BRST-Invariant Constraint Algebra in Terms of Commutators and Quantum Antibrackets

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Abstract

General structure of BRST-invariant constraint algebra is established, in its commutator and antibracket forms, by means of formulation of algebra-generating equations in yet more extended phase space. New ghost-type variables behave as fields and antifields with respect to quantum antibrackets. Explicit form of BRST-invariant gauge algebra is given in detail for rank-one theories with Weyl- and Wick- ordered ghost sector. A gauge-fixed unitarizing Hamiltonian is constructed, and the formalism is shown to be physically equivalent to the standard BRST-BFV approach.

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1 Introduction

When quantizing general gauge theories, a basic principle [1] - [10] is to construct a BRSTcharge Fermionic operator Ω which satisfies the standard BRST algebra $\Omega^2 \equiv \frac{1}{2} [\Omega, \Omega] = 0$, $[G_C, \Omega] = i\hbar\Omega$, with G_C being a ghost number Bosonic operator. As expanded in power series in ghost canonical pairs $(C^{\alpha}, \bar{\mathcal{P}}_{\alpha})$, the operator Ω begins with $\Omega = C^{\alpha}T_{\alpha} + \text{more}$, where T_{α} are original first-class constraints working as gauge algebra generators.

It is remarkable that there exist BRST-invariant modified constraints $\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, \bar{\mathcal{P}}_{\alpha}]$ = T_{α} +more, satisfying $[\Omega, \mathcal{T}_{\alpha}] = 0$ by construction, which determine substantially a new, dynamically prescribed, set of gauge algebra generators. As these generators depend actually on ghost operators $(C^{\alpha}, \bar{\mathcal{P}}_{\alpha})$, they live in an extended phase space, in contrast to original first-class constraints T_{α} .

The main idea of the present paper is to reformulate the standard BRST-BFV quantization scheme directly in terms of BRST-invariant constraints \mathcal{T}_{α} considered as new basic ingredients, by means of further extension of the phase space previously spanned by original phase variables and ordinary ghosts $(C^{\alpha}, \bar{\mathcal{P}}_{\alpha})$.

Our main motivation is that new gauge generators \mathcal{T}_{α} are expected to have, in general, certainly better algebraic properties as compared with original constraints T_{α} . As a particular case, we can mention the well-known situation in Bosonic string theory (see [11] and references therein) where the algebra of original Virasoro generators is centrally extended, while the algebra of the corresponding BRST-invariant generators coincides exactly with the classical one.

It is a characteristic property of BRST-invariant generators \mathcal{T}_{α} that their algebra is closed only if original constraints form a Lie-type algebra with constant structure coefficients. It appears, however, that the algebra spanned by generators \mathcal{T}_{α} and ghost momenta operators $\bar{\mathcal{P}}_{\alpha}$ is always closed by construction.

The above circumstance allows us to formulate closed generating equations of the BRST-invariant gauge algebra in further-extended phase space. With this purpose, we introduce two sets of new ghost-type canonical pairs, $(B^{\alpha}, \Pi_{\alpha})$ and (B^*_{α}, Π^*_*) , which behave as fields and antifields with respect to quantum antibrackets.

It then appears possible to recast the new generating equations of the BRST-invariant gauge algebra to the form of operator-valued master equation [12] - [15] formulated in terms of quantum antibrackets defined originally in [16]. In a natural way, these quantum antibrackets generate operator-valued anticanonical transformations. We represent their general form and the transformation properties of the antibrackets by means of ordinary differential equations in an auxiliary variable.

As a result, we obtain two dual descriptions of BRST-invariant gauge algebra, in terms of standard commutators and quantum antibrackets. To illustrate the dualism in technical respect, we consider in more detail the case of a rank-one gauge theory with Weyl- and Wick- ordered ghost sector.

As usual, $\varepsilon(f) \equiv \varepsilon_f$ denotes the Grassmann parity of a quantity f, while [f, g] stands for the standard supercommutator $[f, g] \equiv fg - (-1)^{\varepsilon_f \varepsilon_g} gf$ of any two operators f and g. It satisfies the standard Leibniz rule, $[fg, h] = f[g, h] + [f, h]g(-1)^{\varepsilon_g \varepsilon_h}$, and Jacobi identity, $[f, [g, h]] (-1)^{\varepsilon_f \varepsilon_h} + \text{cycle}(f, g, h) = 0$. Other notation is clear from the context.

2 Quantum antibrackets and anticanonical transformations

Here we recall main definitions and properties of quantum antibrackets as they formulated in [16] - [18]. Then we define operator-valued anticanonical transformations and derive how quantum antibrackets behave under anticanonical transformations of their entries.

Let Q be a Fermionic nilpotent operator,

$$\varepsilon(Q) = 1, \ Q^2 \equiv \frac{1}{2}[Q,Q] = 0.$$
 (2.1)

Then the general quantum antibracket is defined by the formula

$$(f,g)_Q \equiv \frac{1}{2} \left([f,[Q,g]] - [g,[Q,f]](-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} \right)$$
(2.2)

for any two operators f and g.

It satisfies

$$\varepsilon \left((f,g)_Q \right) = \varepsilon_f + \varepsilon_g + 1,$$
(2.3)

and

$$(f,g)_Q = -(g,f)_Q(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}.$$
 (2.4)

It follows from (2.2) that the modified Leibniz rule

$$(fg,h)_Q - f(g,h)_Q - (f,h)_Q g(-1)^{\varepsilon_g(\varepsilon_h+1)} = \frac{1}{2} \left([f,h][g,Q](-1)^{\varepsilon_h(\varepsilon_g+1)} + [f,Q][g,h](-1)^{\varepsilon_g} \right)$$
(2.5)

and Jacobi identity

$$(f, (g, h)_Q)_Q (-1)^{(\varepsilon_f + 1)(\varepsilon_h + 1)} + \operatorname{cycle}(f, g, h) = -\frac{1}{2} \left[(f, g, h)_Q (-1)^{(\varepsilon_f + 1)(\varepsilon_h + 1)}, Q \right], \quad (2.6)$$

hold, where

$$(f,g,h)_Q \equiv \frac{1}{3}(-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} \left(\left[(f,g)_Q,h \right] (-1)^{\varepsilon_h+(\varepsilon_f+1)(\varepsilon_h+1)} + \operatorname{cycle}(f,g,h) \right) = \frac{1}{3}(-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} \left(\left[f,(g,h)_Q \right] (-1)^{\varepsilon_g+\varepsilon_f(\varepsilon_h+1)} + \operatorname{cycle}(f,g,h) \right)$$
(2.7)

defines the next, 3-antibracket, for any operators f, g, h.

In its own turn, the 3-antibracket (2.7) satisfies the next Jacoby identity involving the next, 4-antibracket, and so on. In [17], [18], this hierarchy of subsequent higher-order quantum antibrackets is determined substancially by means of the corresponding generating mechanism.

Let B be a Bosonic operator, and A is an arbitrary one. Then, it follows from (2.6) that

$$(B, (B, A)_Q)_Q = \frac{1}{2} \left((B, B)_Q, A \right)_Q - \frac{1}{4} \left[(B, B, A)_Q, Q \right],$$
(2.8)

$$(B, B, A)_Q = \frac{1}{3} \left(- \left[A, (B, B)_Q \right] + 2 \left[(A, B)_Q, B \right] \right), \tag{2.9}$$

$$(B, (B, B)_Q)_Q = \frac{1}{6} \left[[B, (B, B)_Q], Q \right].$$
(2.10)

Another important consequence of the definition (2.2) and nilpotence condition (2.1) reads

$$[Q, (f, g)_Q] = [[Q, f], [Q, g]].$$
(2.11)

Now, let us define an operator-valued anticanonical transformation as follows. Let A_0 be an initial operator, and X, $\varepsilon(X) = 1$, be a Fermionic anticanonical generator. Then, the equation

$$\frac{dA}{d\lambda} = (X, A)_Q, \ A|_{\lambda=0} = A_0, \tag{2.12}$$

determines the anticanonical transformation $A_0 \to A$.

It follows from (2.11), (2.12) that

$$\frac{d}{d\lambda}[Q,A] = [[Q,X],[Q,A]],$$
(2.13)

and thereby [Q, X] serves as a canonical Bosonic generator transforming $[Q, A_0]$. The general solution to the parametric differential equation (2.12) reads

$$A = \tilde{A} + [Q, Y], \tag{2.14}$$

where

$$\tilde{A} = \tilde{A}(\lambda) = e^{\lambda[Q,X]} A_0 e^{-\lambda[Q,X]}, \qquad (2.15)$$

and

$$Y = -\frac{1}{2} \int_0^\lambda d\lambda' e^{\frac{1}{2}(\lambda - \lambda')[Q, X]} [X, \tilde{A}(\lambda')] e^{-\frac{1}{2}(\lambda - \lambda')[Q, X]}.$$
 (2.16)

There exists a nice interpretation of the solution (2.14): the first term in r.h.s., A, is just a canonical transform of A_0 with [Q, X] being a generator, while the second term, [Q, Y], is an "exact" form. When taking the formula (2.14) in the first order in X,

$$A - A_0 = \lambda \left([[Q, X], A_0] - \frac{1}{2} [Q, [X, A]] \right) + O(\lambda^2) = \lambda (X, A_0)_Q + O(\lambda^2)$$
(2.17)

we see that the part $[[Q, X], A_0]$ of quantum antibracket $(X, A_0)_Q$ in (2.17) is just an infinitesimal transform of A_0 with [Q, X] being a generator, while the part $-\frac{1}{2}[Q, [X, A]]$ (the "exact" one) represents the nonunimodularity characteristic to anticanonical transformations.

Now, let A and B be anticanonical transforms of A_0 and B_0 in the sense of the equation (2.12). Then the quantum antibracket $(A, B)_Q$ satisfies the equation

$$\frac{d}{d\lambda}(A,B)_Q = (X,(A,B)_Q)_Q + \frac{1}{2}[(X,A,B)_Q,Q], \qquad (2.18)$$

where the modified Jacobi identity (2.6) for f = X, g = A, h = B is taken into account. It follows from (2.18) that the deviation of the antibracket $(A, B)_Q$ from its anticanonical invariance (solution to the homogeneous part of eq. (2.18)) is given by the "exact" form

$$\left[Q, \frac{1}{2} \int_0^\lambda d\lambda' e^{\frac{1}{2}(\lambda - \lambda')[Q, X]} (X, A, B)_Q e^{-\frac{1}{2}(\lambda - \lambda')[Q, X]}\right].$$
(2.19)

Thus we conclude that the appearance of nonzero "exact" form $[(f, g, h)_Q, Q]$, which deviates the modified Jacobi identity from being a strong one, results in a similar deviation of the invariance property of quantum antibracket under anticanonical transformation of its entries.

3 BRST-invariant constraint algebra

Let Ω be a Fermionic operator which satisfies the standard BRST algebra

$$[\Omega, \Omega] = 0, \ [G_C, \Omega] = i\hbar\Omega, \tag{3.1}$$

with G_C being a ghost number Bosonic operator.

For the sake of definiteness, we assume that ghost sector is represented by canonical pairs $(C^{\alpha}, \bar{\mathcal{P}}_{\alpha}), \varepsilon(C^{\alpha}) = \varepsilon(\bar{\mathcal{P}}_{\alpha}) = \varepsilon_{\alpha} + 1$, with the only nonzero commutators

$$\left[C^{\alpha}, \bar{\mathcal{P}}_{\beta}\right] = i\hbar\delta^{\alpha}_{\beta},\tag{3.2}$$

and the BRST operator Ω is $C\overline{\mathcal{P}}$ -ordered. As for the ghost number assignment, we assume that

$$[G_C, C^{\alpha}] = i\hbar C^{\alpha}, \ [G_C, \bar{\mathcal{P}}_{\alpha}] = -i\hbar \bar{\mathcal{P}}_{\alpha}, \tag{3.3}$$

which corresponds to irreducible theories. BRST-invariant constraints are defined as

$$\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, \bar{\mathcal{P}}_{\alpha}], \ [\Omega, \mathcal{T}_{\alpha}] = 0.$$
(3.4)

In terms of quantum antibracket (2.2) with Ω standing for Q we have the following relations

$$(\mathcal{T}_{\alpha}, \mathcal{T}_{\beta})_{\Omega} = 0, \tag{3.5}$$

$$(\bar{\mathcal{P}}_{\alpha}, \bar{\mathcal{P}}_{\beta})_{\Omega} = (i\hbar)^2 U^{\gamma}_{\alpha\beta} \bar{\mathcal{P}}_{\gamma} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta} + \varepsilon_{\gamma}}, \qquad (3.6)$$

$$(\bar{\mathcal{P}}_{\alpha}, \mathcal{T}_{\beta})_{\Omega} = \frac{1}{2} i\hbar [\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}] (-1)^{\varepsilon_{\alpha}}, \qquad (3.7)$$

where structure coefficient operators

$$U^{\gamma}_{\alpha\beta} = -U^{\gamma}_{\beta\alpha}(-1)^{\varepsilon_{\alpha}\varepsilon_{\beta}} \tag{3.8}$$

are, in general, ghost-dependent.

If, in accordance with the ghost number prescriptions (3.1), (3.3), we represent the operator Ω explicitly in the form of a $C\bar{\mathcal{P}}$ -ordered power series expansion in ghosts,

$$\Omega = C^{\alpha}T_{\alpha} + \sum_{n\geq 1} \frac{1}{n!(n+1)!} C^{\alpha_{n+1}} \cdots C^{\alpha_1} \Omega^{\beta_n \cdots \beta_1}_{\alpha_1 \cdots \alpha_{n+1}} \bar{\mathcal{P}}_{\beta_1} \cdots \bar{\mathcal{P}}_{\beta_n}, \qquad (3.9)$$

then the corresponding expansions for \mathcal{T}_{α} and $U^{\gamma}_{\alpha\beta}$ are

$$\mathcal{T}_{\alpha} = T_{\alpha} + \sum_{n \ge 1} \frac{1}{(n!)^2} C^{\alpha_n} \cdots C^{\alpha_1} \Omega^{\beta_n \cdots \beta_1}_{\alpha_1 \cdots \alpha_n \alpha} \bar{\mathcal{P}}_{\beta_1} \cdots \bar{\mathcal{P}}_{\beta_n}, \qquad (3.10)$$

$$U_{\alpha\beta}^{\gamma} = \sum_{n\geq 0} \frac{1}{n!(n+1)!} C^{\alpha_n} \cdots C^{\alpha_1} \Omega_{\alpha_1 \cdots \alpha_n \alpha_\beta}^{\gamma\beta_n \cdots \beta_1} \bar{\mathcal{P}}_{\beta_1} \cdots \bar{\mathcal{P}}_{\beta_n} (-1)^{\varepsilon_\beta + \varepsilon_\gamma}.$$
 (3.11)

Now, due to the property (2.11) and definition (3.4), we get the following commutator algebra

$$[\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}] = i\hbar U^{\gamma}_{\alpha\beta} \mathcal{T}_{\gamma} - [U^{\gamma}_{\alpha\beta}, \Omega] \bar{\mathcal{P}}_{\gamma}, \qquad (3.12)$$

$$[\bar{\mathcal{P}}_{\alpha}, \bar{\mathcal{P}}_{\beta}] = 0, \qquad (3.13)$$

$$[\bar{\mathcal{P}}_{\alpha}, \mathcal{T}_{\beta}] = (i\hbar)^{-1} (\bar{\mathcal{P}}_{\alpha}, \bar{\mathcal{P}}_{\beta})_{\Omega}.$$
(3.14)

Thus we conclude that the BRST-invariant constraints \mathcal{T}_{α} together with ghost momenta $\bar{\mathcal{P}}_{\alpha}$ form two dual operator algebras, namely, the quantum-antibracket algebra (3.5) - (3.7) and commutator algebra (3.12) - (3.14).

4 Generating equations of BRST-invariant constraint algebra.

As we have established that \mathcal{T}_{α} together with $\bar{\mathcal{P}}_{\alpha}$ form two dual algebras, it seems quite natural to formulate the corresponding generating equations, in the line of general ideology of BRST-BFV approach. We can regard \mathcal{T}_{α} and $\bar{\mathcal{P}}_{\alpha}$ as first-class constraints with (3.12) -(3.14) being their involution relations. Moreover, we can rotate these first-class constraints with some (nonsingular) matrices, so that it seems natural to generalize a little bit the definition of \mathcal{T}_{α} .

First of all, let us rotate $\bar{\mathcal{P}}_{\alpha}$ in (3.4),

$$\bar{\mathcal{P}}_{\alpha} \to X_{\alpha} = \Lambda^{\beta}_{\alpha} \bar{\mathcal{P}}_{\beta},$$
(4.1)

so that new \mathcal{T}_{α} read

$$\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, X_{\alpha}], \quad [G_C, X_{\alpha}] = -i\hbar X_{\alpha}.$$
(4.2)

These \mathcal{T}_{α} , however, remain strongly BRST-invariant, $[\Omega, \mathcal{T}_{\alpha}] = 0$. To weaken the invariance, we can modify the definition of \mathcal{T}_{α} yet more,

$$\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, X_{\alpha}] - V_{\alpha}^{\beta} X_{\beta} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta}}, \qquad (4.3)$$

with V^{β}_{α} being a flat BRST connection,

$$R^{\beta}_{\alpha} \equiv (i\hbar)^{-1} [\Omega, V^{\beta}_{\alpha}] - V^{\gamma}_{\alpha} V^{\beta}_{\gamma} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\gamma}} = 0.$$
(4.4)

Then we have a weak BRST invariance,

$$[\Omega, \mathcal{T}_{\alpha}] = i\hbar V_{\alpha}^{\beta} \mathcal{T}_{\beta}, \qquad (4.5)$$

which corresponds to the rotation

$$\mathcal{T}_{\alpha} \to G^{\beta}_{\alpha} \mathcal{T}_{\beta}, \ X_{\alpha} \to G^{\beta}_{\alpha} X_{\beta} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta}},$$

$$(4.6)$$

in (4.2), together with the choice

$$V_{\alpha}^{\beta} = (i\hbar)^{-1} [\Omega, G_{\alpha}^{-1\gamma}] G_{\gamma}^{\beta}.$$

$$(4.7)$$

We expect the above rotations (4.1) - (4.6) to be a part of natural arbitrariness in the general solution to the algebra-generating equations.

Now, let us turn directly to the formulation of generating equations in question. We begin with some operators \mathcal{T}_{α} and X_{α} living in the same extended phase space as a BRST operator Ω does. Their Grassmann parities are

$$\varepsilon(\mathcal{T}_{\alpha}) = \varepsilon_{\alpha}, \ \varepsilon(X_{\alpha}) = \varepsilon_{\alpha} + 1,$$
(4.8)

and their intrinsic ghost number values are given by

$$[G_C, \mathcal{T}_\alpha] = 0, \ [G_C, X_\alpha] = -i\hbar X_\alpha. \tag{4.9}$$

Next, let us extend the phase space yet more by introducing new ghost-type canonical pairs via the correspondence

$$\mathcal{T}_{\alpha} \mapsto (B^{\alpha}, \Pi_{\alpha}), \ X_{\alpha} \mapsto (B^*_{\alpha}, \Pi^{\alpha}_*),$$

$$(4.10)$$

with the only nonzero commutators

$$[B^{\alpha}, \Pi_{\beta}] = i\hbar \delta^{\alpha}_{\beta}, \quad [B^*_{\alpha}, \Pi^{\beta}_*] = i\hbar \delta^{\beta}_{\alpha}. \tag{4.11}$$

Their Grassmann parities are

$$\varepsilon(B^{\alpha}) = \varepsilon(\Pi_{\alpha}) = \varepsilon_{\alpha} + 1, \ \varepsilon(B^*_{\alpha}) = \varepsilon(\Pi^{\alpha}_*) = \varepsilon_{\alpha}.$$
(4.12)

All new operators commute with the intrinsic ghost number operator G_C . However, they have their own ghost number operators G_B and G_{B^*} ,

$$[G_B, B^{\alpha}] = i\hbar B^{\alpha}, \ [G_B, \Pi_{\alpha}] = -i\hbar \Pi_{\alpha}, \tag{4.13}$$

$$[G_{B^*}, B^*_{\alpha}] = i\hbar B^*_{\alpha}, \ [G_{B^*}, \Pi^{\alpha}_*] = -i\hbar \Pi^{\alpha}_*,$$
(4.14)

$$[G_{B^*}, B^{\alpha}] = [G_{B^*}, \Pi_{\alpha}] = [G_B, B^*_{\alpha}] = [G_B, \Pi^{\alpha}_*] = 0, \qquad (4.15)$$

$$[G_B, G_{B^*}] = [G_C, G_B] = [G_C, G_{B^*}] = 0.$$
(4.16)

A total ghost number operator is

$$G = G_C + G_B - 2G_{B^*} = G_{CB^*} + G_{BB^*}, (4.17)$$

where

$$G_{CB^*} = G_C - G_{B^*}, \ G_{BB^*} = G_B - G_{B^*}.$$
(4.18)

Let A be an arbitrary operator. We define the total ghost number value, gh(A), and total degree, deg(A), as

$$[G, A] = i\hbar \operatorname{gh}(A) A, \ [G_{BB^*}, A] = i\hbar \operatorname{deg}(A) A,$$
(4.19)

so that

$$[G_{CB^*}, A] = i\hbar \left(\operatorname{gh}(A) - \operatorname{deg}(A)\right) A.$$
(4.20)

We have, in particular,

$$\varepsilon(\Omega) = 1, \ \operatorname{gh}(\Omega) = 1, \ \operatorname{deg}(\Omega) = 0.$$
 (4.21)

In what follows, it is convenient to use the condensed notation

$$T_A \equiv \{\mathcal{T}_{\alpha}; -X_{\alpha}\}, \qquad (4.22)$$

$$C^{A} \equiv \left\{ B^{\alpha}; \Pi^{\alpha}_{*}(-1)^{\varepsilon_{\alpha}+1} \right\}, \quad \bar{\mathcal{P}}_{A} \equiv \left\{ \Pi_{\alpha}; B^{*}_{\alpha} \right\}, \quad [C^{A}, \bar{\mathcal{P}}_{B}] = i\hbar\delta^{A}_{B}. \tag{4.23}$$

We have

$$\varepsilon(T_A) = \{\varepsilon_{\alpha}; \varepsilon_{\alpha} + 1\}, \quad \operatorname{gh}(T_A) = \{0; -1\}, \quad \operatorname{deg}(T_A) = \{0; 0\}, \quad (4.24)$$

$$\varepsilon(C^A) = \{\varepsilon_{\alpha} + 1; \varepsilon_{\alpha}\}, \quad \operatorname{gh}(C^A) = \{1; 2\}, \quad \operatorname{deg}(C^A) = \{1; 1\}, \quad (4.25)$$

$$\varepsilon(T_A) \equiv \varepsilon_A, \quad \varepsilon(\bar{\mathcal{P}}_A) = \varepsilon(C^A) = \varepsilon_A + 1,$$
(4.26)

$$\operatorname{gh}(\bar{\mathcal{P}}_A) = -\operatorname{gh}(C^A), \quad \operatorname{deg}(\bar{\mathcal{P}}_A) = -\operatorname{deg}(C^A).$$
 (4.27)

In the new extended phase space, spanned by original phase variables, ordinary ghosts and new variables (4.10), let us consider the following set of equations

$$[\Sigma_1, \Sigma_1] = 0, \ [\Delta, \Delta] = 0, \ [\Delta, \Sigma_1] = 0,$$
 (4.28)

$$\varepsilon(\Sigma_1) = 1, \ \operatorname{gh}(\Sigma_1) = 1, \ \operatorname{deg}(\Sigma_1) = 1,$$

$$(4.29)$$

$$\varepsilon(\Delta) = 1, \ \operatorname{gh}(\Delta) = 1, \ \operatorname{deg}(\Delta) = 0,$$

$$(4.30)$$

together with the boundary conditions

$$\Sigma_1 = C^A T_A + \dots, \quad \Delta = \Omega + \dots, \tag{4.31}$$

where dots, ..., mean all possible higher-order terms in $(C^A, \overline{\mathcal{P}}_A)$, allowed by (4.29), (4.30).

We also require for the operator Δ to satisfy the extra condition: the Δ -antibracket matrix $(B^{\alpha}, B^*_{\beta})_{\Delta}$ should be invertible.

We state that the equations (4.28) - (4.31), when expanded in $(C^A, \overline{\mathcal{P}}_A)$, generate a BRST-invariant constraint algebra.

In order to see this, let us consider the $C\bar{\mathcal{P}}$ -ordered expansions for Σ_1 and Δ

$$\Sigma_1 = C^A T_A + \frac{1}{2} C^B C^A U^C_{AB} \bar{\mathcal{P}}_C(-1)^{\varepsilon_B + \varepsilon_C} + \dots, \qquad (4.32)$$

$$\Delta = \Omega + C^A V^B_A \bar{\mathcal{P}}_B(-1)^{\varepsilon_B} + \frac{1}{4} C^B C^A V^{CD}_{AB} \bar{\mathcal{P}}_D \bar{\mathcal{P}}_C(-1)^{\varepsilon_B + \varepsilon_D} + \dots$$
(4.33)

By substituting (4.32) into the first in (4.28), we get, in the second order in C^A , the standard involution relations,

$$[T_A, T_B] = i\hbar U_{AB}^C T_C. \tag{4.34}$$

Next, by substituting (4.33) into the second in (4.28), we get, in the zeroth and first order in C^A ,

$$[\Omega, \Omega] = 0, \tag{4.35}$$

and

$$[V_A^B, \Omega](-1)^{\varepsilon_B} = i\hbar V_A^C V_C^B.$$
(4.36)

In the same way, by substituting (4.32), (4.33) into the third in (4.28), we get, in the first order in C^A ,

$$[T_A, \Omega] = -i\hbar V_A^B T_B. \tag{4.37}$$

In (4.35) we recognize the nilpotence condition for Ω . It is remarkable that (4.36) is nothing but the nilpotence condition for matrix-extended Ω , $\hat{\Omega}_{A}^{B}$,

$$\hat{\Omega}_{A}^{C}\hat{\Omega}_{C}^{B} = 0, \quad \hat{\Omega}_{A}^{B} \equiv (-1)^{\varepsilon_{A}}\delta_{A}^{B}\Omega - i\hbar V_{A}^{B}.$$

$$(4.38)$$

In their turn, the involution relations (4.37) determine T_A to be BRST-invariant constraints in the most general (weak) form.

It is easy to see that the previous (particular) representations (4.2), (4.3), (4.5) follow immediately from (4.37) when choosing V_A^B in the form

$$V_A^B = \begin{pmatrix} V_\alpha^\beta (-1)^{\varepsilon_\alpha} & 0\\ \delta_\alpha^\beta (-1)^{\varepsilon_\alpha} & -V_\alpha^\beta (-1)^{\varepsilon_\beta} \end{pmatrix}.$$
 (4.39)

Now, let us consider the second in (4.28) in the second order in C^A . We get

$$[V_A^C, V_B^D](-1)^{(\varepsilon_B+1)(\varepsilon_C+1)} - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B} = i\hbar (V_A^E V_{EB}^{CD}(-1)^{\varepsilon_B} - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B}) + i\hbar (V_A^C V_{EB}^{CD}(-1)^{\varepsilon_C+\varepsilon_B} - i\hbar (V_A^C V_{EB}^{CD}(-1)^{\varepsilon_C+\varepsilon_B}) + i\hbar (V_A^C V_{$$

$$+i\hbar(V_{AB}^{CE}V_E^D(-1)^{\varepsilon_C} - (C \leftrightarrow D)(-1)^{\varepsilon_C\varepsilon_D}) - [V_{AB}^{CD},\Omega](-1)^{\varepsilon_C+\varepsilon_D} - \frac{1}{2}(i\hbar)^2 V_{AB}^{EF}V_{FE}^{CD}.$$
(4.40)

In the same order in C^A , the third in (4.28) yields

$$([T_A, V_B^C] - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B}) - i\hbar U_{AB}^D V_D^C +$$

$$+i\hbar(V_A^D U_{DB}^C(-1)^{\varepsilon_B} - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B}) + [U_{AB}^C, \Omega](-1)^{\varepsilon_C} + \frac{1}{2}i\hbar V_{AB}^{ED} Z_{DE}^C = 0, \quad (4.41)$$

where

$$Z_{AB}^{C} \equiv T_A \delta_B^C - T_B \delta_A^C (-1)^{\varepsilon_A \varepsilon_B} - i\hbar U_{AB}^C, \qquad (4.42)$$

$$Z_{AB}^C T_C = 0. (4.43)$$

By multiplying (4.41) by T_C from the right, we get, identically, zero due to (4.34), (4.37). There are no more equations up to the third order in C^A .

Given first-class constraints T_A , eqs. (4.34) determine U_{AB}^C . Then, eqs. (4.36), (4.37) determine V_B^A . Then, eqs. (4.40), (4.41) determine V_{AB}^{CD} , and so on.

If generating equations (4.28) allow for Δ linear in C^A ,

$$\Delta = \Omega + C^A V_A^B \bar{\mathcal{P}}_B(-1)^{\varepsilon_B}, \qquad (4.44)$$

which implies, in accordance with (4.40), (4.41), that

$$[V_A^C, V_B^D](-1)^{(\varepsilon_B+1)(\varepsilon_C+1)} - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B} = 0, \qquad (4.45)$$

and

$$([T_A, V_B^C] - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B}) - i\hbar U_{AB}^D V_D^C + +i\hbar (V_A^D U_{DB}^C (-1)^{\varepsilon_B} - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B}) + [U_{AB}^C, \Omega](-1)^{\varepsilon_C} = 0, \qquad (4.46)$$

then, for any quantities f, g, h, depending on $B^{\alpha}, B^{*}_{\alpha}$ only, their Δ -antibrackets satisfy

$$[(f,g)_{\Delta},h] = 0, \tag{4.47}$$

so that their 3-antibrackets vanish

$$(f, g, h)_{\Delta} = 0,$$
 (4.48)

and, thereby, Jacobi identities (2.6) become strong. As these f, g, h commute among themselves, Leibniz rule (2.5) becomes strong as well. Besides, we have

$$\varepsilon(B^{\alpha}) + \varepsilon(B^*_{\alpha}) = 1, \qquad (4.49)$$

$$\operatorname{gh}(B^{\alpha}) + \operatorname{gh}(B^*_{\alpha}) = -1, \qquad (4.50)$$

$$\deg(B^{\alpha}) + \deg(B^*_{\alpha}) = 0, \tag{4.51}$$

by assignment. So, it follows from (4.48) - (4.51) that the variables B^{α} and B^*_{α} behave as normal fields and antifields with respect to Δ -antibracket, provided the conditions (4.45), (4.46) are satisfied.

We emphasize, however, that the conditions (4.45), (4.46) are not required imperatively to be fulfilled in any case. They merely specify a certain basis of constraints T_A and quantities V_A^B , in which the formalism allows for a simple interpretation to the variables B^{α} , B^*_{α} . In the general case, the coefficients V_{AB}^{CD} are nonzero, and the expansions (4.32), (4.33) involve all higher orders in ghosts. Therefore, Δ -antibrackets do not meet, in general, a strong Jacobi identity, even if their entries depend on B^{α} , B^*_{α} only.

In principle, the involution relations (4.34) - (4.37) are the only conditions the lowestorder terms in (4.32), (4.33) should satisfy to. However, we require for r.h.s. in (4.37) to resolve for \mathcal{T}_{α} : this is just the extra condition formulated below (4.31). This condition means that any constraints T_A , satisfying these involution relations, can be rotated with a nonsingular matrix to take the form $T_A = \{(i\hbar)^{-1}[\Omega, \bar{\mathcal{P}}_{\alpha}]; -\bar{\mathcal{P}}_{\alpha}\}.$

5 Generating equations of antibracket algebra

As we have seen above, the variables B^{α} and B^{*}_{α} behave as fields and antifields with respect to Δ -antibracket. It seems quite natural to expect a similar behaviour for momenta Π^{α}_{*} and Π_{α} with respect to some "dual" antibracket.

To put the above idea into effect, let us define the resolvent operator $\overline{\Delta}$ to satisfy the generating equations

$$[\bar{\Delta}, \bar{\Delta}] = 0, \quad [\Delta, \bar{\Delta}] = i\hbar G_{BB^*}, \tag{5.1}$$

$$\varepsilon(\bar{\Delta}) = 1, \quad \operatorname{gh}(\bar{\Delta}) = -1, \quad \operatorname{deg}(\bar{\Delta}) = 0,$$
(5.2)

together with the boundary condition

$$\bar{\Delta} = \bar{\Omega} + \dots, \tag{5.3}$$

where dots, ..., mean all possible higher order terms in the variables $(C^A, \bar{\mathcal{P}}_A)$, allowed by (5.2), while $\bar{\Omega}$ is of the zeroth order.

Let us consider for $\overline{\Delta}$ the $C\overline{\mathcal{P}}$ -ordered power series expansion

$$\bar{\Delta} = \bar{\Omega} + C^A \bar{V}^B_A \bar{\mathcal{P}}_B(-1)^{\varepsilon_B} + \frac{1}{4} C^B C^A \bar{V}^{CD}_{AB} \bar{\mathcal{P}}_D \bar{\mathcal{P}}_C(-1)^{\varepsilon_B + \varepsilon_D} + \dots, \qquad (5.4)$$

similar to (4.33). Then, we get from (5.1), (5.2) the following lowest-order equations for coefficient operators

$$[\bar{\Omega}, \bar{\Omega}] = 0, \tag{5.5}$$

$$[\bar{V}_A^B, \bar{\Omega}](-1)^{\varepsilon_B} = i\hbar \bar{V}_A^C \bar{V}_C^B, \qquad (5.6)$$

$$[\Omega, \bar{\Omega}] = 0, \tag{5.7}$$

$$[V_A^B, \bar{\Omega}](-1)^{\varepsilon_B} + [\bar{V}_A^B, \Omega](-1)^{\varepsilon_B} + i\hbar\delta_A^B = i\hbar V_A^C \bar{V}_C^B + i\hbar \bar{V}_A^C V_C^B.$$
(5.8)

In terms of the operator-valued matrix (4.38) and the same for Δ ,

$$\hat{\bar{\Omega}}_{A}^{B} \equiv (-1)^{\varepsilon_{A}} \delta_{A}^{B} \bar{\Omega} - i\hbar \bar{V}_{A}^{B}, \qquad (5.9)$$

equations (5.5) - (5.8) rewrite as

$$\hat{\Omega}_A^C \hat{\Omega}_C^B = 0, \qquad (5.10)$$

$$\hat{\Omega}_A^C \hat{\bar{\Omega}}_C^B + \hat{\bar{\Omega}}_A^C \hat{\Omega}_C^B = i\hbar \delta_A^B.$$
(5.11)

As for the nilpotent operator $\overline{\Omega}$, it lives in the same phase space as Ω does, and, when expanded in ordinary ghosts $(C^{\alpha}, \overline{\mathcal{P}}_{\alpha})$, begins with $\overline{\Omega} = \overline{T}^{\alpha} \overline{\mathcal{P}}_{\alpha}(-1)^{\varepsilon_{\alpha}} + \ldots$, where \overline{T}^{α} are linear combinations of the first-class constraints T_{α} , dual to $T_{\alpha}, \overline{T}^{\alpha}T_{\alpha} = 0$.

The same as for Δ , if generating equations allow for $\overline{\Delta}$ linear in C^A , then, for any quantities depending on Π^{α}_* , Π_{α} only, $\overline{\Delta}$ -antibracket meets a strong Jacobi identity.

However, now we have, by assignment, a counterpart of (4.49) - (4.51) in the form

$$\varepsilon(\Pi_*^{\alpha}) + \varepsilon(\Pi_{\alpha}) = 1, \qquad (5.12)$$

$$gh(\Pi_*^{\alpha}) + gh(\Pi_{\alpha}) = 1, \qquad (5.13)$$

$$\deg(\Pi_*^{\alpha}) + \deg(\Pi_{\alpha}) = 0, \qquad (5.14)$$

We see that the signs in r.h.s. of (4.50) and (5.13) are opposite, which means that, in contrast to B^{α} , B^*_{α} , the momenta Π^{α}_* and Π_{α} behave as "twisted" fields and antifields [19] - [22].

In what follows, we imply that a solution to the generating equations (4.28) - (4.31) and (5.1) - (5.3) does exist.

Then, by commuting Δ with the third equation in (4.28), and using the third in (4.29) and the second in (5.1), we get

$$\Sigma_1 = (i\hbar)^{-1} [\Delta, S_1], \tag{5.15}$$

where

$$S_1 = (i\hbar)^{-1} [\bar{\Delta}, \Sigma_1] + (i\hbar)^{-1} [\Delta, Y_1], \qquad (5.16)$$

and Y_1 is an arbitrary Fermionic operator with $gh(Y_1) = -1$, $deg(Y_1) = 1$.

For S_1 itself, we have

$$\varepsilon(S_1) = 0, \quad \text{gh}(S_1) = 0, \quad \text{deg}(S_1) = 1.$$
 (5.17)

By substituting (5.15) into the first in (4.28), and using the property (2.11), we get

$$[\Delta, (S_1, S_1)_{\Delta}] = 0. \tag{5.18}$$

In its turn, by commuting $\overline{\Delta}$ with (5.18), we obtain, similarly to (5.15),

$$(S_1, S_1)_{\Delta} = i\hbar[\Delta, S_2], \tag{5.19}$$

where

$$S_2 = \frac{1}{2} (i\hbar)^{-3} [\bar{\Delta}, (S_1, S_1)_{\Delta}] + (i\hbar)^{-1} [\Delta, Y_2], \qquad (5.20)$$

and Y_2 is an arbitrary Fermionic operator with $gh(Y_2) = -1$, $deg(Y_2) = 2$.

For S_2 itself, we have

$$\varepsilon(S_2) = 0, \quad \text{gh}(S_2) = 0, \quad \text{deg}(S_2) = 2.$$
 (5.21)

Now, let us consider the following master equation

$$(S,S)_{\Delta} = i\hbar[\Delta,S], \tag{5.22}$$

for a Bosonic operator S of the form

$$S = \sum_{k \ge 0} S_k, \quad S_0 = G_{CB^*}, \quad \varepsilon(S_k) = 0, \quad gh(S_k) = 0, \quad \deg(S_k) = k.$$
(5.23)

We have

$$[S_0, \Delta] = i\hbar\Delta, \quad [S_0, S_k] = -i\hbar k S_k. \tag{5.24}$$

By substituting (5.23) into (5.22), and using (5.24), we get the following chain of equations

$$F_k = 0, \quad k \ge 2, \tag{5.25}$$

where

$$F_k \equiv R_k - i\hbar(k-1)[\Delta, S_k], \qquad (5.26)$$

$$R_k \equiv \sum_{j=1}^{k-1} (S_j, S_{k-j})_{\Delta}.$$
 (5.27)

At k = 2, (5.25) coincides exactly with (5.19), so that we can identify S_1 and S_2 in (5.23) with (5.16) and (5.20), respectively. Then, by making use of the identity (2.10) for $Q = \Delta$, B = S, it is easy to show that

$$[\Delta, R_k] = 0, \tag{5.28}$$

provided the equations

$$F_m = 0, \quad m = 2, \dots, k - 1,$$
 (5.29)

are satisfied.

Indeed, it follows from (2.10) that

$$6(S, F)_{\Delta} - [[S, F], \Delta] = 4i\hbar[\Delta, F], \qquad (5.30)$$

where

$$F \equiv (S, S)_{\Delta} - i\hbar[\Delta, S].$$
(5.31)

Let the equations (5.29) be satisfied. Then, by taking in (5.30) the sector with degree equal to k, we get

$$6(S_0, F_k)_{\Delta} - [[S_0, F_k], \Delta] = 4i\hbar[\Delta, F_k], \qquad (5.32)$$

which yields immediately

$$(k-2)[\Delta, R_k] = 0. (5.33)$$

Finally, by commuting $\overline{\Delta}$ with (5.28), we obtain (5.25) with

$$S_k = \frac{1}{k(k-1)} (i\hbar)^{-3} [\bar{\Delta}, R_k] + (i\hbar)^{-1} [\Delta, Y_k], \qquad (5.34)$$

$$\varepsilon(Y_k) = 1, \quad \text{gh}(Y_k) = -1, \quad \text{deg}(Y_k) = k. \tag{5.35}$$

Thus, we conclude that all the operators S_k entering the expansion (5.23) for S do exist. Thereby, we have established that master equation (5.22) has a solution generated by Σ_1 via (5.15), (5.19). This solution describes the antibracket algebra generated by BRST-invariant constraints.

Let us consider the simplest case of a rank-one theory, a Lie-type algebra with constant structure coefficients. Then, by choosing $C\bar{\mathcal{P}}$ -ordering in ghost sector, we have the following BRST operator Ω ,

$$\Omega = C^{\alpha}T_{\alpha} + \frac{1}{2}C^{\beta}C^{\alpha}U^{\gamma}_{\alpha\beta}\bar{\mathcal{P}}_{\gamma}(-1)^{\varepsilon_{\beta}+\varepsilon_{\gamma}},$$
(5.36)

with $U^{\gamma}_{\alpha\beta}$ being constant.

Consider the simplest possible form of the operator Δ , which is

$$\Delta = \Omega + \Pi_*^{\alpha} \Pi_{\alpha} (-1)^{\varepsilon_{\alpha} + 1}, \tag{5.37}$$

so that the corresponding resolvent operator $\bar{\Delta}$ reads

$$\bar{\Delta} = \bar{\Omega} - B^{\alpha} B^*_{\alpha}. \tag{5.38}$$

With an operator Δ chosen in the form (5.37), the equation (5.19) has a solution of the form

$$S_1 = \bar{\mathcal{P}}_{\alpha} B^{\alpha} + \frac{1}{2} B^{\beta} B^{\alpha} U^{\gamma}_{\alpha\beta} B^*_{\gamma} (-1)^{\varepsilon_{\beta}}, \qquad (5.39)$$

$$(S_1, S_1)_{\Delta} = 0, \quad S_2 = 0,$$
 (5.40)

while the nilpotent operator Σ_1 in (5.15) is given by the formula

$$\Sigma_{1} = (i\hbar)^{-1}[\Delta, S_{1}] =$$

$$= B^{\alpha} \mathcal{T}_{\alpha} + \frac{1}{2} B^{\beta} B^{\alpha} U^{\gamma}_{\alpha\beta} \Pi_{\gamma} (-1)^{\varepsilon_{\beta} + \varepsilon_{\gamma}} + \Pi^{\alpha}_{*} \bar{\mathcal{P}}_{\alpha} (-1)^{\varepsilon_{\alpha}} - \Pi^{\beta}_{*} B^{\alpha} U^{\gamma}_{\alpha\beta} B^{*}_{\gamma} (-1)^{\varepsilon_{\beta}}, \qquad (5.41)$$

where \mathcal{T}_{α} are $C\bar{\mathcal{P}}$ -ordered BRST-invariant constraints,

=

$$\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, \bar{\mathcal{P}}_{\alpha}] = T_{\alpha} + C^{\beta} U^{\gamma}_{\alpha\beta} \bar{\mathcal{P}}_{\gamma} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\gamma}}.$$
(5.42)

In the general case, it can be shown that the appearance of nonzero S_k , $k \ge 2$, entering the expansion (5.23), is an effect of anticanonical transformation (2.14) - (2.16) applied to the operator S_1 satisfying the homogeneous master equation (5.40). Roughly speaking, we can say that r.h.s. of (5.19) comes just from the deviation (2.19).

Let us also mention that the solution (5.39), (5.40) remains valid even if structure coefficients $U^{\gamma}_{\alpha\beta}$ in (5.36) are not constant but satisfy the quasigroup conditions

$$[U^{\gamma}_{\alpha\beta}, U^{\rho}_{\mu\nu}] = 0, \quad [[T_{\alpha}, U^{\delta}_{\beta\gamma}], U^{\rho}_{\mu\nu}] = 0.$$
 (5.43)

6 BRST-invariant constraint algebra in rank-one theories

Here, we give some explicit formulas potentially useful for practical applications to rankone theories. We consider BRST-invariant algebra in its commutator and antibracket form in the cases of Weyl- and Wick- ordered ghost sector, which are most popular ones.

6.1 Weyl-ordered ghost sector

In the case of Weyl-ordered ghost sector, a rank-one theory is described by the following BRST-operator linear in ghost momenta [23],

$$\Omega = C^{\alpha}T_{\alpha} + \frac{1}{6}C^{\beta}C^{\alpha}U^{\gamma}_{\alpha\beta}\bar{\mathcal{P}}_{\gamma}(-1)^{\varepsilon_{\beta}+\varepsilon_{\gamma}} +$$

$$+\frac{1}{6}C^{\alpha}U^{\gamma}_{\alpha\beta}\bar{\mathcal{P}}_{\gamma}C^{\beta}(-1)^{\varepsilon_{\beta}+\varepsilon_{\gamma}}+\frac{1}{6}\bar{\mathcal{P}}_{\gamma}U^{\gamma}_{\alpha\beta}C^{\beta}C^{\alpha}(-1)^{\varepsilon_{\beta}+\varepsilon_{\gamma}+(\varepsilon_{\alpha}+\varepsilon_{\beta})(\varepsilon_{\gamma}+1)}.$$
(6.1)

Original constraint algebra is given by the involution relation [23],

$$[T_{\alpha}, T_{\beta}] = \frac{i\hbar}{2} \left(U_{\alpha\beta}^{\gamma} T_{\gamma} + T_{\gamma} U_{\alpha\beta}^{\gamma} (-1)^{(\varepsilon_{\alpha} + \varepsilon_{\beta} + 1)\varepsilon_{\gamma}} \right) + \left(\frac{i\hbar}{2}\right)^{2} [U_{\alpha\delta}^{\gamma}, U_{\gamma\beta}^{\delta}] (-1)^{\varepsilon_{\delta}(\varepsilon_{\beta} + 1)}.$$
(6.2)

BRST-invariant constraints are

$$\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, \bar{\mathcal{P}}_{\alpha}] = T_{\alpha} + \frac{1}{2} \left(C^{\beta} U^{\gamma}_{\beta\alpha} \bar{\mathcal{P}}_{\gamma} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\gamma}} + \bar{\mathcal{P}}_{\gamma} U^{\gamma}_{\alpha\beta} C^{\beta} (-1)^{\varepsilon_{\alpha} + (\varepsilon_{\alpha} + \varepsilon_{\beta} + 1)\varepsilon_{\gamma}} \right).$$
(6.3)

Their commutator algebra reads

$$[\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}] = \frac{i\hbar}{2} \left(U^{\gamma}_{\alpha\beta} \mathcal{T}_{\gamma} + \mathcal{T}_{\gamma} U^{\gamma}_{\alpha\beta} (-1)^{(\varepsilon_{\alpha} + \varepsilon_{\beta} + 1)\varepsilon_{\gamma}} \right) + \frac{1}{2} \left([\Omega, U^{\gamma}_{\alpha\beta}] \bar{\mathcal{P}}_{\gamma} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta} + \varepsilon_{\gamma}} - \bar{\mathcal{P}}_{\gamma} [\Omega, U^{\gamma}_{\alpha\beta}] (-1)^{(\varepsilon_{\alpha} + \varepsilon_{\beta})\varepsilon_{\gamma}} \right),$$
(6.4)

$$[\bar{\mathcal{P}}_{\alpha}, \bar{\mathcal{P}}_{\beta}] = 0, \quad [\bar{\mathcal{P}}_{\alpha}, \mathcal{T}_{\beta}] = i\hbar U^{\gamma}_{\alpha\beta} \bar{\mathcal{P}}_{\gamma}(-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta} + \varepsilon_{\gamma}}.$$
(6.5)

We see that the extension, represented by the third term in r.h.s. in (6.2), is absent in (6.4), although we have in (6.4) an admixture of ghost momenta $\bar{\mathcal{P}}_{\alpha}$, instead.

The antibracket algebra, corresponding to (6.4), (6.5), reads

$$(\mathcal{T}_{\alpha}, \mathcal{T}_{\beta})_{\Omega} = 0, \quad (\bar{\mathcal{P}}_{\alpha}, \bar{\mathcal{P}}_{\beta})_{\Omega} = (i\hbar)^2 U^{\gamma}_{\alpha\beta} \bar{\mathcal{P}}_{\gamma} (-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta} + \varepsilon_{\gamma}}, \tag{6.6}$$

$$(\bar{\mathcal{P}}_{\alpha}, \mathcal{T}_{\beta})_{\Omega} = \frac{1}{2} i\hbar[\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}](-1)^{\varepsilon_{\alpha}}.$$
(6.7)

6.2 Wick-ordered ghost sector

As usual, Wick ghost sector is represented by two sets of Wick pairs, $(C^{\alpha}, \bar{C}^{\dagger}_{\alpha})$ and $(\bar{C}_{\alpha}, C^{\dagger \alpha})$, with the only nonzero commutators,

$$[C^{\alpha}, \bar{C}^{\dagger}_{\beta}] = \delta^{\alpha}_{\beta}, \quad [\bar{C}_{\alpha}, C^{\dagger\beta}] = \delta^{\beta}_{\alpha}.$$
(6.8)

In a rank-one theory, Wick-ordered BRST operator reads [23]

$$\Omega = T^{\dagger}_{\alpha}C^{\alpha} + C^{\dagger\alpha}T_{\alpha} +$$

$$+\left(\frac{1}{2}\bar{C}^{\dagger}_{\gamma}U^{\dagger\gamma}_{\alpha\beta}C^{\alpha}C^{\beta}+\frac{1}{2}C^{\dagger\beta}C^{\dagger\alpha}U^{\gamma}_{\alpha\beta}\bar{C}_{\gamma}+C^{\dagger\alpha}\bar{U}^{\gamma}_{\alpha\beta}\bar{C}_{\gamma}C^{\beta}+C^{\dagger\beta}\bar{C}^{\dagger}_{\gamma}\bar{U}^{\dagger\gamma}_{\alpha\beta}C^{\alpha}\right)(-1)^{\varepsilon_{\beta}}.$$
(6.9)

Original constraint algebra is given by the involution relations [23]

$$[T_{\alpha}, T_{\beta}] = U^{\gamma}_{\alpha\beta} T_{\gamma}, \quad [T^{\dagger}_{\beta}, T^{\dagger}_{\alpha}] = T^{\dagger}_{\gamma} U^{\dagger\gamma}_{\alpha\beta}, \tag{6.10}$$

$$[T_{\alpha}, T^{\dagger}_{\beta}] = \bar{U}^{\gamma}_{\alpha\beta}T_{\gamma} + T^{\dagger}_{\gamma}\bar{U}^{\dagger\gamma}_{\beta\alpha} + \bar{U}^{\gamma}_{\alpha\delta}\bar{U}^{\dagger\delta}_{\beta\gamma}(-1)^{\varepsilon_{\gamma}\varepsilon_{\delta}}.$$
(6.11)

BRST-invariant constraints are

$$\mathcal{T}_{\alpha} = [\bar{C}_{\alpha}, \Omega] = T_{\alpha} + C^{\dagger\beta} U^{\gamma}_{\beta\alpha} \bar{C}_{\gamma} (-1)^{\varepsilon_{\alpha}} + \bar{U}^{\gamma}_{\alpha\beta} \bar{C}_{\gamma} C^{\beta} (-1)^{\varepsilon_{\beta}} + \bar{C}^{\dagger}_{\gamma} \bar{U}^{\dagger\gamma}_{\beta\alpha} C^{\beta} (-1)^{\varepsilon_{\alpha}}, \qquad (6.12)$$

$$\mathcal{T}^{\dagger}_{\alpha} = [\Omega, \bar{C}^{\dagger}_{\alpha}] = T^{\dagger}_{\alpha} + \bar{C}^{\dagger}_{\gamma} U^{\dagger\gamma}_{\beta\alpha} C^{\beta}(-1)^{\varepsilon_{\alpha}} + C^{\dagger\beta} \bar{C}^{\dagger}_{\gamma} \bar{U}^{\dagger\gamma}_{\alpha\beta}(-1)^{\varepsilon_{\beta}} + C^{\dagger\beta} \bar{U}^{\gamma}_{\beta\alpha} \bar{C}_{\gamma}(-1)^{\varepsilon_{\alpha}}.$$
 (6.13)

Nonzero relations of their commutator algebra read

$$[\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}] = U^{\gamma}_{\alpha\beta}\mathcal{T}_{\gamma} + [\Omega, U^{\gamma}_{\alpha\beta}]\bar{C}_{\gamma}(-1)^{\varepsilon_{\alpha}+\varepsilon_{\beta}}, \qquad (6.14)$$

$$[\mathcal{T}^{\dagger}_{\beta}, \mathcal{T}^{\dagger}_{\alpha}] = \mathcal{T}^{\dagger}_{\gamma} U^{\dagger\gamma}_{\alpha\beta} + \bar{C}^{\dagger}_{\gamma} [U^{\dagger\gamma}_{\alpha\beta}, \Omega] (-1)^{\varepsilon_{\alpha} + \varepsilon_{\beta}}, \qquad (6.15)$$

$$[\mathcal{T}_{\alpha},\mathcal{T}_{\beta}^{\dagger}] = \bar{U}_{\alpha\beta}^{\gamma}\mathcal{T}_{\gamma} + \mathcal{T}_{\gamma}^{\dagger}\bar{U}_{\beta\alpha}^{\dagger\gamma} + [\Omega,\bar{U}_{\alpha\beta}^{\gamma}]\bar{C}_{\gamma}(-1)^{\varepsilon_{\alpha}+\varepsilon_{\beta}} + \bar{C}_{\gamma}^{\dagger}[\bar{U}_{\beta\alpha}^{\dagger\gamma},\Omega](-1)^{\varepsilon_{\alpha}+\varepsilon_{\beta}}, \qquad (6.16)$$

$$[\bar{C}_{\alpha}, \mathcal{T}_{\beta}] = U^{\gamma}_{\alpha\beta} \bar{C}_{\gamma} (-1)^{\varepsilon_{\beta}}, \quad [\mathcal{T}^{\dagger}_{\beta}, \bar{C}^{\dagger}_{\alpha}] = \bar{C}^{\dagger}_{\gamma} U^{\dagger\gamma}_{\alpha\beta} (-1)^{\varepsilon_{\beta}}, \tag{6.17}$$

$$[\mathcal{T}_{\alpha}, \bar{C}^{\dagger}_{\beta}] = \bar{U}^{\gamma}_{\alpha\beta} \bar{C}_{\gamma} (-1)^{\varepsilon_{\beta}} + \bar{C}^{\dagger}_{\gamma} \bar{U}^{\dagger\gamma}_{\beta\alpha} (-1)^{\varepsilon_{\alpha}}, \qquad (6.18)$$

$$[\bar{C}_{\beta}, \mathcal{T}_{\alpha}^{\dagger}] = \bar{C}_{\gamma}^{\dagger} \bar{U}_{\alpha\beta}^{\dagger\gamma} (-1)^{\varepsilon_{\beta}} + \bar{U}_{\beta\alpha}^{\gamma} \bar{C}_{\gamma} (-1)^{\varepsilon_{\alpha}}.$$
(6.19)

The same as in the case of Weyl-ordered ghost sector, we see that the extension, represented by the third term in r.h.s. of (6.11), is absent in (6.16), although we have in (6.16) an admixture of ghost momenta \bar{C}_{α} and $\bar{C}_{\alpha}^{\dagger}$, instead.

The antibracket algebra, corresponding to (6.14) - (6.19), reads

$$(\mathcal{T}_{\alpha}, \mathcal{T}_{\beta})_{\Omega} = 0, \quad (\mathcal{T}_{\alpha}^{\dagger}, \mathcal{T}_{\beta}^{\dagger})_{\Omega} = 0, \quad (\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}^{\dagger})_{\Omega} = 0,$$
 (6.20)

$$(\bar{C}_{\alpha}, \bar{C}_{\beta})_{\Omega} = U^{\gamma}_{\alpha\beta} \bar{C}_{\gamma}, \quad (\bar{C}^{\dagger}_{\beta}, \bar{C}^{\dagger}_{\alpha})_{\Omega} = \bar{C}^{\dagger}_{\gamma} U^{\dagger\gamma}_{\alpha\beta}, \tag{6.21}$$

$$(\bar{C}_{\alpha}, \bar{C}^{\dagger}_{\beta})_{\Omega} = \bar{U}^{\gamma}_{\alpha\beta} \bar{C}_{\gamma} (-1)^{\varepsilon_{\beta}} + \bar{C}^{\dagger}_{\gamma} \bar{U}^{\dagger\gamma}_{\beta\alpha} (-1)^{\varepsilon_{\alpha}}, \qquad (6.22)$$

$$(\bar{C}_{\alpha}, \mathcal{T}_{\beta})_{\Omega} = \frac{1}{2} [\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}], \quad (\mathcal{T}_{\beta}^{\dagger}, \bar{C}_{\alpha}^{\dagger})_{\Omega} = \frac{1}{2} [\mathcal{T}_{\beta}^{\dagger}, \mathcal{T}_{\alpha}^{\dagger}], \tag{6.23}$$

$$(\bar{C}_{\alpha}, \mathcal{T}_{\beta}^{\dagger})_{\Omega} = \frac{1}{2} [\mathcal{T}_{\alpha}, \mathcal{T}_{\beta}^{\dagger}], \quad (\mathcal{T}_{\beta}, \bar{C}_{\alpha}^{\dagger})_{\Omega} = \frac{1}{2} [\mathcal{T}_{\beta}, \mathcal{T}_{\alpha}^{\dagger}],$$
(6.24)

Explicit formulas given in Subsections 6.1 and 6.2 demonstrate in a transparent way that there exists an obvious dualism, represented via the general correspondence

$$\mathcal{T}, \quad \bar{\mathcal{P}}, \quad [,] \quad \iff \quad \bar{\mathcal{P}}, \quad \mathcal{T}, \quad (,)_{\Omega} \quad ,$$
 (6.25)

between the two alternative forms of BRST-invariant constraint algebra.

7 Conclusion

In previous sections, we have formulated a new approach to quantization of gauge-invariant dynamical systems, which is based substantially on the concept of BRST-invariant constraints.

The hearth of the construction is the new nilpotent "BRST-charge" Σ_1 , which lives in yet more extended phase space. Former extended phase space, spanned by initial phase variables and ordinary ghosts, now becomes a new "initial" space. New canonical pairs $(C^A, \bar{\mathcal{P}}_A)$ (4.23), (4.10) play the role of new "minimal" ghosts, while a new quantum number, the degree, plays the role of a new ghost number. Regarding these new canonical pairs as "minimal" ghosts in effect, we can introduce new antighosts, $(\mathcal{P}^A, \bar{C}_A)$, $\varepsilon(\mathcal{P}^A) = \varepsilon(\bar{C}_A) = \varepsilon_A + 1$, $\operatorname{gh}(\mathcal{P}^A) = -\operatorname{gh}(\bar{C}_A) = \{1; 2\}$, $\operatorname{deg}(\mathcal{P}^A) = -\operatorname{deg}(\bar{C}_A) = \{1; 1\}$, and Lagrange multipliers, (λ^A, π_A) , $\varepsilon(\lambda^A) = \varepsilon(\pi_A) = \varepsilon_A$, $\operatorname{gh}(\lambda^A) = -\operatorname{gh}(\pi_A) = \{0; 1\}$, $\operatorname{deg}(\lambda^A) = -\operatorname{deg}(\pi_A) = \{0; 0\}$, with the only nonzero commutators

$$[\mathcal{P}^A, \bar{C}_B] = i\hbar\delta^A_B, \quad [\lambda^A, \pi_B] = i\hbar\delta^A_B, \tag{7.1}$$

and then construct, in a usual way, a new gauge-fixed unitarizing Hamiltonian.

To realize the above program, we have to construct first a "minimal" Hamiltonian Ξ , which satisfies the equations

$$[\Sigma_1, \Xi] = 0, \tag{7.2}$$

$$\varepsilon(\Xi) = 0, \quad \operatorname{gh}(\Xi) = 0, \quad \operatorname{deg}(\Xi) = 0,$$
(7.3)

and boundary conditions

$$\Xi = \mathcal{H} + \dots, \quad [\mathcal{H}, \Omega] = 0. \tag{7.4}$$

Then, we construct a complete unitarizing Hamiltonian in the standard form,

$$H = \Xi + (i\hbar)^{-1} [\Sigma, \Psi], \qquad (7.5)$$

$$\varepsilon(\Psi) = 1, \quad \operatorname{gh}(\Psi) = -1, \quad \operatorname{deg}(\Psi) = -1,$$
(7.6)

where

$$\Sigma = \Sigma_1 + \pi_A \mathcal{P}^A, \quad \Psi = \bar{C}_A \chi^A + \bar{\mathcal{P}}_A \lambda^A, \tag{7.7}$$

and χ^A are gauge-fixing operators. Original Hamiltonian and first-class constraints are contained in \mathcal{H} and Ω , respectively, in their lowest-order terms, when expanded in power series in ordinary ghost operators $(C^{\alpha}, \bar{\mathcal{P}}_{\alpha})$.

Physical observables commute with Σ , while physical states are annihilated by this operator. Being a physical scalar product defined appropriately, physical matrix elements of physical operators are expected to be gauge independent. If so, one can transit to the unitary limit by choosing a unitary gauge of the form

$$\chi^A = 0, \quad \chi^A \equiv \{\chi^\alpha; C^\alpha\},\tag{7.8}$$

where χ^{α} is an ordinary gauge with respect to original constraints T_{α} , to identify physical transition amplitude (S-matrix) with the one in the standard BRST-BFV approach. However, when using general relativistic gauges, the formalism generalizes essentially the standard one by supporting yet more explicit BRST symmetry of the gauge algebra generating mechanism.

We finish with the following remark. Let us consider the standard form [4], [5] of an unitarizing Hamiltonian in BRST-BFV approach,

$$H = \mathcal{H} + (i\hbar)^{-1}[Q, \Psi], \qquad (7.9)$$

where \mathcal{H} is a minimal Hamiltonian,

$$Q = \Omega + \pi_{\alpha} \mathcal{P}^{\alpha}, \quad \Psi = \bar{C}_{\alpha} \chi^{\alpha} + \bar{\mathcal{P}}_{\alpha} \lambda^{\alpha}, \tag{7.10}$$

$$[\Omega, \Omega] = 0, \quad [\mathcal{H}, \Omega] = 0, \tag{7.11}$$

 Ω is a minimal BRST operator, and Ψ is a gauge-fixing Fermion.

We have

$$H = \mathcal{H} + (i\hbar)^{-1} [\Omega, \bar{\mathcal{P}}_{\alpha}] \lambda^{\alpha} + \bar{\mathcal{P}}_{\alpha} \mathcal{P}^{\alpha} + \pi_{\alpha} \chi^{\alpha} + \bar{C}_{\alpha} (i\hbar)^{-1} [\chi^{\alpha}, \Omega].$$
(7.12)

In the second and third terms in r.h.s. we recognize the constraints \mathcal{T}_{α} and X_{α} in their simplest possible form,

$$\mathcal{T}_{\alpha} = (i\hbar)^{-1} [\Omega, \bar{\mathcal{P}}_{\alpha}], \quad -X_{\alpha} = \bar{\mathcal{P}}_{\alpha}, \tag{7.13}$$

with λ^{α} and $-\mathcal{P}^{\alpha}$ being their respective Lagrange multipliers.

Then, the fourth and fifth terms are gauge-fixing ones with χ^{α} and $(i\hbar)^{-1}[\chi^{\alpha},\Omega]$ being gauge-fixing operators to \mathcal{T}_{α} and X_{α} , respectively, and π_{α} , \bar{C}_{α} being their respective Lagrange multipliers.

So, it appears that the standard Hamiltonian (7.9) - (7.10) is, actually, constructed just in terms of the "standard" BRST-invariant constraints (7.13) and their respective gauge-fixing operators.

However, as compared with general operators T_A , which satisfy (4.34), (4.37) only, the constraints (7.13) are rather special ones in the sense that they relate to a special basis in terms of T_A .

Contrary to that, our new Hamiltonian (7.12), living in yet more extended phase space, is constructed directly in terms of general operators T_A subject to (4.34), (4.37) only. Thus, the involvement of the new variables $(C^A, \bar{\mathcal{P}}_A)$, $(\mathcal{P}^A, \bar{C}_A)$, (λ^A, π_A) is just a price of arbitrariness in choosing possible basis to the general BRST-invariant constraints T_A .

It is also worthy to mention that the step, we have made from (7.9) - (7.11) to (7.2) - (7.7), seems to be only the first one in, possibly infinite, hierarchy of Hamiltonians.

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References

- C. Becchi, A. Rouet and R. Stora, Phys. Lett. B54 (1974) 344; Commun. Math. Phys. 42 (1975) 127; Ann. Phys. (NY) 98 (1976) 287.
- [2] I.V. Tyutin, Lebedev Institute preprint N 39 (1975).
- [3] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B55 (1975) 224; CERN preprint TH 2332-CERN (1977).
- [4] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **B69** (1977) 309.
- [5] E.S. Fradkin and T.E. Fradkina, Phys. Lett. **B72** (1978) 343.
- [6] I.A. Batalin and E.S. Fradkin, Phys. Lett. B128 (1983) 303; Riv. Nuovo Cim. 9 (1986).
- [7] T. Kugo and I. Ojima, Progr. Theor. Phys., Suppl., 66 (1979) 1.
- [8] K. Nishijima, Nucl. Phys. **B238** (1984) 601; Progr. Theor. Phys. **80** (1988) 897, 905.
- [9] M. Henneaux, Phys. Rep., **126** (1985) 1.
- [10] M. Henneaux and C. Teitelboim, "Quantization of Gauge sistems", Princeton University Press, Princeton, 1992.
- [11] M.B. Green, J.H. Shwarz and E. Witten, "Superstring Theory", I, Cambridge University Press, 1987.
- [12] I.A. Batalin and R. Marnelius, Phys. Lett. **B441** (1998) 243, hep-th/9806114.
- [13] I.A. Batalin and R. Marnelius, Nucl. Phys. **B551** (1998) 450, hep-th/9809208.
- [14] I.A. Batalin and R. Marnelius, Mod. Phys. Lett. A14 (1999) 1643, hep-th/9905193.
- [15] I.A. Batalin and R. Marnelius, Int. Journ. Mod. Phys. A15 (2000) 2077, hep-th/9909223.
- [16] I.A. Batalin and R. Marnelius, Phys. Lett. **B434** (1998) 312, hep-th/9805084.
- [17] I.A. Batalin and R. Marnelius, Theor. Math. Phys. **120** (1999) 1115, hep-th/9905083.
- [18] I.A. Batalin and R. Marnelius, Int. Journ. Mod. Phys. A14 (1999) 5049, hepth/9909210.
- [19] A.S. Cattaneo and G. Felder, Commun. Math. Phys. 212 (2000) 591, math.QA/9902090.
- [20] A.S. Cattaneo and G. Felder, Mod. Phys. Lett A16 (2001) 179, hep-th/0102208.
- [21] I.A. Batalin and R. Marnelius, Phys. Lett. **B512** (2001) 225, hep-th/0105190.

- [22] I.A. Batalin and R. Marnelius, "Superfield algorithms for topological field theories", in: Memorial volume dedicated to M. Marinov, edited by M. Olshanetsky and A. Vainshtein, WSPC 2002, hep-th/0110140.
- [23] I.A. Batalin and E.S. Fradkin, Ann. Inst. Henri. Poincare 49 (1988) 145.