

ADELIC HARMONIC OSCILLATOR

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Abstract

Using the Weyl quantization we formulate one-dimensional adelic quantum mechanics, which unifies and treats ordinary and p -adic quantum mechanics on an equal footing. As an illustration the corresponding harmonic oscillator is considered. It is a simple, exact and instructive adelic model. Eigenstates are Schwartz-Bruhat functions. The Mellin transform of a simplest vacuum state leads to the well known functional relation for the Riemann zeta function. Some expectation values are calculated. The existence of adelic matter at very high energies is suggested.

1. Introduction

Since 1987, the application of p -adic numbers has been of interest in string theory [1]-[4], quantum mechanics [5]- [12], and some other areas of theoretical [13] and mathematical [14]- [16] physics (for a review see Refs. [19] and [20]). Many p -adic models have been constructed. In string theory a product (adelic) formula is obtained, i.e. the product of the ordinary four-point amplitude and all p -adic analogs is equal to a constant.

One of the significant achievements in this field is the formulation of p -adic quantum mechanics [5],[7],[8] (see also Refs. [6] and [9]). The corresponding model of the harmonic oscillator is constructed, and its evolution and spectral properties are analyzed [10]-[12]. It is an exactly soluble model.

The connection of p -adic quantum mechanics with ordinary mechanics has so far been an open problem. Problems of this kind are more or less a characteristic of all other p -adic models.

In this article we use the concept of adeles as a basis to unify p -adic and ordinary quantum mechanics. In particular, we investigate the adelic model of the harmonic oscillator. Some aspects of the adelic approach have been earlier considered in p -adic string theory [2]-[4] and p -adic quantum field theory [17]. As we shall see, the adelic harmonic oscillator is an exact model which exhibits many interesting mathematical and physical properties.

In Sec. 2 we present some mathematics of p -adic numbers and adeles (one can also see Refs. [14] and [18]). Section 3 contains the ordinary and p -adic harmonic oscillator in classical and quantum mechanics. In formulation of p -adic quantum mechanics we follow the Vladimirov-Volovich approach [5],[7],[8]. This approach is generalized to adelic quantum mechanics in Sec. 4. As an example the adelic model of the harmonic oscillator is considered. The main mathematical and physical aspects of the adelic quantum approach are discussed in the last section.

2. On Adeles

There is sense in starting with the field of rational numbers \mathbb{Q} . From the physical point of view, \mathbb{Q} contains all experimental data. In mathematics, \mathbb{Q} is the simplest infinite number field. Mathematical models of physical phenomena prefer completions (and algebraic extensions) of \mathbb{Q} rather than \mathbb{Q} itself. Completions of \mathbb{Q} with respect to the usual absolute value $|\cdot|_\infty$ and p -adic norm (valuation) $|\cdot|_p$ give the field of real numbers $\mathbb{R} \equiv \mathbb{Q}_\infty$ and the field of p -adic numbers \mathbb{Q}_p ($p =$ a prime number), respectively. According to the Ostrowski theorem, \mathbb{R} and \mathbb{Q}_p ($p = 2, 3, 5, \dots$) exhaust the possible number fields which can be obtained by completions of \mathbb{Q} .

A series

$$\varepsilon \sum_{k \in \mathbb{Q}} a_k p^k, \quad a_k \in \{0, 1, \dots, p-1\}, \quad (2.1)$$

where $\varepsilon = \pm 1$, and $a_k = 0$ for $k \geq k_0$ is a real number. If $\varepsilon = 1$ and $a_k = 0$ for $k \leq k_0$ the series (2.1) represents a p -adic number in \mathbb{Q}_p . The ring of p -adic integers is $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, i.e. for \mathbb{Z}_p , $k \geq 0$ in (2.1).

Since \mathbb{R} and \mathbb{Q}_p are locally compact groups one can define on them two invariant measures. On the additive groups \mathbb{R}^+ and \mathbb{Q}_p^+ there are translationally invariant measures dx_∞ and dx_p , respectively, where dx_∞ corresponds to the Lebesgue measure normalized by $\int_{|x_\infty|_\infty \leq 1} dx_\infty = 2$, and dx_p is the Haar measure normalized by $\int_{|x_p|_p \leq 1} dx_p = 1$. On the multiplicative groups

\mathbb{R}^* and \mathbb{Q}_p^* there exist the Haar measures d^*x_∞ and d^*x_p , respectively, invariant under multiplication. These two types of measure are connected by the equalities

$$d^*x_\infty = \frac{dx_\infty}{|x_\infty|_\infty}, \quad d^*x_p = \frac{1}{1-p^{-1}} \frac{dx_p}{|x_p|_p}. \quad (2.2)$$

It is significant that real and p -adic numbers can be unified by means of adeles. An adèle is an infinite sequence

$$a = (a_\infty, a_2, \dots, a_p, \dots), \quad (2.3)$$

where $a_\infty \in \mathbb{R}$ and p -adic numbers $a_p \in \mathbb{Z}_p$ for all but a finite number of p . The set of adeles \mathcal{A} may be regarded as a direct topological product $\mathbb{Q}_\infty \times \prod_p \mathbb{Q}_p$ whose elements satisfy the above restriction. \mathcal{A} is a ring under componentwise addition and multiplication. Denote by \mathcal{A}^+ the additive group \mathcal{A} . A multiplicative group of ideles \mathcal{A}^* is a subset of \mathcal{A} with elements $b = (b_\infty, b_2, \dots, b_p, \dots)$ such that $b_\infty \neq 0$ and $b_p \neq 0$ for every p , and $|b_p|_p = 1$ for all but a finite number of p . A principal adèle (idele) is a sequence $(r, r, \dots, r, \dots) \in \mathcal{A}$, where $r \in \mathbb{Q}$ ($r \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$). One can define a product of norms for adeles,

$$|b| = |b_\infty|_\infty \prod_p |b_p|_p, \quad (2.4)$$

which for a principal idele is

$$|r| = |r|_\infty \prod_p |r|_p = 1. \quad (2.5)$$

An additive character on \mathcal{A}^+ is

$$\chi(xy) = \chi_\infty(x_\infty y_\infty) \prod_p \chi_p(x_p y_p) = \exp(-2\pi i x_\infty y_\infty) \prod_p \exp 2\pi i \{x_p y_p\}_p, \quad (2.6)$$

where $x, y \in \mathcal{A}^+$ and $\{a_p\}_p$ is the fractional part of $a_p \in \mathbb{Q}_p$. A multiplicative character on \mathcal{A}^* can be defined as

$$\pi(b) = \pi_\infty(b_\infty) \pi_2(b_2) \cdots \pi_p(b_p) \cdots = |b_\infty|_\infty^s \prod_p |b_p|_p^s = |b|^s, \quad (2.7)$$

where b is an idele and s is a complex number. In fact, only finitely many factors in (2.6) and (2.7) are different from unity. One can show that $\chi(r) = 1$ if r is a principal adele, and $\pi(r) = 1$ if r is a principal idele.

An elementary function on the group of adeles \mathcal{A}^+ is

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_p \varphi_p(x_p), \quad (2.8)$$

where $x \in \mathcal{A}^+$, $\varphi_\infty(x_\infty) \in \mathcal{S}(\mathbb{R})$, $\varphi_p(x_p) \in \mathcal{S}(\mathbb{Q}_p)$. Namely, $\varphi(x)$ is a complex-valued function which satisfies the following conditions: (i) $\varphi_\infty(x_\infty)$ is an infinitely differentiable function on \mathbb{R} and $|x_\infty|_\infty^n \varphi_\infty(x_\infty) \rightarrow 0$ as $|x_\infty|_\infty \rightarrow \infty$ for any $n \in \{0, 1, 2, \dots\}$; (ii) $\varphi_p(x_p)$ is a finite and locally constant function, i.e. φ_p has a compact support and $\varphi_p(x_p + y_p) = \varphi_p(x_p)$ if $|y_p|_p \leq p^{-n}$, $n = n(\varphi_p)$; and (iii) $\varphi_p(x_p) = \Omega(|x_p|_p)$, for all but a finite number of p , where

$$\Omega(u) = \begin{cases} 1, & \text{if } 0 \leq u \leq 1, \\ 0, & \text{if } u > 1. \end{cases} \quad (2.9)$$

All finite linear combinations of elementary functions (2.8) make up the set $\mathcal{S}(\mathcal{A})$ of the Schwartz-Bruhat functions.

The Fourier transform of $\varphi(x) \in \mathcal{S}(\mathcal{A})$ is

$$\begin{aligned} \tilde{\varphi}(y) &= \int_{\mathcal{A}^+} \varphi(x) \chi(xy) dx = \int_{\mathbb{R}} \varphi_\infty(x_\infty) \chi_\infty(x_\infty y_\infty) dx_\infty \\ &\quad \times \prod_p \int_{\mathbb{Q}_p} \varphi_p(x_p) \chi_p(x_p y_p) dx_p, \end{aligned} \quad (2.10)$$

where $dx = dx_\infty dx_2 \cdots dx_p \cdots$ is the Haar measure on \mathcal{A}^+ . The Fourier transform maps $\mathcal{S}(\mathcal{A})$ onto $\mathcal{S}(\mathcal{A})$. The Mellin transform of $\varphi(x) \in \mathcal{S}(\mathcal{A})$ is defined with respect to the multiplicative character $\pi(x) = |x|^s$:

$$\begin{aligned} \Phi(s) &= \int_{\mathcal{A}^*} \varphi(x) |x|^s d^*x = \int_{\mathbb{R}} \varphi_\infty(x_\infty) |x_\infty|_\infty^{s-1} dx_\infty \\ &\quad \times \prod_p \int_{\mathbb{Q}_p} \varphi_p(x_p) |x_p|_p^{s-1} \frac{dx_p}{1-p^{-1}}, \quad \text{Re } s > 1, \end{aligned} \quad (2.11)$$

where $d^*x = d^*x_\infty d^*x_2 \cdots d^*x_p \cdots$ is the Haar measure on \mathcal{A}^* .

The function $\Phi(s)$ can be analytically continued on the whole field of complex numbers, except $s = 0$ and $s = 1$, where it has simple poles with residue $-\varphi(0)$ and $\tilde{\varphi}(0)$, respectively. Let $\Phi(s)$ is the Mellin transform of $\tilde{\varphi}$. One can show [18] that Φ and $\tilde{\Phi}$ are connected by the Tate formula

$$\Phi(s) = \tilde{\Phi}(1 - s). \quad (2.12)$$

Recall that the Hilbert space H is a set such that: (i) H is an infinitely dimensional linear system; (ii) there is a scalar product (f, g) for any $f, g \in H$; and (iii) H is a complete space with respect to the metric $d(f, g) = \|f - g\|$, where the norm $\|f\| = (f, f)^{\frac{1}{2}}$. We use here the following Hilbert spaces.

(1) $L_2(\mathbb{R})$ is a space of complex-valued functions $\psi_1^{(\infty)}(x_\infty), \psi_2^{(\infty)}(x_\infty), \dots$ with scalar product and norm

$$(\psi_1^{(\infty)}, \psi_2^{(\infty)}) = \int_{\mathbb{R}} \bar{\psi}_1^{(\infty)}(x_\infty) \psi_2^{(\infty)}(x_\infty) dx_\infty, \quad (2.13a)$$

$$\|\psi^{(\infty)}\| = (\psi^{(\infty)}, \psi^{(\infty)})^{\frac{1}{2}} < \infty, \quad (2.13b)$$

where $x_\infty \in \mathbb{R}$ and dx_∞ is the Lebesgue measure.

(2) $L_2(\mathbb{Q}_p)$ is a space of complex-valued functions $\psi_1^{(p)}(x_p), \psi_2^{(p)}(x_p), \dots$ with scalar product and norm

$$(\psi_1^{(p)}, \psi_2^{(p)}) = \int_{\mathbb{Q}_p} \bar{\psi}_1^{(p)}(x_p) \psi_2^{(p)}(x_p) dx_p, \quad (2.14a)$$

$$\|\psi^{(p)}\| = (\psi^{(p)}, \psi^{(p)})^{\frac{1}{2}} < \infty, \quad (2.14b)$$

where $x_p \in \mathbb{Q}_p$ and dx_p is the Haar measure on \mathbb{Q}_p .

(3) $L_2(\mathcal{A})$ is a space of complex-valued functions $\psi_1(x), \psi_2(x), \dots$ with scalar product and norm

$$(\psi_1, \psi_2) = \int_{\mathcal{A}} \bar{\psi}_1(x) \psi_2(x) dx, \quad (2.15a)$$

$$\|\psi\| = (\psi, \psi)^{\frac{1}{2}} < \infty, \quad (2.15b)$$

where $x \in \mathcal{A}$ and dx is the Haar measure on \mathcal{A} .

$L_2(\mathbb{R})$, $L_2(\mathbb{Q}_p)$ and $L_2(\mathcal{A})$ are the separable Hilbert spaces.

The bases of the above spaces may be given by the orthonormal eigenfunctions of some operator (e.g. an evolution operator). Let the bases for the evolution operator in the real $[U_\infty(t_\infty)]$ and p -adic $[U_p(t_p)]$ cases be $\{\psi_{nm}^{(\infty)}\}$ and $\{\psi_{\alpha_p\beta_p}^{(p)}\}$, where $n, m = 0, 1, \dots$, and α_p, β_p are indices which characterize energy and its degeneration. Also, let us denote by $\psi_{0\beta_p}^{(p)}$ (vacuum states) eigenfunctions which are invariant under $U_p(t_p)$. We define an orthonormal basis for the corresponding adelic evolution operator,

$$U(t) = U_\infty(t_\infty) \prod_p U_p(t_p), \quad t \in \mathcal{A}, \quad (2.16)$$

as

$$\psi_{\alpha\beta}(x) = \psi_{nm}^{(\infty)}(x_\infty) \prod_p \psi_{\alpha_p\beta_p}^{(p)}(x_p), \quad x \in \mathcal{A}, \quad (2.17)$$

where all but a finite number of $\psi_{\alpha_p\beta_p}^{(p)}(x_p)$ are $\psi_{00}^{(p)}(x_p) = \Omega(|x_p|_p)$ [Eq. (2.9)]. We have to take $\psi_{\alpha\beta}(x)$ in this form because the Fourier transform of the vacuum state $\Omega(|x_p|_p)$ is $\tilde{\Omega} = \Omega(|k_p|_p)$. It enables $\psi_{\alpha\beta}(x)$ and $\tilde{\psi}_{\alpha\beta}(k)$ to be functions of the adeles x and k , where k is a dual (momentum) to x . Note that the Schwartz-Bruhat functions satisfy this condition.

3. Harmonic Oscillator over \mathbb{R} and \mathbb{Q}_p

The harmonic oscillator represents a very simple theoretical model which can be solved exactly classically as well as quantum-mechanically.

Case A: Ordinary and p -adic classical oscillator

The corresponding nonrelativistic classical Hamiltonian is

$$H = \frac{1}{2m}k^2 + \frac{m\omega^2}{2}q^2, \quad m \neq 0, \quad (3.1)$$

where q and k are the position and the momentum, respectively. The classical time evolution of the phase space can be presented in the form

$$\begin{pmatrix} q(t) \\ k(t) \end{pmatrix} = T_t \begin{pmatrix} q \\ k \end{pmatrix} = T_t z, \quad T_t = \begin{pmatrix} \cos \omega t & (m\omega)^{-1} \sin \omega t \\ -m\omega \sin \omega t & \cos \omega t \end{pmatrix}, \quad (3.2)$$

where $q = q(0)$ and $k = k(0)$. In the real case all quantities belong to \mathbb{R} . Analogously, in the p -adic classical case they belong to \mathbb{Q}_p . Convergence

domains for the expansions of p -adic $\cos \omega t$ and $\sin \omega t$ require satisfying the conditions $|\omega t|_p \leq p^{-1}$ for $p \neq 2$ and $|\omega t|_2 \leq 2^{-2}$ for $p = 2$. These domains we denote by G_p , and they are additive groups. One can easily show that $T_t T_{t'} = T_{t+t'}$ and $B(T_t z, T_t z') = B(z, z')$, where

$$B(z, z') = -k q' + q k' \quad (3.3)$$

is the skew-symmetric bilinear form on the phase space.

Case B: Ordinary quantum oscillator

In ordinary quantum mechanics the harmonic oscillator is given by the Schrödinger equation

$$\frac{d^2 \psi^{(\infty)}}{dx^2} + \frac{2m}{\hbar^2} \left(E - \frac{m\omega^2}{2} x^2 \right) \psi^{(\infty)} = 0, \quad (3.4)$$

where $x, m, \omega, \hbar \in \mathbb{R}$ and $\psi^{(\infty)} \in \mathbb{C}$. (For a function containing index ∞ or p we shall often omit such an index for its variable.)

If we introduce a dimensional coordinate

$$\xi = x \sqrt{2\pi} \left(\frac{m\omega}{\hbar} \right)^{\frac{1}{2}} \quad (3.5)$$

Eq. (3.4) becomes

$$\frac{d^2 \psi^{(\infty)}}{d\xi^2} + \left(\frac{4\pi E}{\hbar\omega} - \xi^2 \right) \psi^{(\infty)} = 0. \quad (3.6)$$

from now on we make simplification using $m = \omega = \hbar = 1$. Physical solutions to (3.6) represent an orthonormal basis of an $L_2(\mathbb{R})$ and since $\xi = x\sqrt{2\pi}$ we have

$$\psi_n^{(\infty)}(x) = \frac{2^{\frac{1}{4}}}{(2^n n!)^{\frac{1}{2}}} e^{-\pi x^2} H_n(x\sqrt{2\pi}), \quad n = 0, 1, \dots, \quad (3.7)$$

where $H_n(x\sqrt{2\pi})$ are the Hermite polynomials. One can easily show that $\psi_n^{(\infty)}(x)$ satisfies the above condition (i) for an elementary function (2.8).

Ordinary quantum mechanics can also be given by a triple [7, 21]

$$(L_2(\mathbb{R}), W_\infty(z_\infty), U_\infty(t_\infty)), \quad (3.8a)$$

where $L_2(\mathbb{R})$ is the Hilbert space defined in the preceding section, z_∞ is a point of real classical phase space, $W_\infty(z_\infty)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathbb{R})$, and $U_\infty(t_\infty)$ is a unitary representation of the evolution operator on $L_2(\mathbb{R})$.

Recall that the Heisenberg-Weyl group consists of elements (z, α) with the group product

$$(z, \alpha)(z', \alpha) = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z')), \quad (3.9)$$

where $B(z, z')$ is as defined by (3.3) and α is a parameter. The corresponding unitary representation in the real case is

$$\chi_\infty(\alpha) W_\infty(z), \quad (3.10)$$

and $W_\infty(z)$ satisfies the Weyl relation

$$W_\infty(z) W_\infty(z') = \chi_\infty\left(\frac{1}{2}B(z, z')\right) W_\infty(z + z'). \quad (3.11a)$$

The operator $W_\infty(z)$ acts on $\psi_n^{(\infty)}(x)$ in the following way:

$$\begin{aligned} W_\infty(z) \psi_n^{(\infty)}(x) &= \chi_\infty\left(-\frac{kq}{2}\right) U_q \chi_\infty(kx) \psi_n^{(\infty)}(x) \\ &= \chi_\infty\left(\frac{kq}{2} + kx\right) \psi_n^{(\infty)}(x + q), \end{aligned} \quad (3.12a)$$

where $U_q \psi(x) = \psi(x + q)$.

The evolution operator $U_\infty(t)$ is defined by

$$U_\infty(t) \psi_n^{(\infty)}(x) = \int_{\mathbb{R}} \mathcal{K}_t^{(\infty)}(x, y) \psi_n^{(\infty)}(y) dy, \quad (3.13a)$$

where the kernel $\mathcal{K}_t^{(\infty)}(x, y)$ for the harmonic oscillator is

$$\mathcal{K}_t^{(\infty)}(x, y) = \lambda_\infty(2 \sin t) |\sin t|_\infty^{-\frac{1}{2}} \exp 2\pi i \left(\frac{x^2 + y^2}{2 \tan t} - \frac{xy}{\sin t} \right), \quad (3.14a)$$

$$\mathcal{K}_0^{(\infty)}(x, y) = \delta_\infty(x - y). \quad (3.13b)$$

In (3.14a) the function $\lambda_\infty(a)$ is

$$\lambda_\infty(a) = \frac{1}{\sqrt{2}} (1 - i \operatorname{sign} a). \quad (3.15a)$$

The unitary operator $U_\infty(t)$ satisfies the group relation

$$U_\infty(t+t') = U_\infty(t)U_\infty(t') \quad (3.16a)$$

and consequently for the kernel one has

$$\mathcal{K}_{t+t'}^{(\infty)}(x, y) = \int_{\mathbb{R}} \mathcal{K}_t^{(\infty)}(x, z) \mathcal{K}_{t'}^{(\infty)}(z, y) dz. \quad (3.17a)$$

The operators $U_\infty(t)$ and $W_\infty(z)$ are connected by the relation

$$U_\infty(t)W_\infty(z) = W_\infty(T_t z)U_\infty(t). \quad (3.18a)$$

Note that (3.7) are eigenfunctions of the evolution operator $U_\infty(t)$, i.e.

$$U_\infty(t)\psi_n^{(\infty)}(x) = \exp(-2\pi i E_n^{(\infty)} t) \psi_n^{(\infty)}(x), \quad (3.19)$$

where $E_n^{(\infty)} = (n + \frac{1}{2}) \frac{1}{2\pi}$ is the corresponding energy. The right hand side of (3.19) represents eigenfunctions of the energy operator $\hat{E} = \frac{i}{2\pi} \frac{\partial}{\partial t}$.

Let \mathcal{D}_∞ be an observable and $\hat{\mathcal{D}}_\infty$ the corresponding operator which acts in $L_2(\mathbb{R})$. An expectation (average) value of \mathcal{D}_∞ in a state $\psi^{(\infty)}(x)$ is

$$\langle \mathcal{D}_\infty \rangle = (\psi^{(\infty)}, \hat{\mathcal{D}}_\infty \psi^{(\infty)}). \quad (3.20a)$$

It is also of interest to have a knowledge of the mean square deviation $\Delta \mathcal{D}_\infty$, which is a measure of the dispersion around $\langle \mathcal{D}_\infty \rangle$:

$$\Delta \mathcal{D}_\infty = [(\langle (\mathcal{D}_\infty - \langle \mathcal{D}_\infty \rangle)^2 \rangle)]^{1/2} = (\langle \mathcal{D}_\infty^2 \rangle - \langle \mathcal{D}_\infty \rangle^2)^{1/2}. \quad (3.21a)$$

In the vacuum state

$$\psi_0^{(\infty)}(x) = 2^{\frac{1}{4}} e^{-\pi x^2}, \quad \tilde{\psi}_0^{(\infty)}(k) = 2^{\frac{1}{4}} e^{-\pi k^2}, \quad (3.22)$$

where $\tilde{\psi}_0^{(\infty)} = \psi_0^{(\infty)}$ is the Fourier transform of $\psi_0^{(\infty)}$, one obtains

$$\langle x_\infty \rangle = \langle k_\infty \rangle = 0, \quad (3.23a)$$

$$\langle |x_\infty|^s \rangle = \langle |k_\infty|^s \rangle = \sqrt{2} \Gamma\left(\frac{s+1}{2}\right) (2\pi)^{-\frac{s+1}{2}}, \quad (2.23b)$$

$$\Delta x_\infty = \Delta k_\infty = \frac{1}{2\sqrt{\pi}}, \quad \Delta x_\infty \Delta k_\infty = \frac{1}{4\pi}, \quad (3.23c)$$

$$\Delta|x_\infty|_\infty = \Delta|k_\infty|_\infty = \frac{1}{2\sqrt{\pi}} \left(1 - \frac{2}{\pi}\right)^{1/2}. \quad (3.23d)$$

Case C: p -Adic quantum oscillator

In p -adic quantum mechanics, which is the subject of this subsection, canonical variables are p -adic numbers and wave functions are complex-valued. Since wave functions and their variables belong to differently valued number fields, the usual (canonical) quantization does not work. However, one can make a p -adic generalization of the above Weyl representation.

According to the Vladimirov-Volovich approach [5, 7, 8], p -adic quantum mechanics is given by a triple

$$(L_2(\mathbb{Q}_p), W_p(z_p), U_p(t_p)), \quad (3.8b)$$

where $L_2(\mathbb{Q}_p)$ is the p -adic Hilbert space defined in the preceding section, z_p is a point of p -adic classical phase space, $W_p(z_p)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathbb{Q}_p)$, and $U_p(t_p)$ is a p -adic evolution operator which realizes a unitary representation on $L_2(\mathbb{Q}_p)$ of a group G_p .

Analogously to the real case, the operator $W_p(z_p)$ is the unitary representation of the Heisenberg-Weyl group (3.9) with p -adic values of z and α . It also satisfies the relations

$$W_p(z) W_p(z') = \chi_p\left(\frac{1}{2}B(z, z')\right) W_p(z + z'), \quad (3.11b)$$

$$W_p(z) \psi^{(p)}(x) = \chi_p\left(\frac{kq}{2} + kx\right) \psi^{(p)}(x + q). \quad (3.12b)$$

A p -adic evolution operator is given by

$$U_p(t) \psi^{(p)}(x) = \int_{\mathbb{Q}_p} \mathcal{K}_t^{(p)}(x, y) \psi^{(p)}(y) dy, \quad (3.13b)$$

where the kernel for the harmonic oscillator is

$$\mathcal{K}_t^{(p)}(x, y) = \lambda_p(2t) |t|_p^{-\frac{1}{2}} \chi_p\left(\frac{xy}{\sin t} - \frac{x^2 + y^2}{2 \tan t}\right), \quad t \in G_p \setminus \{0\}, \quad (3.24a)$$

$$\mathcal{K}_0^{(p)}(x, y) = \delta_p(x - y), \quad (3.24b)$$

where $\delta_p(x - y)$ is a p -adic analog of the Dirac δ -function. For canonical expansion

$$a = p^\nu (a_0 + a_1 p + a_2 p^2 + \dots), \quad \nu \in \mathbb{Z}, \quad a_0 \neq 0, \quad 0 \leq a_i \leq p - 1, \quad (3.25)$$

the number-theoretic function $\lambda_p(a)$ is

$$\lambda_p(a) = \begin{cases} 1, & \nu = 2k, & p \neq 2, \\ \left(\frac{a_0}{p}\right), & \nu = 2k + 1, & p \equiv 1 \pmod{4}, \\ i\left(\frac{a_0}{p}\right), & \nu = 2k + 1, & p \equiv 3 \pmod{4}, \end{cases} \quad (3.15b)$$

$$\lambda_2(a) = \begin{cases} \frac{1}{\sqrt{2}} [1 + (-1)^{a_1} i], & \nu = 2k, \\ \frac{1}{\sqrt{2}} (-1)^{a_1 + a_2} [1 + (-1)^{a_1} i], & \nu = 2k + 1, \end{cases} \quad (3.15c)$$

where $\left(\frac{a_0}{p}\right)$ is the Legendre symbol and $k \in \mathbb{Z}$.

The operator $U_p(t)$ and its kernel $\mathcal{K}_t^{(p)}(x, y)$ satisfy the group relations

$$U_p(t + t') = U_p(t) U_p(t'), \quad (3.16b)$$

$$\mathcal{K}_{t+t'}^{(p)}(x, y) = \int_{\mathbb{Q}_p} \mathcal{K}_t^{(p)}(x, z) \mathcal{K}_{t'}^{(p)}(z, y) dz. \quad (3.17b)$$

To prove (3.17b) we use

$$\int_{\mathbb{Q}_p} \chi_p(\alpha x^2 + \beta x) dx = \lambda_p(\alpha) |2\alpha|_p^{-\frac{1}{2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0,$$

with properties of $\lambda_p(a)$: $\lambda_p(0) = 1$, $\lambda_p(a^2 b) = \lambda_p(b)$, $\lambda_p(a) \lambda_p(b) = \lambda_p(a + b) \lambda_p(a^{-1} + b^{-1})$. The operators $U_p(t)$ and $W_p(z)$ are connected in the same way as in the real case:

$$U_p(t) W_p(z) = W_p(T_t z) U_p(t). \quad (3.18b)$$

A spectral problem in p -adic quantum mechanics is related to an investigation of the eigenvalues and eigenfunctions of the evolution operator. According to Ref. 11, a character $\chi_p(\alpha t)$ is an eigenvalue of $U_p(t)$ for the harmonic oscillator if and only if α takes one of the values

$$\alpha = 0, \quad (3.26a)$$

$$\alpha = p^{-\nu} (\alpha_0 + \alpha_1 p + \dots + \alpha_{\nu-2} p^{\nu-2}), \quad \alpha_0 \neq 0, \quad 0 \leq \alpha_i \leq p - 1, \quad (3.26b)$$

where (i) $\nu \geq 2$ for $p \equiv 1 \pmod{4}$, (ii) $\nu = 2n$ ($\nu \in \mathbb{N}$) for $p \equiv 3 \pmod{4}$, and (iii) $\nu \geq 3$ for $p = 2$ [with $\alpha_{\nu-3} p^{\nu-3}$ as a last term in (3.26b)]. The set of α in Eqs. (3.26) we denote by I_α . We can introduce

$$E_\alpha^{(p)} = \alpha_p, \quad \alpha_p \in I_p, \quad (3.27)$$

which may be regarded as a discrete p -adic energy of the harmonic oscillator.

The corresponding eigenstates satisfy the equation

$$U_p(t) \psi_{\alpha\beta}^{(p)}(x) = \chi_p(\alpha t) \psi_{\alpha\beta}^{(p)}(x), \quad (3.28)$$

where the index β differentiates eigenfunctions for degenerate states. In particular, $\alpha = 0$ corresponds to a vacuum state which is invariant under $U_p(t)$, i.e.

$$U_p(t) \psi_{0\beta}^{(p)}(x) = \psi_{0\beta}^{(p)}(x). \quad (3.29)$$

The Hilbert space $L_2(\mathbb{Q}_p)$ can be presented as

$$L_2(\mathbb{Q}_p) = \bigoplus_{\alpha \in I_p} H_\alpha^{(p)}, \quad (3.30)$$

which is a direct sum of mutually orthogonal subspaces $H_\alpha^{(p)}$. The dimensions of $H_\alpha^{(p)}$ are as follows:

- (i) When $p \equiv 1 \pmod{4}$, $\dim H_\alpha^{(p)} = \infty$ for every $\alpha \in I_p$;
- (ii) When $p \equiv 3 \pmod{4}$, $\dim H_0^{(p)} = 1$ but $\dim H_\alpha^{(p)} = p + 1$ for $|\alpha|_p \geq p^{2n}$ ($n \in \mathbb{N}$);
- (iii) When $p = 2$, $\dim H_0^{(2)} = \dim H_\alpha^{(2)} = 2$ for $|\alpha|_2 = 2^3$ and $\dim H_\alpha^{(2)} = 4$ for $|\alpha|_2 \geq 2^4$. These dimensions determine the number of linearly independent eigenfunctions for any eigenvalue $\chi_p(\alpha t)$. The eigenfunctions $\psi_{\alpha\beta}^{(p)}(x)$ are obtained [11, 12] in an explicit form for the vacuum state ($\alpha = 0$) and for some excited states $\alpha \neq 0$. All $\psi_{\alpha\beta}^{(p)}(x)$ satisfy the condition (ii) of the elementary functions. The orthonormal vacuum eigenfunctions of $U_p(t)$ for the harmonic oscillator are:

$$(i) \text{ For } p \equiv 1 \pmod{4}, \psi_{00}^{(p)}(x) = \Omega(|x|_p), \psi_{0\nu}^{(p)}(x) \\ = p^{-\frac{\nu}{2}} (1 - p^{-1})^{-\frac{1}{2}} \chi_p(\tau x^2) \delta(p^\nu - |x|_p), \nu \in \mathbb{N}, \tau^2 = -1; \quad (3.31a)$$

$$(ii) \text{ For } p \equiv 3 \pmod{4}, \psi_{00}^{(p)}(x) = \Omega(|x|_p); \quad (3.31b)$$

$$(iii) \text{ For } p = 2, \psi_{00}^{(2)}(x) = \Omega(|x|_2), \psi_{01}^{(2)}(x) = 2 \Omega(2|x|_2) - \Omega(|x|_2). \quad (3.31c)$$

In (3.31a) $\delta(p^\nu - |x|_p)$ is an elementary function defined by

$$\delta(p^\nu - |x|_p) = \begin{cases} 1, & |x|_p = p^\nu, \\ 0, & |x|_p \neq p^\nu. \end{cases} \quad (3.32)$$

One can generalize (3.20a) and (3.21a) to the p -adic case. Thus one obtains

$$\langle \mathcal{D}_p \rangle = (\psi^{(p)}, \hat{\mathcal{D}}_p \psi^{(p)}), \quad (3.20b)$$

$$\Delta \mathcal{D}_p = [(\langle \mathcal{D}_p \rangle - \langle \mathcal{D}_p \rangle^2)]^{1/2} = (\langle \mathcal{D}_p^2 \rangle - \langle \mathcal{D}_p \rangle^2)^{1/2}. \quad (3.21b)$$

Note that $\langle x_p \rangle$ and $\langle k_p \rangle$ have no meaning. For the simplest vacuum state

$$\psi_{00}^{(p)}(x) = \Omega(|x|_p), \quad \tilde{\psi}_{00}^{(p)}(x) = \Omega(|k|_p), \quad (3.33)$$

where $\tilde{\psi}_{00}^{(p)} = \psi_{00}^{(p)}$ is the Fourier transform of $\psi_{00}^{(p)}(x)$, we get

$$\langle |x|_p^s \rangle = \langle |k|_p^s \rangle = \frac{1 - p^{-1}}{1 - p^{-s-1}}, \quad \text{Re } s > -1, \quad (3.34a)$$

$$\Delta |x|_p = \Delta |k|_p = \left(\frac{1 - p^{-1}}{1 - p^{-3}} \right)^{\frac{1}{2}} \left[1 - \frac{(1 - p^{-1})(1 - p^{-3})}{(1 - p^{-2})^2} \right]^{\frac{1}{2}}. \quad (3.34b)$$

4. Harmonic Oscillator over Adeles

Let us define adelic quantum mechanics as a triple

$$(L_2(\mathcal{A}), W(z), U(t)), \quad (4.1)$$

where \mathcal{A} is the additive group of adeles, z is an adelic point of a classical phase space, and t is an adelic time. $L_2(\mathbb{Q}_p)$ is the Hilbert space of complex-valued square integrable functions on \mathcal{A} (see Sec. 2), $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathcal{A})$, and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathcal{A})$.

The Heisenberg-Weyl group (3.9) is generalized to the adelic case by taking $z \in \mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$, $\alpha \in \mathcal{A}$, and $B(z, z') \in \mathcal{A}$. In this case the unitary representation of (3.9) is

$$\chi(\alpha) W(z) = \chi_\infty(\alpha_\infty) W_\infty(z_\infty) \prod_p \chi_p(\alpha_p) W_p(z_p), \quad (4.2)$$

where χ is defined by (2.6), and the Weyl relation

$$W(z)W(z') = \chi\left(\frac{1}{2}B(z, z')\right)W(z+z') \quad (4.3)$$

is satisfied. On the basis of (3.12a) and (3.12b) we have

$$W(z)\psi(x) = \chi\left(\frac{kq}{2} + kx\right)\psi(x+q). \quad (4.4)$$

When x, q, k are principal adeles (adelic rational variables) then $\chi(\frac{kq}{2} + kx) = 1$ and

$$W(z)\psi(x) = \psi(x+q). \quad (4.5)$$

Let the evolution operator $U(t)$ be defined by

$$U(t)\psi(x) = \int_{\mathcal{A}} \mathcal{K}_t(x, y)\psi(y)dy, \quad (4.6)$$

where $t \in G \subset \mathcal{A}$, $x, y \in \mathcal{A}$, and $\psi(x) \in L_2(\mathcal{A})$. Also $U(t) = U_\infty(t_\infty) \prod_p U_p(t_p)$ and

$$\mathcal{K}_t(x, y) = \mathcal{K}_{t_\infty}^{(\infty)}(x_\infty, y_\infty) \prod_p \mathcal{K}_{t_p}^{(p)}(x_p, y_p), \quad (4.7)$$

where $\mathcal{K}_{t_\infty}^{(\infty)}(x_\infty, y_\infty)$ and $\mathcal{K}_{t_p}^{(p)}(x_p, y_p)$ for the harmonic oscillator are given by (3.14) and (3.24). Denoting $\lambda(a) = \lambda_\infty(a_\infty) \prod_p \lambda_p(a_p)$ one can write

$$\mathcal{K}_t(x, y) = \lambda(2 \sin t) |\sin t|^{-\frac{1}{2}} \chi\left(\frac{xy}{\sin t} - \frac{x^2 + y^2}{2 \tan t}\right), \quad (4.8)$$

which resembles the form of real and p -adic kernels. The infinite products (4.7) and (4.8) are divergent, and they represent generalized functions which make definite sense under adelic integration. By virtue of (3.16a), (3.16b) and (3.17a), (3.17b), $U(t)$ and $\mathcal{K}_t(x, y)$ satisfy group relations

$$U(t+t') = U(t)U(t'), \quad (4.9)$$

$$\mathcal{K}_{t+t'}(x, y) = \int_{\mathcal{A}} \mathcal{K}_t(x, z)\mathcal{K}_{t'}(z, y)dz. \quad (4.10)$$

Note that (4.10) does not represent a product of (3.17a), (3.17b) with integration over the whole \mathbb{Q}_p for every p . Such product would be inconsistent with the adelic approach. Adelic integration means that (4.10) consists of

$$\mathcal{K}_{t+t'}^{(p)}(x, y) = \int_{|z|_p \leq 1} \mathcal{K}_t^{(p)}(x, z)\mathcal{K}_{t'}^{(p)}(z, y)dz \quad x, y \in \mathbb{Z}_p, \quad (4.11)$$

for all but a finite number of p . This restricted integration is in a close connection with the vacuum state $\Omega(|x|_p)$. Equation (4.11) can be derived using the Gauss integral [11]

$$\int_{|x|_p \leq 1} \chi_p(\alpha x^2 + \beta x) dx = \lambda_p(\alpha) |2\alpha|_p^{-\frac{1}{2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right) \Omega\left(\left|\frac{\beta}{2\alpha}\right|_p\right), \quad |\alpha|_p > 1. \quad (4.12)$$

The group G for the adelic harmonic oscillator is

$$G = \mathbb{R} \times G_2 \times \cdots \times G_p \times \cdots,$$

where $G_2 = \{t \in \mathbb{Q}_2 : |t|_2 \leq 2^{-2}\}$ and $G_p = \{t \in \mathbb{Q}_p : |t|_p \leq p^{-1}\}$ for $p \neq 2$. One can easily see that an adelic time cannot be a principal idele.

Using (3.18a), (3.18b) and factorization of $U(t)$ and $W(z)$ on real and p -adic parts, one can show that

$$U(t) W(z) = W(T_t z) U(t). \quad (4.13)$$

By virtue of (3.19) and (3.28) the eigenfunctions $\psi_{\alpha\beta}$ of the adelic harmonic oscillator must satisfy the equation

$$U(t) \psi_{\alpha\beta}(x) = \chi(Et) \psi_{\alpha\beta}(x), \quad (4.14)$$

where α and β are adelic indices of the form

$$\alpha = (n, \alpha_2, \cdots, \alpha_p, \cdots), \quad (4.15a)$$

$$\beta = (0, \beta_2, \cdots, \beta_p, \cdots). \quad (4.15b)$$

According to the definition (2.17), it follows that any eigenstate of the adelic harmonic oscillator is

$$\psi_{\alpha\beta}(x) = \frac{2^{\frac{1}{4}}}{(2^n n!)^{\frac{1}{2}}} e^{-\pi x^2} H_n(x\sqrt{2\pi}) \prod_{p \in \Gamma_{\alpha\beta}} \psi_{\alpha_p \beta_p}(x_p) \prod_{p \notin \Gamma_{\alpha\beta}} \Omega(|x_p|_p), \quad (4.16)$$

where $\Gamma_{\alpha\beta}$ is a finite set of the primes for which at least one of the indices α_p and β_p is different from zero. It is obvious that any eigenstate (4.16) is a Schwartz-Bruhat function. Any finite linear combination of these eigenstates is also a Schwartz-Bruhat function. The eigenfunctions (4.16) are orthonormal and any $\psi(x) \in L_2(\mathcal{A})$ can be presented as

$$\psi(x) = \sum C_{\alpha\beta} \psi_{\alpha\beta}(x), \quad (4.17)$$

where $C_{\alpha\beta} = (\psi_{\alpha\beta}, \psi)$ and $\sum |C_{\alpha,\beta}|_{\infty}^2 = 1$.

According to (3.20a), (3.20b) and (3.21a), (3.21b) we define

$$\langle \mathcal{D} \rangle = (\psi, \hat{\mathcal{D}} \psi), \quad \psi \in L_2(\mathcal{A}), \quad (4.18)$$

$$\Delta \mathcal{D} = [(\langle \mathcal{D} - \langle \mathcal{D} \rangle)^2]^{1/2} = (\langle \mathcal{D}^2 \rangle - \langle \mathcal{D} \rangle^2)^{1/2}, \quad (4.19)$$

where $\hat{\mathcal{D}} = \hat{\mathcal{D}}_{\infty} \prod_p \hat{\mathcal{D}}_p$ is the adelic operator of the observable \mathcal{D} . For a normalized state (4.17) we have

$$\langle \mathcal{D} \rangle = \sum |C_{\alpha\beta}|_{\infty}^2 \langle \mathcal{D} \rangle_{\alpha\beta},$$

$$\langle \mathcal{D} \rangle_{\alpha\beta} = (\psi_{\alpha\beta}, \hat{\mathcal{D}} \psi_{\alpha\beta}) = \langle \mathcal{D}_{\infty} \rangle_n \prod_p \langle \mathcal{D}_p \rangle_{\alpha_p \beta_p}. \quad (4.20)$$

Let us denote

$$\psi_{00}(x) = \psi_0^{(\infty)}(x_{\infty}) \prod_p \psi_{00}^{(p)}(x_p) = 2^{\frac{1}{4}} e^{-\pi x_{\infty}^2} \prod_p \Omega(|x_p|_p), \quad (4.21a)$$

$$|x|_{(p_n)}^s = |x_{\infty}|_{\infty}^s \prod_{p=2}^{p_n} |x_p|_p^s, \quad |k|_{(p_n)}^s = |k_{\infty}|_{\infty}^s \prod_{p=2}^{p_n} |k_p|_p^s, \quad (4.22)$$

where $s \in \mathbb{C}$ and p_n is an n th prime. As a consequence of (3.22) and (3.33) the Fourier transform of (4.21a) is

$$\tilde{\psi}_{00}(k) = 2^{\frac{1}{4}} e^{-\pi k_{\infty}^2} \prod_p \Omega(|k_p|_p). \quad (4.21b)$$

Expectation values of (4.22) in the simplest vacuum state ψ_{00} are

$$\langle |x|_{(p_n)}^s \rangle = \langle |k|_{(p_n)}^s \rangle = \sqrt{2} \Gamma\left(\frac{s+1}{2}\right) (2\pi)^{-\frac{s+1}{2}} \prod_{p=2}^{p_n} \frac{1-p^{-1}}{1-p^{-s-1}}. \quad (4.23)$$

For the corresponding ($s=1$) $\Delta|x|_{(p_n)}$ and $\Delta|k|_{(p_n)}$ one gets

$$\Delta|x|_{(p_n)} = \Delta|k|_{(p_n)} = \left(\frac{1}{4\pi} \prod_{p=2}^{p_n} \frac{1-p^{-1}}{1-p^{-3}}\right)^{\frac{1}{2}} \left[1 - \frac{2}{\pi} \prod_{p=2}^{p_n} \frac{(1-p^{-1})(1-p^{-3})}{(1-p^{-2})^2}\right]^{\frac{1}{2}}. \quad (4.24)$$

When $p_n \rightarrow \infty$ one obtains

$$\langle |x|^s \rangle = \langle |k|^s \rangle = \Delta|x| = \Delta|k| = 0. \quad (4.25)$$

Note that the vacuum state $\psi_{00}(x)$ [Eq. (4.21a)] is the simplest elementary function (2.8) defined on adèles. Applying the Mellin transform (2.11) to $\psi_{00}(x)$ one gets

$$\Phi(s) = \sqrt{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s). \quad (4.26)$$

Since $\tilde{\psi}_{00} = \psi_{00}$ it follows that $\tilde{\Phi}(s) = \Phi(s)$. If we replace Φ by $\tilde{\Phi}$ in the Tate formula (2.12) by (4.26), there appears the well-known functional relation for the Riemann zeta function:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \quad (4.27)$$

The expectation value of $U(t)$ in an adelic eigenstate (4.16) is equal to its eigenvalue:

$$U(t) = \chi(Et) = e^{2\pi i\{Et\}} \quad (4.28)$$

where

$$\{Et\} = -E_\infty t_\infty + \sum_p \{E_p t_p\}_p \quad (4.29)$$

is an adelic dynamical phase.

5. Discussion and Concluding Remarks

For one-dimensional dynamical systems, whose classical time evolution leaves invariant the symplectic bilinear form (3.3), we have formulated adelic quantum mechanics. As an illustration we considered the corresponding model of a harmonic oscillator. Let us discuss this model and some general features of quantum mechanics on adèles.

The adelic harmonic oscillator is a natural illustration of mathematical analysis over adèles. It is a simple, exact and instructive adelic model. All eigenstates, as well as all their finite linear combinations, are Schwartz-Bruhat functions. The simplest vacuum state is also the simplest elementary function on adèles and its form is invariant under the Fourier transformation. The Mellin transform of this state is the same function on configuration and momentum space.

Recall that ordinary and p -adic quantum mechanics are theories on real and p -adic spaces, respectively. Adelic quantum mechanics may be regarded as a theory on adelic space \mathcal{A} .

One may generalize the concept of real matter, and introduce p -adic and adelic matter, as well. Real and p -adic particles belong to different (real and p -adic) parts of the adelic space and they do not mutually interact. They are distinct low energy limits of the adelic matter which exists at very high energies. Adelic particles are unstable and decay to real and p -adic ones, which belong to their own spaces.

According to the above assumption adelic quantum theory for real particles should be equivalent to ordinary quantum theory. This is really the case for the harmonic oscillator. For example, according to (4.21a), (4.21b) and $\Omega(|0|_p) = 1$, the simplest adelic vacuum state for a real harmonic oscillator $x_p = 0$ becomes the corresponding vacuum state within ordinary quantum mechanics.

One might get impression that in solving the adelic problem one has to collect real and p -adic solutions and take their product. It is not quite correct. Namely, when adelic solution is a product it is a very restricted one. One has also to take care of convergence and adelic consistence. Moreover, there are solutions which are not a product of real and p -adic parts [see e.g. (4.24)].

By virtue of the above discussion the adelic harmonic oscillator is a system of two particles whose interaction in \mathcal{A} is presented by the potential

$$V(x) = \left(\frac{x_\infty^2}{2}, \frac{x_2^2}{2}, \dots, \frac{x_p^2}{2}, \dots \right), \quad m = \omega = 1.$$

Uncertainty relations in the simplest vacuum state are given by (3.23c), (3.23d) for the real coordinates, and by (3.34b) for the p -adic ones. We also calculated some expectation values of the product of norms for position and momentum.

Finally, the above nonrelativistic mode may be regarded as a first step towards a more profound relativistic adelic quantum theory.

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