.e, the cohomology of* δ *is empty in antighost number strictly greater than zero.*

Proof: From [42], we know that $H_i(\delta_1) = 0$. Let be $a \in \mathcal{A}$ a δ-closed local function of antighost number $i > 0$. We decompose a according to the degree N

$$a = a_1 + \ldots + a_m. \tag{10.8}$$

The expansion stops because a is polynomial in the antifields and the derivatives. Furthermore, $a_0 = 0$ because $antigh(a) = i > 0$. The equation $\delta a = 0$ gives in N-degree $m + 1$: $\delta_1 a_m = 0$. But $H_i(\delta_1) = 0$, hence $a_m = \delta_1 b_{m-1}$. We can define an a' as being

$$a' = a - \delta b_{m-1} = a_1 + \ldots + a_{m-2} + a'_{m-1}, \tag{10.9}$$

with $a'_{m-1} = a_{m-1} - \delta_0 b_{m-1}$. We can proceed in the same way as before with a', whose component of higher N-degree is of degree less than m. We will then find a new a' of highest degree less than $m - 1$, and so on, each time lowering the N-degree. After a finite number of steps, we arrive at $a' = a'_1 = a - \delta b$. Then, $\delta a = 0$ implies $\delta_1 a'_1 = 0$. Hence, $a'_1 = \delta_1 b_0 = \delta b_0$ because $\delta_0 \Phi^M = 0$. In conclusion $a = \delta b$, with $b = b_0 + \ldots + b_{m-1}$. $\square$

Of course, this theorem is really a consequence of general known results on the cohomology of the Koszul-Tate differential. It simply confirms, in a sense, that we

have correctly taken into account all gauge symmetries and reducibility identities in constructing the antifield spectrum.

The cohomological space $H_k^{5,inv}(\delta \mid \tilde{d})$ is defined as $H_k^5(\delta \mid \tilde{d})$ in the space of local spatial forms that belongs to $\mathcal{I}$, i.e., that are invariant. We want to compute it for k even and $\neq 0$. To do this, we will proceed as in the proof of theorem 10.1. We first prove the requested result for δ_1; we then use "cohomological perturbation" techniques to extend the result to δ.

Lemma 10.1 *For $k = 2, 4, \ldots$*

$$H_k^{5,inv}(\delta_1 \mid \tilde{d}) = 0. \tag{10.10}$$

Again, this result is simply a particular case of more general results, which were previously known, but for completeness, we prove it here.

Proof: Firstly, the theorem 9.1 of [43] says that for a linear gauge theory of reducibility order p in n dimensions $H_k^n(\delta \mid d) = 0$ for $k > p + 2$. A system of abelian spatial 2-forms in 5 dimensions is a linear gauge theory of reducibility order 1 (see section 3), thus, we can state that $H_k^5(\delta_1 \mid \tilde{d}) = 0$ for $k > 3$.

Secondly, the theorem 7.4 of [42] gives here : $H_2^5(\delta_1 \mid \tilde{d}) = 0$.

Finally, the theorem 10.1 of [42] says that for a system of space-time p-form gauge fields of the same degree $H_k^n(\delta \mid d) \cong H_k^{n,inv}(\delta \mid d)$ for $k > 0$. For the system under consideration here, this can be translated into: $H_k^5(\delta_1 \mid \tilde{d}) \cong H_k^{5,inv}(\delta_1 \mid \tilde{d})$ for $k > 0$. Putting all these results together completes the proof. $\square$

Let be $a^5(\chi)$ a local spatial 5-form in $\mathcal{I}$ of strictly positive and even antighost number, satisfying

$$\delta a^5(\chi) + \tilde{d}b^4(\chi) = 0. \tag{10.11}$$

We can decompose a^5 and b^4 according to the degree N

$$a^5 = a_1^5 + \ldots + a_n^5, \tag{10.12}$$
$$b^4 = b_1^4 + \ldots + b_m^4. \tag{10.13}$$

$a_0^5 = 0$ and $b_0^4 = 0$ because a^5 and b^4 are of antighost number > 0. We can always suppose $m \leq n$ because if $m > n$, (10.11) gives in N-degree $m + 1$: $\tilde{d}b_m^4 = 0$. Using the invariant Poincaré lemma, this yields $b_m^4 = \tilde{d}c_{m-1}^3$. Hence, b_m^4 only contributes to b^4 by a $\tilde{d}$-trivial term which can be eliminated. Proceeding in the same way until $m = n$, we arrive at the equation

$$\delta_1 a_n^5(\chi) + \tilde{d}b_n^4(\chi) = 0. \tag{10.14}$$

It has already been noticed above that the algebra $\mathcal{I}$ without dependence on dx^0 is the same as for a system of spatial 2-forms. We can thus use the lemma 10.1 in (10.14) to find that

$$a_n^5(\chi) = \delta_1 e_{n-1}^5(\chi) + \tilde{d} f_{n-1}^4(\chi). \tag{10.15}$$

Therefore, $a'^5 = a^5 - \delta e_{n-1}^5 - \tilde{d} f_{n-1}^4$ satisfies the same properties as a^5, except that its component of highest N-degree is of degree $< n$. We can now apply the same reasoning as before to a'^5, and so on, until we arrive at

$$a'^5 = a_1'^5 = a^5 - \delta(\sum_{i=1}^{n-1} e_i^5) - \tilde{d}(\sum_{i=1}^{n-1} f_i^4) \tag{10.16}$$

This leads to

$$a_1'^5 = \delta_1 e_0^5(\chi) + \tilde{d} f_0^4(\chi). \tag{10.17}$$

But $\delta_1 e_0^5 = \delta e_0^5$ because $\delta_0 \Phi^M = 0$. Eventually, we have $a^5 = \delta e^5(\chi) + \tilde{d} f^4(\chi)$, with $e^5 = \sum_{i=0}^{n-1} e_i^5$ and $f^4 = \sum_{i=0}^{n-1} f_i^4$. This gives the awaited theorem:

Theorem 10.2 *For $k = 2, 4, \ldots$*

$$H_k^{5,inv}(\delta \mid \tilde{d}) = 0. \tag{10.18}$$

11 Decomposition of the Wess-Zumino equation

We now have all the necessary tools to solve the Wess-Zumino consistency condition that controls the consistent deformations (to first-order) of the action,

$$sa^6 + db^5 = 0, \tag{11.1}$$

where a^6 and b^5 are local forms of respective form degrees 6 and 5, and ghost number 0 and 1. These forms are defined up to the following allowed redefinitions

$$a^6 \to a^6 + sf^6 + dg^5 \tag{11.2}$$
$$b^5 \to b^5 + sg^5 + dh^4, \tag{11.3}$$

which preserve (11.1). We can decompose a^6 and b^5 according to antighost number, which gives

$$a^6 = a_0^6 + \ldots + a_k^6, \tag{11.4}$$
$$b^5 = b_0^5 + \ldots + b_q^5, \tag{11.5}$$

with $a_k^6 \neq 0$.

We suppose $k > 0$ and we will show that a_k^6 can be eliminated if we redefine a^6 in an appropriate way. In antighost number k, the equation (11.1) just reads

$$\gamma a_k^6 + d b_k^5 = 0. \tag{11.6}$$

We can always assume $k \geq q$ because if $q > k$, the equation (11.1) gives in highest antighost number $d b_q^5 = 0$. Using the algebraic Poincaré lemma, we find that $b_q^5 = d c_q^4$. Hence, we can remove the component b_q^5 up to a d-trivial redefinition of b^5.

From the theorems 7.1 and 8.2, we know that Eq. (11.6) implies

$$a_k^6 = \sum_I P_I(\chi) \omega^I + \gamma f_k^6 + d g_k^5. \tag{11.7}$$

The γ modulo d trivial part of a_k^6 can be eliminated by redefining a^6 in the following way

$$a^6 \rightarrow a^6 - s f_k^6 - d g_k^5. \tag{11.8}$$

We notice that $H_k^{6,0}(\gamma)$ is non trivial only in even antighost number k (because η is of pureghost number 2). This implies that we can assume k to be even.

The Wess-Zumino consistency condition in antighost number $k - 1$ is

$$\gamma a_{k-1}^6 + \delta a_k^6 + d b_{k-1}^5 = 0. \tag{11.9}$$

The term b_{k-1}^5 is invariant because (11.9) implies $d(\gamma b_{k-1}^5) = 0$. Therefore, the algebraic Poincaré lemma gives $\gamma b_{k-1}^5 + d c_{k-1}^4 = 0$ because $k > 1$. From ¿From the theorem 8.2 we know that we can suppose $\gamma b_{k-1}^5 = 0$ without affecting a^6. Furthermore, if $b_{k-1}^5 = \gamma c_{k-1}^5$ we can eliminate b_{k-1}^5 by redefining b^5 in the following way: $b^5 \rightarrow b^5 - s c_{k-1}^5$, which does not modify a_k^6.

Therefore, we can assume

$$a_k^6 = \sum_I dx^0 \tilde{P}_I^5 \omega^I, \tag{11.10}$$

$$b_{k-1}^5 = \sum_I (\tilde{Q}_I^5 + dx^0 \tilde{R}_I^4) \omega^I. \tag{11.11}$$

The $\tilde{P}_I^5$, $\tilde{Q}_I^5$, and $\tilde{R}_I^4$ are local spatial forms belonging to $\mathcal{I}$.

Inserting (11.10) and (11.11) in (11.9), we find

$$\gamma a_{k-1}^6 = \sum_I \{ -\tilde{d} \tilde{Q}_I^5 \omega^I - \gamma [(\tilde{Q}_I^5 + dx^0 \tilde{R}_I^4) \hat{\omega}^I] \tag{11.12}$$

$$+ dx^0 [(\delta \tilde{P}_I^5 + \tilde{d} \tilde{R}_I^4 - \partial_0 \tilde{Q}_I^5) \omega^I - \tilde{Q}_I^5 \partial_0 \omega^I] \}, \tag{11.13}$$

with $\tilde{d} \omega^I = \gamma \hat{\omega}^I$. This implies that

$$\sum_I [(\delta \tilde{P}_I^5 + \tilde{d} \tilde{R}_I^4 - \partial_0 \tilde{Q}_I^5) \omega^I - \tilde{Q}_I^5 \partial_0 \omega^I] = \gamma \beta. \tag{11.14}$$

If we analyse this equation in the same way as the equation (8.8), we can prove that $\tilde{Q}_I^5 = \delta\tilde{P}_I^5 + \tilde{d}\tilde{R}_I^4$ (or simply vanishes). Inserting these equations in (11.11), we find that b_{k-1}^5 is of the form

$$b_{k-1}^5 = \delta c_k^5 + de_{k-1}^4 + \gamma f_{k-1}^5 + dx^0 \sum_I \tilde{R}_I'^4(\chi)\omega^I, \tag{11.15}$$

where c_k^5 and e_{k-1}^4 belong to $H(\gamma)$. In conclusion, we can eliminate $\tilde{Q}_I^5$ from b_{k-1}^5 by redefining a^6 and b^5 in the following way

$$a^6 \to a^6 - d(c_k^5 + f_{k-1}^5), \tag{11.16}$$
$$b^5 \to b^5 - s(c_k^5 + f_{k-1}^5) - de_{k-1}^4, \tag{11.17}$$

which does not affect the condition $\gamma a_k^6 = 0$, because $\gamma c_k^5 = 0$.

Therefore, we can finally assume

$$a_k^6 = \sum_I dx^0 \tilde{P}_I^5(\chi)\omega^I, \quad b_{k-1}^5 = \sum_I dx^0 \tilde{R}_I^4(\chi)\omega^I. \tag{11.18}$$

The equation (11.9) becomes

$$\gamma a_{k-1}' + dx^0 \sum_I (\delta\tilde{P}_I^5(\chi) + \tilde{d}\tilde{R}_I^4(\chi))\omega^I = 0, \tag{11.19}$$

which implies that $\delta\tilde{P}_I^5(\chi) + \tilde{d}\tilde{R}_I^4(\chi) = 0$. We know that we are in even antighost number, thus we can use the theorem 10.2 to find that $\tilde{P}_I^5 = \delta\tilde{S}_I^5(\chi) + \tilde{d}\tilde{T}_I^4(\chi)$. Hence,

$$a_k^6 = sf_{k+1}^6 + dg_k^5 + \gamma h_k^6, \tag{11.20}$$

where we have defined

$$f_{k+1}^6 = -dx^0 \sum_I \tilde{S}_I^5\omega^I, \quad g_k^5 = -dx^0 \sum_I \tilde{T}_I^4\omega^I, \tag{11.21}$$
$$h_k^6 = dx^0 \sum_I \tilde{T}_I^4\hat{\omega}^I, \quad \tilde{d}\omega^I = \gamma\hat{\omega}^I. \tag{11.22}$$

Thus a_k^6 can be completely eliminated by redefing a^6 as

$$a'^6 = a^6 - s(f_{k+1}^6 + h_k^6) - dg_k^5, \tag{11.23}$$

which only affects the components of antighost number $< k$. Repeating the argument at lower antighost numbers enables one to remove successively a_{k-1}, a_{k-2}, ..., up to a_1. This completes the proof of the fact that there is no non trivial dependence on the antifields for the elements of $H^{6,0}(s \mid d)$.

For antifield-independent local forms, the cocycle condition $H^{6,0}(s \mid d)$ reduces to the cocycle condition for $H^{6,0}(\gamma \mid d)$. Furthermore, γ-exact (mod-d) solutions are

also s-exact. Thus, we are led to consider $H^{6,0}(\gamma \mid d)$. This cohomology is given by the theorem 9.1. [The terms in that cohomology that vanish on-shell are trivial in the s-cohomology.] Thus, the only consistent deformations of the free action for a system of abelian chiral 2-forms are either functions of the curvatures or of the Chern-Simons type. In both cases, the integrated deformations are off-shell gauge invariant and yield no modification of the gauge transformations.

12 Final comments and conclusions

We have shown that the most general first-order consistent deformation of a set of free chiral 2-forms cannot modify (non trivially) the original gauge transformations and a fortiori, their algebra, which remains abelian. Thus, there is no room for a non-abelian, local, generalization of the theory analogous to the Yang-Mills construction.

This result holds in fact to all orders, since the allowed deformations involve the gauge-invariant curvatures or Chern-Simons terms. The addition of such terms to the original action yields a new action which is evidently gauge-invariant under the original gauge transformations to all orders.

One can show along identical lines that the rigidity of the gauge symmetries is actually valid for a set of chiral $2p$-forms in $2p + 2$ dimensions, for any $p > 0$. If one includes other fields, one may deform the gauge transformations, but the possibilities are severely limited [21]. For instance, in 10 dimensions, the only couplings of a chiral 4-form to 2-forms are those present in type IIB supergravity.

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