

On the limiting procedure by which $SDiff(T^2)$ and $SU(\infty)$ are associated

John Swain

Department of Physics, Northeastern University, Boston, MA 02115, USA

email: john.swain@cern.ch

(April 29, 2004)

ABSTRACT

There have been various attempts to identify groups of area-preserving diffeomorphisms of 2-dimensional manifolds with limits of $SU(N)$ as $N \rightarrow \infty$. We discuss the particularly simple case where the manifold concerned is the two-dimensional torus T^2 and argue that the limit, even in the basis commonly used, is ill-behaved and that the large- N limit of $SU(N)$ is much larger than $SDiff(T^2)$.

I. INTRODUCTION

Groups of area-preserving diffeomorphisms and their Lie algebras have recently been the focus of much attention in the physics literature. Hoppe [1] has shown that in a suitable basis, the Lie algebra of the group $SDiff(S^2)$ of area-preserving diffeomorphisms of a sphere tends to that of $SU(N)$ as $N \rightarrow \infty$. Similar arguments have been made associating various infinite limits of Lie algebras of classical groups with Lie algebras of groups of area-preserving

diffeomorphisms of 2-dimensional surfaces. This has obvious interest in connection with gauge theories of $SU(N)$ for large N . The use of $SU(N)$ for finite N as an approximation to groups of area-preserving diffeomorphisms has also been used in studies of supermembranes [2–4] and in particular has been used to argue for their instability. The authors of references [3] and [4] have especially emphasized the difficulties in relating such infinite limits with Lie algebras of area-preserving diffeomorphisms. Various authors have considered special limits and/or large- N limits of other classical Lie algebras [6–10] as relevant for 2-manifolds other than spheres. The purpose of this Letter is to clarify the nature of the limiting procedure by which $SU(\infty)$ has been related to $SDiff(T^2)$.

II. THE LIE ALGEBRAS OF $SDIFF(T^2)$

We follow here the treatment of [7], which is particularly clear. The torus T^2 is represented by the plane \mathbb{R}^2 with coordinates x and y and the identifications

$$(x, y) = (x + 2\pi, y) \tag{1}$$

and

$$(x, y) = (x, y + 2\pi) \tag{2}$$

A basis for functions on the torus is chosen as

$$Y_{mn}(x, y) = \exp[i(mx + ny)] \tag{3}$$

with m, n running over all integers. The local area-preserving diffeomorphisms are then generated by the vector fields

$$L_{mn} = (\epsilon^{ab} \partial_b Y_{mn}) \partial_a = i \exp[i(mx + ny)] (n \partial_x - m \partial_y) \quad (4)$$

with indices $a, b = 1, 2$. In other words, the divergence-free vector fields are those which are the curl of something else.

The generators clearly close under commutation, with the commutator

$$[L_{mn}, L_{m',n'}] = (mn' - m'n) L_{m+m',n+n'} \quad (5)$$

III. THE LIE ALGEBRA OF $SU(N)$

To construct the Lie algebra of $SU(N)$, again following [7], we sketch the basic idea. Fix a positive integer N and a complex number ω such that $\omega^N = 1$ but $\omega^r \neq 1$ for $0 < r < N$. ω is called a primitive root of unity. Then we have $\omega = \exp(2\pi i k/N)$ for some k relatively prime to N . Now we find unitary, traceless matrices g and h such that

$$hg = \omega gh \quad (6)$$

Then the set of matrices

$$J_{m,n} = \omega^{mn/2} g^m h^n \quad (7)$$

for $0 \leq m, n < N$ are linearly independent and are a basis for the $N \times N$ matrices. $J_{0,0} = 1$, and all the other $J_{m,n}$ are traceless and satisfy $J_{m,n}^\dagger = J_{-m,-n}$. Leaving out $J_{0,0}$, the scaled matrices $J'_{m,n} = iN/(2k\pi) J_{m,n}$ generate $SU(N)$ with the commutation relations

$$[J'_{m,n}, J'_{m',n'}] = \frac{N}{k\pi} \sin\left(\frac{k\pi}{N}(mn' - m'n)\right) J'_{m+m',n+n'} \quad (8)$$

IV. THE $N \rightarrow \infty$ LIMIT

The claim now is that in the limit $N \rightarrow \infty$ that the commutation relations in equation III go over to those in equation II. Naively, of course, one would like to argue that as $N \rightarrow \infty$,

$$\frac{N}{k\pi} \sin \left(\frac{k\pi}{N} (mn' - m'n) \right) = (mn' - m'n) + O(1/N^2) \quad (9)$$

and drop the terms of order $1/N^2$ and higher. However, let us keep the next term and examine whether or not it can indeed be taken to be small.

$$\frac{N}{k\pi} \sin \left(\frac{k\pi}{N} (mn' - m'n) \right) = (mn' - m'n) - \frac{1}{3!} \frac{(k\pi)^2}{N^2} (mn' - m'n)^3 + \dots \quad (10)$$

Now consider any choice of $(m, n) = (N/a, 0)$ and $(m', n') = (0, N/b)$ where a and b are arbitrary integers that divide N (including one). Then

$$\frac{(k\pi)^2}{N^2} (mn' - m'n)^3 = \frac{(k\pi)^2}{a^3 b^3} N^4 \quad (11)$$

which is clearly *not* negligible as $N \rightarrow \infty$. It would seem that there are many elements of the Lie algebra of $SU(N)$ which do not belong to $SDiff(T^2)$.

This is in keeping with ideas raised in [11] suggesting that $SU(\infty)$ is much larger than the group of area-preserving diffeomorphisms of a surface, and perhaps describes some sort of theory including topology change. Other work demonstrating that topologically, $SDiff(T^2)$, and indeed all the area-preserving diffeomorphism groups, are inequivalent to $SU(\infty)$ is in [12].

V. ACKNOWLEDGEMENT

The author would like to my colleagues at Northeastern University and the National Science Foundation for their support.

-
- [1] J. Hoppe, Proceedings, Constraints Theory and Relativistic Dynamics (Florence, 1986) eds. G. Longhi and L Lusana (World Scientific, Singapore, 1987) p. 267; Phys. Lett. **B215** (1988) 706.
- [2] B. deWit, J. Hoppe, and H. Nicolai, Nucl. Phys. **B305** (1988) 545.
- [3] B. deWit, M. Lüscher, and H. Nicolai, Nucl. Phys. **B320** (1989) 133.
- [4] B. de Wit and H. Nicolai, Proceedings, 3rd Hellenic School on Elementary Particle Physics (Corfu, 1989) eds. E. N. Argyres et al. (World Scientific, Singapore) p777.
- [5] J. Hoppe and P. Schaller, Phys. Lett. **B237** (1990) 407.
- [6] C. N. Pope and K. S. Stelle, Phys. Lett. **B226** (1989) 257.
- [7] C. N. Pope and L. J. Romans, Class. Quantum. Grav. **7** (1990) 97.
- [8] A. Wolski and J. S. Dowker, J. Math. Phys. **32** (1991) 2304.
- [9] B. deWit, U. Marquard, and H. Nicolai, Comm. Math. Phys. **128** (1990) 39.
- [10] D. V. Vassilevich, Class. Quantum. Grav. **8** (1991) 2163.
- [11] J. Swain, “The Majorana representation of spins and the relation between $SU(\infty)$ and $SDiff(S^2)$ ”, <http://arXiv.org/abs/hep-th/0405004>.
- [12] J. Swain, “The Topology of $SU(\infty)$ and the Group of Area-Preserving Diffeomorphisms of a Compact 2-manifold”, <http://arXiv.org/abs/hep-th/0405003>.