

# Low-dimensional sisters of Seiberg–Witten effective theory <sup>1</sup>

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## Abstract

We consider the theories obtained by dimensional reduction to  $D = 1, 2, 3$  of  $4D$  supersymmetric Yang–Mills theories and calculate there the effective low-energy lagrangia describing moduli space dynamics — the low-dimensional analogs of the Seiberg–Witten effective lagrangian. The effective theories thus obtained are rather beautiful and interesting from mathematical viewpoint. In addition, their study allows one to understand better some essential features of  $4D$  supersymmetric theories, in particular — the nonrenormalisation theorems.

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## 1 Introduction

Ian’s scientific style had two attractive features: *(i)* his works were using more often than not rather nontrivial modern mathematical constructions; *(ii)* they were always based on a solid and clear physical idea. This text also represents an exercise (a review of exercises) on “physical mathematics” involving an interplay between purely mathematical geometric constructions and a simple physical notion of effective lagrangian.

Effective lagrangia/hamiltonia arise naturally in theories involving two energy scales. Integrating out the “fast” variables (the degrees of freedom with large characteristic excitation energy), one obtains the effective lagrangian involving only “slow” variables and describing low-energy dynamics. The classical example is the Born-Oppenheimer effective hamiltonian describing nuclei dynamics in a molecule and obtained after integrating out the electron degrees of freedom. The Euler-Heisenberg effective lagrangian describing nonlinear soft photon interactions, the effective chiral lagrangian in QCD, the Wilsonian renormalized effective lagrangian (where modes with high frequency up to  $\Lambda_{UV}$  are integrated out) all belong to this class.

The same concerns the famous Seiberg–Witten effective lagrangian [1]. Let us remind its salient features. Consider pure 4D  $\mathcal{N} = 2$  supersymmetric Yang–Mills theory. The lagrangian written in terms of  $\mathcal{N} = 1$  superfields is<sup>3</sup>

$$\mathcal{L} = \frac{1}{g^2} \text{Tr} \left\{ \int d^2\theta W^\alpha W_\alpha + 2 \int d^2\theta d^2\bar{\theta} \Phi e^{-V} \bar{\Phi} e^V \right\} \quad (1)$$

In bosonic sector, it includes the gauge field  $A_\mu$  and a complex scalar  $\phi$  belonging to the adjoint representation of the gauge group,

$$g^2 \mathcal{L} = -\frac{1}{2} \text{Tr} \{ F_{\mu\nu}^2 \} + 2 \text{Tr} \{ \mathcal{D}_\mu \bar{\phi} \mathcal{D}_\mu \phi \} - \text{Tr} \{ [\bar{\phi}, \phi]^2 \} + \text{fermions} \quad (2)$$

The lagrangian is most economically expressed as (see e.g. [3])

$$\mathcal{L} = \frac{1}{g^2} \text{Tr} \int d^2\theta d^2\tilde{\theta} \mathcal{W}^2, \quad (3)$$

where  $\mathcal{W}(x_L, \theta_\alpha, \tilde{\theta}_\alpha)$  is a  $\mathcal{N} = 2$  chiral superfield

$$\mathcal{W} = \Phi + i\sqrt{2}\tilde{\theta}^\alpha W_\alpha - \frac{\tilde{\theta}^2}{4} \bar{D}^2 (e^{-V} \bar{\Phi} e^V). \quad (4)$$

Superfields  $V, W_\alpha, \Phi, \bar{\Phi}$  live in ordinary superspace  $(x, \theta, \bar{\theta})$ . Besides the chirality conditions,  $\bar{D}_\alpha^a \mathcal{W} = 0$ , the superfield (4) satisfies the constraints

$$D^{a\alpha} D_\alpha^b \mathcal{W} = \bar{D}_a^{\dot{a}} \bar{D}^{b\dot{a}} \bar{\mathcal{W}}, \quad (5)$$

where  $a, b = (\textit{nothing}, \textit{tilde})$  are the global  $SU(2)$  indices. The superfield  $\mathcal{W}$  can be naturally expressed in the framework of harmonic superspace approach (see the monography [4] and also recent [5]), but do not themselves depend on harmonics in the chosen basis.

This theory has (infinitely) many different classical vacua. Supersymmetric vacuum has zero energy. At the classical level, it has zero potential energy. Note now that the potential commutator term in (2) vanishes when  $[\bar{\phi}, \phi] = 0$ , which implies that  $\phi$  belongs to the Cartan subalgebra of the corresponding Lie algebra.

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<sup>3</sup> Our convention is close to that of Ref. [2],  $\theta^2 = \theta^\alpha \theta_\alpha \bar{\theta}^2 = \theta_{\dot{\alpha}} \theta^{\dot{\alpha}}$ ,  $\int d^2\theta \theta^2 = \int d^2\bar{\theta} \bar{\theta}^2 = 1$ . In the following we will also use  $(\sigma^\mu)_{\alpha\dot{\beta}} = \{1, \boldsymbol{\tau}\}_{\alpha\dot{\beta}}$ ,  $(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} = \{1, -\boldsymbol{\tau}\}^{\dot{\beta}\alpha}$ . But our Minkowski metric  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  differs in sign from Wess and Bagger’s conventions and we include the extra factor 2 in the definition of  $V$ .

Factorizing over gauge transformations, this gives  $r$  physical complex parameters ( $r$  is the rank of the group) characterizing the classical vacuum moduli space. When quantum corrections are taken into account, one could in principle expect the appearance of a nontrivial effective potential on the moduli space so that the energy would generically be shifted from zero. It is specific for supersymmetric theories that quantum corrections *vanish* in any order of perturbation theory. It is specific for  $\mathcal{N} = 2$  theory that also nonperturbative corrections to the effective potential vanish. However, the corrections to the kinetic part of the lagrangian need not vanish and they do not. The relevant slow variables are  $r$  complex parameters  $\phi^A$  mentioned above and their  $\mathcal{N} = 2$  superpartners involving fermions and also  $r$  Abelian gauge fields. In the simplest  $SU(2)$  case, they can be combined in one  $\mathcal{N} = 2$  superfield  $\mathcal{W} = \phi + \dots$ .<sup>4</sup> The effective Seiberg–Witten lagrangian has the form

$$\mathcal{L} = \int d^4\theta F(\mathcal{W}) + \text{c.c.} \quad (6)$$

Being expressed in components, this gives a nontrivial metric on the moduli space. Now,  $F(\mathcal{W})$  is a nontrivial elliptic function taking account of the instanton contributions, etc. Its large  $\mathcal{W}$  asymptotics is simple, however:  $F(\mathcal{W}) = \frac{\mathcal{W}^2}{4\pi^2} \ln \mathcal{W}$ . This takes into account only the perturbative corrections, which appear only at the one-loop level.

This paper is devoted to evaluation of the effective lagrangians in the theories obtained by the dimensional reduction of (2) and also by the dimensional reduction of  $\mathcal{N} = 1$  SYM theories.

Let us start with discussing the latter. In four dimensions, pure SYM theories do not involve vacuum moduli space. The number of quantum vacua is finite there coinciding with the dual Coxeter number (or, which is the same, the adjoint Casimir operator  $c_V$ ) of the gauge group [6]. However, moduli space appears after dimensional reduction. Consider first the theory reduced to  $(0 + 1)$  dimensions. In such a theory, new gauge invariants made of the spatial components of gauge potential appear. The simplest such invariant is  $\text{Tr}\{A_i^2\}$ . Indeed, gauge transformation of  $A_i^a$  is reduced now to multiplication by a group matrix  $O_{ab}$  and does not involve the derivative term. The tree potential term  $\propto \text{Tr}[A_i, A_j]^2$  vanishes when  $[A_i, A_j] = 0$ , i.e. when  $A_i$  belongs to the Cartan subalgebra. For  $SU(2)$ , this means that  $A_i$  can be gauge rotated to the form  $c_i t^3$ . Three variables  $c_i$  characterize the vacuum moduli space. For an arbitrary gauge group, the moduli space is characterized by  $3r$  parameters.

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<sup>4</sup>No spinor or matrix indices there!

Consider now the reduction to  $(1+1)$  dimensions. Only two components of  $A_i$  do not involve the derivative term in their gauge transformation law and we have  $2r$  physical moduli space parameter. When reducing to  $D = 3$ , only one component of the vector potential for each unit of rank is left, but there are also  $r$  Abelian gauge fields which are dual in three dimensions to scalars,  $\epsilon_{ijk}F_{jk} \leftrightarrow \partial_i\Psi$ . Thus, in three dimensions we have  $r + r = 2r$  parameters in the vacuum moduli space.

For  $\mathcal{N} = 2$  theories, the counting is basically the same, only we have to add  $2r$  parameters associated with the scalar fields. In other words, the corresponding effective lagrangia involve  $5r$  bosonic degrees of freedom in the  $1D$  case and  $4r$  degrees of freedom in the  $2D$  and  $3D$  cases.

The paper is organized as follows. In the next section, we describe the supersymmetric quantum mechanical models representing effective lagrangia for the theories obtained after reduction to  $(0+1)$  dimensions. They represent nonstandard (so called *symplectic*) supersymmetric sigma models. They are characterized by a mismatch between the number of bosonic and fermionic degrees of freedom: for example, in the symplectic  $\sigma$  models of the first kind (obtained from  $\mathcal{N} = 1$  theories), we have 3 bosonic and 2 fermionic degrees of freedom for each unit of rank, while for symplectic  $\sigma$  models of the second kind (obtained from  $\mathcal{N} = 2$  theories), we have  $5r$  bosonic and  $4r$  fermionic degrees of freedom. We hasten to mention right now that the number of bosonic and fermionic *quantum states* is still equal as is dictated by supersymmetry. We will explain later why the existence of such an unusual  $\mathcal{N} = 2$  sigma model<sup>5</sup> (it is not Kähler !) does not contradict the no-go theorem proven in [7].

In Sect. 3, we discuss 2-dimensional effective theories. The theories obtained from  $\mathcal{N} = 1$   $4D$  SYM represent conventional Kähler sigma models. For extended SYM, the effective theories are more interesting — they enjoy  $\mathcal{N} = 4$  supersymmetry, but are not hyper-Kähler belonging to the class of so called twisted sigma models [8].

Sect. 4 is devoted to  $3D$  effective theories. They are hyper-Kähler sigma models. In the simplest  $SU(2)$  case, the corresponding target space is the known Atiyah–Hitchin manifold [the  $(0+1)$  version of this sigma model describes also the dynamic of two BPS monopoles]. In the  $SU(N)$  case, the target space represents a generalized Atiyah–Hitchin manifold associated with the dynamics of  $N$  BPS monopoles.<sup>6</sup> For

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<sup>5</sup>Our counting of  $\mathcal{N}$  always refers to a number of minimal supercharge representations in a given dimension. Thus, for  $D = 1$ ,  $\mathcal{N}$  counts the number of *complex* supercharges, for  $D = 4$  it counts the number of Weyl spinors, etc.

<sup>6</sup>To avoid confusion, we note that they are the standard monopoles of  $O(3)$  Georgi–Glashow model characterized by spatial position and a single  $U(1)$  phase.

an arbitrary gauge group, the corresponding hyper-Kähler manifolds (not studied by mathematicians before) are obtained after certain factorizations (hyper-Kähler reductions) of generalized AH manifolds.

In Sect. 5 we discuss the relationship between effective lagrangia in different dimensions and discuss in details the *nonrenormalization theorems* for  $D = 1, 2, 3$  and their relationship to the conventional nonrenormalization theorems in four dimensions.

## 2 $D = 1$ : symplectic sigma models

### 2.1 $\mathcal{N} = 1$

Consider the simplest example, the massless  $\mathcal{N} = 1$  4D SQED with the lagrangian

$$\mathcal{L} = \frac{1}{2e^2} \int d^2\theta W^2 + \int d^4\theta [\bar{S} e^V S + \bar{T} e^{-V} T] , \quad (7)$$

( $S, T$  are chiral multiplets carrying the opposite electric charges) reduced to  $(0+1)$  dimensions. The effective lagrangian (determined in [9]) depends on the gauge potentials  $A_i(t)$  and their superpartners, the photino fields  $\psi_\alpha(t)$ ,  $\alpha = 1, 2$ . The charged scalar and spinor fields represent fast variables that should be integrated over. Now,  $A_i$ , the auxiliary field  $D$  and the spinor fields  $\psi_\alpha$  can be combined in a single  $\mathcal{N} = 2$  1D superfield [10] (see also [11])<sup>7</sup>

$$\begin{aligned} \Gamma_k &= A_k + \bar{\theta}\sigma_k\psi + \bar{\psi}\sigma_k\theta + \epsilon_{kjp}\dot{A}_j\bar{\theta}\sigma_p\theta + D\bar{\theta}\sigma_k\theta \\ &\quad + i(\bar{\theta}\sigma_k\dot{\psi} - \dot{\bar{\psi}}\sigma_k\theta)\bar{\theta}\theta + \frac{\ddot{A}_k}{4}\theta^2\bar{\theta}^2 . \end{aligned} \quad (8)$$

The field (8) satisfies the constraints

$$D_{(\alpha}\Gamma_{\beta\gamma)} = 0, \quad \bar{D}_{(\alpha}\Gamma_{\beta\gamma)} = 0 , \quad (9)$$

where  $\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha} = i(\sigma_k)_\alpha{}^\gamma\epsilon_{\beta\gamma}\Gamma_k$  and

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}_\alpha\frac{\partial}{\partial t}, \quad \bar{D}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta_\alpha\frac{\partial}{\partial t} \quad (10)$$

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<sup>7</sup>We follow the notations of Ref.[5],

$$\bar{\theta}^\alpha = (\theta_\alpha)^\dagger, \quad \bar{\theta}\theta = \bar{\theta}^\alpha\theta_\alpha, \quad \bar{\theta}\sigma_k\theta = \bar{\theta}^\alpha(\sigma_k)_\alpha{}^\beta\theta_\beta$$

and the indices are raised and lowered with the help of the invariant tensors  $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$ .

are the covariant derivatives. Actually,  $\Gamma_k$  are nothing but the spatial components of the former  $4D$  superconnections

$$\Gamma_\mu = \frac{1}{4}(\bar{\sigma}_\mu)^{\dot{\beta}\alpha}\bar{D}_{\dot{\beta}}D_\alpha V = A_\mu + \dots, \quad (11)$$

the covariant background derivatives having the form  $\nabla_\mu = \partial_\mu - i\Gamma_\mu$  [12]. In one-dimensional theory,  $\Gamma_k$  is gauge invariant,

$$\delta\Gamma_{\alpha\beta} \sim (D_\alpha\bar{D}_\beta + D_\beta\bar{D}_\alpha)(\Lambda - \bar{\Lambda}) = 0$$

as follows from (anti)chirality of  $\Lambda(\bar{\Lambda})$  and the  $1D$  relationship  $\{D_\alpha, \bar{D}_\beta\} = 2i\epsilon_{\alpha\beta}\partial_t$ .

The effective supersymmetric and gauge-invariant action is presented in the form

$$S = \int dt \int d^2\theta d^2\bar{\theta} F(\Gamma_k) \quad (12)$$

By construction it enjoys  $\mathcal{N} = 2$  supersymmetry. The lagrangian is expressed in components as follows

$$\begin{aligned} \mathcal{L} = & \frac{h}{2}\dot{A}_k\dot{A}_k + \frac{ih}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) + \frac{\partial_k h}{2}\epsilon_{kjp}\dot{A}_j\bar{\psi}\sigma_p\psi \\ & + \frac{hD^2}{2} - \frac{D\partial_k h}{2}\bar{\psi}\sigma_k\psi - \frac{\partial^2 h}{8}\bar{\psi}^2\psi^2, \end{aligned} \quad (13)$$

where

$$h(\mathbf{A}) = -\frac{1}{2}\partial^2 F(\mathbf{A}). \quad (14)$$

This is a supersymmetric sigma model with conformally flat  $3D$  target space,  $ds^2 = hd\mathbf{A}^2$ . However, it is not the conventional supersymmetric sigma model associated with the de Rahm complex. The latter has only one pair of complex supercharges  $(Q, Q^\dagger) \equiv (d, d^\dagger)$ . When the target space represents a Kähler manifold, one can define an extra pair of supercharges (three such extra pairs for hyper-Kähler manifolds), but in our case the target space is 3-dimensional and definitely not Kähler.

One can also notice that the number of bosonic and fermionic degrees of freedom are not matched in a usual way: in a conventional sigma model one has a complex fermion for each boson while the lagrangian (13) involves three bosonic dynamic variables and only two fermionic. In field theory, where each field is associated with the asymptotic quantum state, such a mismatch would not be allowed by

supersymmetry. But in supersymmetric quantum mechanics there are no problems with the mismatch of this kind: for each nonzero eigenvalue of the hamiltonian we still have two bosonic and two fermionic degenerate states  $|n\rangle = \Phi_n(\mathbf{A}, \psi_\alpha)$ .<sup>8</sup>

A reader might be somewhat confused at this point. The widely known theorem [7] seems to assert that  $\mathcal{N} = 2$  sigma models can be defined *only* on Kähler manifolds (and  $\mathcal{N} = 4$  models — only on hyper-Kähler manifolds). However, this theorem was proven under two assumptions : (i) the theory considered should be a real field theory with at least 2 spacetime dimensions and (ii) the kinetic term should have a standard form  $\propto g_{ab}\partial_\mu\phi^a\partial_\mu\phi^b$ . For quantum mechanics the first condition is not satisfied and there are no restrictions whatsoever.

In a standard sigma model, fermions are vectors in the tangent space. In our case, they belong to the spinor representation of  $SO(3) \equiv Sp(2)$ . We will call this model symplectic sigma model of the first kind (the second kind is coming soon).

In our case, the function  $F(\Gamma_k)$  has a certain particular form. At the tree level,  $F(\Gamma) = -\Gamma^2/(3e^2)$  and  $h = 1/e^2$ . This gives the lagrangian of dimensionally reduced photodynamics. Let us evaluate one-loop correction to the metric. To this end, we need to calculate the loops of charged superfields  $S, T$  in gauge background. It is convenient to do it in components. We choose the background  $A_i = C_i + E_it$ ,  $\psi_\alpha = 0$  and calculate the charged scalar and fermion loops. The corresponding contributions to the effective action have the form

$$\Delta S_{\text{eff}} = -i \ln \frac{\det^{\frac{1}{2}}(-D^2 I + \frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu})}{\det^{\frac{1}{2}}(-D^2 I)}, \quad (15)$$

where  $I$  is the  $4 \times 4$  unity matrix and the identity

$$(i\not{D})^2 = -D^2 I + \frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu}, \quad (16)$$

$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ , was used. We observe that a nonzero correction is due solely to magnetic interactions  $\propto \sigma_{\mu\nu}F_{\mu\nu}$ . Were the latter absent, the fermion and scalar contributions would exactly cancel. This feature is common for all supersymmetric gauge theories, non-Abelian and Abelian (see [14] for more details). And this fact is related to another known fact that, when supersymmetric  $\beta$  function is calculated in the *instanton* background, only the contribution of the zero modes survives [15].

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<sup>8</sup> As is well known, vacuum states with zero energy need not be paired. In Ref. [13], we considered SQM models with nonstandard “weak” supersymmetric algebra. For such models, the exact pairing is absent also for the first excited state. But the algebra of all models considered in this paper is standard.



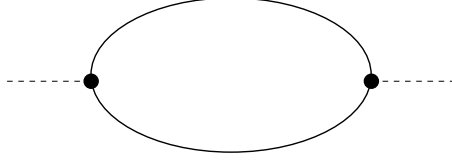


Figure 1: One-loop renormalization of the kinetic term in SQED. The internal lines are Green's functions of the operator  $(-D^2)$  with constant  $A_i = C_i$ . The vertices involve the magnetic interaction  $\propto \sigma_{0i} E_i$ .

In the lowest order in  $F_{\mu\nu}$  (or  $E_i$ ), the contribution (15) can be represented by the graph in Fig.1. Constant background  $\mathbf{C}$  gives the “mass” to the charged fields and the Euclidean propagator has the form  $1/(\omega^2 + \mathbf{C}^2)$ . The calculation gives

$$\Delta S_{\text{eff}} = -i \cdot i_{\text{Wick}} \cdot i^2 \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot E_j E_k \text{Tr}\{\sigma_{0j}\sigma_{0k}\} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \mathbf{C}^2)^2} = \frac{\mathbf{E}^2}{4|\mathbf{C}|^3}, \quad (17)$$

where the factor  $1/2$  is the power of the determinant and the factor  $-1/2$  comes from the expansion

$$\ln \det \|I + \alpha\| = \ln \left[ 1 - \frac{1}{2} \text{Tr} \alpha^2 + \frac{1}{3} \text{Tr} \alpha^3 + \dots \right] \approx -\frac{1}{2} \text{Tr} \alpha^2$$

( $\text{Tr} \alpha = 0$  in our case). This immediately gives

$$e^2 h(\mathbf{C}) = 1 + \frac{e^2}{2|\mathbf{C}|^3} + \dots \quad (18)$$

Let us discuss now non-Abelian theories. In the simplest case of the group  $SU(2)$ , the moduli space involves the variables  $c_k = A_k^3$  and their superpartners, which are combined in the superfield  $\Gamma_k^3$ . The effective action has, again, the form (12), but the function  $F(\mathbf{\Gamma}^3)$  is now different. Like in Abelian case it can be determined by calculating the loops of gauge and fermion fields in Abelian background  $A^{\text{cl}}(t) = (C_i + E_i t)t^3$  (where  $C$  stands not only for “constant”, but also for “Cartan”).

The graphs are conveniently calculated in the background gauge. We represent  $A_\mu = A_\mu^{\text{cl}} + \mathcal{A}_\mu$ , where  $\mathcal{A}_\mu$  is the quantum fluctuation and add to the Lagrangian the gauge-fixing term

$$-\frac{1}{2g^2}(D_\mu^{\text{cl}}\mathcal{A}_\mu)^2, \quad (19)$$

where  $D_\mu^{\text{cl}} = \partial_\mu - i[A_\mu^{\text{cl}}, \cdot]$ . The coefficient chosen in Eq. (19) defines the ‘‘Feynman background gauge’’, which is simpler and more convenient than others. Adding (19) to the lagrangian and integrating by parts, we obtain for the gauge-field-dependent part of the Lagrangian

$$\mathcal{L}_A = -\frac{1}{2g^2} \text{Tr} (F_{\mu\nu}^2) + \frac{1}{g^2} \text{Tr} \left\{ \mathcal{A}_\mu (D^2 g_{\mu\nu} \mathcal{A}_\nu - 2i [F_{\mu\nu}, \mathcal{A}_\nu]) \right\} + \dots, \quad (20)$$

where the dots stand for the terms of higher order in  $\mathcal{A}_\mu$ . The ghost part of the Lagrangian is

$$\mathcal{L}_{ghost} = -2 \text{Tr} (\bar{c} D^2 c) + \text{higher order terms} \quad (21)$$

Now we can integrate over the quantum fields  $\mathcal{A}_\mu$ ,  $c$ , and over the fermions using the relation (16). We obtain

$$\delta S_{\text{eff}} = -i \ln \left( \frac{\det^{\frac{1}{4}} (-D^2 I + \frac{i}{2} \sigma_{\mu\nu} [F_{\mu\nu}, \cdot]) \det^{1/4} (-D^2 I)}{\det^{\frac{1}{2}} (-D^2 g_{\mu\nu} + 2i [F_{\mu\nu}, \cdot])} \right). \quad (22)$$

Again, the result would be zero in the absence of magnetic interactions. In this case, besides the fermion loop also the gauge field loop should be taken into account. Nonzero commutators  $[F_{\mu\nu}, \mathcal{A}_\nu]$ ,  $[F_{\mu\nu}, \lambda_\alpha]$  imply that the quantum fields are charged with respect to the background, i.e. their color indices  $a$  acquire the values 1, 2.

The fermion loop gives the same contribution to  $S_{\text{eff}}$  as in Abelian theory: the power of the determinant is now 1/4 rather than 1/2 (the theory involves a Weyl rather than Dirac fermion), but this is compensated by the extra color factor 2. One can be convinced that the gauge boson loop contribution involves the factor -4 compared to the fermion one (the factor -2 coming from the power of determinant -1/2 vs. 1/4 is explicitly seen in (22) and another factor 2 comes from spin). This gives

$$g^2 h^{SU(2)}(\mathbf{C}) = 1 - \frac{3g^2}{2\mathbf{C}^2} + \dots \quad (23)$$

One can notice at this point that exactly the same graphs determine one-loop renormalization of the effective charge in the corresponding 4D theories. The only difference is that in four dimensions we have to substitute

$$\int \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \mathbf{C}^2)^2} \longrightarrow \int \frac{d^2p}{(2\pi)^4} \frac{1}{(p^2 + \mathbf{C}^2)^2} \propto \ln \frac{\Lambda}{|\mathbf{C}|} .$$

In other words, the coefficients in (18), (23) are rigidly related to the one-loop  $\beta$  function coefficients in the parent 4D theories. Indeed, the  $\beta$  function in non-Abelian SYM theory with  $SU(2)$  gauge group involves the factor  $-3$  compared to SQED. We will return to the discussion of this point in the last section.

The metrics (18), (23) and the relation (14) allow one to restore the corresponding prepotentials:

$$\begin{aligned} e^2 F^{\text{SQED}}(\mathbf{\Gamma}) &= -\frac{\mathbf{\Gamma}^2}{3} + \frac{e^2 \ln |\mathbf{\Gamma}|}{|\mathbf{\Gamma}|} + \dots \\ g^2 F^{SU(2)}(\mathbf{\Gamma}) &= -\frac{\mathbf{\Gamma}^2}{3} - \frac{3g^2 \ln |\mathbf{\Gamma}|}{|\mathbf{\Gamma}|} + \dots \end{aligned} \quad (24)$$

(we posed  $\mathbf{\Gamma}^3 \rightarrow \mathbf{\Gamma}$  in the non-Abelian case). Consider now an arbitrary simple compact Lie group. The classical potential energy vanishes when  $[A_j, A_k] = 0$ , which implies that  $A_j$  lies in the Cartan subalgebra (is effectively Abelian). This gives  $3r$  bosonic variables in the effective lagrangian. They are supplemented by  $2r$  Abelian gluino variables. These variables are organized in  $r$  superfields  $\mathbf{\Gamma}^{A=1, \dots, r}$  defined like in Eqs. (8), (9).  $\mathbf{\Gamma}^A$  represent dimensionally reduced Abelian superconnections. Thus, the effective lagrangian has the form<sup>9</sup>  $\int d^4\theta F(\mathbf{\Gamma}^A)$  and the only question is what is the function  $F(\mathbf{\Gamma}^A)$ . Again, we have choose an Abelian gauge field background and perform the calculation over quantum fields. The latter must have nonzero commutators with the background. They are classified according to the *roots* of the corresponding Lie algebra. Actually, we have to add the contributions of the loops corresponding to each such (positive) root. The result is (see [17] for more details)

$$g^2 F(\mathbf{\Gamma}^A) = -\sum_j \left[ \frac{2}{3c_V} (\mathbf{\Gamma}^{(j)})^2 + \frac{3g^2}{|\mathbf{\Gamma}^{(j)}|} \ln |\mathbf{\Gamma}^{(j)}| \right] , \quad (25)$$

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<sup>9</sup>It is interesting that such a lagrangian describes also the dynamics of  $r$  extremal Reissner-Nordström black holes (representing classical solutions in  $\mathcal{N} = 2$  4D supergravity) [16].

where  $\Gamma^j = \alpha_j(\Gamma^A)$  and  $\alpha_j$  are the roots. For example, for  $SU(3)$  we have the sum of three terms with

$$\alpha_1(\Gamma^A) = \Gamma^3, \quad \alpha_2(\Gamma^A) = \frac{-\Gamma^3 + \sqrt{3}\Gamma^8}{2}, \quad \alpha_3(\Gamma^A) = \frac{\Gamma^3 + \sqrt{3}\Gamma^8}{2}. \quad (26)$$

## 2.2 $\mathcal{N} = 2$

The same program can be carried out for SQM models obtained by dimensional reduction from  $\mathcal{N} = 2$  4D theories. Consider first Abelian theory.  $\mathcal{N} = 2$  SQED has the same charged matter content as  $\mathcal{N} = 1$  theory, but involves an extra neutral chiral multiplet  $\Phi$ . The lagrangian acquires two new terms

$$\Delta L = \int d^4\theta \bar{\Phi}\Phi + \left[ \sqrt{2}e \int d^2\theta \Phi ST + \text{c.c.} \right]. \quad (27)$$

The lowest component of  $\Phi$  gives two extra degrees of freedom in the vacuum moduli space, which becomes thereby 5-dimensional.<sup>10</sup> The vector superfield  $V$  and the chiral superfield  $\Phi$  can be unified in a single  $\mathcal{N} = 4$  (in SQM sense) harmonic gauge superfield and the effective lagrangian can be formulated in the terms of the latter [5]. We use here a more conventional approach using  $\mathcal{N} = 2$  superfields. The effective action depends on  $\Gamma_J = (\Gamma, \sqrt{2}\text{Re}\{\Phi\}, \sqrt{2}\text{Im}\{\Phi\})$  (interpreted as superconnection in the “grandmother” 6D theory) and must have the form

$$S = \int dt \int d^2\theta d^2\bar{\theta} \mathcal{K}(\Gamma, \bar{\Phi}, \Phi). \quad (28)$$

Now,  $\mathcal{N} = 2$  symmetry is manifest here. The action (28) is invariant under *additional*  $\mathcal{N} = 2$  supersymmetry transformations

$$\begin{aligned} \delta\bar{\Phi} &= \frac{2i}{3}\epsilon^\alpha(\sigma_k)_\alpha{}^\beta D_\beta\Gamma_k, \\ \delta\Phi &= \frac{2i}{3}\bar{\epsilon}_\alpha(\sigma_k)_\beta{}^\alpha \bar{D}^\beta\Gamma_k, \\ \delta\Gamma_k &= -i\epsilon^\alpha(\sigma_k)_\alpha{}^\beta D_\beta\Phi - i\bar{\epsilon}_\alpha(\sigma_k)_\beta{}^\alpha \bar{D}^\beta\bar{\Phi}, \end{aligned} \quad (29)$$

provided that

$$\frac{\partial^2\mathcal{K}}{\partial\Gamma_k^2} + 2\frac{\partial^2\mathcal{K}}{\partial\bar{\Phi}\partial\Phi} \equiv \frac{\partial^2\mathcal{K}}{\partial\Gamma_J^2} = 0, \quad (30)$$

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<sup>10</sup>The moduli can be represented as spatial components of the gauge potential in 6D SQED, from which the  $\mathcal{N} = 2$  4D theory is obtained by dimensional reduction.

i.e.  $\mathcal{K}$  is a 5-dimensional harmonic function [18]. Unifying  $\mathbf{A}$  and  $\phi, \bar{\phi}$  in a single 5-dimensional vector  $A_J$  and two spinors from the multiplets  $\Gamma_k$  and  $\Phi$  in a single 4-component complex spinor  $\eta_\alpha$  lying in the fundamental (spinor) representation of  $SO(5) \equiv Sp(4)$ , we can write the following component expression for the lagrangian [19]

$$\begin{aligned} \mathcal{L} = & h \left[ \frac{1}{2} \dot{A}_J^2 + \frac{i}{2} (\bar{\eta} \dot{\eta} - \dot{\bar{\eta}} \eta) \right] + \frac{i}{2} \partial_J h \dot{A}_K \bar{\eta} \sigma_{JK} \eta + \\ & \frac{1}{24} \left( 2 \partial_J \partial_K h - \frac{3}{h} \partial_J h \partial_K h \right) (\bar{\eta} \gamma_J \eta \bar{\eta} \gamma_K \eta - \eta C \gamma_J \eta \bar{\eta} \gamma_K C \bar{\eta}) , \end{aligned} \quad (31)$$

where  $\gamma_K$  are 5-dim Dirac matrices,  $\sigma_{JK} = (1/2)(\gamma_J \gamma_K - \gamma_K \gamma_J)$  and  $C$  is the anti-symmetric matrix of charge conjugation,  $C \gamma_J^T = -\gamma_J C$ . The metric  $h$  is related to  $\mathcal{K}$  as  $h = -(1/2) \partial^2 \mathcal{K} / \partial \mathbf{A}^2$ .

The lagrangian (31) describes a sigma model defined on a conformally flat 5-dimensional target space. We will call it symplectic sigma model of the second kind. A generalized symplectic model of the second kind depends in this approach on  $r$  sets of  $\mathcal{N} = 2$  superfields  $\Gamma_J \equiv (\mathbf{\Gamma}^A, \Phi^A, \bar{\Phi}^A)$ . The action

$$S = \int dt \int d^2 \theta d^2 \bar{\theta} \mathcal{K}(\Gamma_J^A) . \quad (32)$$

enjoys extended  $\mathcal{N} = 4$  supersymmetry provided the following generalized harmonicity conditions are satisfied [5]

$$\frac{\partial^2 \mathcal{K}}{\partial \Gamma_I^A \partial \Gamma_I^B} = 0 , \quad \frac{\partial^2 \mathcal{K}}{\partial \Gamma_I^{[A} \partial \Gamma_J^{B]}} = 0 . \quad (33)$$

In Abelian case, the effective action has the form (31) with the *same* metric  $h$  as in the  $\mathcal{N} = 1$  4D SQED case discussed above. Indeed, we can choose the background with zero  $\phi$  in which case the effective action is given by the graph drawn in Fig.1. Now,  $O(5)$  invariance dictates that the metric has the form (18) also in a generic background  $C_J$  with  $\mathbf{C}^2$  being substituted by  $C_J^2$ . The prepotential can be chosen as

$$e^2 \mathcal{K} = -\frac{R^2}{3} + \frac{\rho^2}{2} + \frac{e^2}{R} \ln \left( R + \sqrt{R^2 + \rho^2} \right) , \quad (34)$$

where  $R^2 = \mathbf{\Gamma}^2$  and  $\rho^2 = 2\bar{\Phi}\Phi$ . Note that  $\mathcal{K}$  need not be and is not  $O(5)$  invariant.

In non-Abelian  $\mathcal{N} = 2$  SYM theory with  $SU(2)$  gauge group, the calculations are readily done in the same way as before. The only modification is that there are two Weyl fermions now and an additional adjoint scalar. The ghost determinant is exactly canceled by the adjoint scalar determinant and we obtain

$$\delta S_{\text{eff}} = -i \ln \left( \frac{\det^{\frac{1}{2}} \left( -D^2 I + \frac{i}{2} \sigma_{\mu\nu} [F_{\mu\nu}, \cdot] \right)}{\det^{\frac{1}{2}} \left( -D^2 g_{\mu\nu} + 2i [F_{\mu\nu}, \cdot] \right)} \right). \quad (35)$$

This gives the expression

$$g^2 h_{\mathcal{N}=2}^{SU(2)}(C_J) = 1 - \frac{g^2}{|C_J|^3} \quad (36)$$

for the metric. The respective coefficients in the correction in Abelian and non-Abelian case conform with the respective coefficients in the corresponding  $4D$  beta functions.

The structure of the expressions (36) and (23) is similar, but there is one essential difference. Eq. (36) does not involve dots! The expression for the metric is *exact* and higher loop corrections vanish. The proof of this *nonrenormalization theorem* is simple. Dimensional counting tells us that an  $n$ -loop correction to the metric should be proportional to  $(A_J A_J)^{-3n/2}$ . But this is not harmonic for  $n \geq 2$  and is excluded by supersymmetry requirements. We will discuss the relationship of this nonrenormalization theorem to the  $4D$  nonrenormalization theorem (in  $\mathcal{N} = 2$  theories two and higher loop contributions to the beta function vanish) in Sect. 5.

We want to emphasize that the absence of the corrections to the metric does not mean the absence of the corrections to the effective lagrangian. The latter involves higher derivative corrections, which do not vanish neither at one-loop nor at two and higher loop level [20]. Thus, singularity of the metric at  $A_J^2 = 0$  has no great physical meaning: anyway the effective lagrangian involves uncontrollable higher-derivative corrections there.

The effective lagrangian can also be found for an arbitrary gauge group. Again, we have to sum over all positive roots. The prepotential is

$$g^2 \mathcal{K} = - \sum_j \left\{ \frac{2}{3c_V} \left[ (R^{(j)})^2 - \frac{3}{2} (\rho^{(j)})^2 \right] + \frac{2g^2}{R^{(j)}} \ln \left[ R^{(j)} + \sqrt{(R^{(j)})^2 + (\rho^{(j)})^2} \right] \right\}, \quad (37)$$

where  $(R^{(j)})^2 = (\mathbf{\Gamma}^{(j)})^2$ ,  $(\rho^{(j)})^2 = 2\bar{\Phi}^{(j)}\Phi^{(j)}$  and  $\mathbf{\Gamma}^{(j)} = \alpha_j(\mathbf{\Gamma}^A)$ ,  $\Phi^{(j)} = \alpha_j(\Phi^A)$ .

### 3 $D = 2$ : Kähler and twisted

#### 3.1 $\mathcal{N} = 1$ : Unfolding the ring

Consider first Abelian theory. As was noted, in two dimensions we have two rather than three moduli representing the components of the gauge potential in the reduced dimensions. The bosonic part of the effective lagrangian can be evaluated in the same way as in the  $1D$  case by calculating the loop diagram in Fig.1. The only difference is that the loop integral is two-dimensional now. We obtain

$$e^2 \mathcal{L}_{\text{eff}}^{\text{bos}} = \frac{(\partial_\alpha A_j)^2}{2} \left[ 1 + \frac{e^2}{2\pi A_j^2} + \dots \right], \quad (38)$$

$\alpha = 1, 2$  and  $j = 1, 2$ . This describes a sigma model on a 2-dimensional target space. One can, of course, introduce the complex coordinate  $\sigma = (A_1 + iA_2)/\sqrt{2}$ .

The full effective lagrangian involves besides  $A_j$  their supersymmetric partners — two-component photino fields. We see that in this case there is perfect matching between the number of bosonic and fermionic degrees of freedom. Actually, the Alvarez–Gaume–Freedman theorem [7] dictates that the only two-dimensional  $\mathcal{N} = 2$  supersymmetric theory with standard sigma-model kinetic term like in (38) is the supersymmetric Kähler sigma model. The Kähler potential  $\mathcal{K}$  ( $\mathcal{L} = \int d^4\theta \mathcal{K}$ ) can be restored from the metric. In the case under consideration it can be chosen as

$$e^2 \mathcal{K}(\bar{\Phi}, \Phi) = \bar{\Phi}\Phi + \frac{e^2}{4\pi} \ln \Phi \ln \bar{\Phi}. \quad (39)$$

Now,  $\Phi$  is a chiral superfield that is related to the gauge-invariant superconnections  $\Gamma_j$  in reduced dimensions in the following way. Consider the superfield  $\Sigma = (\Gamma_1 + i\Gamma_2)/\sqrt{2}$ . From the definition (11) and the  $2D$  anticommutation relations between  $D_\alpha$  and  $\bar{D}_{\dot{\alpha}}$ , we deduce that  $\Sigma$  satisfies the constraints

$$\bar{\mathcal{D}}_1 \Sigma = \mathcal{D}_2 \Sigma = 0 \quad (40)$$

and represented a so called *twisted* chiral multiplet. It differs from the standard one by a pure convention:  $\Sigma$  is obtained from  $\Phi$  by interchanging  $\theta_2$  and  $\bar{\theta}^2$ . This means that the change  $\Phi \rightarrow \Sigma$  in any standard action involving  $\Phi$  would not change anything except the sign due to the change of sign of  $d^4\theta$ . For example, the tree Lagrangian is expressed as

$$e^2 \mathcal{L}_{\text{tree}}^{2d} = -\frac{1}{2} \int d^4\theta (\Gamma_1^2 + \Gamma_2^2) = -\int d^4\theta \bar{\Sigma} \Sigma \equiv \int d^4\theta \bar{\Phi} \Phi \quad (41)$$

It is very instructive to derive the effective  $2D$  lagrangian *directly* elucidating its relationship with the SQM effective lagrangian (13) discussed in the previous section. To do it, consider the original theory not on  $R^2$  and not on  $R^1$ , but rather on  $R^1 \times S^1$ . Playing with the length  $L$  of the circle, one can interpolate between  $1D$  and  $2D$  pictures [14].

The Lagrangian (13) was obtained after integrating out the charged fields in  $1D$  theory. Thinking in  $1D$  terms, we have now an infinite number of charged fields representing the coefficients in the Fourier series

$$f(z, t) = \sum_{n=-\infty}^{\infty} f_n(t) e^{inz/L} . \quad (42)$$

The relevant variables in the effective Lagrangian are still zero Fourier modes of the vector potential  $\mathbf{A} \equiv (A_{j=1,2}, A_3)$  and its superpartners. The expression (13) is replaced by the infinite sum<sup>11</sup>

$$e_1^2 \mathcal{L} = \left[ 1 + \sum_{n=-\infty}^{\infty} \delta h \left( A_j, A_3 + \frac{2\pi n}{L} \right) \right] (\dot{A}_j^2 + \dot{A}_3^2) + \text{other terms} \quad (43)$$

In the limit  $L \rightarrow 0$ , only one term in the sum survives and we are reproducing the previous  $1D$  result (with  $\delta h = e_1^2 / (2|\mathbf{A}|^3)$ ). But for large  $L \gg e_1^{-2/3}$  all terms are essential. In the limit  $L \rightarrow \infty$ , we can actually replace the sum by an integral,  $\sum_n \rightarrow \frac{L}{2\pi} \int dA_3$ . This integral depends on  $A_j$ , but not on  $A_3$ : the expression in square brackets in Eq. (43) gives

$$\tilde{h} = 1 + \frac{e_2^2}{2\pi A_j^2} \quad (44)$$

with  $e_2^2 = e_1^2 L$ . This agrees, of course, with (38). Actually, in the limit  $L \rightarrow \infty$ , the effective lagrangian cannot depend on  $A_3$ . For large  $L$ , the range where  $A_3$  changes is very small,  $0 \leq A_3 \leq 2\pi/L$ , and the eigenmodes of the hamiltonian  $\Psi_n(A_3) \sim \exp\{inA_3\}$  with  $n \neq 0$  acquire large energy and decouple; only the mode  $n = 0$  survives. To do this limit carefully, we cannot just set  $A_3 = 0$  in Eq. (43), however, but should perform the functional integral of  $e^{iS}$  [ $S$  is obtained from Eq. (13) by substituting  $\tilde{h}$  for  $h$ ] over  $\prod_t dA_3(t)$  first. Doing this and integrating

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<sup>11</sup> The notation  $e_1$  indicates that we are dealing with the coupling constant in  $1D$  theory,  $[e_1] \sim m^{3/2}$ .



out also the auxiliary field  $D$ , we arrive at the result

$$e_2^2 \mathcal{L}_{2D} = \frac{1}{2} g_{jk} \dot{A}^j \dot{A}^k + \frac{i\tilde{h}}{2} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) + i\tilde{h} \omega_j^{ab} \dot{A}^j \bar{\psi} \sigma^{ab} \psi + \frac{1}{8\tilde{h}} [(\partial_j \tilde{h})^2 - \tilde{h}(\partial^2 \tilde{h})] (\bar{\psi})^2 (\psi)^2, \quad (45)$$

where we have raised the index of the vector  $A^j$  indicating its contravariant nature,  $g_{jk} = \tilde{h} \delta_{jk}$ ,  $\sigma^{ab} = \frac{i}{2} \epsilon^{abc} \sigma^c = \frac{i}{2} \epsilon^{ab} \sigma^3$  ( $a, b = 1, 2$ ) is the generator of rotations in the tangent space, and

$$\omega_i^{ab} = \frac{1}{2} [\delta_i^a \partial^b \log(\tilde{h}) - \delta_i^b \partial^a \log(\tilde{h})] \quad (46)$$

is the spin connection on a conformally flat manifold with the natural choice of the zweibein,  $e_j^a = \sqrt{\tilde{h}} \delta_j^a$ .

When deriving (45), we went over from the lagrangian  $\mathcal{L}_{1D}$  to  $2D$  lagrangian density  $\mathcal{L}_{2D} = \mathcal{L}_{1D}/L$ . (Normally,  $\mathcal{L}_{1D}$  is a spatial integral of  $\mathcal{L}_{2D}$ , but we have dealt up to now only with the terms depending on zero spatial Fourier modes, in which case the spatial integral is reduced to multiplication by  $L$ .)

The lagrangian (45) coincides with the standard lagrangian of Kähler supersymmetric sigma model [21] in the QM limit. In particular, the coefficient of the 4-fermion term represents a  $2D$  scalar curvature.<sup>12</sup> The full  $(1+1)$  effective lagrangian could be obtained if taking into account the higher Fourier harmonics  $\propto \exp\{inz/L\}$  of  $A_j(z, t)$  and  $\psi_\alpha(z, t)$  in the background.

The result (39) can be readily generalized for an arbitrary non-Abelian gauge group. The Kähler potential depends on  $r$  complex chiral superfields  $\Phi^A$  and has the same sum-over-the-roots structure as the  $1D$  prepotential in Eq. (25),

$$g^2 \mathcal{K}(\Phi^A) = \sum_j \left[ \frac{2}{c_V} \bar{\Phi}^{(j)} \Phi^{(j)} - \frac{3g^2}{4\pi} \ln \bar{\Phi}^{(j)} \ln \Phi^{(j)} \right], \quad (47)$$

where  $\Phi^{(j)} = \alpha_j(\Phi^A)$ .

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<sup>12</sup> Incidentally, though the bifermion term in (13) can be interpreted in terms of  $3D$  spin connection, the 4-fermion term there (before or after integrating out  $D$ ) is *not* expressed in terms of  $3D$  curvature.

### 3.2 $\mathcal{N} = 2$ : Twisted sigma model

We start again with analyzing Abelian theory. The effective lagrangian involves now two complex bosonic variables

$$\sigma = (A_1 + iA_2)/\sqrt{2}, \quad \phi = (A_4 + iA_5)/\sqrt{2}. \quad (48)$$

One loop calculation brings about a nontrivial metric in the target space  $(\sigma, \bar{\sigma}, \phi, \bar{\phi})$ . This metric can be related to the SQM 5-dimensional metric by integrating the latter over  $A_3$  by the same token as the Kähler metric (38) was obtained from the metric of the SQM model in the  $\mathcal{N} = 1$  case.

$$e^2 ds_{1+1}^2|_{\mathcal{N}=2} = \left(1 + \int_{-\infty}^{\infty} \frac{dA_3}{2\pi} \delta h_{0+1}\right) = \left[1 + \frac{e^2}{4\pi(\bar{\phi}\phi + \bar{\sigma}\sigma)}\right] (2d\bar{\sigma}d\sigma + 2d\bar{\phi}d\phi). \quad (49)$$

We expect the effective action to have the  $\sigma$  model form. One could worry at this point because the metric (49) is not hyper-Kähler (the Ricci tensor and the scalar curvature do not vanish), while hyper-Kähler nature of the metric was shown to be necessary for the standard (1+1)  $\sigma$  model to enjoy  $\mathcal{N} = 4$  supersymmetry [7]. In our case,  $\mathcal{N} = 4$  supersymmetry is there but the metric is not hyper-Kählerian, and this seems to present a paradox. The resolution is that the  $\sigma$  model in hand is not standard [19]. Indeed, the bosonic part of the Lagrangian involves besides the standard kinetic term  $h(\partial_\alpha \bar{\sigma} \partial_\alpha \sigma + \partial_\alpha \bar{\phi} \partial_\alpha \phi)$  also the "twisted" term  $\propto \epsilon_{\alpha\beta} \partial_\alpha \sigma \partial_\beta \phi$  and  $\propto \epsilon_{\alpha\beta} \partial_\alpha \bar{\sigma} \partial_\beta \bar{\phi}$ . To understand where the twisted term comes from, consider a charged fermion loop in the background

$$\sigma = \sigma_0 + \sigma_\tau \tau + \sigma_z z, \quad \phi = \phi_0 + \phi_\tau \tau + \phi_z z \quad (50)$$

( $\tau$  is the Euclidean time). The contribution to the effective action is  $\propto \ln \det \|\mathfrak{D}\|$ , where  $\mathfrak{D}$  is the 6-dimensional Euclidean Dirac operator, which can be written in the form

$$\mathfrak{D} = i \frac{\partial}{\partial \tau} + \gamma_3 \frac{\partial}{\partial z} - i(\gamma_1 A_1 + \gamma_2 A_2 + \gamma_4 A_4 + \gamma_5 A_5). \quad (51)$$

Now, if  $A_4$  and  $A_5$  were absent, we could write  $\mathfrak{D} = \gamma_4 (i\tilde{\gamma}_\mu \mathcal{D}_\mu)$ , with

$$\mu = 1, 2, 3, 4; \quad \mathcal{D}_4 = \frac{\partial}{\partial \tau}, \quad \mathcal{D}_3 = \frac{\partial}{\partial z}, \quad \mathcal{D}_{1,2} = -iA_{1,2}; \quad \tilde{\gamma}_4 = \gamma_4, \quad \tilde{\gamma}_{1,2,3} = -i\gamma_4 \gamma_{1,2,3}$$

and then use the squaring trick

$$\det \|\mathfrak{D}\| = \det \|i\tilde{\gamma}_\mu \mathcal{D}_\mu\| = \det^{1/2} \left\| -\mathcal{D}^2 + \frac{i}{2} \tilde{\sigma}_{\mu\nu} F_{\mu\nu} \right\| , \quad (52)$$

with  $F_{14} = -\partial A_1 / \partial \tau$ , etc. The effective action would be proportional to

$$\text{Tr}\{\sigma_{\mu\nu}\sigma_{\alpha\beta}\} F_{\mu\nu} F_{\alpha\beta} \int \frac{d^2 p}{4\pi^2} \frac{1}{(p^2 + 2\bar{\sigma}\sigma)^2} \propto F_{\mu\nu}^2 , \quad (53)$$

which gives the renormalization of the kinetic term while the twisted term does not appear. The squaring trick works also in the case where  $A_{4,5}$  are nonzero, but do not depend on  $\tau, z$ . Then  $2\bar{\phi}\phi$  is just added to  $-\mathcal{D}^2$  in Eq. (52) and to  $2\bar{\sigma}\sigma$  in Eq. (53) leading to Eq. (49). But in the generic case the fermion determinant cannot be reduced to  $\det^{1/2} \left\| -\mathcal{D}^2 + \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu} \right\|$ . The basic reason for this impasse is that one cannot adequately “serve” six components of the gradient with only five  $\gamma$  matrices.<sup>13</sup> As a result, the extra twisted term in the determinant appears.

We need not perform an explicit calculation here as the twisted and all other terms in the Lagrangian are fixed by supersymmetry. The twisted  $\mathcal{N} = 4$  supersymmetric  $\sigma$  model was constructed almost 20 years ago [8]. At that time it did not attract much attention. Recently, there is some revival of interest in the GHR model: it happened to pop up in some string-related problems [22, 23]. It also pops up as the effective (1 + 1) Lagrangian in the case under study.

It was shown that, for  $\mathcal{N} = 4$  supersymmetric generalization to be possible, the conformal factor in the metric  $h(\bar{\sigma}, \sigma, \bar{\phi}, \phi)$  should satisfy the harmonicity condition

$$\frac{\partial^2 h}{\partial \bar{\sigma} \partial \sigma} + \frac{\partial^2 h}{\partial \bar{\phi} \partial \phi} = 0 . \quad (54)$$

Obviously, (49) satisfies it everywhere besides the origin. The relationship of (54) to the 5-dimensional harmonicity condition for the metric in the effective SQM model (31) is also obvious. Indeed, integrating a  $D$ -dimensional harmonic function over one of the coordinates like in (49), we always arrive at a  $(D - 1)$ -dimensional harmonic function.

To construct the full action, consider along with the standard chiral multiplet  $\Phi$  satisfying the conditions  $\mathcal{D}_\alpha \Phi = 0$  also a *twisted* chiral multiplet  $\Sigma$  which satisfies

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<sup>13</sup>By the same reason, the squaring trick does not work for Weyl 2-component fermions in 4 dimensions: three Pauli matrices that are available in that case are not enough to do the job.

the constraints (40). As we have seen, the action depending on only  $\bar{\Sigma}$  and  $\Sigma$  can be expressed in terms of standard chiral multiplets. However, one can write nontrivial Lagrangians involving *both*  $\Phi$  and  $\Sigma$ . The twisted  $\sigma$  model is determined by the expression

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \mathcal{K}(\bar{\Phi}, \Phi; \bar{\Sigma}, \Sigma), \quad (55)$$

where the prepotential  $\mathcal{K}$  satisfies the harmonicity condition,

$$\frac{\partial^2 \mathcal{K}}{\partial \bar{\Sigma} \partial \Sigma} + \frac{\partial^2 \mathcal{K}}{\partial \bar{\Phi} \partial \Phi} = 0. \quad (56)$$

The condition (56) is required if we want the theory to be  $\mathcal{N} = 4$  supersymmetric. This is best seen if expressing the lagrangian in components [19] and observing that the lagrangian is symmetric under interchange of fermionic variables entering the twisted and untwisted multiplets only for a harmonic  $\mathcal{K}$ . The composition of this discrete symmetry and  $\mathcal{N} = 2$  supersymmetry that is manifest in (55) brings about two extra supersymmetries mixing  $\phi$  and  $\sigma$  with the fermion components of “alien”  $\mathcal{N} = 2$  multiplets.

One of the possible choices for  $\mathcal{K}$  (two functions  $\mathcal{K}$  and

$$\mathcal{K}' = \mathcal{K} + f(\bar{\sigma}, \phi) + \bar{f}(\sigma, \bar{\phi}) + g(\sigma, \phi) + \bar{g}(\bar{\sigma}, \bar{\phi})$$

result, up to a total derivative, in one and the same lagrangian.) leading to the metric (49) is [24, 23]

$$e^2 \mathcal{K} = \bar{\Sigma} \Sigma - \bar{\Phi} \Phi + \frac{e^2}{4\pi} \left[ F \left( \frac{\bar{\Sigma} \Sigma}{\bar{\Phi} \Phi} \right) - \ln \Phi \ln \bar{\Phi} \right], \quad (57)$$

where

$$F(\eta) = \int_1^\eta \frac{\ln(1 + \xi)}{\xi} d\xi \quad (58)$$

is the Spence function. This gives besides (49) a twisted term

$$\mathcal{L}^{\text{twisted}} = -\frac{e^2}{4\pi(\bar{\sigma}\sigma + \bar{\phi}\phi)} \left[ \frac{\sigma}{\bar{\phi}} \epsilon_{\alpha\beta} (\partial_\alpha \bar{\sigma})(\partial_\beta \bar{\phi}) + \frac{\bar{\sigma}}{\phi} \epsilon_{\alpha\beta} (\partial_\alpha \sigma)(\partial_\beta \phi) \right] \quad (59)$$

in the lagrangian. The twisted term is a 2-form  $F$ . Its external derivative  $dF$  can be associated with the torsion. The above-mentioned freedom of choice of  $\mathcal{K}$

corresponds to adding to  $F$  the external derivative of the 1-form “organized” from the functions  $f, \bar{f}, g, \bar{g}$ . The torsion is invariant under such a change.

Consider now a generic non-Abelian case. For a simple Lie group of rank  $r$ , the effective Lagrangian is

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \mathcal{K}(\bar{\Phi}^A, \Phi^A; \bar{\Sigma}^A, \Sigma^A), \quad (60)$$

where  $A = 1, \dots, r$  and the expression for  $\mathcal{K}$  is derived exactly in the same way as in previous cases. We have

$$g^2 \mathcal{K} = \sum_j \left\{ \frac{2}{c_V} [\bar{\Sigma}^{(j)} \Sigma^{(j)} - \bar{\Phi}^{(j)} \Phi^{(j)}] - \frac{g^2}{2\pi} \left[ F \left( \frac{\bar{\Sigma}^{(j)} \Sigma^{(j)}}{\bar{\Phi}^{(j)} \Phi^{(j)}} \right) - \ln \Phi^{(j)} \ln \bar{\Phi}^{(j)} \right] \right\}, \quad (61)$$

where  $\Sigma^{(j)} = \alpha_j(\Sigma^A)$ , etc. The prepotential (61) satisfies a generalized harmonicity condition

$$\frac{\partial^2 \mathcal{K}}{\partial \bar{\Sigma}^A \partial \Sigma^B} + \frac{\partial^2 \mathcal{K}}{\partial \bar{\Phi}^A \partial \Phi^B} = 0 \quad (62)$$

for all  $A, B$ .

## 4 $D = 3$ : Kähler and hyper-Kähler

### 4.1 $\mathcal{N} = 1$ : Dual photon

The effective lagrangian for  $3D$ ,  $\mathcal{N} = 2$  (in  $3d$  sense) SQED depends on only one gauge invariant superconnection in the reduced dimension  $\Gamma_3$ . Its component expansion (in Wess-Bagger notation) is

$$\Gamma_3 = A_3 - \frac{1}{2} \epsilon_{\mu\rho\alpha} F_{\mu\rho} \theta \sigma_\alpha \bar{\theta} - D\theta \sigma_3 \bar{\theta} + \frac{1}{4} (\partial^2 A_3) \theta^2 \bar{\theta}^2 + \text{fermion terms}, \quad (63)$$

where  $F_{\mu\rho}$  is  $3D$  electromagnetic field ( $\mu, \rho = 0, 1, 2$ ). The bosonic terms in the effective lagrangian are

$$\mathcal{L} = \int \mathcal{F}(\Gamma_3) d^4\theta = h(A_3) \left[ \frac{1}{2} (\partial_\mu A_3)^2 - \frac{1}{4} F_{\mu\rho} F_{\mu\rho} + \frac{D^2}{2} \right], \quad (64)$$

where  $h = -\mathcal{F}''/2$ .

It is convenient to perform now the *duality transformation*. To this end, present the functional integral corresponding to the lagrangian (64) in the form

$$\int \prod dF d\Psi \exp \left\{ i \int d^3x \left( \mathcal{L} + \frac{1}{2} \epsilon_{\mu\rho\alpha} F_{\mu\rho} \partial_\alpha \Psi \right) \right\} \quad (65)$$

Integrating this over  $\prod d\Psi$  brings about the Bianchi constraints  $\epsilon_{\mu\rho\alpha} \partial_\alpha F_{\mu\rho} = 0$ , which are resolved leading to the standard relation  $F_{\mu\rho} = \partial_{[\mu} A_{\rho]}$ . But let us do the integral in Eq. (65) over  $\prod dF$  first. We are left with

$$\prod d\Psi \exp \left\{ i \int d^3x \left[ \frac{h}{2} (\partial_\mu A_3)^2 + \frac{1}{2h} (\partial_\mu \Psi)^2 \right] \right\} . \quad (66)$$

The integrand in the exponent is the dual lagrangian (the bosonic part thereof). The scalar field  $\Psi$  is dual photon.

Let us now introduce the field

$$B = -\frac{\mathcal{F}'(A_3)}{2} \quad (67)$$

so that  $\partial_\mu B = h \partial_\mu A_3$ . Introducing a complex variable  $\phi = (B + i\Psi)/\sqrt{2}$  we can write  $\mathcal{L}_{\text{dual}}$  in the Kähler form  $\int d^4\theta \mathcal{K}(\bar{\Phi}, \Phi)$ . The relation of the Kähler potential  $\mathcal{K}$  to the function  $\mathcal{F}$  can be inferred from Eq. (67).

For the effective lagrangian of 3D SQED, the particular form of the metric and prepotentials  $\mathcal{F}, \mathcal{K}$  can be found in the framework of “unfolding the ring” procedure. We have to integrate the one-loop correction to the 2D metric in (44) over one of the components  $A_j$  is the same way as we earlier integrated the correction to the 1D metric to derive (44). We obtain

$$e^2 h_{3D} = 1 + \frac{e^2}{4\pi|A_3|} + \dots \quad (68)$$

(with 3-dimensional charge  $e$ ). The metric is singular at  $A_3 = 0$ . This point separates here two completely independent sectors in the moduli space (and in the theory !) with positive and negative  $A_3$ . We will assume for definiteness  $A_3$  to be positive. The prepotential entering (64) can be restored from the metric as

$$-e^2 \mathcal{F}(\Gamma_3) = \Gamma_3^2 + \frac{e^2}{2\pi} \Gamma_3 \ln \Gamma_3 + \dots . \quad (69)$$

The Kähler potential depends only on  $\Delta = (\bar{\Phi} + \Phi)/\sqrt{2}$  and is given at the one-loop level by a similar formula

$$\mathcal{K} = e^2 \left[ \Delta^2 + \frac{1}{2\pi} \Delta \ln \Delta \right] . \quad (70)$$

The generalization to non-Abelian case is straightforward. The generalized Kähler potential is

$$\mathcal{K}(\Delta^A) = g^2 \sum_j \left\{ \frac{2}{c_V} (\Delta^{(j)})^2 - \frac{3}{2\pi} \Delta^{(j)} \ln \Delta^{(j)} \right\} + \dots , \quad (71)$$

where  $\Delta^A = (\bar{\Phi}^A + \Phi^A)/\sqrt{2}$  and  $\Delta^{(j)} = \alpha_j(\Delta^A)$

This is not yet the end of the story, however. Considerations of supersymmetry alone do not exclude the presence of *superpotential*  $\sim \text{Re} \int d^2\theta F(\Delta^A)$  on top of the Kähler potential in the effective lagrangian. Indeed, such a superpotential is generated in non-Abelian 3D theories by *nonperturbative* mechanism [25]. The mechanism is roughly the same as the known instanton mechanism for generating superpotential in 4D  $\mathcal{N} = 1$  SYM theory with matter [26]. In three dimensions, instantons are t' Hooft–Polyakov monopoles. They have two fermion “legs” (zero modes) which leads to generation of gluino condensate. The superpotential can be restored from the condensate. In the simplest  $SU(2)$  case it has the form

$$F(\Delta) \sim g^4 \exp \left\{ -2\sqrt{2}\pi\Phi \right\} . \quad (72)$$

The superpotential (72) lifts the degeneracy on the valley. Actually, the scalar potential  $U \sim \exp \left\{ -2\sqrt{2}\pi(\phi + \bar{\phi}) \right\}$  corresponding to the superpotential (72) (the exponent  $2\sqrt{2}\pi(\phi + \bar{\phi}) = 4\pi A_3/g^2$  is nothing but the 3D instanton action) does *not* have a minimum at a finite value of  $\phi$  (as it does not in the massless  $\mathcal{N} = 1$  supersymmetric QCD — this is a typical “runaway vacuum” phenomenon). In 4-dimensional SQCD, this can be cured by giving a mass to the matter fields: supersymmetric vacuum would then occur at a finite value of  $\phi$ . But in the framework of the game we are playing here, the form of the lagrangian of the descendants is dictated by the original 4D theory, and we have to conclude that the 3D  $\mathcal{N} = 1$  sister simply does not exist as a consistent theory.

In principle, this all could happen also in the 2D  $\mathcal{N} = 1$  theory, where the appearance of superpotential is also not excluded by the symmetry considerations. Moreover, nonperturbative solutions, the instantons, exist also in the 2D case [27, 28].

They appear in any  $2D$  gauge theory involving only adjoint fields due to nontrivial  $\pi_1(\text{gauge group})$  by the same token as ordinary BPST instantons in four dimensions appear due to nontrivial  $\pi_3(G)$ . Now,  $\pi_1[SU(N)] = 0$ , but if only adjoint fields are present, the gauge group is globally  $SU(N)/Z_N$  and involves  $N - 1$  topologically distinct noncontractible loops and, correspondingly,  $N - 1$  different types of instantons. In the theories involving only adjoint fermions considered in [28].

However, these instantons do not generate superpotential in this case by two reasons.

- The minima of classical action are realized on delocalized constant gauge field strength configurations (like in Schwinger model). The non-Abelian  $2D$  instantons do not know about the scalar fields (like monopoles do) and cannot lift the degeneracy on the moduli space.
- As was shown in [28], the instantons involve  $N - 1$  *pairs* of fermion zero modes for each fermion flavor. In our case, there are two flavors and an instanton involves all together  $4(N - 1)$  instanton legs, which is too much to generate the fermion condensate and superpotential.

## 4.2 $\mathcal{N} = 2$ : Taub–NUT, Atiyah–Hitchin, and their relatives

The effective lagrangian involves here  $4r$  moduli: three components of vector-potential in reduced dimensions and a dual photon for each unit of the rank. One of the ways to derive the effective lagrangian there is to determine first the effective lagrangian for the theory defined on  $R^2 \times S^1$  (it represents a twisted sigma model involving an infinite sequence of the Fourier modes associated with the circle) and unfold the circle as was explained in details in Sect. 3. When the length  $L$  of the circle becomes large, we can replace the sum over the modes by the integral. In the Abelian case, we obtain [29]

$$e_3^2 \mathcal{L} = \left( \frac{1}{2} + \frac{e^2}{8\pi|\mathbf{A}|} \right) [(\partial_\mu \mathbf{A})^2 + (\partial_\mu \tau)^2] - \frac{ie^2}{4\pi} \boldsymbol{\omega}(\mathbf{A}) \epsilon_{\mu\nu} \partial_\mu \tau \partial_\nu \mathbf{A} , \quad (73)$$

where  $g_3^2 = g_2^2 L$  is the 3-dimensional gauge coupling constant,  $\mu = 1, 2$  (we have not yet added excited Fourier modes of the slow variables), and  $\boldsymbol{\omega}(\mathbf{A}) \equiv \omega(\mathbf{A}) = \cos\theta d\phi$  is a 1-form which coincides with the Abelian connection describing a Dirac monopole



in space of  $\mathbf{A}$ . The variables  $\mathbf{A}$  live on  $R^3$  whereas the variable  $\tau$  lives on the dual circle,  $0 \leq \tau \leq 2\pi/L$ . The second term in Eq. (73) came from the twisted term (59).

When  $L$  is very large, the size of the dual circle is very small which would normally imply that the excitations related to nonzero Fourier modes of  $\tau$  would become heavy and decouple. This is exactly what happened when we reconstructed with this method the  $2D$  effective Lagrangian from the  $1D$  one in Sect. 3. But in the case under consideration it would not be correct just to cross out the terms involving  $\partial_\mu \tau$ . The presence of the twisted term  $\propto \epsilon_{\mu\nu}$  prevents us to do it.

To understand it, consider a trivial toy model,

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + Bx\dot{y} \implies H = \frac{1}{2} \left[ p_x^2 + (p_y - Bx)^2 \right], \quad (74)$$

where  $x \in R^1$ , while  $y$  is restricted to lie on a small circle,  $0 \leq y \leq \alpha$ . The Lagrangian (74) describes a particle living on a cylinder and moving in a constant magnetic field. Now, if the magnetic field  $B$  were absent, the higher Fourier modes of the variable  $y$  would be heavy and the low-energy spectrum would be continuous corresponding to free motion along  $x$  direction. When  $B \neq 0$ , for *each* Fourier mode of the variable  $y$ , we obtain the *same* oscillatorial spectrum. Only the position of the center of the orbit and not the energy depends on  $p_y^{(n)} = 2\pi n/\alpha$ .

Thus, we cannot suppress the variable  $y$  in the Lagrangian (74). Likewise, we cannot suppress the variable  $\tau$  in Eq. (73). What we can do, however, is to trade it to another variable using the duality trick. Performing the same transformations as in the  $\mathcal{N} = 1$  case, we arrive at the lagrangian of the sigma model. living on the target space with the metric

$$ds^2 = \left( 1 + \frac{e^2}{4\pi|\mathbf{A}|} \right) d\mathbf{A}^2 + \frac{\left( d\Psi - \frac{e^2}{4\pi}\omega \right)^2}{\left( 1 + \frac{e^2}{4\pi|\mathbf{A}|} \right)}. \quad (75)$$

The dual variable  $\Psi$  describes the dual photon.

We want to emphasize that the effective lagrangian thus obtained represents a *conventional* sigma model — the twisted term disappears after duality transformation. A conventional  $\mathcal{N} = 4$  sigma model must be hyper-Kähler. Indeed, the metric (75) describes a well-known hyper-Kähler Taub-NUT manifold [30].

Consider now SYM theory and let us start with the case of  $SU(2)$ . The result is immediately written by substituting  $-2g^2$  for  $e^2$  in all above formulae. The metric thus obtained (Taub-NUT with negative mass term) is also hyper-Kähler. However,

in contrast to the regular Taub–NUT metric, it is singular at  $|\mathbf{A}| = 2\pi/g^2$  and represents an *orbifold*. Note that the singularity occurs at small values of  $|\mathbf{A}|$  where the Born–Oppenheimer approximation is not valid anymore. As was explained earlier, we cannot neglect higher–derivative terms in the effective lagrangian in this region. Moreover, the very notion of the effective lagrangian makes no sense anymore. Still, the presence of singularity in the double–derivative part of the lagrangian is somewhat irritating.

The remarkable fact is that the singularity actually disappears when taking into account *nonperturbative* effects associated with instantons (coinciding with ’t Hooft–Polyakov monopoles). Instantons bring about the corrections to the metric  $\sim \exp\{-4\pi n|\mathbf{A}|/g^2\}$  ( $n$  is the topological charge). They are irrelevant in the asymptotics, but are hundred percent important for small values of  $|\mathbf{A}|$ : their resummation gives a *smooth* hyper–Kähler Atiyah–Hitchin metric.<sup>14</sup> There exist an explicit expression for the AH metric. It involves elliptic functions and is not so simple. An interested reader may look it up in [33] and be convinced that its asymptotics coincide with Taub–NUT, indeed, and that the corrections to this asymptotics are exponential.

Another remarkable fact is that the same AH metric describes the low–energy dynamics of two BPS monopoles [34]. In this case, the vector  $\mathbf{A}$  acquires the meaning of the monopole separation  $\mathbf{r}$  and  $\Psi$  of their relative phase. The (smoothened) singularity occurs when the distance between the monopoles is of the same order as the size of the monopole cores. The classical trajectories of the monopoles represent geodesics on the AH manifold.

By the same token as we did it before, we can write the effective lagrangian for an arbitrary gauge group,

$$g^2\mathcal{L} = \frac{1}{2}(\partial_\mu\mathbf{A}^A)(\partial_\mu\mathbf{A}^B)Q_{AB} + \frac{1}{2}J_\mu^A Q_{AB}^{-1}J_\mu^B, \quad (76)$$

where

$$\begin{aligned} Q_{AB} &= \delta_{AB} - \frac{g^2}{2\pi} \sum_j \frac{\alpha_j^A \alpha_j^B}{|\mathbf{A}^{(j)}|}, \\ J_\mu^A &= \partial_\mu\Psi^A + \frac{g^2}{2\pi} \sum_j \boldsymbol{\omega}(\mathbf{A}^{(j)}) \partial_\mu\mathbf{A}^{(j)} \alpha_j^A, \end{aligned} \quad (77)$$

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<sup>14</sup>This was the conjecture of Ref. [31] confirmed by direct evaluation of the one-instanton contribution to the metric in Ref. [32]. (Multiinstanton contributions need not be calculated. Hyper–Kähler nature of the metric fixes them once the one–instanton contribution is known.)

$$\mathbf{A}^{(j)} = \alpha_j(\mathbf{A}^A) \equiv \alpha_j^A \mathbf{A}^A.$$

These are asymptotical expressions for the metric. They involve singularities and their structure is complicated. However, resummation of instanton corrections should patch up this singularities. The result of such a resummation gives a smooth hyper-Kähler manifold. Let me give arguments in favor of this conclusion.

(i) Consider first the unitary groups [35]. The Cartan subalgebra of  $SU(N)$  consists of traceless  $N \times N$  diagonal matrices. As far as the effective lagrangian (76) is concerned, we have four such matrices:  $\text{diag}(\mathbf{A}_1, \dots, \mathbf{A}_N)$  and  $\text{diag}(\Psi_1, \dots, \Psi_N)$ ,  $\sum_m \mathbf{A}_m = \sum_m \Psi_m = 0$ . There are  $N(N-1)/2$  positive roots,  $\alpha_{ml}(\mathbf{A}) = \mathbf{A}_m - \mathbf{A}_l, m < l = 1, \dots, N$ . Substituting it in Eq. (76), we obtain the metric

$$ds^2 = A_{ml} d\mathbf{A}_m d\mathbf{A}_l + A_{ml}^{-1} \Lambda_m \Lambda_l, \quad (78)$$

where  $A$  is the following  $N \times N$  matrix:

$$\begin{aligned} A_{mm} &= 1 - \frac{g^2}{4\pi} \sum_{l \neq m} \frac{1}{|\mathbf{A}_m - \mathbf{A}_l|} && \text{(no summation over } m), \\ A_{ml} &= \frac{g^2}{4\pi |\mathbf{A}_m - \mathbf{A}_l|}, && (m \neq l), \end{aligned} \quad (79)$$

and

$$\Lambda_m = d\Psi_m + \frac{g^2}{4\pi} \sum_{l \neq m} \omega(\mathbf{A}_m - \mathbf{A}_l).$$

This metric happens to describe the dynamics of  $N$  well-separated BPS monopoles [34].  $\mathbf{A}_m \equiv \mathbf{r}_m$  and  $\Psi_i$  are interpreted as positions and phases of individual monopoles. The condition  $\sum_m \mathbf{r}_m = 0$  and similarly for phases means that the trivial center of mass motion is separated out. The classical dynamics is described

by the following equations of motion<sup>15</sup>

$$\begin{aligned}
\ddot{\mathbf{r}}_l - \frac{g^2}{4\pi} \sum_{m \neq l}^N \frac{\ddot{\mathbf{r}}_{lm}}{\mathbf{r}_{ml}} + \frac{g^2}{8\pi} \sum_{l \neq m=1}^N \frac{2 [\dot{\mathbf{r}}_{ml} \times \mathbf{r}_{ml}] \cdot \dot{\mathbf{r}}_{ml} - \mathbf{r}_{ml}(\dot{\mathbf{r}}_{ml}^2)}{r_{ml}^3} \\
- \frac{g}{4\pi} \sum_{m \neq l} (q_{ml}) \dot{\mathbf{r}}_{ml} \times \frac{\mathbf{r}_{ml}}{r_{ml}^3} + \frac{1}{8\pi} \sum_{m \neq l} \frac{q_{ml}^2 \mathbf{r}_{ml}}{r_{ml}^3} = 0 , \\
q_l = g A_{lm}^{-1} \left[ \dot{\Psi}_m + \frac{g^2}{4\pi} \sum_{n \neq m} \boldsymbol{\omega}(\mathbf{r}_{nm}) \dot{\mathbf{r}}_{nm} \right] = \text{const} , \tag{80}
\end{aligned}$$

where  $\mathbf{r}_{ml} = \mathbf{r}_m - \mathbf{r}_l$ ,  $q_{ml} = q_m - q_l$ . The equations of motion for the effective lagrangian (76) have a similar form, with time derivatives being replaced by  $\partial_\mu$  (and  $\mathbf{r}$  by  $\mathbf{A}$ ).

The metric (78) is singular for certain small values of the distances between the monopoles  $|\mathbf{r}_{ml}|$ . These singularities can be patched, however, and with all probability *are* patched by the instanton corrections. A following conjecture of existence and uniqueness can now be formulated: there is only one smooth hyper-Kähler manifold of dimension  $4(N-1)$  ( a *generalized* Atiyah–Hitchin manifold) with the asymptotics (78). (I bet there is though, as far as I know, this has not yet been proven mathematically in an absolutely rigorous way.) An explicit expression for the generalized AH metric is not known.

(ii)  $Sp(2r)$ . There are  $r$  long positive roots  $\alpha_m(\mathbf{r}) = \mathbf{r}_m$  and  $r(r-1)$  short positive roots  $\alpha_{ml}(\mathbf{r}) = (\mathbf{r}_m \pm \mathbf{r}_l)/2$  ( $m < l = 1, \dots, r$ ;  $\mathbf{r}_m$  are mutually orthogonal and linearly independent). The metric reads

$$\begin{aligned}
ds^2 &= \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{4\pi} \sum_{\pm} \sum_{m < l} \frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} - \frac{g^2}{2\pi} \sum_m \frac{(d\mathbf{r}_m)^2}{r_m} + \text{phase part} \\
&\equiv Q_{ml} d\mathbf{r}_m d\mathbf{r}_l + \text{phase part} . \tag{81}
\end{aligned}$$

The full metric is restored from Eqs. (76, 77).

An important observation is that the corresponding effective Lagrangian (the QM version thereof) is obtained from the effective Lagrangian describing the dynamics

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<sup>15</sup> Here  $g$  is interpreted as the monopole magnetic charge. The equations (80) are classical as far as the variables  $\mathbf{r}_l$  are concerned, but the quantization of the dynamic variables  $\Psi_l$  is already carried out. The spectral parameters  $q_l$  are quantized to (integer)/ $g$  and are interpreted as the electric charges of the corresponding dyons.

of  $2r + 1$  BPS monopoles numbered by the integers  $j = -r, \dots, r$  by imposing the constraints

$$\begin{aligned} \mathbf{r}_{-r} + \mathbf{r}_r &= \dots = \mathbf{r}_{-1} + \mathbf{r}_1 = 2\mathbf{r}_0 = 0, \\ \Psi_{-r} + \Psi_r &= \dots = \Psi_{-1} + \Psi_1 = 2\Psi_0 = 0. \end{aligned} \quad (82)$$

We *are* allowed to impose these constraints because they are compatible with the equations of motion (80). The corresponding metric is hyper-Kähler. It has to be, due to  $\mathcal{N} = 4$  supersymmetry, absence of the twisted term and the theorem [7]. One can also prove it more directly reproducing the result (81) by the hyper-Kähler reduction procedure worked out in [36].

(iii)  $SO(2r + 1)$ . The system of roots is the same as for  $Sp(2r)$  only the long and short roots are interchanged: there are now  $r(r - 1)$  long roots  $(\mathbf{r}_m \pm \mathbf{r}_l)/\sqrt{2}$  and  $r$  short roots  $\mathbf{r}_m/\sqrt{2}$ . The metric reads

$$ds^2 = \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{2\pi\sqrt{2}} \left[ \sum_{\pm} \sum_{m < l} \frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} + \sum_m \frac{(d\mathbf{r}_m)^2}{r_m} \right] + \text{phase part}. \quad (83)$$

This metric is obtained from the Gibbons-Manton type metric for  $2r$  BPS monopoles numbered by the integers  $j = -r, \dots, r$ ,  $j \neq 0$  by imposing the constraints

$$\begin{aligned} \mathbf{r}_{-r} + \mathbf{r}_r &= \dots = \mathbf{r}_{-1} + \mathbf{r}_1 = 0, \\ \Psi_{-r} + \Psi_r &= \dots = \Psi_{-1} + \Psi_1 = 0 \end{aligned} \quad (84)$$

and rescaling  $ds^2$  and  $g^2$ . The constraints (84) are compatible with the equations of motion.

Note that we obtained the effective Lagrangian for  $Sp(2r)$  out of that for  $SU(2r + 1)$  and not  $SU(2r)$ , as one could naively expect in view of the embedding  $Sp(2r) \subset SU(2r)$ . Likewise, the moduli space for  $SO(2r + 1)$  is obtained out of  $SU(2r)$  and not  $SU(2r + 1)$ . This is due to the fact that magnetic charges are coupled to coroots rather than roots.

The effective lagrangian for  $SO(2r)$  can also be readily written. It can also be interpreted in monopole terms and the corresponding manifold is related to a generalized AH manifold for the system of  $2r$  monopoles by hyper-Kähler reduction accompanied by a certain mass *deformation* (suppressing interactions between certain monopoles) [29].

The constraints (82), (84) have the form of “mirrors” in the monopole configuration space. In the  $Sp(2r)$  case, the mirror passes through one of the monopoles while in the  $SO(2r+1)$  case it does not. Such “mirrors” appeared earlier in string-related problems and were christened *orientifolds* [37]. For me, they just represent graphical pictures describing embedding of the symplectic and orthogonal groups into unitary ones.

(iv)  $G_2$ . This is the simplest exceptional group. There are three long ( $\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_3, \mathbf{r}_2 - \mathbf{r}_3$ ) and three short ( $\mathbf{r}_{1,2,3}$ ) positive roots (the constraint  $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0$  being imposed). The metric reads

$$ds^2 = \sum_{m=1}^3 d\mathbf{r}_m^2 - \frac{g^2}{2\pi} \left( \sum_{m>l=1}^3 \frac{(d\mathbf{r}_m - d\mathbf{r}_l)^2}{|\mathbf{r}_m - \mathbf{r}_l|} + 3 \sum_{m=1}^3 \frac{d\mathbf{r}_m^2}{|\mathbf{r}_m|} \right) + \dots \quad (85)$$

It can be obtained out of the metric for  $Sp(6)$

$$ds^2 = \sum_{m=1}^3 d\mathbf{r}_m^2 - \frac{g^2}{4\pi} \left( \sum_{\pm} \sum_{m>l=1}^3 \frac{(d\mathbf{r}_m \pm d\mathbf{r}_l)^2}{|\mathbf{r}_m \pm \mathbf{r}_l|} + 2 \sum_{m=1}^3 \frac{d\mathbf{r}_m^2}{|\mathbf{r}_m|} \right) + \dots \quad (86)$$

by rescaling and imposing the [compatible with  $Sp(6)$  equations of motion] constraints

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0, \quad \Psi_1 + \Psi_2 + \Psi_3 = 0. \quad (87)$$

This “3-fold mirror” is (can be called) an orientifold of new type. Again, we obtained  $\mathcal{L}_{\text{eff}}$  for  $G_2$  out of  $\mathcal{L}_{\text{eff}}$  for  $Sp(6)$ , though  $G_2$  is embedded not into  $Sp(6)$ , but into the dual algebra  $SO(7)$ .

The effective Lagrangian for  $F_4$  can be related to the moduli space of 26 monopoles. (26 is the lowest dimension of a unitary group where  $F_4$  can be embedded. This follows from the fact that the representation **26** of  $F_4$  has the lowest dimension.)  $E_6$  can be embedded into  $SU(27)$  and hence the corresponding effective Lagrangian is related to the moduli space of 27 monopoles. Now, the shortest representation in  $E_7$  has the dimension 56 and we need at least 56 monopoles in this case. Finally,  $E_8 \subset SU(248)$  and we need 248 monopoles. The moduli space of 248 monopoles can also be used as a universal starting point to describe the dynamics of  $F_4, E_6$  and  $E_7$ , if following the chain of embeddings  $F_4 \subset E_6 \subset E_7 \subset E_8 \subset SU(248)$ .

The explicit formulae we have written refer to the asymptotic region where non-perturbative effects are suppressed. The corresponding metrics involve singularities

at small  $|\mathbf{r}^{(j)}|$ . Like for the  $SU(N)$  case, a very reasonable conjecture is that these singularities are sewn up by instantons for any simple Lie group giving a unique smooth hyper-Kähler metric with the asymptotics

$$ds^2 = d\mathbf{r}^A Q_{AB} d\mathbf{r}^B + \dots \quad (88)$$

It is natural to conjecture that this metric is obtained from the multimonopole Atiyah-Hitchin metrics by the same hyper-Kähler reduction procedure as above. To the best of my knowledge, hyper-Kähler manifolds thus obtained were not studied before by mathematicians.

## 5 Non-renormalization theorems

There are several proofs of the well-known fact that two- and higher-order corrections to the  $\beta$  function in the  $4D \mathcal{N} = 2$  supersymmetric Yang-Mills theory vanish. We will discuss here two such proofs: (i) diagrammatic (historically, this was the first) and (ii) the one following from holomorphy.

*Supergraphs.* The diagrammatic proof is based on the technique of supergraphs. The simplest nonrenormalization theorem states that all loop corrections to the *superpotential* (the term  $\int d^2\theta F(\Phi_i)$  in the lagrangian,  $\Phi_i$  are chiral superfields) vanish. We refer the reader to the textbooks [2, 12, 38] for its proof, recall in some more details how it is done for gauge couplings (following Refs.[39]), the subject of our interest here.

Consider for simplicity SQED.<sup>16</sup> We explained above [see Eq. (15)] how the one-loop correction to the effective action in any dimension can be evaluated. It does not vanish here. The effective charge in  $\mathcal{N} = 1$  theory is given by

$$\frac{1}{e_{\text{phys}}^2} = \frac{1}{e_0^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda_{UV}}{m_0} + \dots, \quad (89)$$

where  $m_0$  is the *bare* charged fields mass.

The relevant two-loop graph is drawn in Fig. 2 (there are actually two such supergraphs giving the same contribution with the superfields  $S$  or  $T$  in the loop). According to the supergraph Feynman rules [12, 38], each vertex involves the integral  $\int d^8z = \int d^4x d^2\theta d^2\bar{\theta}$  and the whole contribution of the graph in Fig. 2 is

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<sup>16</sup> Generalization to non-Abelian case is relatively straightforward, but it involves some subtleties associated with infrared singularities of the theory [39, 40] which we do not like to discuss here.

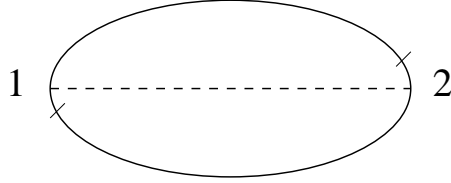


Figure 2: Two-loop contribution to the effective action. Solid lines are chiral field superpropagators  $\langle\Phi\bar{\Phi}\rangle$  evaluated in the classical gauge background and the dashed line is the propagator of the (quantum part of the) vector superfield  $V$ . The bar on the solid line marks the  $\bar{\Phi}$  end.

$\int d^4x_1 d^2\theta_1 d^2\bar{\theta}_1 K$ , where

$$\mathcal{K} = \frac{i}{2} \int d^8z_2 \langle\Phi_1\bar{\Phi}_2\rangle \langle\Phi_2\bar{\Phi}_1\rangle \langle v_1 v_2 \rangle .$$

Here  $\Phi$  stands for charged chiral superfields  $S, T$ ,  $v$  is quantum vector superfield and  $\langle\Phi_1\bar{\Phi}_2\rangle$ ,  $\langle v_1 v_2 \rangle$  are quantum superpropagators evaluated in external background  $V_{cl}$ . Now,  $\langle v_1 v_2 \rangle$  does not depend on external field and on its gauge. The charged field propagators are gauge-dependent:

$$\begin{aligned} \langle S_1 \bar{S}_2 \rangle &\rightarrow e^{i\Lambda_1} \langle S_1 \bar{S}_2 \rangle e^{-i\bar{\Lambda}_2} \\ \langle T_1 \bar{T}_2 \rangle &\rightarrow e^{-i\Lambda_1} \langle T_1 \bar{T}_2 \rangle e^{i\bar{\Lambda}_2} . \end{aligned} \tag{90}$$

The point is, however, that the *integrand*  $K$  is gauge-independent and should thereby be *locally*<sup>17</sup> expressed via the gauge-invariant superfield  $W_\alpha$ . But  $W_\alpha$  is a chiral superfield and the integral over  $d^2\bar{\theta}$  of any function of  $W$  vanishes. Therefore  $\int d^2\theta d^2\bar{\theta} \mathcal{K} = 0$  Q.E.D. The same reasoning apply also to an arbitrary multiloop graph.<sup>18</sup>

We hasten to comment that this does *not* mean that multiloop contributions to  $\beta$  function in  $\mathcal{N} = 1$  supersymmetric QED vanish. Higher loops appear when

<sup>17</sup> Locality follows from the presence of an infrared cutoff (nonzero mass) in the theory.

<sup>18</sup> To be quite precise, the integrand could depend on *both*  $W$  and  $\bar{W}$ , in which case the integral  $\int d^2\theta d^2\bar{\theta} \mathcal{K}$  need not vanish. One can be convinced, however, that such contribution is a supersymmetric generalization of higher derivative  $\sim F^4$  terms in the Euler-Heisenberg effective lagrangian. *Such* corrections to the effective lagrangian are present, indeed, but this does not affect the renormalization of the gauge coupling.



expressing  $m_0$  entering Eq. (89) into  $m_{\text{phys}}$ . As was already mentioned above, physical mass *is* renormalized in spite of the fact that the mass term in the Lagrangian is not. Indeed, the physical mass is defined as the pole of the fermion propagator  $\propto 1/(Z\not{p} - m_0)$ , where  $Z$  describes the renormalization of the *kinetic* term

$$\propto \int d^2\theta d^2\bar{\theta} (\bar{S}e^V S + \bar{T}e^{-V} T) .$$

We have  $m_0 = Zm_{\text{phys}}$  what leads to an exact relation expressing the charge renormalization via the matter  $Z$  factor,

$$\frac{1}{e_{\text{phys}}^2} = \frac{1}{e_0^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda}{m_{\text{phys}}} - \frac{1}{4\pi^2} \ln Z . \quad (91)$$

In particular, using knowledge of  $Z$  at the one-loop level

$$Z = 1 - \frac{e_0^2}{4\pi^2} \ln \frac{\Lambda}{m_{\text{phys}}} \quad (92)$$

we obtain the two-loop renormalization of the charge

$$\frac{1}{e_{\text{phys}}^2} = \frac{1}{e_0^2} + \frac{1}{4\pi^2} \ln \frac{\Lambda}{m_{\text{phys}}} + \frac{e_0^2}{16\pi^4} \ln \frac{\Lambda}{m_{\text{phys}}} + \dots \quad (93)$$

Now,  $\mathcal{N} = 2$  supersymmetric electrodynamics involves an extra neutral chiral superfield  $\Upsilon$ . The Lagrangian involves its kinetic term and the extra superpotential term  $\propto \int d^2\theta \Upsilon ST$ . The latter is not renormalized: this is the standard  $F$  term nonrenormalization theorem. The point is that this superpotential term is related by extended supersymmetry to the charged field kinetic term. Hence, nonrenormalization of the superpotential *implies* in  $\mathcal{N} = 2$  theory nonrenormalization of the kinetic term, which implies the absence of the mass renormalization. In other words, in  $\mathcal{N} = 2$  theory,  $m_{\text{phys}} = m_0$  and hence only the first term in the  $\beta$  function survives. All  $\mathcal{N} = 2$  theories with vanishing 1-loop contribution to the  $\beta$  function are finite. The  $\mathcal{N} = 4$  SYM theory belongs to this class.

The proof just given uses the formalism of  $\mathcal{N} = 1$  supergraphs.  $\mathcal{N} = 2$  symmetry is used indirectly via requirement of equality of renormalization factors for standard the kinetic and the superpotential terms. One can also define and calculate supergraphs in  $\mathcal{N} = 2$  (harmonic) superspace. In this case, the absence of the corrections is manifest [4].

*Holomorphy.* An alternative elegant proof comes from the analysis of the Seiberg–Witten effective lagrangian. As was mentioned in the beginning,  $\mathcal{N} = 2$  supersymmetry dictates the form (6) for the effective lagrangian where  $F(\mathcal{W})$  is a *holomorphic* function of the  $\mathcal{N} = 2$  superfield (4) (the Abelian version thereof). Consider this function for large  $\mathcal{W}$ . Going around the large circle ( multiplying  $\mathcal{W}$  by  $e^{2i\pi}$  ), we should obtain the *same* theory.<sup>19</sup> Knowing that in the asymptotics

$$\mathcal{L}_{\text{eff}} \sim \text{Re} \int d^4\theta \frac{1}{2g^2(\mathcal{W})} \mathcal{W}^2 \quad (94)$$

(That was written in non–Abelian  $SU(2)$  theory. The same expression with  $e^2(\mathcal{W})$  substituted for  $g^2(\mathcal{W})$  holds in  $\mathcal{N} = 2$  SQED), only two possibilities are allowed: (i)  $g^2(\mathcal{W})$  is a constant (this possibility is realized for  $\mathcal{N} = 4$  non–Abelian gauge theories) or (ii)  $g^{-2}(\mathcal{W})$  involves a term  $\sim \ln \mathcal{W}$ , which corresponds to one–loop renormalization. When multiplying  $\mathcal{W}$  by  $e^{2i\pi}$ , the logarithm is shifted by an imaginary constant. This gives a change  $\sim \text{Im} \int d^4\theta \mathcal{W}^2$  in the lagrangian, which is a total derivative ( $\theta$  term). In Abelian theory,  $\theta$  term is never relevant. In non–Abelian theory it *might* have been relevant, but it does not in this case: one can be convinced that multiplying  $\mathcal{W}^2$  by  $e^{2i\pi}$  amounts to the shift  $\theta \rightarrow \theta + 4\pi$ .

Higher–order coefficients  $\beta_2, \beta_3$ , etc should vanish. Were e.g.  $\beta_2$  nonzero, the coefficient  $g^{-2}(\mathcal{W})$  in  $\mathcal{L}_{\text{eff}}$  would involve the contribution  $\sim \ln |1 + c \ln(\Lambda_{\text{UV}}/\mathcal{W})|$ , which is not holomorphic and not allowed. On the other hand, nothing prevents the function  $f(\mathcal{W})$  from having contributions  $\sim \mathcal{W}^{-n}$ , which vanish in the asymptotics. Indeed,  $f(\mathcal{W})$  *does* involve such contributions brought about by instantons [1].

We have seen that low-dimensional sisters of  $4D \mathcal{N} = 2$  have similar properties: the perturbative corrections to the effective lagrangia vanish beyond one loop. In sect. 2.2 we explained why: extended supersymmetry dictates a special form of prepotentials. In  $1D$ , resp.  $2D$  theories, the prepotentials in (28), resp. (55) living in  $\mathbf{R}^5$ , resp.  $\mathbf{R}^4$  must be harmonic functions of their arguments. For  $3D$  theories, extended supersymmetry requires the metric to be hyper–Kähler. In the asymptotics, the metric (75) involves, indeed a harmonic function  $1 + e^2/(4\pi|\mathbf{A}|)$  living in  $\mathbf{R}^3$ . A more detailed analysis shows that the harmonicity follows from the hyper–Kähler nature of the metric (i.e. extended supersymmetry) and from its  $U(1)$  isometry corresponding to shifting the phase  $\Psi$  (this isometry shows up in the asymptotics). Actually, in the cases when such an isometry is present, the Kähler

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<sup>19</sup> Actually, it is sufficient to multiply  $\mathcal{W}$  by  $e^{i\pi}$ . It is  $\mathcal{W}^2$  rather than  $\mathcal{W}$  which has direct physical meaning, the lowest component of  $\mathcal{W}^2$  coinciding with the true moduli  $u = \text{Tr} \phi^2$ .

potential of a hyper-Kähler metric can be obtained from a certain 3-dim harmonic prepotential by Legendre (physically - by duality) transformation [24, 29].<sup>20</sup>

Now, in  $4D$  theories the moduli space is  $\mathbf{R}^2 \equiv \mathbf{C}^1$  and harmonicity there is the same as analyticity! In other words, the proof of the  $4D$  nonrenormalization theorem based on holomorphy has direct low-dimensional counterparts. Nonrenormalizability is a family property of all sisters.

What about the diagrammatic proof?

The  $4D$  diagrammatic proof quoted above involved two parts: (i) the  $\mathcal{N} = 1$  nonrenormalization theorem and (ii) the  $\mathcal{N} = 2$  relationship between the kinetic and superpotential terms. This relationship holds also in low dimensions, but there is no  $\mathcal{N} = 1$  nonrenormalization theorem anymore. Indeed, the theorem was based on the fact that the  $4D$  effective lagrangian had the form  $\int d^2\theta W^2$ , while the two and higher loop supergraphs suggested the form  $\int d^2\theta d^2\bar{\theta} X(W, \bar{W})$ , which could be only reconciled if  $X = 0$ . But in low dimensions, the effective lagrangian does not have a chiral form, but represents an integral  $\int d^2\theta d^2\bar{\theta}$  of a local density depending not on  $W$ , but rather on superconnections  $\Gamma_k$  in reduced dimensions [see e.g. Eq. (12)]. This can well be reconciled with what follows from the diagram in Fig. 2.

Indeed, direct component calculations of the two-loop corrections to the effective action in the  $D = 1$ ,  $\mathcal{N} = 1$  Abelian theory showed that they do not vanish [42]. One obtains instead of Eq. (18)

$$e^2 h(\mathbf{C}) = 1 + \frac{e^2}{2|\mathbf{C}|^3} - \frac{3e^4}{4|\mathbf{C}|^6} + \dots \quad (95)$$

for the metric. In addition, the two-loop contribution is not related to any  $Z$ -factor, like it is the case in four dimensions: the latter just cannot be defined in quantum mechanics.

In Ref.[42], the result (95) was obtained after rather cumbersome calculations where the contribution of several graphs was added. Using  $\mathcal{N} = 1$  supergraph technique, only one graph in Fig. 2 should be evaluated and the calculation is rather simple [40]. In  $\mathcal{N} = 2$  theory, a similar graph with the exchange of  $\Upsilon$  field should be added. It has exactly the same structure and gives exactly the same contribution, but with the opposite sign. The cancellation is manifest. Unfortunately, this simple

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<sup>20</sup> The full Atiyah-Hitchin metric, which involves besides one loop also nonperturbative instanton corrections does not have this isometry and cannot be expressed via a 3-dim harmonic function. It can be expressed, however, via certain more complicated generalized harmonic functions [41]. Their physical meaning is yet to be revealed.

cancellation pattern does not hold at the 3-loop level and higher. Again, one obtains zero after adding up several different supergraphs. In other words, it is hardly possible to prove nonrenormalization of  $\mathcal{N} = 2$  theories using the formalism of  $\mathcal{N} = 1$  supergraphs.

On the other hand, it is very reasonable to suppose that the diagrammatic proof of non-renormalizability based on the technique of harmonic supergraphs [4] can be extended to low dimensions. This question is currently under study.

## 6 Conclusions

To make distinction with the Introduction where main results concerning the nature and character of different sisters were outlined in words and the main body of the paper where the relevant formulae were written, we give here the same information in the form of a table.

Table 1: Pure SYM: the family of effective theories.

|         | $\mathcal{N} = 1$  | $\mathcal{N} = 2$                            |
|---------|--|--|
| $D = 1$ | Symplectic $\sigma$ model of the first kind                | Symplectic $\sigma$ model of the second kind |
| $D = 2$ | Kähler $\sigma$ model                                      | Twisted $\sigma$ model (GHR)                 |
| $D = 3$ | Kähler $\sigma$ model with superpotential. Run-away vacuum | Hyper-Kähler $\sigma$ model                  |
| $D = 4$ | No moduli space. Discrete vacua                            | SW effective theory                          |

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