Magnetic Interactions

in

Relativistic Two-Particle Systems

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Abstract

The magnetic interactions of the two electrons in helium-like ions are studied in detail within the framework of Relativistic Schrödinger Theory (RST). The general results are used to compute the ground-state interaction energy of some highly-ionized atoms ranging from germanium (Z = 32) up to bismuth (Z = 83). When the magnetic interaction energy is added to its electric counterpart resulting from the electrostatic approximation, the present RST predictions reach a similar degree of precision (relative to the experimental data) as the other theoretical approaches known in the literature. However since the RST magnetism is then treated only in lowest-order approximation, further improvements of the RST predictions seem possible.

I. INTRODUCTION AND SURVEY OF RESULTS

The overwhelming success of relativistic quantum field theory (especially quantum electrodynamics) could tempt one to belief that any relativistic form of quantum theory must necessarily include the virtual degrees of freedom of matter. Indeed it is well-known that the virtual processes show up experimentally in form of radiation corrections, vacuum polarization, pair creation and annihilation etc. [1]. If such a viewpoint would be true, it would logically not be possible to construct a consistent relativistic quantum mechanics for interacting N-particle systems whose non relativistic description presents no problem at all [2]. However it seems that there actually exists no convincing argument why such a relativistic quantum mechanics for systems of fixed particle number N could not exist. Quite on the contrary, since the virtual processes of relativistic field theory (e.g. the Lamb shift) induce only small corrections of the results obtained from the first-quantized quantum mechanics theories, one concludes that a consistent relativistic quantum mechanics would correctly describe the main relativistic effects occurring in the many-particle systems (e.g. heavy atoms) but would leave the small corrections due to the virtual processes to the impact of quantum field theory.

In this context, one could now think that the desired relativistic quantum mechanics should be deduced as a certain kind of limit process from quantum field theory itself, namely through fixing the particle number by neglection of just the virtual processes. Such a deduction has been tried already long ago by Bethe and Salpeter [3, 4] but the resulting N-particle wave equations are plagued by many deficiencies and interpretation problems [5, 6]. Therefore a new approach to relativistic quantum mechanics for many-particle systems has recently been established in form of Relativistic Schrödinger Theory (RST) [7, 8, 9, 10]. This approach is a relativistic gauge theory based upon the gauge group U(N). The gauge group is broken down to the abelian subgroup U(1)×U(1)...×U(1) which then describes the electromagnetic interaction of N particles, whereas the frozen gauge degrees of freedom are responsible for the exchange interactions. The non-relativistic limit of this theory coincides with the well-known Hartree-Fock approach [11].

The present paper presents a test of the practical usefulness of RST, namely by elaborating its predictions for the ground-state interaction energy of the two electrons in highlyionized helium-like atoms where relativistic effects play a non-negligible role. In contrast to the two-particle ground-state energy itself, the corresponding *interaction* energy is directly accessible to the experiments [12] which thus provide sufficient data for testing various theoretical approaches: relativistic 1/Z expansion [13] multiconfiguration, Dirac-Fock method (MCDF) [14, 15, 16], relativistic many-body perturbation theory (MBPT) [17, 18], all-order technique for relativistic MBPT [19]. These four theoretical approaches have been chosen to be opposed to the experimental data; and the present paper adds now the corresponding RST results which are found to describe the experimental data with a similar degree of precision as these other theoretical approaches (table II). This success is attained by going beyond the electrostatic approximation ([20]) and taking into account also the magnetic interactions which implies an improvement of the RST predictions by (roughly) one order of magnitude. However one should observe here that these RST results have been obtained in a preliminary way by a very rough approximation technique, namely by neglecting the non-abelian character of RST through linearizing the gauge field equations (Sect. 4B) and furthermore by treating the magnetic interactions only in their non-relativistic limit. Thus the potentiality of RST is not yet fully exploited and further improvements of the RST predictions seem still possible.

The procedure is as follows: In Sect. II we present a brief survey of RST in order that the main arguments of the subsequent discussions can be understood without looking up the whole development of the theory in the preceding papers. Here the emphasis is laid upon the construction of an energy functional $E_{\rm T}$, namely as the integral (over all 3-space) of an energy density $T_{00}(\vec{r})$, see equation (2.44) below. Clearly the reason is that, with such a functional at hand, the energy level system of the bound N-particle systems may be determined as the value of that functional $E_{\rm T}$ upon the stationary bound solutions of the RST field equations.

This kind of solutions is then described in great detail in Sect. III with an explicit presentation of the mass-eigenvalue equations (see equations (3.4a)-(3.4b) below). Since this eigenvalue problem is too difficult to be solved exactly, one has to resort to appropriate approximation techniques (as is mostly the case in quantum field theory). Here the logical structure of RST suggests to first neglect the *magnetic* interactions between the particles (*"electrostatic approximation"*) which leaves us with a simpler eigenvalue problem (see equations (3.6a)-(3.6b) below). This truncated system is then solved numerically for the ground state of the two-particle system in the Coulomb field ^(ex) $A_0(\vec{r})$ (3.9) where the nuclear charge numbers $(z_{\rm ex})$ range from $z_{\rm ex} = 32$ (germanium) up to $z_{\rm ex} = 83$ (bismuth). The corresponding ground-state interaction energy $\Delta E_{\rm RST}^{(e)}$ (in the electrostatic approximation) is then compared to the corresponding experimental values $\Delta E_{\rm exp}$ (see table I); and it is found that there is a discrepancy between the theoretical (RST) and experimental values extending from 1.7 eV for germanium up to 11.5 eV for bismuth. However some intuitive arguments indicate that the observed discrepancy should actually be due to the neglection of the magnetic forces. When the latter forces are taken into account, there arises a structure equation which specifies the desired interaction energy ΔE in terms of the electromagnetic coupling constant ($z_{\rm ex}\alpha_{\rm S}$) and two functions ε_* and f_*^2 which are only weakly depending upon that coupling constant, see equation (3.59) and the last two columns of table I.

In Sect. IV the hypothesis of the magnetic origin of the discrepancy between the electrostatic RST predictions $\Delta E_{\rm RST}^{(e)}$ and the experimental data $\Delta E_{\rm exp}$ is inspected very thoroughly by working out the RST field theory of atomic magnetism. Here it seems reasonable to consider first the magnetic effects in the lowest-order approximation, i.e. we linearize the non-abelian gauge field equations and additionally we resort to the non-relativistic limit of the electrostatic wave functions for calculating the magnetic energy contributions. It is very striking to observe that the magnetic and electric contributions display certain dissimilarities which can however be revealed as necessary consequences of the principle of minimal coupling and Lorentz invariance of the theory. Though one obtains a very plausible result for the magnetic energy contribution $\Delta E_{\rm T}^{(\rm mg)}$, see equation (4.52) below, one nevertheless may wish to become convinced by an independent argument which supports our claim that magnetism is correctly incorporated RST by the present approach.

And indeed, such an additional argument in favor of the RST picture of atomic magnetism can be supplied, namely by considering the interaction of the bound particles with an *external* magnetic field \vec{H}_{ex} (Sect. V). Here it can be shown that the interaction energy between the particles and the external magnetic field \vec{H}_{ex} exactly agrees with the conventional results for the Zeeman effects; i.e. the magnetic RST energy coincides with the expectation value of the conventional Zeeman Hamiltonian, see equation (5.2) below.

Being thus convinced of the physical correctness of the RST picture of atomic magnetism, one can return to the hypothesis of the magnetic origin of the numerical gap between the electrostatic RST predictions $\Delta E_{\text{RST}}^{(e)}$ and the experimental data ΔE_{exp} of table I. In Sect. VI we apply the general results of Sect. IV in order to once more calculate the ground-state interaction energy for the helium-like ions from $z_{\rm ex} = 32$ (germanium) up to $z_{\rm ex} = 83$ (bismuth), but now with inclusion of the magnetic interactions. Here we are satisfied for the moment with their *linearized* description in the *non-relativistic* limit. Amazingly enough, this simple treatment of atomic magnetism is sufficient in order to close the gap between the RST predictions $\Delta E_{\rm RST}^{(e)}$ and the experimental data $\Delta E_{\rm exp}$ up to less than 0.5%, see table II in comparison to table I. This must be considered as a rather convincing argument in favor of the "magnetic" hypothesis for the observed discrepancy between $\Delta E_{\text{BST}}^{(e)}$ and ΔE_{exp} . It is also very satisfying to observe that, with the inclusion of the magnetic effects, the corresponding RST predictions $\Delta E_{\rm RST}^{\rm (emg)}$ are now as close to the experimental values $\Delta E_{\rm exp}$ as is the case with the other approximation methods known in the literature: Relativistic Many-Body Perturbation Theory [17, 18], all-order technique for MBPT [19], Multi-Configuration Dirac-Fock method (MCDF) [14, 15, 16] and relativistic 1/Z expansion [13], see table II. (The predictions of these theoretical approaches have been listed in table III of ref. [12]. Clearly one expects that the RST predictions will be further improved by taking into account also the *relativistic* effects of atomic magnetism and retaining the non-linear terms due to the non-abelian character of the gauge group U(N) (separate paper).

Finally it should be stressed that the numerical success of the magnetic hypothesis is mainly due to the application of the non-abelian U(2). The reason is that after the "abelian" symmetry-breaking" (Sect. 2) the frozen gauge degrees of freedom of U(2) imply the existence of a "magnetic" exchange vector potential $\vec{B}(\vec{r})$ which plays an analogous part with respect to the x- and y-axis as the magnetostatic potentials $\vec{A}_a(\vec{r})$ with respect to the z-axis of the coordinate system. Indeed when the spins of the two ground-state electrons are oriented along the z-axis, the corresponding spin-spin interaction energy $\Delta E_{\rm T}^{(z)}$ (see equation (6.39) below) is well suited in order to contribute to closing the gap between $\Delta E_{\rm RST}^{(e)}$ and $\Delta E_{\rm exp}$, however only one third of the gap could be closed in this way. The other two thirds of the gap must be filled by the spin-spin interaction energies $\Delta E_{\rm T}^{({\rm x})}$ (6.41a) and $\Delta E_{\rm T}^{({\rm y})}$ (6.41b) due to the spin orientation along the x- and y-axis (see fig. 1). But the latter interactions are described by just that exchange vector potential $B(\vec{r})$ being due to the frozen gauge degrees of freedom of U(2). Thus each axis of spin orientation contributes the same magnetic energy, and this is nothing else than a demonstration of the *isotropic* geometry of the ground-state. In this sense the choice of the non-abelian U(2) is seen to be necessary just in order to guarantee the isotropy of the ground state.

II. RELATIVISTIC SCHRÖDINGER THEORY

In order to let the subsequent elaboration of arguments appear sufficiently self - contained, we first present a brief sketch of the **R**elativistic **S**chrödinger **T**heory (RST), which places emphasis on a closer inspection of the energy functional, since this will ultimately yield the energy level system of the bound many - particle systems. For more details and deductions, the interested reader is referred to the preceding papers, e.g. refs. ([7])-([11]).

A. RST Dynamics

As for any field theory of matter, the fundamentals of RST consist in a basic system of field equations, which is subdivided into three coupled subystems: matter dynamics, Hamiltonian dynamics and gauge field dynamics.

(i) Matter Dynamics

When matter occurs in form of pure state Ψ , its distribution over space-time is governed by the **R**elativistic Schrödinger Equation (RSE)

$$i\hbar c\mathcal{D}_{\mu}\Psi = \mathcal{H}_{\mu}\Psi$$
 (2.1)

If it is more adequate to describe matter by a mixture, being characterized by an intensity matrix \mathcal{I} , one requires \mathcal{I} to obey the **R**elativistic von **N**eumann **E**quation (RNE)

$$\mathcal{D}_{\mu}\mathcal{I} = \frac{i}{\hbar c} \left[\mathcal{I} \cdot \bar{\mathcal{H}}_{\mu} - \mathcal{H}_{\mu} \cdot \mathcal{I} \right] .$$
(2.2)

Clearly, a pure state Ψ can be considered as a special type of mixture, namely that for which the intensity matrix degenerates to the tensor product of Ψ and its Hermitian conjugate $\overline{\Psi}$, i.e.

$$\mathcal{I} \Rightarrow \Psi \otimes \bar{\Psi} . \tag{2.3}$$

In the present paper we are restricting ourselves to the investigation of stationary bound systems being described by pure states. Since a pure state Ψ of an N-fermion system has 4N components, we have to consider a \mathbb{C}^{4N} -realization of RST.

(ii) Hamiltonian Dynamics

The Hamiltonian \mathcal{H}_{μ} , occuring in the matter field equations (2.1) and (2.2), is itself a dynamical object obeying its own field equations, namely the *integrability condition*

$$\mathcal{D}_{\mu}\mathcal{H}_{\nu} - \mathcal{D}_{\nu}\mathcal{H}_{\mu} + \frac{i}{\hbar c}\left[\mathcal{H}_{\mu}, \mathcal{H}_{\nu}\right] = i\hbar c\mathcal{F}_{\mu\nu}$$
(2.4)

and the *conservation equation* which reads for fermions

$$\mathcal{D}^{\mu}\mathcal{H}_{\mu} - \frac{i}{\hbar c}\mathcal{H}^{\mu} \cdot \mathcal{H}_{\mu} = i\hbar c \left(\frac{Mc}{\hbar}\right)^{2} \cdot \mathbf{1} - i\hbar c \Sigma^{\mu\nu}\mathcal{F}_{\mu\nu} .$$
(2.5)

For bosons the spin term (last term on the right-hand side) is omitted. The meaning of the integrability condition (2.4) is to ensure the validity of the bundle identities

$$\left[\mathcal{D}_{\mu}\mathcal{D}_{\nu}-\mathcal{D}_{\nu}\mathcal{D}_{\mu}\right]\Psi=\mathcal{F}_{\mu\nu}\cdot\Psi\tag{2.6a}$$

$$\left[\mathcal{D}_{\mu}\mathcal{D}_{\nu}-\mathcal{D}_{\nu}\mathcal{D}_{\mu}\right]\mathcal{I}=\left[\mathcal{F}_{\mu\nu},\mathcal{I}\right] .$$
(2.6b)

Similarly, the conservation equation (2.5) guarantees the existence of certain conservation laws, e.g. the charge conservation

$$\nabla^{\mu} j_{\mu} \equiv 0 , \qquad (2.7)$$

or the energy-momentum conservation for free particles (Dirac particles here)

$$\nabla^{\mu(S)} T_{\mu\nu} = 0 , \qquad (2.8)$$

whose energy-momentum density is denoted by ${}^{(S)}T_{\mu\nu}$.

Clearly, if the considered N-particle system is interacting with an external field ${}^{(ex)}\mathcal{F}_{\mu\nu}$

$${}^{(\text{ex})}\mathcal{F}_{\mu\nu} = -i {}^{(\text{ex})}F_{\mu\nu} \cdot \mathbf{1} \equiv \mathcal{F}_{\mu\nu} - {}^{(S)}\mathcal{F}_{\mu\nu} , \qquad (2.9)$$

the energy-momentum density ${}^{(S)}T_{\mu\nu}$ cannot obey the conservation law (2.8) without being complemented by the energy-momentum density ${}^{(es)}T_{\mu\nu}$ due to the external interaction. Consequently, the conservation law (2.8) has to be replaced by the balance equation

$$\nabla^{\mu} \left({}^{(S)}T_{\mu\nu} + {}^{(es)}T_{\mu\nu} \right) = -{}^{(xe)}f_{\nu} , \qquad (2.10)$$

where the Lorentz - force density ${}^{(xe)}f_{\nu}$ is given by

$${}^{(\mathrm{xe})}f_{\nu} = -\hbar c F_{\mu\nu}{}^{(\mathrm{ex})}j^{\mu} . \qquad (2.11)$$

Here, ${}^{(ex)}j_{\mu}$ is the external four-current generating the external field ${}^{(ex)}F_{\mu\nu}$ according to Maxwell's equations

$$\nabla^{\mu(ex)}F_{\mu\nu} = 4\pi\alpha_{\rm S}^{(ex)}j_{\nu} \qquad (2.12)$$
$$\left(\alpha_{\rm S} = \frac{e^2}{\hbar c}\right),$$

and $F_{\mu\nu}$ is the coherent electromagnetic field strength generated by all the particles of the system:

$$F_{\mu\nu} = \frac{i}{N-1} \operatorname{tr}\left\{{}^{(S)}\mathcal{F}_{\mu\nu}\right\} . \qquad (2.13)$$

This means that the source of this total field $F_{\mu\nu}$ is the total current j_{μ} of the N-particle system

$$\nabla^{\mu}F_{\mu\nu} = -4\pi\alpha_{\rm S}j_{\nu} , \qquad (2.14)$$

which obeys the strict conservation law (2.7). If one prefers to think of a closed system, one adds the energy-momentum density ${}^{(ex)}T_{\mu\nu}$ of the external source

$$\nabla^{\mu(\mathrm{ex})}T_{\mu\nu} = {}^{(\mathrm{xe})}f_{\nu} \tag{2.15}$$

and then finds the desired closedness relation by combining equations (2.10) and (2.15)

$$\nabla^{\mu} \left({}^{(S)}T_{\mu\nu} + {}^{(es)}T_{\mu\nu} + {}^{(es)}T_{\mu\nu} \right) = 0 . \qquad (2.16)$$

Finally, let us also remark that the conservation equation (2.5) is equivalent to the following condition upon the Hamiltonian:

$$\Gamma^{\mu} \cdot \mathcal{H}_{\mu} = Mc^2 \mathbf{1} . \tag{2.17}$$

Here, the total velocity operator Γ^{μ} generates a (4N)-dimensional representation of the Clifford algebra C(1,3), i.e.

$$\Gamma_{\mu} \cdot \Gamma_{\nu} + \Gamma_{\nu} \cdot \Gamma_{\mu} = 2g_{\mu\nu} \cdot \mathbf{1} , \qquad (2.18)$$

with the corresponding generators $\Sigma_{\mu\nu}$ of the group Spin(1,3) being given by

$$\Sigma_{\mu\nu} = \frac{1}{4} \left[\Gamma_{\mu}, \Gamma_{\nu} \right] , \qquad (2.19)$$

see the right-hand side of the conservation equation 2.5. The physical meaning of Γ_{μ} is to build up the *total current* j_{μ} according to

$$j_{\mu} \doteq \operatorname{tr} \left\{ \mathcal{I} \cdot \Gamma_{\mu} \right\} , \qquad (2.20)$$

i.e. for a pure state (2.3)

$$j_{\mu} = \bar{\Psi} \cdot \Gamma_{\mu} \cdot \Psi . \qquad (2.21)$$

Since by means of the latter form (2.17) of the conservation equation the RSE (2.1) can be converted to the N-particle Dirac equation

$$i\hbar\Gamma^{\mu}\mathcal{D}_{\mu}\Psi = Mc\Psi , \qquad (2.22)$$

it thus becomes a simple matter to verify directly the charge conservation law (2.7). This form of the charge conservation does not only work for pure states but equally well for mixtures, which can be proven by resorting to the RNE (2.2) instead of the RSE (2.1). Observe also that the non-relativistic limit of the N-particle Dirac equation (2.22) coincides with the well-known Hartree-Fock equations [11].

(iii) Gauge Field Dynamics

In order to have a closed dynamical system, one finally has to specify a field equation for the bundle curvature ("field strength") $\mathcal{F}_{\mu\nu}$. This is the non-abelian Maxwell equation

$$\mathcal{D}^{\mu}\mathcal{F}_{\mu\nu} = -4\pi i\alpha_{\rm S}\mathcal{J}_{\nu} \ . \tag{2.23}$$

Similarly, as it was done for the field strenght $\mathcal{F}_{\mu\nu}$ (2.9), the current operator \mathcal{J}_{μ} may also be split into an external and a system part

$$\mathcal{J}_{\mu} = {}^{(\mathrm{ex})}\mathcal{J}_{\mu} + {}^{(\mathrm{S})}\mathcal{J}_{\mu} \tag{2.24a}$$

$${}^{(\text{ex})}\mathcal{J}_{\mu} = {}^{(\text{ex})}j_{\mu} \cdot \mathbf{1}$$
(2.24b)

so that the non-abelian Maxwell equation (2.23) can be required to decay analogously into two parts

$$\mathcal{D}^{\mu(\mathrm{ex})}\mathcal{F}_{\mu\nu} = -4\pi i\alpha_{\mathrm{S}}^{(\mathrm{ex})}\mathcal{J}_{\nu} \tag{2.25a}$$

$$\mathcal{D}^{\mu(S)}\mathcal{F}_{\mu\nu} = -4\pi i\alpha_{\rm S}{}^{\rm (S)}\mathcal{J}_{\nu} . \qquad (2.25b)$$

Here the external part (2.25a) gives the former (abelian) Maxwell equation (2.12), whereas the "total" Maxwell equation (2.14) emerges as the trace part of (2.25b) with

$$j_{\mu} = -\frac{1}{N-1} \operatorname{tr} \{\mathcal{J}_{\mu}\}$$
 (2.26)

The essential point with the gauge structure of RST refers to the right choice of the gauge group. In conventional quantum (field) theory, the group U(1) is evoked in order to describe the electromagnetic interactions. However, in RST the N-particle bundle of wave functions Ψ is the Whitney sum of the corresponding single - particle bundles which suggests to adopt the N-fold product group $U(1) \times U(1) \times \ldots \times U(1)$ as the adequate subgroup of U(N) for the description of the RST gauge interactions. Though the *electromagnetic* interactions are thus adequately incorporated into RST by means of this "abelian symmetry breaking", one needs the residual structure of the embedding group U(N) in order to take into account also the *exchange* interactions between the N particles.

Considering for instance the two-fermion case (N = 2), one has to deal with two "electromagnetic generators" τ_a (a = 1, 2) and two "exchange generators" $\chi, \bar{\chi}$ in order to span the gauge algebra $\mathfrak{u}(2)$. The bundle connection ("gauge potential") \mathcal{A}_{μ} may then again be splitted up into an external (ex) and internal system part (S)

$$\mathcal{A}_{\mu} = {}^{(ex)}\mathcal{A}_{\mu} + {}^{(S)}\mathcal{A}_{\mu} \tag{2.27a}$$

$${}^{(ex)}\mathcal{A}_{\mu} = -i{}^{(ex)}\mathcal{A}_{\mu} \cdot \mathbf{1} , \qquad (2.27b)$$

where the internal part ${}^{(S)}\mathcal{A}_{\mu}$ decomposes with respect to the $\mathfrak{u}(2)$ basis $\{\tau_a, \chi, \bar{\chi}\}$ as follows

$${}^{(S)}\mathcal{A}_{\mu} = A^{a}{}_{\mu}\tau_{a} + B_{\mu}\chi - B^{*}_{\mu}\bar{\chi} . \qquad (2.28)$$

Clearly, the connection \mathcal{A}_{μ} generates its curvature $\mathcal{F}_{\mu\nu}$ as usual

$$\mathcal{F}_{\mu\nu} = \nabla_{\mu}\mathcal{A}_{\nu} - \nabla_{\nu}\mathcal{A}_{\mu} + [\mathcal{A}_{\mu}, \mathcal{A}_{\nu}] , \qquad (2.29)$$

and by decomposing the internal part ${}^{(S)}\mathcal{F}_{\mu\nu}$ (2.9) in a similar way

$${}^{(S)}\mathcal{F}_{\mu\nu} = F^{a}_{\ \mu\nu}\tau_{a} + G_{\mu\nu}\chi - G^{*}_{\mu\nu}\bar{\chi} , \qquad (2.30)$$

the curvature components $F^a_{\ \mu\nu}, G_{\mu\nu}$ are found in terms of the connection components $A^a_{\ \mu}, B_\mu$ as follows:

$${}^{(\text{ex})}F_{\mu\nu} = \nabla_{\mu}{}^{(\text{ex})}A_{\nu} - \nabla_{\nu}{}^{(\text{ex})}A_{\mu}$$
(2.31a)

$$F^{1}_{\ \mu\nu} = \nabla_{\mu}A^{1}_{\ \nu} - \nabla_{\nu}A^{1}_{\ \mu} + i\left[B_{\mu}B^{*}_{\nu} - B_{\nu}B^{*}_{\mu}\right]$$
(2.31b)

$$F^{2}_{\ \mu\nu} = \nabla_{\mu}A^{2}_{\ \nu} - \nabla_{\nu}A^{2}_{\ \mu} - i\left[B_{\mu}B^{*}_{\nu} - B_{\nu}B^{*}_{\mu}\right]$$
(2.31c)

$$G_{\mu\nu} = \nabla_{\mu}B_{\nu} - \nabla_{\nu}B_{\mu} + i\left[A^{1}_{\ \mu} - A^{2}_{\ \mu}\right] \cdot B_{\nu} - i\left[A^{1}_{\ \nu} - A^{2}_{\ \nu}\right] \cdot B_{\mu}$$
(2.31d)

$$G_{\mu\nu}^{*} = \nabla_{\mu}B_{\nu}^{*} - \nabla_{\nu}B_{\mu}^{*} - i\left[A_{\ \mu}^{1} - A_{\ \mu}^{2}\right] \cdot B_{\nu}^{*} + i\left[A_{\ \nu}^{1} - A_{\ \nu}^{2}\right] \cdot B_{\mu}^{*} .$$
(2.31e)

The total field strength $F_{\mu\nu}$ (2.13) appears now as the sum of the "electromagnetic" field strengths $F^{a}_{\mu\nu}$ (2.31b)-(2.31c), i.e.

$$F_{\mu\nu} = F^{1}_{\ \mu\nu} + F^{2}_{\ \mu\nu} , \qquad (2.32)$$

and therefore $F_{\mu\nu}$ is generated by the "total potential" A_{μ}

$$A_{\mu} = A^{1}{}_{\mu} + A^{2}{}_{\mu} \tag{2.33}$$

in a strictly abelian way

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} . \qquad (2.34)$$

Recalling also the abelian character of the "total" Maxwell equations (2.14) we see that the total objects of the system $A_{\mu}, F_{\mu\nu}, j_{\mu}$ obey a strictly abelian structure due to the "total" group U(1), just as it is the case in conventional (quantum) electrodynamics.

In contrast to these "total" properties, describing the N-particle system as a whole, the "internal" gauge structure is truly non-abelian on account of the exchange interactions. More concretely, when the current operator ^(S) \mathcal{J}_{μ} (2.24a) is decomposed as

$${}^{(S)}\mathcal{J}_{\mu} = i\left\{-k^{a}_{\ \mu}\tau_{a} + h^{*}_{\mu}\chi - h_{\mu}\bar{\chi}\right\} , \qquad (2.35)$$

the "internal" Maxwell equations (2.25b) read in component form

$$\nabla^{\mu} F^{1}_{\ \mu\nu} + i \left[B^{\mu} G^{*}_{\mu\nu} - B^{*\mu} G_{\mu\nu} \right] = -4\pi \alpha_{\rm S} k^{1}_{\ \nu} \tag{2.36a}$$

$$\nabla^{\mu}F^{2}_{\ \mu\nu} - i\left[B^{\mu}G^{*}_{\mu\nu} - B^{*\mu}G_{\mu\nu}\right] = -4\pi\alpha_{\rm S}k^{2}_{\ \nu} \qquad (2.36b)$$

$$\nabla^{\mu}G_{\mu\nu} + i\left[A^{1\mu} - A^{2\mu}\right]G_{\mu\nu} - iB^{\mu}\left[F^{1}_{\ \mu\nu} - F^{2}_{\ \mu\nu}\right] = 4\pi\alpha_{\rm S}h^{*}_{\nu}$$
(2.36c)

$$\nabla^{\mu}G^{*}_{\mu\nu} - i\left[A^{1\mu} - A^{2\mu}\right]G^{*}_{\mu\nu} + iB^{*\mu}\left[F^{1}_{\ \mu\nu} - F^{2}_{\ \mu\nu}\right] = 4\pi\alpha_{\rm S}h_{\nu} \ . \tag{2.36d}$$

Adding here both equations (2.36a) and (2.36b) yields just the "total" Maxwell equations (2.14) with the total current j_{μ} now being found in terms of the electromagnetic single-particle currents $k^{a}_{\ \mu}$ as

$$j_{\mu} = k^{1}_{\ \mu} + k^{2}_{\ \mu} \,. \tag{2.37}$$

Finally, one may also consider the non-abelian "charge conservation". Observe here, that the curvature $\mathcal{F}_{\mu\nu}$ must obey the following bundle identity (valid in any space-time with vanishing torsion)

$$\mathcal{D}^{\mu}\mathcal{D}^{\nu}\mathcal{F}_{\mu\nu} \equiv 0. \qquad (2.38)$$

Therefore differentiate once more the internal Maxwell equations (2.25b) and find the following source equation in operator form

$$\mathcal{D}^{\mu(S)}\mathcal{J}_{\mu} \equiv 0. \qquad (2.39)$$

Written in component form, this source relation reads

$$\nabla^{\mu}k^{1}_{\ \mu} = i \left[B^{\mu}h_{\mu} - B^{*\mu}h_{\mu}^{*} \right]$$
(2.40a)

$$\nabla^{\mu}k_{\ \mu}^{2} = -i\left[B^{\mu}h_{\mu} - B^{*\mu}h_{\mu}^{*}\right]$$
(2.40b)

$$\nabla^{\mu}h_{\mu} = i \left[A^{1\mu} - A^{2\mu}\right]h_{\mu} + iB^{*\mu}\left[k^{1}_{\ \mu} - k^{2}_{\ \mu}\right]$$
(2.40c)

$$\nabla^{\mu} h_{\mu}^{*} = -i \left[A^{1\mu} - A^{2\mu} \right] h_{\mu}^{*} - i B^{\mu} \left[k_{\ \mu}^{1} - k_{\ \mu}^{2} \right] . \qquad (2.40d)$$

As a consistency test, one may add up the first two source relations (2.40a) - (2.40b) in order to find just the former charge conservation law (2.7) for the total current j_{μ} (2.37).

Clearly, the crucial point with this non-abelian gauge structure of RST is now whether one can identify certain experimental facts which support the non-abelian hypothesis. Since the latter aims at the *exchange* interactions of many-particle systems, one may consider the stationary states of bound N-particle systems (e.g. many-electron atoms) which occur in form of entangled states implying the emergence of "exchange energy". Or in other words, verification of the RST predictions may be obtained by inspection of the atomic energy level systems, which are determined to a considerable extent by the exchange interactions. In order to prepare such a test of RST, it is very instructive to consider first an appropriate energy functional ($E_{\rm T}$, say) for the RST field configurations.

B. Energy Functional

Within the context of a relativistic field theory, one would like to define the energy $E_{\rm T}$ as the space integral over the time component ${}^{({\rm T})}T_{00}$ of an appropriate energy-momentum tensor ${}^{({\rm T})}T_{\mu\nu}$. Equation (2.16) suggests to take for ${}^{({\rm T})}T_{\mu\nu}$ the sum of the internal part ${}^{({\rm S})}T_{\mu\nu}$ of the system and the interaction part ${}^{({\rm es})}T_{\mu\nu}$ with respect to the external source

$${}^{(\mathrm{T})}T_{\mu\nu} = {}^{(\mathrm{S})}T_{\mu\nu} + {}^{(\mathrm{es})}T_{\mu\nu} . \qquad (2.41)$$

Since every N-particle system is composed of matter and gauge fields, mediating the interactions of matter, one will build up the internal density ${}^{(S)}T_{\mu\nu}$ by a Dirac matter part ${}^{(D)}T_{\mu\nu}$ and a gauge part ${}^{(G)}T_{\mu\nu}$:

$${}^{(S)}T_{\mu\nu} = {}^{(D)}T_{\mu\nu} + {}^{(G)}T_{\mu\nu} . \qquad (2.42)$$

Thus the total density ${}^{(T)}T_{\mu\nu}$ (2.41) of the N-particle system consists of three parts

$${}^{(\mathrm{T})}T_{\mu\nu} = {}^{(\mathrm{D})}T_{\mu\nu} + {}^{(\mathrm{G})}T_{\mu\nu} + {}^{(\mathrm{es})}T_{\mu\nu} . \qquad (2.43)$$

Consequently, the total energy $E_{\rm T}$

$$E_{\rm T} = \int d^3 \vec{r} \,^{\rm (T)} T_{00}(\vec{r}) \tag{2.44}$$

will also be built up by three constituents

$$E_{\rm T} = E_{\rm D} + E_{\rm G} + E_{\rm es} ,$$
 (2.45)

with the self-evident definitions

$$E_{\rm D} = \int {\rm d}^3 \vec{r}^{\,({\rm D})} T_{00}(\vec{r}) \tag{2.46a}$$

$$E_{\rm G} = \int {\rm d}^3 \vec{r}^{\,\rm (G)} T_{00}(\vec{r}) \tag{2.46b}$$

$$E_{\rm es} = \int d^3 \vec{r} \,^{\rm (es)} T_{00}(\vec{r}) \,. \tag{2.46c}$$

Here the simplest part is $E_{\rm es}$ (2.46c), because the corresponding energy-momentum density ${}^{\rm (es)}T_{\mu\nu}$ is in a bilinear way composed of the the external field strength ${}^{\rm (ex)}F_{\mu\nu}$ (2.9) and the total field strength $F_{\mu\nu}$ (2.13):

$${}^{(\text{es})}T_{\mu\nu} = -\frac{\hbar c}{4\pi\alpha_{\text{S}}} \left\{ {}^{(\text{ex})}F_{\mu\lambda} \cdot F_{\nu}{}^{\lambda} + F_{\mu\lambda} \cdot {}^{(\text{ex})}F_{\nu}{}^{\lambda} - \frac{1}{2}g_{\mu\nu}{}^{(\text{ex})}F_{\sigma\lambda} \cdot F^{\sigma\lambda} \right\} .$$
(2.47)

Consequently, the external interaction energy $E_{\rm es}$ (2.46c) is given by

$$E_{\rm es} = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \left\{ \vec{E}_{\rm ex} \cdot \vec{E} + \vec{H}_{\rm ex} \cdot \vec{H} \right\} \doteq E_{\rm es}^{(e)} + E_{\rm es}^{(m)} , \qquad (2.48)$$

where the three-vectors $\vec{E}(\vec{r}) = \{E^j\}$ and $\vec{H}(\vec{r}) = \{H^j\}$ of electric and magnetic field strengths are given in terms of the components of the field strength tensor $F_{\mu\nu}$ (2.32) by

$$E^j \doteqdot F_{0j} \tag{2.49a}$$

$$H^{j} \doteq \frac{1}{2} \varepsilon^{jk}{}_{l} F_{k}{}^{l} , \text{etc.}$$
 (2.49b)

If the external fields \vec{E}_{ex} , \vec{H}_{ex} are adopted to be constant (as it is usually done for treating the Zeeman and Stark effects), the external energy E_{es} (2.48) becomes infinite and one has

to substract the infinite contribution due to the external source generating the homogeneous fields \vec{E}_{ex} , \vec{H}_{ex} (see the discussion below equations (3.34) and (5.8)).

Next, considering the gauge field density ${}^{(G)}T_{\mu\nu}$, one finds this object to appear as the difference of the electromagnetic energy - momentum density ${}^{(R)}T_{\mu\nu}$ and the exchange density ${}^{(C)}T_{\mu\nu}$

$${}^{(G)}T_{\mu\nu} = {}^{(R)}T_{\mu\nu} - {}^{(C)}T_{\mu\nu}$$
(2.50)

with the energy-momentum content of the electromagnetic modes being given by

$${}^{(\mathrm{R})}T_{\mu\nu} = -\frac{\hbar c}{4\pi\alpha_{\mathrm{S}}} \left\{ F^{1}_{\ \mu\lambda}F^{2}_{\ \nu}{}^{\lambda} + F^{2}_{\ \mu\lambda}F^{1}_{\ \nu}{}^{\lambda} - \frac{1}{2}g_{\mu\nu}F^{1}_{\ \sigma\lambda}F^{2\sigma\lambda} \right\} , \qquad (2.51)$$

and similarly for the exchange modes of the gauge field system

$$^{(C)}T_{\mu\nu} = -\frac{\hbar c}{4\pi\alpha_{\rm S}} \left\{ G_{\mu\lambda}G_{\nu}^{*\lambda} + G_{\mu\lambda}^{*}G_{\nu}^{\ \lambda} - \frac{1}{2}g_{\mu\nu}G_{\sigma\lambda}^{*}G^{\sigma\lambda} \right\} .$$
(2.52)

Naturally, according to this subdivision, the gauge field energy $E_{\rm G}$ (2.46b) is also split up into two parts

$$E_{\rm G} = E_{\rm R} - E_{\rm C} , \qquad (2.53)$$

where the electromagnetic energy $E_{\rm R}$ is given by

$$E_{\rm R} = \int d^3 \vec{r}^{\,(\rm R)} T_{00}(\vec{r}) = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int d^3 \vec{r} \left\{ \vec{E}_1 \cdot \vec{E}_2 + \vec{H}_1 \cdot \vec{H}_2 \right\} \,, \tag{2.54}$$

and similarly for the exchange energy $E_{\rm C}$

$$E_{\rm C} = \int d^3 \vec{r}^{\,(\rm C)} T_{00}(\vec{r}) = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int d^3 \vec{r} \left\{ \vec{X}^* \cdot \vec{X} + \vec{Y}^* \cdot \vec{Y} \right\} \,. \tag{2.55}$$

Clearly, the "electric" and "magnetic" exchange three-vectors $\vec{X} = \{X^j\}$ and $\vec{Y} = \{Y^j\}$ are defined analogously to their electromagnetic counterparts (2.49a)-(2.49b)

$$X^{j} \doteqdot G_{0j} \tag{2.56a}$$

$$Y^{j} \doteq \frac{1}{2} \varepsilon^{jkl} G_{kl} . \tag{2.56b}$$

Now it is evident that both the electromagnetic energy $E_{\rm R}$ (2.54) and the exchange energy $E_{\rm C}$ (2.55) may be split up further into their "electric" and "magnetic" constituents, i.e. we put

$$E_{\rm R} = E_R^{\rm (e)} + E_R^{\rm (m)}$$
 (2.57a)

$$E_{\rm C} = E_C^{\rm (h)} + E_C^{\rm (g)} ,$$
 (2.57b)

with the "electric" parts being defined in an obvious way through

$$E_R^{(e)} \doteq \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \, \vec{E}_1 \cdot \vec{E}_2 \tag{2.58a}$$

$$E_C^{(\mathrm{h})} \doteq \frac{\hbar c}{4\pi\alpha_{\mathrm{S}}} \int \mathrm{d}^3 \vec{r} \, \vec{X}^* \cdot \vec{X} \,, \qquad (2.58\mathrm{b})$$

and similarly for the "magnetic" parts

$$E_R^{(\mathrm{m})} \doteq \frac{\hbar c}{4\pi\alpha_{\mathrm{S}}} \int \mathrm{d}^3 \vec{r} \, \vec{H}_1 \cdot \vec{H}_2 \tag{2.59a}$$

$$E_C^{(g)} \doteq \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \, \vec{Y}^* \cdot \vec{Y} \,. \tag{2.59b}$$

The important point with this subdivision into "electric" and "magnetic" parts of the gauge energy is, that the "electric" contributions $E_R^{(e)}$ and $E_C^{(h)}$ will turn out to be much larger than their "magnetic" counterparts $E_R^{(m)}$ and $E_C^{(g)}$, so that the latter ones may be treated as small perturbations. The main effect is therefore due to the "electric" contributions and represents thus by itself an acceptable first approximation to the experimental data (*Electrostatic Approximation*). In this sense we may rearrange the gauge field energy E_G (2.53) as

$$E_{\rm G} = E_R^{\rm (e)} - E_C^{\rm (h)} + \left(E_R^{\rm (m)} - E_C^{\rm (g)}\right) , \qquad (2.60)$$

so that the "magnetic" contributions (in brackets) can be omitted for the electrostatic approximation. This approximation is then based solely on the truncated gauge field energy $E_G^{(eh)}$

$$E_G^{(\mathrm{eh})} \doteqdot E_R^{(\mathrm{e})} - E_C^{(\mathrm{h})} \tag{2.61}$$

and thus misses the magnetic energy $E_G^{\rm (mg)}$

$$E_G^{(mg)} \doteq E_R^{(m)} - E_C^{(g)} ,$$
 (2.62)

which will be considered as the "magnetic correction" of the electrostatic energy $E_G^{(eh)}$ (2.61).

Finally, the matter energy $E_{\rm D}$ (2.46a) has to be considered. The corresponding energymomentum density ${}^{(\rm D)}T_{\mu\nu}$ carried by the wave function Ψ is given by

$$^{(D)}T_{\mu\nu} = \frac{i\hbar c}{4} \left\{ \bar{\Psi}\Gamma_{\mu} \left(\mathcal{D}_{\nu}\Psi \right) - \left(\mathcal{D}_{\nu}\bar{\Psi} \right)\Gamma_{\mu}\Psi + \bar{\Psi}\Gamma_{\nu} \left(\mathcal{D}_{\mu}\Psi \right) - \left(\mathcal{D}_{\mu}\bar{\Psi} \right)\Gamma_{\nu}\Psi \right\} .$$
(2.63)

Intuitively, one expects the associated matter energy $E_{\rm D}$ to consist of the rest mass energy $2Mc^2$ of both particles (take for the moment N = 2) plus their kinetic energy E_{kin} , i.e. one

expects

$$E_{\rm D} = 2Mc^2 + \sum_{a=1}^{2} E_{kin\,(a)} \,. \tag{2.64}$$

Indeed, this result can actually be deduced in the non-relativistic limit [21]. In order to have a brief sketch of the proof, eliminate the time derivative $\mathcal{D}_0 \Psi$ from the matter energy density ${}^{(D)}T_{00}$

$$^{(D)}T_{00} = \frac{i\hbar c}{2} \left\{ \bar{\Psi} \Gamma_0 \left(\mathcal{D}_0 \Psi \right) - \left(\mathcal{D}_0 \bar{\Psi} \right) \Gamma_0 \Psi \right\}$$
(2.65)

by means of the Dirac equation (2.22) and find

$${}^{(\mathrm{D})}T_{00} = Mc^2 \bar{\Psi}\Psi + \frac{i\hbar c}{2} \left\{ \left(\mathcal{D}_j \bar{\Psi} \right) \Gamma^j \Psi - \bar{\Psi} \Gamma^j \mathcal{D}_j \Psi \right\} .$$

$$(2.66)$$

Integrating now over all three-space and carrying out the non-relativistic limit in a consistent way yields just the claimed result (2.64), where the correspondence of rest mass terms and kinetic energy terms should be obvious by comparing both equations (2.64) and (2.66).

Observe also, that for the stationary bound field configurations (as solutions of the mass eigenvalue equations) the mass eigenvalues should appear somewhere in the energy functional. For one-particle systems, one may even expect that the field energy $E_{\rm T}$ has to coincide with the mass-energy eigenvalue (M_*c^2 , say) of the bound particle [10]:

$$E_{\rm T} = M_* c^2 \,.$$
 (2.67)

The relationship between the field energy $E_{\rm T}$ and the mass eigenvalues for few-particle systems has now to be studied in more detail, together with their dependence on the "electric" and "magnetic" interactions.

III. ELECTROSTATIC APPROXIMATION

The relative magnitude of the "electric" and "magnetic" interparticle interactions in a stationary bound system strongly depends on the value of the electromagnetic coupling constant $\alpha_{\rm S}$. In view of its smallness ($\alpha_{\rm S} \approx \frac{1}{137}$), the "electric" interactions clearly dominate the "magnetic" interactions, because their relative magnitude is of the order $(z_{ex}\alpha_{\rm S})^2$, where z_{ex} is the number of (positive) charge units located at the center of the binding force, e.g. the nucleus [20]. Therefore, at least for the light atoms ($z_{ex} \leq 10, ..., 20$,

say), one may speculate that the complete neglection of the interactions of the "magnetic" type ("electrostatic approximation") yields a first useful estimate of the relativistic energy levels of the bound system. The reason for this expectation is that, as a consequence of their dominance over the magnetic interactions, the electric interactions will also contribute the main part of the relativistic effects. If this presumption is true (to be verified readily), one will not only be able to treat the magnetic interactions as a small perturbation but furthermore the non-relativistic limit of this perturbation will be all that is needed for taking into account the magnetic effect to first order. In chapter VI we will demonstrate that this apparently rough approximation scheme gets the RST predictions closer to the experimental data of ref [12] ending up with less than 0, 5% of relative deviation (see table II).

Thus we will proceed as follows:

 (i) elaborating in this chapter the relativistic stationary field configurations as far as they are needed for the electrostatic approximation

and

(ii) computing the magnetic corrections in the non-relativistic limit in the subsequent chapters.

A. Stationary States

The stationary bound states are not completely time-independent but have such a temporal behaviour that the observable quantities of the theory (i.e. eigenvalues and densities of charge, current, energy etc.) become truly time-independent. Thus the wave functions of the considered two-particle system (N = 2) vary with time as

$$\psi_1(\vec{r},t) = \exp\left[-i\frac{M_1c^2}{\hbar}t\right] \cdot \psi_1(\vec{r})$$
(3.1a)

$$\psi_2(\vec{r},t) = \exp\left[-i\frac{M_2c^2}{\hbar}t\right] \cdot \psi_2(\vec{r}) , \qquad (3.1b)$$

where M_a (a = 1, 2) denote the mass eigenvalues. This stationary form of the wave functions implies the time-independence of the electromagnetic objects, i.e. the currents $k^a_{\ \mu}$, potentials $A^a{}_\mu$ and field strengths \vec{E}_a, \vec{H}_a :

$$\left\{k^{a}_{\ \mu}\right\} \Rightarrow \left\{{}^{(a)}k_{0}(\vec{r}); \vec{k}_{a}(\vec{r})\right\}$$
(3.2a)

$$\left\{A^{a}_{\ \mu}\right\} \Rightarrow \left\{{}^{(a)}A_{0}(\vec{r}); \vec{A}_{a}(\vec{r})\right\}$$
(3.2b)

$$\vec{E}_a \Rightarrow \vec{E}_a(\vec{r})$$
 (3.2c)

$$\vec{H}_a \Rightarrow \vec{H}_a(\vec{r}) , \qquad (3.2d)$$

whereas their exchange counterparts adopt the following time dependence:

$$h_{\mu}(\vec{r},t) = \exp\left[i\frac{M_1 - M_2}{\hbar}c^2t\right] \cdot h_{\mu}(\vec{r})$$
(3.3a)

$$h_{\mu}^{*}(\vec{r},t) = \exp\left[-i\frac{M_{1}-M_{2}}{\hbar}c^{2}t\right] \cdot h_{\mu}^{*}(\vec{r})$$
 (3.3b)

$$B_{\mu}(\vec{r},t) = \exp\left[-i\frac{M_1 - M_2}{\hbar}c^2t\right] \cdot B_{\mu}(\vec{r})$$
(3.3c)

$$B^*_{\mu}(\vec{r},t) = \exp\left[i\frac{M_1 - M_2}{\hbar}c^2t\right] \cdot B^*_{\mu}(\vec{r})$$
(3.3d)

$$\vec{X}(\vec{r},t) = \exp\left[-i\frac{M_1 - M_2}{\hbar}c^2t\right] \cdot \vec{X}(\vec{r})$$
(3.3e)

$$\vec{Y}(\vec{r},t) = \exp\left[-i\frac{M_1 - M_2}{\hbar}c^2t\right] \cdot \vec{Y}(\vec{r}) . \qquad (3.3f)$$

The relativistic eigenvalue problem for the determination of the mass eigenvalues M_a is obtained now by substituting the stationary form of the wave functions into the two-particle Dirac equation (2.22) which yields the following coupled system for the spatial parts $\psi_a(\vec{r})$ of the stationary wave functions $\psi_a(\vec{r}, t)$ (3.1a)-(3.1b):

$$i\gamma^{j}\mathbb{D}_{j}\psi_{1}(\vec{r}) + \left[{}^{(\text{ex})}A_{0}(\vec{r}) + {}^{(2)}A_{0}(\vec{r}) \right]\gamma_{0}\psi_{1}(\vec{r}) + B_{0}(\vec{r})\gamma_{0}\psi_{2}(\vec{r}) = \frac{M - M_{1}\gamma_{0}}{\hbar}c\,\psi_{1}(\vec{r}) \qquad (3.4a)$$

$$i\gamma^{j}\mathbb{D}_{j}\psi_{2}(\vec{r}) + \left[{}^{(\text{ex})}A_{0}(\vec{r}) + {}^{(1)}A_{0}(\vec{r}) \right]\gamma_{0}\psi_{2}(\vec{r}) + B_{0}^{*}(\vec{r})\gamma_{0}\psi_{1}(\vec{r}) = \frac{M - M_{2}\gamma_{0}}{\hbar}c\,\psi_{2}(\vec{r}) \,. \tag{3.4b}$$

Here the time derivatives of the wave functions have brought in the mass eigenvalues M_a , and the remaining spatial derivations \mathbb{D}_j are defined as follows:

$$\mathbb{D}_{j}\psi_{1}(\vec{r}) \doteq \partial_{j}\psi_{1}(\vec{r}) - i\left[{}^{(\text{ex})}A_{j}(\vec{r}) + {}^{(2)}A_{j}(\vec{r})\right]\psi_{1}(\vec{r}) - iB_{j}(\vec{r})\psi_{2}(\vec{r})$$
(3.5a)

$$\mathbb{D}_{j}\psi_{2}(\vec{r}) \doteq \partial_{j}\psi_{2}(\vec{r}) - i\left[{}^{(\text{ex})}A_{j}(\vec{r}) + {}^{(1)}A_{j}(\vec{r})\right]\psi_{2}(\vec{r}) - iB_{j}^{*}(\vec{r})\psi_{1}(\vec{r}) .$$
(3.5b)

Now, by its very definition, the *electrostatic approximation* consists in neglecting the interactions of the "magnetic" type, i.e. one puts to zero all three-vector potentials $\vec{A_a}(\vec{r}), \vec{B}(\vec{r})$ and thereby ends up with the following truncated system:

$$i\gamma^{j}\partial_{j}\psi_{1}(\vec{r}) + \left[{}^{(\text{ex})}A_{0}(\vec{r}) + {}^{(2)}A_{0}(\vec{r}) \right]\gamma_{0}\psi_{1}(\vec{r}) + B_{0}(\vec{r})\gamma_{0}\psi_{2}(\vec{r}) = \frac{M - \tilde{M}_{1}\gamma_{0}}{\hbar}c\,\psi_{1}(\vec{r}) \qquad (3.6a)$$

$$i\gamma^{j}\partial_{j}\psi_{2}(\vec{r}) + \left[{}^{(\text{ex})}A_{0}(\vec{r}) + {}^{(1)}A_{0}(\vec{r}) \right]\gamma_{0}\psi_{2}(\vec{r}) + B_{0}^{*}(\vec{r})\gamma_{0}\psi_{1}(\vec{r}) = \frac{M - M_{2}\gamma_{0}}{\hbar}c\,\psi_{2}(\vec{r}) \,. \tag{3.6b}$$

Logically, both Dirac spinors $\psi_a(\vec{r})$ (a = 1, 2) do couple here exclusively to the remaining time components ${}^{(a)}A_0(\vec{r})$, $B_0(\vec{r})$ of the corresponding four - vector potentials $A^a{}_\mu$, B_μ . Therefore we have to complement the RST - Dirac system (3.6a) - (3.6b) merely by the Poisson equations for these time components, which of course have to be deduced from the Maxwell equations (2.36a) - (2.36d) by neglecting again the spatial components of the four - vector potentials:

$$\Delta^{(a)} A_0(\vec{r}) = 4\pi \alpha_{\rm S}{}^{(a)} k_0(\vec{r}) \tag{3.7a}$$

$$\Delta B_0(\vec{r}) = -4\pi \alpha_{\rm S} h_0^*(\vec{r}) . \qquad (3.7b)$$

Observing here that the currents $k^a{}_{\mu}, h_{\mu}$ are generated by the wave functions through

$$k^a{}_\mu(\vec{r}) = \bar{\psi}_a(\vec{r})\gamma_\mu\psi_a(\vec{r}) \tag{3.8a}$$

$$h_{\mu}(\vec{r}) = \bar{\psi}_1(\vec{r})\gamma_{\mu}\psi_2(\vec{r})$$
, (3.8b)

one arrives at a coupled but closed Dirac-Poisson system (3.6a)-(3.7b) whose solutions $\left\{\tilde{\psi}_a(\vec{r}), \tilde{M}_a\right\}$ constitute what we consider to be the "electrostatic approximation".

Subsequently, we want to carry out a test of the usefulness of RST by opposing its theoretical predictions for the two-particle ground-state in the Coulomb potential

$${}^{(\text{ex})}A_0(\vec{r}) = z_{ex}\frac{\alpha_{\rm S}}{|\vec{r}|} \tag{3.9}$$

to the experimental data. Since the ground-state is a member of the para-sytem with the highest possible symmetry, we try the following ansatz for the Dirac spinors $\tilde{\psi}_a(\vec{r})$:

$$\tilde{\psi}_a(\vec{r}) = \begin{pmatrix} {}^{(a)}\!\phi_+(\vec{r})\\ {}^{(a)}\!\phi_-(\vec{r}) \end{pmatrix}$$
(3.10)

with the first particle having spin up, i.e. the two component Pauli spinors are chosen as

$${}^{(1)}\phi_{+}(\vec{r}) = {}^{(1)}R_{+}(r) \cdot \zeta_{0}^{\frac{1}{2},\frac{1}{2}}$$
(3.11a)

$${}^{(1)}\phi_{-}(\vec{r}) = -i \,{}^{(1)}R_{-}(r) \cdot \zeta_{1}^{\frac{1}{2},\frac{1}{2}} \,. \tag{3.11b}$$

Here the one-particle eigenspinors $\zeta_{l}^{j,m}$ of the angular momentum operators obey the relations [22, 23]

$$\vec{J}^{2} \zeta_{l}^{j,m} = j(j+1)\hbar^{2} \cdot \zeta_{l}^{j,m}$$
(3.12a)

$$J_z \,\zeta_l^{j,m} = m\hbar \cdot \zeta_l^{j,m} \tag{3.12b}$$

$$\vec{L}^{2} \zeta_{l}^{j,m} = l(l+1)\hbar^{2} \cdot \zeta_{l}^{j,m}$$
(3.12c)

$$\vec{S}^{2} \zeta_{l}^{j,m} = \frac{1}{2} \left(\frac{1}{2} + 1 \right) \hbar^{2} \cdot \zeta_{l}^{j,m} , \qquad (3.12d)$$

with the composition law for spin - $\frac{1}{2}$ particles

$$j = l \pm \frac{1}{2} \,. \tag{3.13}$$

Since the two spins are anti-parallel in the ground-state, we put for the second particle

$${}^{(2)}\phi_{+}(\vec{r}) = {}^{(2)}R_{+}(r) \cdot \zeta_{0}^{\frac{1}{2},-\frac{1}{2}}$$
(3.14a)

$${}^{(2)}\phi_{-}(\vec{r}) = -i \,{}^{(2)}R_{-}(r) \cdot \zeta_{1}^{\frac{1}{2},-\frac{1}{2}} \,. \tag{3.14b}$$

Now, for the ground state, it is reasonable to assume that the radial parts ${}^{(a)}R_{\pm}(r)$ of both wave functions $\tilde{\psi}_a(\vec{r})$ are identical

$${}^{(1)}R_{+}(r) \equiv {}^{(2)}R_{+}(r) \doteqdot R_{+}(r)$$
(3.15a)

$${}^{(1)}R_{-}(r) \equiv {}^{(2)}R_{-}(r) \doteqdot R_{-}(r) , \qquad (3.15b)$$

and similarly for the mass eigenvalues

$$\tilde{M}_1 = \tilde{M}_2 \doteqdot \tilde{M}_0 . \tag{3.16}$$

Thus the original RST eigenvalue system (3.6a) - (3.6b) becomes simplified to two coupled equations for the two radial ansatz functions $R_{\pm}(r)$:

$$\frac{\mathrm{d}R_{+}(r)}{\mathrm{d}r} + \left[{}^{(\mathrm{ex})}A_{0}(r) + A_{0}(r) \right] \cdot R_{-}(r) = -\frac{\tilde{M}_{0} + M}{\hbar} c \cdot R_{-}(r)$$
(3.17a)

$$\frac{\mathrm{d}R_{-}(r)}{\mathrm{d}r} + \frac{2}{r}R_{-}(r) - \left[{}^{(\mathrm{ex})}A_{0}(r) + A_{0}(r) \right] \cdot R_{+}(r) = \frac{\tilde{M}_{0} - M}{\hbar} c \cdot R_{+}(r) .$$
(3.17b)

Observe here that for the para-ansatz (3.14a)-(3.14b) the time component $h_0(\vec{r})$ of the exchange current h_{μ} (3.8b) vanishes ($h_0(\vec{r}) \equiv 0$) so that one can take a vanishing exchange potential ($B_0(\vec{r}) \equiv 0$) as solution of the Poisson equation (3.7b). Therefore the radial

functions $R_{\pm}(r)$ do couple only to the electrostatic potentials ${}^{(a)}A_0(r)$ which of course now have to be identical, too

$${}^{(1)}A_0(r) \equiv {}^{(2)}A_0(r) \doteq A_0(r) \tag{3.18}$$

and obey the Poisson equation (3.7a)

$$\Delta A_0(r) = 4\pi \alpha_{\rm S} k_0(r) , \qquad (3.19)$$

with the coinciding electric charge densities ${}^{(a)}k_0(\vec{r})$ (3.2a) being given by

$${}^{(1)}k_0(r) \equiv {}^{(2)}k_0(r) \doteq k_0(r) = \frac{(R_+(r))^2 + (R_-(r))^2}{4\pi} \,. \tag{3.20}$$

Thus the formal solution of the Poisson equation (3.19) is found to be

$$A_0(r) = -\frac{\alpha_{\rm S}}{4\pi} \int {\rm d}^3 \vec{r}' \, \frac{(R_+(r'))^2 + (R_-(r'))^2}{|\vec{r} - \vec{r}'|} \,. \tag{3.21}$$

B. Non-relativistic Limit

Since the electrostatic approximation is required to take fully into account the relativistic effects, the coupled system (3.17a) - (3.20) has to be solved numerically and then yields the basis for the RST results $\Delta E_{RST}^{(e)}$ (third row of table I). However, for the calculation of the magnetic corrections we may restrict ourselves to the lowest-order approximation of the solutions $\tilde{\psi}_a(\vec{r})$. This means that we

(i) neglect the interaction between both particles (i.e. putting $A_0(r) \Rightarrow 0$)

and

(ii) we are satisfied with the non-relativistic approximation of the corresponding solutions $R_{\pm}(r)$.

Now, by virtue of the first assumption, the interactive system (3.17a) - (3.17b) becomes decoupled and truncated to the simpler form

$$\frac{\mathrm{d}R_{+}(r)}{\mathrm{d}r} + {}^{(\mathrm{ex})}A_{0}(r) \cdot R_{-}(r) = -\frac{M_{*} + M}{\hbar} c \cdot R_{-}(r)$$
(3.22a)

$$\frac{\mathrm{d}R_{-}(r)}{\mathrm{d}r} + \frac{2}{r}R_{-}(r) - {}^{(\mathrm{ex})}A_{0}(r) \cdot R_{+}(r) = \frac{M_{*} - M}{\hbar} c \cdot R_{+}(r) , \qquad (3.22\mathrm{b})$$

whose solutions are given by [24, 25]

$$R_{+}(r) = N_{*}\sqrt{M + M_{*}} r^{\nu} \exp\left[-\frac{z_{ex}r}{a_{\rm B}}\right]$$
 (3.23a)

$$R_{-}(r) = N_{*}\sqrt{M - M_{*}} r^{\nu} \exp\left[-\frac{z_{ex}r}{a_{\rm B}}\right]$$

$$\left(a_{\rm B} = \frac{\hbar^{2}}{Me^{2}}\dots \text{Bohr radius}\right).$$
(3.23b)

Here the parameter ν is given by

$$\nu = -1 + \sqrt{1 - (z_{ex}\alpha_{\rm S})^2} \,. \tag{3.24}$$

Furthermore the mass eigenvalue \tilde{M}_0 of the coupled two-particle system (3.17a)-(3.17b) degenerates to the well-known one-particle result M_* [24, 25]

$$M_* = M\sqrt{1 - (z_{ex}\alpha_{\rm S})^2};$$
 (3.25)

and finally the normalization constant N_* is computed via the relativistic normalization condition

$$\int d^3 \vec{r} \, k_0(\vec{r}) = 1 \tag{3.26}$$

as

$$N_*^2 = \frac{1}{2M} \frac{\left(\frac{2z_{ex}}{a_{\rm B}}\right)^{3+2\nu}}{\Gamma(3+2\nu)} \,. \tag{3.27}$$

The second assumption (of non-relativistic approximation) says that the radial function $R_{-}(r)$ (3.23b) is neglected against $R_{+}(r)$ and for the latter function (3.23a) one takes its non-relativistic limit $\mathring{R}_{+}(r)$, i.e.

$$R_{+}(r) \Rightarrow \overset{\circ}{R}_{+}(r) = 2\sqrt{\left(\frac{z_{ex}}{a_{\rm B}}\right)^{3}} \exp\left[-\frac{z_{ex}r}{a_{\rm B}}\right]$$
(3.28)

which obeys the non-relativistic version of the normalization condition (3.26)

$$\int_{0}^{\infty} \mathrm{d}r \; r^2 \left(\overset{\circ}{R}_{+}(r) \right)^2 = 1 \; . \tag{3.29}$$

Clearly, this non-relativistic limit $\overset{\circ}{R}_+(r)$ is the ground-state solution of the ordinary Schrödinger equation

$$-\frac{\hbar^2}{2M}\Delta\psi(\vec{r}) - \hbar c^{(\text{ex})}A_0(\vec{r})\psi(\vec{r}) = E_0\psi(\vec{r}) \qquad (3.30)$$
$$\left(E_0 = -\frac{z_{ex}^2 e^2}{2a_{\text{B}}}\right)$$

and therefore it may appear somewhat amazing that the ordinary Schrödinger equation is sufficient in order to calculate the magnetic corrections in the lowest-order approximation. However, as we shall readily see, relativity nevertheless enters the "magnetic" exchange energy, namely in form of the exchange current $\vec{h}(\vec{r})$ which is built up by the radial functions ${}^{(a)}R_{\pm}(r)$ through the combination

$$\mathbb{R}_{+}(r) \doteq {}^{(1)}R_{+}(r) \cdot {}^{(2)}R_{-}(r) + {}^{(2)}R_{+}(r) \cdot {}^{(1)}R_{-}(r) , \qquad (3.31)$$

see equation (6.26) below. Observing here the coincidence of the radial functions ${}^{(a)}R_{\pm}(r)$ for the two-particle ground-state (3.15a)-(3.15b), one deduces from the non-relativistic approximations (3.23a)-(3.23b) the corresponding approximation for $\mathbb{R}_+(r)$ (3.31) as

$$\mathbb{R}_{+}(r) \Rightarrow 4\left(z_{ex}\alpha_{\rm S}\right)\frac{z_{ex}^{3}}{a_{\rm B}^{3}}\exp\left[-2\frac{z_{ex}r}{a_{\rm B}}\right] \,. \tag{3.32}$$

C. Mass Eigenvalues M_a and Field Energy E_T

Intuitively, one should suppose that the total energy $E_{\rm T}$ (2.45) is related to the mass eigenvalues M_a in some way. Indeed, this desired relation can easily be obtained by inserting the stationary wave functions $\psi_a(\vec{r})$ (see the right - hand sides of (3.1a) - (3.1b)), potentials $A^a{}_{\mu}$ (3.2b) and B_{μ} (3.3c) - (3.3d) into the matter density ${}^{(\rm D)}T_{00}$ (2.65) and integrating over all three-space. In this way, one finds quite generally the following result for the matter energy $E_{\rm D}$ (2.46a) in terms of the mass eigenvalues M_a

$$E_{\rm D} = \hat{z}_1 \cdot M_1 c^2 + \hat{z}_2 \cdot M_2 c^2 - E_{\rm es}^{(e)}$$

$$+ \hbar c \int d^3 \vec{r} \left\{ {}^{(1)}A_0(\vec{r}) \cdot {}^{(2)}k_0(\vec{r}) + {}^{(2)}A_0(\vec{r}) \cdot {}^{(1)}k_0(\vec{r}) \right\}$$

$$+ \hbar c \int d^3 \vec{r} \left\{ B_0(\vec{r}) \cdot h_0(\vec{r}) + B_0^*(\vec{r}) \cdot h_0^*(\vec{r}) \right\} .$$
(3.33)

Here it is presumed that the external interaction energy of the electric type $E_{es}^{(e)}$ (2.48) can be recasted into the following form by means of Gauß' integral theorem and the Poisson equations (3.7a):

$$E_{\rm es}^{(e)} \Rightarrow -\hbar c \int d^3 \vec{r}^{\,(\rm ex)} A_0(\vec{r}) \left\{ {}^{(1)} k_0(\vec{r}) + {}^{(2)} k_0(\vec{r}) \right\} \,. \tag{3.34}$$

This presumption implies that the external field $\vec{E}_{ex}(\vec{r})$ is well-localized; if this is not true (e.g. by considering a homogeneous field \vec{E}_{ex}), the Gauß' surface term becomes infinite at

infinity and has to be omitted. The reason is that such an infinite surface term has to be considered as part of the (infinite) energy of the external non-localized source generating the homogeneous field \vec{E}_{ex} . Moreover, the former normalization condition (3.26) upon the wave functions $\psi_a(\vec{r})$, being adequate for the electrostatic approximation, has to be generalized in the presence of magnetic interactions to

$$\int d^3 \vec{r}^{(a)} k_0(\vec{r}) = \hat{z}_a , \qquad (3.35)$$

with the real numbers \hat{z}_a still close to unity (for the relativistic normalization of wave functions in the general case see ref.[20, 21]).

Now one substitutes the present result for the matter energy $E_{\rm D}$ (3.33) into the total field energy $E_{\rm T}$ (2.45) which then reappears in the following form:

$$E_{\rm T} = \hat{z}_1 \cdot M_1 c^2 + \hat{z}_2 \cdot M_2 c^2 + E_{\rm es}^{(m)}$$

$$+ E_R^{(e)} + \hbar c \int d^3 \vec{r} \left\{ {}^{(1)}A_0(\vec{r}) \cdot {}^{(2)}k_0(\vec{r}) + {}^{(2)}A_0(\vec{r}) \cdot {}^{(1)}k_0(\vec{r}) \right\}$$

$$- E_C^{(h)} + \hbar c \int d^3 \vec{r} \left\{ B_0(\vec{r}) \cdot h_0(\vec{r}) + B_0^*(\vec{r}) \cdot h_0^*(\vec{r}) \right\}$$

$$+ E_R^{(m)} - E_C^{(g)} .$$
(3.36)

However, the gauge objects emerging here (namely the potentials ${}^{(a)}A_0(\vec{r})$ and the charge densities ${}^{(a)}k_0(\vec{r})$) are not independent, but rather are connected by the Poisson equations (3.7a); and this establishes the following relationship among the electrostatic energy contributions

$$E_R^{(e)} = -\frac{\hbar c}{2} \int d^3 \vec{r} \left\{ {}^{(1)}A_0(\vec{r}) \cdot {}^{(2)}k_0(\vec{r}) + {}^{(2)}A_0(\vec{r}) \cdot {}^{(1)}k_0(\vec{r}) \right\} .$$
(3.37)

This result again comes about by applying Gauß' integral theorem to the internal energy functional $E_R^{(e)}$ (2.58a) and using also the Poisson equations (3.7a). Clearly, there exists an analogous relationship for the "electric" exchange energy $E_C^{(h)}$ (2.58b), namely

$$E_C^{(h)} = \frac{\hbar c}{2} \int d^3 \vec{r} \left\{ B_0(\vec{r}) \cdot h_0(\vec{r}) + B_0^*(\vec{r}) \cdot h_0^*(\vec{r}) \right\} + \frac{\hbar c}{4\pi\alpha_{\rm S} a_{\rm M}^2} \int d^3 \vec{r} \, \vec{B}(\vec{r}) \cdot \vec{B}^*(\vec{r}) \,, \quad (3.38)$$

which will be discussed in greater detail below. By means of these relationships the total energy $E_{\rm T}$ (3.36) can now be simplified to the plausible result

$$E_{\rm T} = \hat{z}_1 \cdot M_1 c^2 + \hat{z}_2 \cdot M_2 c^2 + E_{\rm es}^{(m)} - \Delta E_G^{\rm (eh)} + E_G^{\rm (mg)} .$$
(3.39)

Here the "magnetic" contribution $E_G^{(mg)}$ (2.62) reads in terms of the vector potentials $\vec{A}_a(\vec{r})$ and $\vec{B}(\vec{r})$

$$E_{G}^{(\mathrm{mg})} = -\frac{1}{2}\hbar c \int \mathrm{d}^{3}\vec{r} \left\{ \vec{k}_{1}(\vec{r}) \cdot \vec{A}_{2}(\vec{r}) + \vec{k}_{2}(\vec{r}) \cdot \vec{A}_{1}(\vec{r}) + \vec{h}(\vec{r}) \cdot \vec{B}(\vec{r}) + \vec{h}^{*}(\vec{r})\vec{B}^{*}(\vec{r}) \right\} (3.40) - \frac{\hbar c}{4\pi\alpha_{\mathrm{S}}a_{\mathrm{M}}^{2}} \int \mathrm{d}^{3}\vec{r} \, \vec{B}^{*}(\vec{r}) \cdot \vec{B}(\vec{r}) ,$$

and similarly its "electric" counterpart is defined through

$$\Delta E_G^{(\text{eh})} \doteq E_G^{(\text{eh})} + \frac{\hbar c}{2\pi\alpha_{\text{S}}a_{\text{M}}^2} \int d^3\vec{r} \,\vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r})$$

$$= -\frac{\hbar c}{2} \int d^3\vec{r} \, \left\{ {}^{(1)}A_0(\vec{r}) \cdot {}^{(2)}k_0(\vec{r}) + {}^{(2)}A_0(\vec{r}) \cdot {}^{(1)}k_0(\vec{r}) + B_0(\vec{r}) \cdot h_0(\vec{r}) + B_0^*(\vec{r})h_0^*(\vec{r}) \right\}$$

$$+ \frac{\hbar c}{4\pi\alpha_{\text{S}}a_{\text{M}}^2} \int d^3\vec{r} \, \vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r}) \,.$$
(3.41)

This now is a very interesting result, because it provides us with much insight into the difference between the "electric" and the "magnetic" interactions. First observe that either of the two mass eigenvalues M_a (a = 1, 2) already completely incorporates the "electric" plus "magnetic" interactions of one particle with the other, respectively. This can be most clearly seen from the original form of the eigenvalue equations (3.4a)-(3.4b), namely by multiplying those equations by $\bar{\psi}_1$ or $\bar{\psi}_2$, resp., from the left and integrating over all three-space and finally resolving for the eigenvalues M_a . Therefore the sum of the mass-energy eigenvalues $\hat{z}_a \cdot M_a c^2$, occurring in the total energy $E_{\rm T}$ (3.39), counts the electromagnetic interaction energy between both particles *twice* whereby the "electric" parts enter with a positive sign and the "magnetic" parts with a negative sign. Now, one of the pleasant features of the RST energy functional $E_{\rm T}$ (3.39) is that this double-counting is corrected by *subtraction* of the "electric" interaction part $\Delta E_G^{(\rm eh)}$ and *addition* of the "magnetic" interaction part $E_G^{(\rm mg)}$. The relatively good agreement of this theoretical picture with the experimental data (see table I and II) may be taken as a support of both the RST philosophy and of its practical consequences which will be considered readily.

As far as the method of *electrostatic approximation* is concerned, it is self-suggesting to omit the magnetic terms in the total energy $E_{\rm T}$ (3.39), quite similarly to the way the electrostatic eigenvalue system (3.6a)-(3.6b) was obtained from the general sytem (3.4a)-(3.4b) by omitting the "magnetic" potentials $\vec{A}_a(\vec{r})$ and $\vec{B}(\vec{r})$. This leads us to define the electrostatic approximation \tilde{E}_T of the total $E_{\rm T}$ by

$$\tilde{E}_T = \tilde{M}_1 c^2 + \tilde{M}_2 c^2 - \Delta E_G^{(eh)} ,$$
 (3.42)

where the mass eigenvalues \tilde{M}_a are the eigenvalues of the truncated system (3.6a)-(3.6b) and the "electric" interaction functional $\Delta E_G^{(\text{eh})}$ is to be taken upon the solutions $\tilde{\psi}_a(\vec{r})$ of this truncated system:

$$\Delta E_G^{(\text{eh})} \Rightarrow -\frac{\hbar c}{2} \int \mathrm{d}^3 \vec{r} \left\{ {}^{(1)}A_0(\vec{r}) \cdot {}^{(2)}k_0(\vec{r}) + {}^{(2)}A_0(\vec{r}) \cdot {}^{(1)}k_0(\vec{r}) + B_0(\vec{r}) \cdot h_0(\vec{r}) + B^{0*}(\vec{r}) \cdot h_0^*(\vec{r}) \right\}$$
(3.43)

The corresponding numerical results for the ground-state interaction energy $\Delta E_{\text{RST}}^{(e)}$ are presented in the third column of table I.

D. Numerical Results

Within the framework of the electrostatic approximation all prerequisites are now at hand in order to test the quality of this type of approximation. The numerical solution of the electrostatic eigenvalue system (3.17a) - (3.17b) provides us with the radial ground-state functions $R_{\pm}(r)$ together with the mass eigenvalue \tilde{M}_0 . Consequently, these results may now be substituted into the electrostatic energy functional \tilde{E}_T (3.42), whose value upon the calculated ground-state solutions $\{R_{\pm}(r); \tilde{M}_0\}$ appears then as

$$\tilde{E}_T \Rightarrow 2\tilde{M}_0 c^2 + \hbar c \int d^3 \vec{r} A_0(r) \cdot k_0(r)$$
(3.44)

where the electrostatic potential $A_0(r)$ and the charge density $k_0(r)$ are given by equations $(3.20) \cdot (3.21)$. Observe here also that the exchange potential $B_0(r)$ as a solution of the Poisson equation (3.7b) vanishes because the exchange current $h_{\mu}(\vec{r})$ (3.8b) has a vanishing time-component ($h_0(\vec{r}) \equiv 0$) for the stationary field configurations being defined through equations $(3.11a) \cdot (3.11b)$ and $(3.14a) \cdot (3.14b)$.

Now the problem is how to judge of the present RST prediction $E_{\rm T}$ (3.44) for the twoelectron ground-state energy when this is not directly observable. However, what can be *directly* measured is the interaction energy $\Delta E_{\rm exp}$ between the two electrons in the groundstate. This was done for six highly ionized elements, ranging from germanium ($z_{ex} = 32$) up to bismuth ($z_{ex} = 83$), see ref.[12]. On the other hand, this interaction energy of the two ground-state electrons arises within the framework of RST as the difference $\Delta E_{\rm RST}^{(e)}$ of the ground-state energy $\tilde{E}_{\rm T}$ (3.44) and the double value ($2M_*c^2$) of the single-particle energy eigenvalues (3.25), i.e.

$$\Delta E_{\rm RST}^{(e)} \doteq \tilde{E}_{\rm T} - 2M_*c^2 \qquad (3.45)$$
$$= 2\left(\tilde{M}_0 - M_*\right)c^2 + \hbar c \int d^3\vec{r} A_0(r) \cdot k_0(r) .$$

The comparison of this RST prediction $\Delta E_{\text{RST}}^{(e)}$ with the experimental values ΔE_{exp} displays some very instructive features of both the electric and the magnetic interactions (see table I).

Element (z_{ex})	$\Delta E_{\rm exp} \ [{\rm eV}]$	$\Delta E_{\rm RST}^{\rm (e)} \ [{\rm eV}]$	$\Delta = \frac{\Delta E_{\rm exp} - \Delta E_{\rm RST}^{\rm (e)}}{\Delta E_{\rm exp}} \ [\%]$	f_{*}^{2}	ε_*
Ge (32)	$562,5{\pm}1,6$	553,0	1,7	0,297	16,8
Xe (54)	$1027,2{\pm}3,5$	974,3	$5,\!1$	0,295	$16,\!58$
Dy (66)	$1341,6\pm4,3$	1232	8,2	0,294	$16,\!36$
W (74)	1568 ± 15	1423	9,3	0,247	$16,\!18$
Bi (83)	1876 ± 14	1661	11,5	0,223	$15,\!92$

TABLE I: Comparison of experimental values $\Delta E_{\rm exp}$ (second column) [12] with the RST predictions $\Delta E_{\rm RST}^{(e)}$ (3.45) (third column) for the ground-state interaction energy of helium-like ions. The last two columns display the geometric factor f_*^2 (3.58) for the magnetic interactions and the reference energy ε_* (3.51) for the electric interactions. Both limit values $f_0^2 = 0,4$ (6.43) and $\varepsilon_* \approx 17 \text{eV}$ (3.52) for small values of $z_{\text{ex}} \alpha_{\text{S}} (\ll 1)$ appear to be consistent with the numerical data.

For an intuitive interpretation of the results of table I it is important to first observe that the relative derivation Δ of experimental data ΔE_{exp} and electrostatic RST predictions $\Delta E_{RST}^{(e)}$ (fourth column) increases from 1,7% up to 11,5% when the nuclear charge z_{ex} ranges from $z_{ex} = 32$ (germanium) up to $z_{ex} = 83$ (bismuth). This effect can easily be understood in terms of the different relativistic behaviour of the electric and magnetic interaction energies. Considering first the electric type, one should recall that the *single - particle* energy $E_{\rm T}$ (2.67) equals the mass eigenvalue M_*c^2 (3.25) which splits up into the matter energy $E_{\rm D}$ (2.46a) and external interaction energy $E_{\rm es}$ (2.46c) according to

$$M_*c^2 = Mc^2 \sqrt{1 - (z_{ex}\alpha_S)^2} = E_D + E_{es}$$
(3.46a)

$$E_{\rm D} = \frac{Mc^2}{\sqrt{1 - \left(z_{ex}\alpha_{\rm S}\right)^2}} \tag{3.46b}$$

$$E_{\rm es} = -\frac{(z_{ex}\alpha_{\rm S})^2}{\sqrt{1 - (z_{ex}\alpha_{\rm S})^2}}Mc^2 = \frac{1}{\sqrt{1 - (z_{ex}\alpha_{\rm S})^2}} \overset{\circ}{E}_{\rm es} , \qquad (3.46c)$$

see ref. [21]. Here the non-relativistic limit \mathring{E}_{es} of the external interaction energy in the Coulomb field ${}^{(ex)}A_0(\vec{r})$ (3.9) is given by

$$\hat{E}_{es} = \langle \psi_0 |^{(ex)} A_0 | \psi_0 \rangle$$

$$= \hbar c \int d^3 \vec{r} \, \psi_0^*(\vec{r}) \frac{z_{ex} \alpha_S}{r} \psi(\vec{r}) = - (z_{ex} \alpha_S)^2 \, M c^2 \,,$$
(3.47)

where the non-relativistic ground-state function $\psi_0(\vec{r})$ of the single-particle Schrödinger problem (3.30) coincides of course with the radial wave function $\overset{\circ}{R}_+(r)$ (3.28)

$$\psi_0(\vec{r}) = \frac{1}{\sqrt{4\pi}} \mathring{R}_+(r) = \sqrt{\frac{1}{\pi} \left(\frac{z_{ex}}{a_{\rm B}}\right)^3} \exp\left[-\frac{z_{ex}r}{a_{\rm B}}\right] \,. \tag{3.48}$$

From this result for the *external* interaction one may conclude that also the *internal* electrostatic interaction energy $E_R^{(e)}$ (2.58a) of both ground-state electrons is of the following form

$$E_{R}^{(e)} = \frac{1}{\sqrt{1 - (z_{ex}\alpha_{\rm S})^{2}}} \mathring{E}_{R}^{(e)} \approx \Delta E_{\rm RST}^{(e)}$$
(3.49)

where $\overset{\circ}{E}_{R}^{(e)}$ is again the non-relativistic limit ($z_{ex}\alpha_{\rm S} \to 0$) of the internal electrostatic interaction energy $E_{R}^{(e)}$. This non-relativistic limit may be simply determined by considering the electrostatic interaction energy of the two charge clouds due to the single-particle groundstate $\psi_0(\vec{r})$ (3.48) of the ordinary Schrödinger problem (3.30):

$$\hat{E}_{R}^{(e)} = \iint d^{3}\vec{r} \, d^{3}\vec{r'} \, \frac{|\psi_{0}(\vec{r})|^{2} \cdot |\psi_{0}(\vec{r'})|^{2}}{|\vec{r} - \vec{r'}|}
= \frac{5}{8} z_{ex} \frac{e^{2}}{a_{\rm B}} \approx 17,00725 \cdot z_{ex} \, [\text{eV}] .$$
(3.50)

Combining this result with the former hypothesis (3.49) for the electrostatic interaction energy $\Delta E_{\rm RST}^{(e)}$ says that the "reference energy" ε_* defined through

$$\varepsilon_* \doteq \frac{\sqrt{1 - (z_{ex}\alpha_{\rm S})^2}}{z_{\rm ex}} \cdot \Delta E_{\rm RST}^{\rm (e)}$$
(3.51)

should adopt the value of (roughly)17eV for all values of nuclear charge $z_{\rm ex}$. Now it is just this estimate which is confirmed by our numerical calculations (table I) within an error of a few percent when the elements from $z_{\rm ex} = 32$ (germanium) up to $z_{\rm ex} = 83$ (bismuth) are considered. Indeed, the value ε_0 to be obtained by a combination of both equations (3.49) and (3.50)

$$\varepsilon_0 = \frac{5}{8} \frac{e^2}{a_{\rm B}} = 17,00725\dots[\text{eV}]$$
 (3.52)

appears as the upper limit for our RST calculations of ε_* (3.51), see table I if the nuclear charge z_{ex} is adopted to become smaller (up to $z_{\text{ex}} = 2$ for the neutral helium; see a separate paper).

A similar estimate enlightens also the mechanism of the magnetic interactions, whose energy-momentum content is characterized by $E_R^{(m)}$ (2.59a). In order to get the magnitude of the internal magnetic fields $\vec{H}_a(\vec{r})$, we observe that when a point charge (emitting an electrostatic field \vec{E} at rest) is moving with velocity \vec{v} , then there arises a magnetic field of magnitude

$$\left|\vec{H}\right| \sim \frac{\frac{v}{c}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \left|\vec{E}\right| . \tag{3.53}$$

For an extended charge, one would include here some geometric factor f_* being characteristic for the special charge distribution, and thus one would arrive at the following relationship between the magnetic and electric field strengths

$$\left|\vec{H}\right|^2 \sim f_*^2 \frac{\left(\frac{v}{c}\right)^2}{1 - \left(\frac{v}{c}\right)^2} \cdot \left|\vec{E}\right|^2 \,. \tag{3.54}$$

Now the kinetic energy part $E_{\rm D}$ (3.46b) of the single-particle eigenvalue M_*c^2 (3.46a) suggests to take for the particle velocity in the ground-state

$$\frac{v}{c} \sim z_{\rm ex} \alpha_{\rm S} , \qquad (3.55)$$

and this yields the following relation between electric and magnetic energies $\Delta E_{\rm RST}^{(e)} \approx E_R^{(e)}$ (2.58a) and $\Delta E_{\rm T}^{(\rm mg)}$

$$\Delta E_{\rm T}^{\rm (mg)} \approx f_*^2 \frac{\left(z_{\rm ex} \alpha_{\rm S}\right)^2}{1 - \left(z_{\rm ex} \alpha_{\rm S}\right)^2} \cdot \Delta E_{\rm RST}^{\rm (e)} .$$
(3.56)

Here it is very natural to assume that the magnetic interaction energy $\Delta E_{\rm T}^{(\rm mg)}$ is responsible for the discrepancy of the experimental value $\Delta E_{\rm exp}$ and the electrostatic RST prediction $\Delta E_{\rm RST}^{(e)}$, i.e. we rewrite equation (3.56) as

$$\Delta E_{\rm T}^{\rm (mg)} \equiv \Delta E_{\rm exp} - \Delta E_{\rm RST}^{\rm (e)} = f_*^2 \frac{\left(z_{\rm ex}\alpha_{\rm S}\right)^2}{1 - \left(z_{\rm ex}\alpha_{\rm S}\right)^2} \cdot \Delta E_{\rm RST}^{\rm (e)} . \tag{3.57}$$

This then leads us to the expectation that the geometric factor f_*^2

$$f_*^2 = \frac{1 - (z_{\rm ex}\alpha_{\rm S})^2}{(z_{\rm ex}\alpha_{\rm S})^2} \cdot \frac{\Delta E_{\rm exp} - \Delta E_{\rm RST}^{(e)}}{\Delta E_{\rm RST}^{(e)}}$$
(3.58)

will depend only very weakly, if at all, upon the coupling constant $(z_{\text{ex}}\alpha_{\text{S}})$. And indeed, if we insert the experimental values ΔE_{exp} and our electrostatic RST results $\Delta E_{\text{RST}}^{(e)}$ into the right-hand side of equation (3.58), one just finds the expected weak dependence of f_*^2 , see for this table I.

Finally, combining both estimates for the electrostatic part $\Delta E_{\rm RST}^{(e)}$ (3.51) and the magnetic part $\Delta E_{\rm T}^{(\rm mg)}$ (3.56), we arrive at the general form of the interaction energy $\Delta E_{\rm exp}$

$$\Delta E_{\text{exp}} = \Delta E_{\text{RST}}^{(e)} + \Delta E_{\text{RST}}^{(\text{mg})}$$

$$= \frac{z_{\text{ex}}}{\sqrt{1 - (z_{\text{ex}}\alpha_{\text{S}})^2}} \left\{ 1 + f_*^2 \cdot \frac{(z_{\text{ex}}\alpha_{\text{S}})^2}{1 - (z_{\text{ex}}\alpha_{\text{S}})^2} \right\} \cdot \varepsilon_* .$$
(3.59)

This general result contains two slowly varying functions of the coupling constant $(z_{\text{ex}}\alpha_{\text{S}})$, which adopt their maximum values for small nuclear charge numbers $(z_{\text{ex}} \rightarrow 2)$. The maximum value of ε_0 of the reference energy ε_* has already been determined in equation (3.52), but for determining the maximal value (f_0^2, say) for the geometric factor f_*^2

$$f_0^2 = 0,4\tag{3.60}$$

one has to go deeply into the details of the magnetic interaction mechanism (see the deducation of equation (6.43) below). As we shall readily see, the crucial point here is the fact that the magnetic exchange forces do not vanish since there exists a non-trivial exchange current $\vec{h}(\vec{r})$, producing the magnetic exchange effects such as the magnetic exchange energy $E_C^{(g)}$ (2.59b), whereas for the ground-state there is no *electric* exchange energy $E_C^{(h)}$ (2.58b) on behalf of the vanishing of the scalar exchange potential $B_0(\vec{r})$!

IV. MAGNETIC CORRECTIONS

The preceding estimate of the magnetic energy contributions hinted strongly upon a magnetic explanation for the discrepancy between the experimental values ΔE_{exp} and the

electrostatic RST predictions $\Delta E_{\rm RST}^{(e)}$ (table I). Therefore it appears now mandatory to explicitly compute the magnetic corrections, thought to be responsible for the observed discrepancy. Through this procedure one will then obtain also more accurate RST predictions. In the course of such an elaboration of the magnetic corrections it should also become clear in which way the upper limit f_0^2 (3.60) of the slowly varying function f_*^2 (3.58) comes about. Furthermore, the role of the magnetic exchange effects, being induced by the "magnetic" exchange potential $\vec{B}(\vec{r})$, has to be clarified. Indeed, it will readily turn out that this "magnetic" exchange potential $\vec{B}(\vec{r})$ cannot vanish for the reason of ground - state symmetry (i.e. isotropy), which is in sharp contrast to the missing of the "electric" exchange potential $\vec{B}_0(\vec{r})$. Actually, the "magnetic" exchange energy due to $\vec{B}(\vec{r})$ will turn out to be *twice* the magnetostatic contribution due to $\vec{A}_a(\vec{r})$, which yields a higher precision of the RST predictions by roughly one order of magnitude (compare the RST results of tables I and II). Consequently, the experimentally supported inclusion of the "magnetic" exchange effects due to $\vec{B}(\vec{r})$ must be viewn as a strong confirmation of the non-abelian RST construction. Thus it becomes now necessary to work out the field theory of atomic magnetism in great detail.

The desired magnetic corrections do emerge in two places:

- (i) as a small change $M_a^{(mg)}$ of the mass eigenvalues M_a (3.4a)-(3.4b),
- (ii) as an additional contribution $E_G^{(mg)}$ (2.62) of the interaction energy E_G (2.60).

Therefore the magnetic perturbation scheme must consist in first expressing the mass changes $M_a^{(mg)}$ and energy change $E_G^{(mg)}$ in terms of the wave functions $\psi_a(\vec{r})$ as solutions of the relativistic eigenvalue problem (3.4a) - (3.4b), and then substituting for these solutions their electrostatic approximations $\tilde{\psi}_a(\vec{r})$ as solutions of the truncated system (3.6a) -(3.6b). In other words, one computes the value of the total energy functional $E_{\rm T}$ upon the approximative solutions $\tilde{\psi}_a(\vec{r})$ in place of the exact solutions $\psi_a(\vec{r})$ of the original system (3.4a) - (3.4b).

A. Magnetic Mass Corrections $M_a^{(mg)}$

The point of departure for obtaining the "magnetic" mass corrections is the original exact form (3.4a) - (3.4b) of the two-particle problem. The corresponding eigenvalues M_a are

splitted into their electrostatic approximations \tilde{M}_a and their magnetic corrections $M_a^{(mg)}$

$$M_a = \tilde{M}_a + M_a^{(mg)}. \tag{4.1}$$

Quite generally, the exact mass eigenvalues M_a can be expressed in terms of the corresponding eigenfunctions $\psi_a(\vec{r})$ by multiplying both sides of the eigenvalue equations (3.4a) - (3.4b) by $\bar{\psi}_a(\vec{r})$ and integrating, observing also the normalization relations (3.35). The magnetic mass corrections will then be due to those terms containing the vector potentials $\vec{A}_a(\vec{r}) = \{^{(a)}A^j(\vec{r})\}\$ and $\vec{B}(\vec{r}) = \{B^j(\vec{r})\}\$, which have been omitted for the electrostatic approximation. However, for discussing these magnetic terms it is instructive to reunite them with the electrostatic interaction terms and thus to consider the interelectronic interactions as a whole. This admits us to resort to the covariant Maxwell equations for carrying out the magnetic perturbation procedure where afterwards the separation of the electric and magnetic effects for obtaining the desired result $M_a^{(mg)}$ will present no problem.

In this sense, the united mass corrections ΔM_a may be deduced from the eigenvalue problem (3.4a)-(3.4b) in the following form:

$$\hat{z}_1 \cdot \Delta M_1 c^2 = -\hbar c \int d^3 \vec{r} \left\{ {}^{(1)}\!k_\mu(\vec{r}) \cdot {}^{(2)}\!A^\mu(\vec{r}) + h_\mu(\vec{r}) \cdot B^\mu(\vec{r}) \right\}$$
(4.2a)

$$\hat{z}_2 \cdot \Delta M_2 c^2 = -\hbar c \int d^3 \vec{r} \left\{ {}^{(2)}k_\mu(\vec{r}) \cdot {}^{(1)}A^\mu(\vec{r}) + h^*_\mu(\vec{r}) \cdot B^{\mu*}(\vec{r}) \right\} .$$
(4.2b)

Obviously the mass corrections are built up by an electromagnetic part $M_a^{(em)}$ and an exchange contribution $M_a^{(hg)}$

$$\Delta M_a = M_a^{(em)} + M_a^{(hg)} , \qquad (4.3)$$

with the self-evident arrangements

$$\hat{z}_1 \cdot M_1^{(em)} c^2 = -\hbar c \int d^3 \vec{r} \,^{(1)} k_\mu(\vec{r}) \cdot {}^{(2)} A^\mu(\vec{r}) \tag{4.4a}$$

$$\hat{z}_2 \cdot M_2^{(em)} c^2 = -\hbar c \int d^3 \vec{r} \,^{(2)} k_\mu(\vec{r}) \cdot {}^{(1)} A^\mu(\vec{r}) \tag{4.4b}$$

$$\hat{z}_1 \cdot M_1^{(hg)} c^2 = -\hbar c \int d^3 \vec{r} \ h_\mu(\vec{r}) \cdot B^\mu(\vec{r})$$
(4.4c)

$$\hat{z}_2 \cdot M_2^{(hg)} c^2 = -\hbar c \int d^3 \vec{r} \ h^*_\mu(\vec{r}) \cdot B^{\mu *}(\vec{r}) \ . \tag{4.4d}$$

Here, the *electromagnetic* mass corrections $M_a^{(em)}$ can be further split up according to

$$M_a^{(em)} = M_a^{(e)} + M_a^{(m)}$$
(4.5)

with the electric part $M_a^{(e)}$ being given by

$$\hat{z}_1 \cdot M_1^{(e)} c^2 = -\hbar c \int d^3 \vec{r} \,^{(1)} k_0(\vec{r}) \cdot {}^{(2)} A_0(\vec{r}) \tag{4.6a}$$

$$\hat{z}_2 \cdot M_2^{(e)} c^2 = -\hbar c \int d^3 \vec{r} \,^{(2)} k_0(\vec{r}) \cdot {}^{(1)} A_0(\vec{r}) , \qquad (4.6b)$$

and similarly for the magnetic part ${\cal M}_a^{(m)}$

$$\hat{z}_1 \cdot M_1^{(m)} c^2 = -\hbar c \int d^3 \vec{r} \,^{(1)} k_j(\vec{r}) \cdot {}^{(2)} A^j(\vec{r}) \equiv \hbar c \int d^3 \vec{r} \,\,\vec{k}_1(\vec{r}) \cdot \vec{A}_2(\vec{r}) \tag{4.7a}$$

$$\hat{z}_2 \cdot M_2^{(m)} c^2 = -\hbar c \int d^3 \vec{r} \ ^{(2)}k_j(\vec{r}) \cdot {}^{(1)}A^j(\vec{r}) \equiv \hbar c \int d^3 \vec{r} \ \vec{k}_2(\vec{r}) \cdot \vec{A}_1(\vec{r}) .$$
(4.7b)

In a similar way, the exchange corrections $M_a^{(hg)}$ (4.4c)-(4.4d) can also be subdivided into their "electric" parts $M_a^{(h)}$ and "magnetic" parts $M_a^{(g)}$

$$M_a^{(hg)} = M_a^{(h)} + M_a^{(g)} , (4.8)$$

i.e. we put

$$\hat{z}_1 \cdot M_1^{(h)} c^2 = -\hbar c \int d^3 \vec{r} \ h_0(\vec{r}) \cdot B^0(\vec{r})$$
(4.9a)

$$\hat{z}_2 \cdot M_2^{(h)} c^2 = -\hbar c \int d^3 \vec{r} \ h_0^*(\vec{r}) \cdot B^{0*}(\vec{r})$$
(4.9b)

$$\hat{z}_1 \cdot M_1^{(g)} c^2 = -\hbar c \int d^3 \vec{r} \ h_j(\vec{r}) \cdot B^j(\vec{r}) \equiv \hbar c \int d^3 \vec{r} \ \vec{h}(\vec{r}) \cdot \vec{B}(\vec{r})$$
(4.9c)

$$\hat{z}_2 \cdot M_2^{(g)} c^2 = -\hbar c \int d^3 \vec{r} \ h_j^*(\vec{r}) \cdot B^{j*}(\vec{r}) \equiv \hbar c \int d^3 \vec{r} \ \vec{h}^*(\vec{r}) \cdot \vec{B}^*(\vec{r}) \ .$$
(4.9d)

The physical meaning of these mass corrections becomes now evident when they are written in terms of the curvature components $F^a_{\mu\nu}$ and $G_{\mu\nu}$. Such a reformulation of the mass corrections can easily be attained by eliminating the currents $k_{a\mu}$ and h_{μ} from the original definitions (4.4a)-(4.4d) in favour of the curvature components, namely just by means of the covariant Maxwell equations (2.36a)-(2.36d). Now, at this point we resort to an *additional* approximation which considerably simplifies our magnetic perturbation approach, namely the *linearization* of just that gauge field dynamics (2.36a)-(2.36d). This linearization means concretely that

(i) the cuvature components $F^a_{\mu\nu}$, $G_{\mu\nu}$ (2.31a)-(2.31e) become truncated into

$$F^a_{\ \mu\nu} = \partial_\mu A^a_{\ \nu} - \partial_\nu A^a_{\ \mu} \tag{4.10a}$$

$$G_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu} \tag{4.10b}$$

by simply omitting the non-linear terms due to the non-abelian character of the original gauge group U(2);

and

 (ii) the non-abelian Maxwell equations (2.36a)-(2.36d) become similarly truncated to their linear form

$$\partial^{\mu}F^{a}_{\ \mu\nu} = -4\pi\alpha_{S} k^{a}_{\ \nu} \tag{4.11a}$$

$$\partial^{\mu}G_{\mu\nu} = 4\pi\alpha_{S} h_{\nu}^{*} . \qquad (4.11b)$$

As a consequence of these simplifying assumptions, one finds for those four - vector products dertermining the mass corrections $M_a^{(em)}$ and $M_a^{(hg)}$ (4.4a) - (4.4d):

$${}^{(1)}k_{\mu}(\vec{r}) \cdot {}^{(2)}A^{\mu}(\vec{r}) = -\frac{1}{4\pi\alpha_{\rm S}}\partial^{\mu} \Big\{ {}^{(1)}F_{\mu\nu}(\vec{r}) \cdot {}^{(2)}A^{\nu}(\vec{r}) \Big\} + \frac{1}{8\pi\alpha_{\rm S}} {}^{(1)}F_{\mu\nu}(\vec{r}) \cdot {}^{(2)}F^{\mu\nu}(\vec{r}) \qquad (4.12a)$$

$${}^{(2)}k_{\mu}(\vec{r}) \cdot {}^{(1)}A^{\mu}(\vec{r}) = -\frac{1}{4\pi\alpha_{\rm S}}\partial^{\mu} \Big\{ {}^{(2)}F_{\mu\nu}(\vec{r}) \cdot {}^{(1)}A^{\nu}(\vec{r}) \Big\} + \frac{1}{8\pi\alpha_{\rm S}} {}^{(2)}F_{\mu\nu}(\vec{r}) \cdot {}^{(1)}F^{\mu\nu}(\vec{r}) \qquad (4.12b)$$

$$h_{\mu}(\vec{r}) \cdot B^{\mu}(\vec{r}) = \frac{1}{4\pi\alpha_{\rm S}} \partial^{\mu} \Big\{ G^{*}_{\mu\nu}(\vec{r}) \cdot B^{\nu}(\vec{r}) \Big\} - \frac{1}{8\pi\alpha_{\rm S}} G^{*\mu\nu}(\vec{r}) \cdot G_{\mu\nu}(\vec{r}) \ . \tag{4.12c}$$

When this is substituted back into the correction formulae (4.4a) - (4.4d), they appear in a new form by means of Gauss' integral theorem, namely as

$$\hat{z}_1 \cdot M_1^{(em)} c^2 = \hat{z}_2 \cdot M_2^{(em)} c^2 = -\frac{\hbar c}{8\pi\alpha_{\rm S}} \int d^3 \vec{r} \ ^{(1)}F_{\mu\nu}(\vec{r}) \cdot \ ^{(2)}F^{\mu\nu}(\vec{r})$$
(4.13a)

$$\hat{z}_1 \cdot M_1^{(hg)} c^2 = \hat{z}_2 \cdot M_2^{(hg)} c^2 = \frac{\hbar c}{8\pi\alpha_{\rm S}} \int d^3\vec{r} \ G^{*\mu\nu}(\vec{r}) \cdot G_{\mu\nu}(\vec{r}) \ . \tag{4.13b}$$

This result explicitly demonstrates that the mass corrections ΔM_a (4.2a)-(4.2b) for both particles (a = 1, 2) are actually identical ($\hat{z}_1 \cdot \Delta M_1 \equiv \hat{z}_2 \cdot \Delta M_2$).

But once that "Lorentz invariant" form (4.13a) - (4.13b) of the mass corrections is known, it is self-suggesting to split them up into two contributions, which are offered by themselves through the introduction of the electrostatic and magnetostatic field strengths $\vec{E}_a(\vec{r}) = \{{}^{(a)}E^j(\vec{r})\}$ and $\vec{H}_a(\vec{r}) = \{{}^{(a)}H^j(\vec{r})\}$, i.e. we put

$${}^{(a)}E^{j}(\vec{r}) \doteq {}^{(a)}F_{0j}(\vec{r}) = -\partial_{j}{}^{(a)}A_{0}(\vec{r})$$
(4.14a)

$${}^{(a)}H^{j}(\vec{r}) \doteq \frac{1}{2} \varepsilon^{jk}{}_{l}{}^{(a)}F_{k}{}^{l}(\vec{r}) = \varepsilon^{jk}{}_{l}\partial_{k}{}^{(a)}A^{l}(\vec{r}) , \qquad (4.14b)$$

or in three-vector notation

$$\vec{E}_a(\vec{r}) = -\vec{\nabla}^{(a)} A_0(\vec{r})$$
 (4.15a)

$$\vec{H}_a(\vec{r}) = \vec{\nabla} \times \vec{A}_a(\vec{r}) . \tag{4.15b}$$

Now by this arrangement, the Lorentz invariant product of curvature components determining the electromagnetic mass corrections $M_a^{(em)}$ (4.13a) reads in three-vector notation

$${}^{(1)}F_{\mu\nu}(\vec{r}) \cdot {}^{(2)}F^{\mu\nu}(\vec{r}) = 2\left[\vec{H}_1(\vec{r}) \cdot \vec{H}_2(\vec{r}) - \vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r})\right] , \qquad (4.16)$$

and clearly this yields a natural splitting of those electromagnetic mass corrections (4.13a), namely just the former equation (4.5) with the following identifications for the electric part

$$\hat{z}_1 \cdot M_1^{(e)} c^2 = \hat{z}_2 \cdot M_2^{(e)} c^2 = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \ \vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r}) , \qquad (4.17)$$

and similarly for the magnetic part

$$\hat{z}_1 \cdot M_1^{(m)} c^2 = \hat{z}_2 \cdot M_2^{(m)} c^2 = -\frac{\hbar c}{4\pi\alpha_{\rm S}} \int {\rm d}^3 \vec{r} \ \vec{H}_1(\vec{r}) \cdot \vec{H}_2(\vec{r}) \ . \tag{4.18}$$

Observe here the curious fact that the magnetic part (4.18) enters the electromagnetic mass correction $M_a^{(em)}$ (4.5) with the *opposite* sign in comparison to the electric part (4.17) which is a consequence of the Lorentz invariance of the product of the field strengths (4.16)!

Clearly, it is self-suggesting now to treat the exchange corrections $M_a^{(hg)}$ (4.13b) in a quite similar way. This means that one introduces an "electric" exchange vector field $\vec{X}(\vec{r}) = \{X^j(\vec{r})\}$ (as the exchange counterpart of $\vec{E}(\vec{r})$ (4.14a)) and also a "magnetic" exchange field $\vec{Y}(\vec{r}) = \{Y^j(\vec{r})\}$ (as the exchange counterpart of $\vec{H}(\vec{r})$ (4.14b)) through

$$X^{j}(\vec{r}) \doteq G_{0j}(\vec{r}) = -\partial_{j}B_{0}(\vec{r}) - \frac{i}{a_{\rm M}}B_{j}(\vec{r})$$
 (4.19a)

$$Y^{j}(\vec{r}) \doteq \frac{1}{2} \varepsilon^{jkl} G_{kl}(\vec{r}) = \varepsilon^{jk}{}_{l} \partial_{k} B^{l}(\vec{r}) , \qquad (4.19b)$$

where the exchange length parameter $a_{\rm M}$ is given by

$$a_{\rm M} \doteq \frac{\hbar}{(M_1 - M_2)c^2} \,. \tag{4.20}$$

In three-vector notation, the relations (4.19a)-(4.19b) read

$$\vec{X}(\vec{r}) = -\vec{\nabla}B_0(\vec{r}) + \frac{i}{a_{\rm M}}\vec{B}(\vec{r})$$
 (4.21a)

$$\vec{Y}(\vec{r}) = \vec{\nabla} \times \vec{B}(\vec{r}) , \qquad (4.21b)$$

and thus obviously represent the exchange analogue of the corresponding electromagnetic relations (4.15a)-(4.15b), however with the difference that there emerges now a typical length parameter $a_{\rm M}$ (4.20) which gives an inherent measure for the spatial range of the exchange effects. In terms of these new exchange fields, the Lorentz scalar for the exchange mass corrections $M_a^{(hg)}$ (4.13b) reads

$$G^{*\mu\nu}(\vec{r}) \cdot G_{\mu\nu}(\vec{r}) = 2 \left[\vec{Y}(\vec{r})^* \cdot \vec{Y}(\vec{r}) - \vec{X}(\vec{r})^* \cdot \vec{X}(\vec{r}) \right] , \qquad (4.22)$$

which is of course again the exchange analogue of the corresponding electromagnetic relation (4.16). As a result, the exchange mass corrections $M_a^{(hg)}$ (4.13b) can ultimately be written as

$$\hat{z}_1 \cdot M_1^{(hg)} c^2 = \hat{z}_2 \cdot M_2^{(hg)} c^2 = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \left[\vec{Y}(\vec{r})^* \cdot \vec{Y}(\vec{r}) - \vec{X}(\vec{r})^* \cdot \vec{X}(\vec{r}) \right] \,. \tag{4.23}$$

Though this result looks formally quite analogous to the electromagnetic case, one nevertheless cannot relate here the "electric" exchange masses $M_a^{(h)}(4.9a) - (4.9b)$ to the "electric" exchange vector $\vec{X}(\vec{r})$ and analogously the "magnetic" corrections $M_a^{(g)}(4.9c) - (4.9d)$ to the "magnetic" exchange vector $\vec{Y}(\vec{r})$ as it was done for the electromagnetic situation (4.17) - (4.18). Instead, the correct relationships for the "electric" exchange mass must look as follows:

$$\hat{z}_1 \cdot M_1^{(h)} c^2 = \hat{z}_2 \cdot M_2^{(h)} c^2 = -\frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \ \vec{X}(\vec{r})^* \cdot \vec{X}(\vec{r}) + \frac{\hbar c}{4\pi\alpha_{\rm S}} \frac{1}{a_{\rm M}^2} \int \mathrm{d}^3 \vec{r} \ \vec{B}(\vec{r})^* \cdot \vec{B}(\vec{r}) \ . \tag{4.24}$$

Thus both "electric" exchange corrections are again identical, as it is the case also for the electromagnetic subsystem (4.17). However, they do not coincide with the "electric" part due to $\vec{X}(\vec{r})$ of the total exchange corrections $M_a^{(hg)}$ (4.23).

A similar effect occurs also with the "magnetic" parts $M_a^{(g)}$ of the exchange mass corrections $M_a^{(hg)}$ (4.23). Starting from the original definitions (4.9c)-(4.9d) one finds again that both "magnetic" contributions are identical

$$\hat{z}_1 \cdot M_1^{(g)} c^2 = \hat{z}_2 \cdot M_2^{(g)} c^2 = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \ \vec{Y}(\vec{r})^* \cdot \vec{Y}(\vec{r}) - \frac{\hbar c}{4\pi\alpha_{\rm S}} \frac{1}{a_{\rm M}^2} \int \mathrm{d}^3 \vec{r} \ \vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r}) \ , \ (4.25)$$

but they do again not coincide with the "magnetic" part (due to $\vec{Y}(\vec{r})$) of the total exchange masses $M_a^{(hg)}$ (4.23). However, if both the "electric" and the "magnetic", contributions (4.24) and (4.25) are added up separately for either particle (a = 1, 2), one just finds the total exchange corrections $M_a^{(hg)}$ (4.23). Thus the electromagnetic and exchange subsystems are found to differ in the way in which the total corrections are distributed upon the "electric" and the "magnetic" subsystems.

B. Linearized Gauge Field Equations

Obviously, the introduction of the three-vector notation enables one to separate uniquely the electric corrections (being already included in the electrostatic approximation) from the magnetic corrections which will be treated subsequently by use of an adequate perturbation approach. Consequently, one would like to write down now the gauge field equations (2.36a) - (2.36d) in three - vector notation where one simultaneously restricts oneself to their linearized form (4.11a) - (4.11b). This implies that the magnetic corrections will be taken into account only in their lowest-order approximation. However, the consistent linearization of the gauge field equations represents a certain problem which requires now an extra discussion.

First consider the electromagnetic subsystem, for which one concludes from the linearized Maxwell equations (4.11a) that the three-current $\vec{k}_a(\vec{r}) = \{{}^{(a)}k^j(\vec{r})\}$ must have vanishing divergence (a = 1, 2)

$$\vec{\nabla} \cdot \vec{k}_a(\vec{r}) = 0. \qquad (4.26)$$

Thus the linearization of the (non-Abelian)Maxwell equations (2.36a)-(2.36b) must imply the neglection of the right-hand sides of the source equations (2.40a)-(2.40b). Furthermore, when the curvature components $F^a_{\mu\nu}$ are expressed through the connection components A^a_{μ} (4.10a), that equation (4.11a) yields the corresponding Poisson equations

$$\Delta^{(a)}A_0(\vec{r}) = 4\pi\alpha_{\rm S} \cdot {}^{(a)}k_0(\vec{r}) \tag{4.27a}$$

$$\Delta \vec{A}_a(\vec{r}) = 4\pi\alpha_{\rm S} \cdot \vec{k}_a(\vec{r}) , \qquad (4.27b)$$

provided one subjects the magnetic vector potentials $\vec{A}_a(\vec{r})$ (a = 1, 2) to the Coulomb gauge condition

$$\vec{\nabla} \cdot \vec{A}_a(\vec{r}) = 0 . \tag{4.28}$$

Observing here the usual boundary conditions at infinity $(r \to \infty)$, the solutions of the Poisson equations (4.27a)-(4.27b) are easily found as

$${}^{(a)}A_0(\vec{r}) = -\alpha_{\rm S} \int {\rm d}^3 \vec{r}' \;\; \frac{{}^{(a)}k_0(\vec{r}')}{|\vec{r} - \vec{r'}|} \tag{4.29a}$$

$$\vec{A}_{a}(\vec{r}) = -\alpha_{\rm S} \int {\rm d}^{3} \vec{r}' \; \frac{\vec{k}_{a}(\vec{r}')}{|\vec{r} - \vec{r'}|} \;, \tag{4.29b}$$

where the spatial part (4.29b) is easily seen to meet with the gauge condition (4.28). Thus it is evident that the linearized electromagnetic subsystem is governed by the usual (Abelian)Maxwellian structure, which may be expressed also through the field equations for the field strenghts $\vec{E}_a(\vec{r})$, $\vec{H}_a(\vec{r})$ (4.15a)-(4.15b):

$$\vec{\nabla} \cdot \vec{E}_a(\vec{r}) = -4\pi\alpha_{\rm S} \cdot {}^{(a)}k_0(\vec{r}) \tag{4.30a}$$

$$\vec{\nabla} \cdot \vec{H}_a(\vec{r}) = 0 \tag{4.30b}$$

$$\vec{\nabla} \times \vec{E}_a(\vec{r}) = 0 \tag{4.30c}$$

$$\vec{\nabla} \times \vec{H}_a(\vec{r}) = -4\pi\alpha_{\rm S} \cdot \vec{k}_a(\vec{r}) . \qquad (4.30d)$$

However, the exchange subsystem has a somewhat different structure. This became already obvious through the introduction of the "electric" exchange field $\vec{X}(\vec{r})$ (4.21a) which is not simply the gradient of the "electric" exchange potential $B_0(\vec{r})$ but contains also the "magnetic" exchange potential $\vec{B}(\vec{r})$! Nevertheless, one finds from the linearized equations (4.11b) again the ordinary Poisson equation for $B_0(\vec{r})$

$$\Delta B_0(\vec{r}) = -4\pi\alpha_{\rm S}h_0^*(\vec{r}) \tag{4.31a}$$

$$B_0(\vec{r}) = \alpha_{\rm S} \int {\rm d}^3 \vec{r}' \frac{h_0^*(\vec{r}')}{|\vec{r} - \vec{r}'|} , \qquad (4.31b)$$

which is thus revealed as not being affected by taking into account the magnetic corrections (similar as for the electric potentials ${}^{(a)}A_0(\vec{r})$ (4.27a)). Observe also, that for the deduction of the exchange Poisson equation (4.31a) we imposed a Coulomb-like gauge condition upon the "magnetic" exchange potential $\vec{B}(\vec{r})$

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0 , \qquad (4.32)$$

in close analogy to the magnetic counterpart $\vec{A}_a(\vec{r})$ (4.28). Strictly speaking, such gauge fixing conditions as (4.28) must not be imposed upon the fields B_{μ} because they lost their geometric status as gauge potentials in the process of Abelian symmetry breaking and therefore obey a *homogeneous* transformation law [21]. However, it is easy to see that the requirement (4.32) is consistent with all the other static field equations for the exchange variables $B_0(\vec{r})$, $\vec{B}(\vec{r})$.

It is true, the "electric" part of the exchange field equations (4.31a) - (4.31b) is not too much different from its electric counterpart (4.27a) and (4.29a); however, for the exchange

vector potential $\vec{B}(\vec{r})$ one finds from the linear equations (4.11b)

$$\Delta \vec{B}(\vec{r}) + \frac{1}{a_{\rm M}^2} \cdot \vec{B}(\vec{r}) = -4\pi\alpha_{\rm S}\vec{h}^*(\vec{r}) - \frac{i}{a_{\rm M}}\vec{\nabla}B_0(\vec{r}) . \qquad (4.33)$$

As a consistency test of this equation one forms here the divergence of the left- and righthand side and arrives just at the Poisson equation (4.31a) for the exchange potential $B_0(\vec{r})$, provided the divergence relation for $\vec{B}(\vec{r})$ (4.32) is respected together with the following source equation for the exchange current $\vec{h}(\vec{r})$:

$$\vec{\nabla} \cdot \vec{h}^*(\vec{r}) = \frac{i}{a_{\rm M}} h_0^*(\vec{r}) \ .$$
 (4.34)

However, this is nothing else than the continuity equation for the exchange four-current h^*_{μ}

$$\partial^{\mu}h^*_{\mu}(\vec{r}) = 0 \tag{4.35}$$

which itself is a consequence of the *linear* exchange field equations (4.11b) and, on the other hand, coincides with the former source equations (2.40c) - (2.40d) if the right-hand sides vanish (to be justified below). In this way, one actually attains a consistent linear approximation of the non-Abelian (and therefore non-linear) gauge field equations.

Of course, the solutions $\vec{B}(\vec{r})$ of the exchange field equation (4.33) must have a somewhat other shape than their magnetic counterparts $\vec{A}_a(\vec{r})$ (4.29b):

$$\vec{B}(\vec{r}) = \alpha_{\rm S} \int d^3 \vec{r}' \frac{1}{|\vec{r} - \vec{r}'|} \cdot \cos\left(\frac{|\vec{r} - \vec{r}'|}{a_{\rm M}}\right) \cdot \left\{\vec{h}^*(\vec{r}') + \frac{i}{4\pi\alpha_{\rm S}a_{\rm M}} \cdot \vec{\nabla}' B_0(\vec{r}')\right\} .$$
(4.36)

Thus the effect of the exchange length $a_{\rm M}$ is to suppress the magnetic exchange vector potential $\vec{B}(\vec{r})$ if the exchange current $\vec{h}(\vec{r})$ is smoothly spread over a spatial domain much larger than the exchange length $a_{\rm M}$. But if $\vec{h}(\vec{r})$ is well-localized within such an "exchange domain", the vector potential $\vec{B}(\vec{r})$ is non-zero, but fades away as $r^{-1} \cdot \cos\left(\frac{r}{a_{\rm M}}\right)$. On the other hand, for $a_{\rm M} \to \infty$ the "magnetic" exchange equation (4.33) degenerates to an ordinary Poisson equation of the kind (4.31a) or (4.27a)-(4.27b) with the corresponding behaviour of the solutions. Thus the "magnetic" exchange effects are found to be of a rather different type which may again be expressed more concisely by the following "exchange equations":

$$\vec{\nabla} \cdot \vec{X}(\vec{r}) = 4\pi \alpha_{\rm S} h_0^*(\vec{r}) \tag{4.37a}$$

$$\vec{\nabla} \times \vec{X}(\vec{r}) = \frac{i}{a_{\rm M}} \vec{Y}(\vec{r})$$
(4.37b)

$$\vec{\nabla} \cdot \vec{Y}(\vec{r}) = 0 \tag{4.37c}$$

$$\vec{\nabla} \times \vec{Y}(\vec{r}) = 4\pi \alpha_{\rm S} \cdot \vec{h}^*(\vec{r}) + \frac{1}{a_{\rm M}^2} \cdot \vec{B}(\vec{r}) + \frac{i}{a_{\rm M}} \cdot \vec{\nabla} B_0(\vec{r}) ,$$
 (4.37d)

which adopt the ordinary Maxwellian form (4.30a) - (4.30d) for infinite exchange length $(a_{\rm M} \rightarrow \infty)$. As a consistency test for the linearization procedure, apply the divergence operation to the last equation (4.37d) and find just the source equation for the exchange current $\vec{h}(\vec{r})$ (4.34) by use of the Poisson equation (4.31a).

C. Mass Corrections and Densities

Once both the electromagnetic potentials $\{{}^{(a)}A_0(\vec{r}); \vec{A}_a(\vec{r})\}\$ and the exchange potentials $\{B_0(\vec{r}); \vec{B}(\vec{r})\}\$ are known in terms of the charge and current densities, it becomes possible to eliminate these potentials completely from the mass corrections and to express the latter objects exclusively in terms of these physical densities. Clearly, this then represents a considerable technical simplification because one can express the desired mass corrections directly in terms of the wave functions whose link to the current densities is well-known, see equations (3.8a) - (3.8b). Thus to begin with, reconsider the electric corrections $M_a^{(e)}$ (4.6a) - (4.6b) and substitute therein for the electrostatic potentials ${}^{(a)}A_0(\vec{r})$ their general form (4.29a) in order to arrive at

$$\hat{z}_1 \cdot M_1^{(e)} c^2 = \hat{z}_2 \cdot M_2^{(e)} c^2 = e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \frac{{}^{(1)} k_0(\vec{r}) \cdot {}^{(2)} k_0(\vec{r}')}{|\vec{r} - \vec{r}'|} \,. \tag{4.38}$$

Of course, the charge densities ${}^{(a)}k_0(\vec{r})$ may here be further expressed in terms of the Dirac wave functions $\psi_a(\vec{r})$ according to the former relations (3.8a). Observe also, that the present result of the electrostatic interaction energy $\hat{z}_a \cdot M_a^{(e)}c^2$ now admits two different interpretations, namely either as an instantaneous Coulomb interaction between the two charge clouds ${}^{(a)}k_0(\vec{r})$ or as the *interaction* energy (*not* self-energy!) of the electric field modes $\vec{E}_a(\vec{r})$ emitted by the charge clouds, see equation (4.17). This bilinear (instead of quadratic) construction for the electromagnetic interaction energy of the particles has a certain tradition in the literature and emerges also quite naturally in the RST formalism [26]. A similar effect is obtained also for the magnetic corrections $M_a^{(m)}$ (4.7a)-(4.7b) which by means of the vector potentials $\vec{A}_a(\vec{r})$ (4.29b) may be recast into the following form

$$\hat{z}_1 \cdot M_1^{(m)} c^2 = \hat{z}_2 \cdot M_2^{(m)} c^2 = -e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \frac{\vec{k}_1(\vec{r}) \cdot \vec{k}_2(\vec{r}')}{|\vec{r} - \vec{r}'|} \,. \tag{4.39}$$

Thus the magnetostatic interaction energy $\hat{z}_a \cdot M_a^{(m)} c^2$ can also be interpreted as being due to either an instantaneous direct interaction of the currents $\vec{k}_a(\vec{r})$ or as the interaction energy of the magnetic field modes $\vec{H}_a(\vec{r})$ (4.18). It is also interesting to remark, that the the sum of the electric and magnetic corrections appears as the integral of a four-vector product, namely by adding both equations (4.38) and (4.39):

$$\hat{z}_{a} \cdot M_{a}^{(e)}c^{2} + \hat{z}_{a} \cdot M_{a}^{(m)}c^{2} \equiv \hat{z}_{a} \cdot M_{a}^{(em)}c^{2} \qquad (4.40)$$

$$= e^{2} \iint d^{3}\vec{r} d^{3}\vec{r}' \frac{{}^{(a)}k_{\mu}(\vec{r}) \cdot {}^{(a)}k^{\mu}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$(a = 1 \text{ or } 2).$$

Now it seems natural to expect that the exchange field system will display an analogous structure. This however is true only for its "electric" component, but not for the "magnetic" one. In order to see this more clearly, first recall that the "electric" exchange potential $B_0(\vec{r})$ (4.31a)-(4.31b) is of the same structure as its electric counterpart ^(a) $A_0(\vec{r})$ (4.29a) and therefore yields for the "electric" exchange mass $M_a^{(h)}$ (4.9a)-(4.9b)

$$\hat{z}_1 \cdot M_1^{(h)} c^2 = \hat{z}_2 \cdot M_2^{(h)} c^2 = -e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \frac{h_0(\vec{r}) \cdot h_0^*(\vec{r}')}{|\vec{r} - \vec{r}'|} \,. \tag{4.41}$$

Indeed, this looks very similar to the electrostatic corrections $M_a^{(e)}$ (4.38) where the charge densities ${}^{(a)}k_0(\vec{r})$ play the part of the exchange densities $h_0(\vec{r})$, $h_0^*(\vec{r})$ and a change of sign does occur additionally. This *lowering* of the electrostatic interaction energy (4.38) by the exchange energy (4.41) is due to the fact that the present RST formalism is equivalent to the *antisymmetrized* product states of the conventional Hartree-Fock approach.

However, the "magnetic" exchange system appears in a somewhat different shape which traces back to the modified Poisson equation (4.33) for the "magnetic" exchange potential $\vec{B}(\vec{r})$. More concretly, introducing the solution (4.36) for $\vec{B}(\vec{r})$ into the exchange correction

formulae (4.9c) - (4.9d) yields

$$\hat{z}_{1} \cdot M_{1}^{(g)} c^{2} = e^{2} \iiint d^{3}\vec{r} d^{3}\vec{r}' \frac{\cos\left(\frac{|\vec{r}-\vec{r}'|}{a_{\rm M}}\right)\vec{h}(\vec{r}) \cdot \vec{h}^{*}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{i\hbar c}{4\pi a_{\rm M}} \iint d^{3}\vec{r} d^{3}\vec{r}' \frac{\cos\left(\frac{|\vec{r}-\vec{r}'|}{a_{\rm M}}\right)\vec{h}(\vec{r}) \cdot \vec{\nabla}' B_{0}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$
(4.42a)

$$\hat{z}_{2} \cdot M_{2}^{(g)} c^{2} = e^{2} \iint d^{3}\vec{r} \, d^{3}\vec{r}' \frac{\cos\left(\frac{|\vec{r}-\vec{r}'|}{a_{\rm M}}\right) \vec{h}(\vec{r}) \cdot \vec{h}^{*}(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad (4.42b)$$
$$-\frac{i\hbar c}{4\pi a_{\rm M}} \iint d^{3}\vec{r} \, d^{3}\vec{r}' \frac{\cos\left(\frac{|\vec{r}-\vec{r}'|}{a_{\rm M}}\right) \vec{h}^{*}(\vec{r}) \cdot \vec{\nabla}' B_{0}^{*}(\vec{r}')}{|\vec{r}-\vec{r}'|} .$$

Obviously, the presence of the terms containing the gradient of the "electric" exchange potential $B_0(\vec{r})$ does not allow us to recast this result again into the form of an integral over the four-vector product $h^*_{\mu}(\vec{r}) \cdot h^{\mu}(\vec{r})$, as it was the case for the electromagnetic analogue (4.40).

After the energy contributions of the electric and magnetic type have been discussed in detail, one can now render more precise the meaning of the "magnetic corrections". Obviously, the purely electric corrections $M_a^{(eh)}$ (to be included into the electrostatic approximation) are given by all the mass corrections of the "electric" type (4.38) and (4.41)

$$M_a^{(eh)} \doteq M_a^{(e)} + M_a^{(h)} , \qquad (4.43)$$

whereas the magnetic corrections (to be superimposed as perturbations over the results of the electrostatic approximation) are given by the correction terms of the "magnetic" type (4.39) and (4.42a)-(4.42b):

$$M_a^{(mg)} \doteq M_a^{(m)} + M_a^{(g)} . ag{4.44}$$

The lowest - order approximation of such a perturbation scheme consists then in inserting the solutions $\tilde{\psi}_a(\vec{r})$ of the electrostatic approximation system (3.6a) - (3.6b) into these magnetic correction functionals (4.44).

D. Magnetic Energy of the Gauge Field

Besides the magnetic contributions $M_a^{(mg)}$ as part of the mass eigenvalues M_a , there occur further magnetic constituents of the total energy $E_{\rm T}$, namely those being due to the gauge field energy $E_{\rm G}$ (2.53). Since the gauge field energy $E_{\rm G}$ (2.53) itself splits up into the electromagnetic part $E_{\rm R}$ (2.54) and exchange part $E_{\rm C}$ (2.55), one may separate again both parts into the electric and magnetic type (2.57a)-(2.57b). Here the electric part $E_R^{(e)}$ (2.58a) is naturally included in the electrostatic approximation and is found now to agree just with the electrostatic mass corrections $M_a^{(e)}$ (4.17):

$$E_R^{(e)} \equiv \hat{z}_a \cdot M_a^{(e)} c^2 \,, \, (a = 1 \text{ or } 2).$$
 (4.45)

This is the reason why the *subtraction* of $E_G^{(eh)}$ from the sum of mass eigenvalues rescinds the double counting of the electrostatic energy $E_R^{(e)}$ for the total energy E_T (3.39). Similarly, the magnetic part $E_R^{(m)}$ of the electromagnetic gauge field energy E_R (2.57a) reads in terms of the magnetic field strengths $\vec{H}_a(\vec{r})$

$$E_R^{(m)} = \frac{\hbar c}{4\pi\alpha_S} \int d^3 \vec{r} \, \vec{H}_1(\vec{r}) \cdot \vec{H}_2(\vec{r})$$
(4.46)

and thus is the *negative* of the magnetostatic mass corrections $M_a^{(m)}$ (4.18)

$$E_R^{(m)} = -\hat{z}_a \cdot M_a^{(m)} c^2 \quad (a = 1 \text{ or } 2) , \qquad (4.47)$$

in contrast to the corresponding electric counterpart (4.45). Therefore the *addition* of $E_G^{(mg)}$ to the sum of mass eigenvalues rescinds the double counting of the magnetostatic energy $E_R^{(m)}$ for the total energy E_T (3.39).

In an analoguous way, the exchange energy $E_{\rm C}$ (2.57b) may also be split up into the contributions of "electric" and "magnetic" type according to $E_C^{\rm (h)}$ (2.58b) and $E_C^{\rm (g)}$ (2.59b). The "electric" part $E_C^{\rm (h)}$ however cannot be identified here with the mass corrections $M_a^{(h)}$ (4.9a)-(4.9b), as it was possible for the electrostatic approximation, but the exchange vector potential $\vec{B}(\vec{r})$ has to be retained so that one first finds

$$E_C^{(h)} = \frac{\hbar c}{4\pi\alpha_{\rm S}a_{\rm M}^2} \int d^3\vec{r} \,\vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r}) + \frac{\hbar c}{4\pi\alpha_{\rm S}} \int d^3\vec{r} \,\vec{\nabla}B_0^*(\vec{r}) \cdot \vec{\nabla}B_0^*(\vec{r}) \,. \tag{4.48}$$

Clearly, when the exchange vector potential $\vec{B}(\vec{r})$ is neglected here, one returns to the truncated form due to the electrostatic approximation. On the other hand, eliminating the

"electric" exchange potential $B_0(\vec{r})$ from the present result for $E_C^{(h)}$ (4.48) in favour of $\vec{X}(\vec{r})$ by means of its definition (4.21a) yields now the generalization of the former electrostatic approximation, namely for both a = 1 or a = 2:

$$E_{C}^{(h)} = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int d^{3}\vec{r} \,\vec{X}^{*}(\vec{r}) \cdot \vec{X}(\vec{r})$$

$$= -\hat{z}_{a} \cdot M_{a}^{(h)}c^{2} + \frac{\hbar c}{4\pi\alpha_{\rm S}a_{\rm M}^{2}} \int d^{3}\vec{r} \,\vec{B}^{*}(\vec{r}) \cdot \vec{B}(\vec{r}) .$$
(4.49)

Thus it is seen that the "electric" exchange energy $E_C^{(h)}$ receives an additional contribution in comparison to its mass correction counterpart $M_a^{(h)}$ (4.9a)-(4.9b) when the "magnetic" exchange potential $\vec{B}(\vec{r})$ is not zero. This additional contribution (i.e. the integral on the right-hand side of (4.49)) must be taken into account for building up the "magnetic" energy correction $\Delta E_G^{(mg)}$ below; whereas the first contribution, i.e. the "electric" exchange mass $M_a^{(h)}$ (4.41), is already contained in the electrostatic approximation of \tilde{E}_T (3.42) and must therefore be omitted for the magnetic part of $\Delta E_G^{(mg)}$.

In a quite analogous manner one can treat the "magnetic" exchange energy $E_C^{(g)}$ (2.59b) which is nothing else than the exchange analogue of the magnetostatic field energy $E_R^{(m)}$ (4.46). Comparing this again to the "magnetic" exchange masses $M_a^{(g)}$ (4.25) yields the relationship

$$E_C^{(g)} = \hat{z}_a \cdot M_a^{(g)} c^2 + \frac{\hbar c}{4\pi\alpha_S a_M^2} \int d^3 \vec{r} \, \vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r}) \,. \tag{4.50}$$

Here it is interesting to observe that the energy corrections of the *electric* type, namely $E_R^{(e)}$ (4.45) and $E_C^{(h)}$ (4.49), contain the corresponding mass corrections $M_a^{(e)}$ and $M_a^{(h)}$ with opposite sign so that they actually become subtracted from the sum of mass eigenvalues M_a in order to build up the total energy E_T (3.39). As a result, the double counting of these corrections of the "electric" type is actually avoided, as explained in connection with the electrostatic approximation (see the discussion below equation (3.41).

However, in contrast to this, the corrections of the magnetic type, i.e. $E_R^{(m)}$ (4.47) and $E_C^{(g)}$ (4.50) both contain the "magnetic" mass corrections $M_a^{(m)}$ and $M_a^{(g)}$ with the "false" sign; but since $E_G^{(mg)}$ (2.62) has to be *added* to the same terms occuring in the sum of mass corrections M_a for building up the total energy E_T (3.39) the double counting of the "magnetic" energy contributions is rescinded in just the same way as it is the case with the

"electric" contributions. For the magnetic energy correction $\Delta E_{\rm G}^{(mg)}$ we therefore find

$$\Delta E_{\rm G}^{(mg)} = E_{R}^{(m)} - E_{C}^{(g)} - \frac{\hbar c}{4\pi\alpha_{\rm S}} \int {\rm d}^{3}\vec{r} \,\vec{B}^{*}(\vec{r}) \cdot \vec{B}(\vec{r}) \qquad (4.51)$$
$$= -\hat{z}_{a} \cdot M_{a}^{(m)}c^{2} - \hat{z}_{a} \cdot M_{a}^{(g)}c^{2} - \frac{\hbar c}{2\pi\alpha_{\rm S}a_{\rm M}^{2}} \int {\rm d}^{3}\vec{r} \,\vec{B}^{*}(\vec{r}) \cdot \vec{B}(\vec{r}) \,.$$

Observe here that half of the integral on the right - hand side emerges for both the "electric" part $E_C^{(h)}$ (4.49) and the magnetic part $E_C^{(g)}$ (4.50) and thus must enter the energy correction $\Delta E_G^{(mg)}$ (4.51) with its double value.

Summarizing, the total energy correction $\Delta E_{\rm T}^{\rm (mg)}$ of the "magnetic" type is now found as the sum of all those terms in the total energy $E_{\rm T}$ (3.39) which were omitted for the electrostatic approximation \tilde{E}_T (3.42), i.e.

$$\Delta E_{\rm T}^{(\rm mg)} \doteq \sum_{a=1}^{2} \hat{z}_{a} M_{a}^{(mg)} c^{2} + \Delta E_{\rm G}^{(mg)}$$
$$= \frac{1}{2} \sum_{a=1}^{2} \hat{z}_{a} M_{a}^{(mg)} c^{2} - \frac{\hbar c}{2\pi\alpha_{\rm S} a_{\rm M}^{2}} \int d^{3}\vec{r} \vec{B}^{*}(\vec{r}) \cdot \vec{B}(\vec{r}) . \qquad (4.52)$$

Here the "magnetic" masses $M_a^{(mg)}$ are the proper sum of the magnetostatic mass $M_a^{(m)}$ (4.39) and its exchange counterpart $M_a^{(g)}$ (4.42a)-(4.42b), see equation (4.44). Clearly, the expectation is now that, when the magnetic energy correction $\Delta E_{\rm T}^{(mg)}$ (4.52) is added to the electrostatic approximation \tilde{E}_T (3.42) one will arrive at a more precise numerical prediction for the atomic energy levels $E_{\rm T}$. However, before this expectation receives its validation, it is very instructive to convince oneself of the physical correctness of the RST picture of the atomic magnetism by considering the interaction with an *external* magnetic field (Zeeman effects).

V. EXTERNAL MAGNETISM

It may seem somewhat strange that the magnetic contributions $E_G^{(mg)}$ are entering the total energy E_T (3.39) with the opposite sign in comparison to the electric contributions $\Delta E_G^{(eh)}$. This is the more amazing as the magnetostatic energy content $E_R^{(m)}$ (2.59a) is also given by the *positive* product of the magnetostatic fields $\vec{H}_a(\vec{r})$, quite analogously to the electrostatic counterpart $E_R^{(e)}$ (2.58a). However, the formalism of minimal coupling (3.5a)-(3.5b) together with Lorentz invariance inevitably leads to that minus sign for

the magnetostatic corrections as part of the total energy $E_{\rm T}$; and therefore one may wish to have some additional supporting argument that the magnetic interactions are actually implemented in the RST dynamics in the right way. Such an additional argument can be put forward by considering the interaction of either of the two electrons with an external constant magnetic field \vec{H}_{ex} . Indeed, the covariant derivatives (3.5a) - (3.5b) most clearly display the fact that the corresponding vector potential $\vec{A}_{ex}(\vec{r})$

$$\vec{\nabla} \times \vec{A}_{ex}(\vec{r}) = \vec{H}_{ex} \tag{5.1}$$

acts upon the *a*-th particle in just the same way as does the vector potential ${}^{(b)}\vec{A}(\vec{r})$ $(b \neq a)$ due to the other particle. Therefore, we can test the correctness of the magnetostatic interelectronic interactions by simply inspecting the interaction with an external source emitting the constant field \vec{H}_{ex} (Zeeman effect, see e.g. [27]).

Now it is well known that the interaction of a bound electronic system with an external magnetic field is phenomenologically described by including into the (non-relativistic) Hamiltonian \hat{H} an interaction term \hat{H}_{int} of the following form

$$\hat{H}_{int} = -\hat{\vec{\mu}}_{\rm J} \cdot \vec{H}_{ex} . \tag{5.2}$$

Here the operator of the total magnetic moment $\hat{\vec{\mu}}_{J}$ of the system is composed additively of the orbital and spin parts

$$\hat{\vec{\mu}}_{\rm J} = \hat{\vec{\mu}}_{\rm L} + \hat{\vec{\mu}}_{\rm S} , \qquad (5.3)$$

which themselves are proportional to the corresponding angular momentum operators $\hat{\vec{L}}$ and $\hat{\vec{S}}$ resp., i.e.

$$\hat{\vec{\mu}}_{\rm L} = -\frac{\mu_{\rm B}}{\hbar} \hat{\vec{L}} \tag{5.4a}$$

$$\hat{\vec{\mu}}_{\rm S} = -2\frac{\mu_{\rm B}}{\hbar}\hat{\vec{S}} \tag{5.4b}$$

$$\left(\mu_{\rm B} \doteq \frac{e_* \hbar}{2Mc}, \text{Bohr magneton}\right)$$
.

Thus, e.g. when the Russell-Saunders coupling occurs, the total magnetic moment $\hat{\vec{\mu}}_{J}$ (5.3) can be written as

$$\hat{\vec{\mu}}_{\rm J} = -g_{\rm J} \cdot \frac{\mu_{\rm B}}{\hbar} \hat{\vec{J}} \,, \tag{5.5}$$

where the gyromagnetic ratio g_J is given by the Landé g-factor. In any case, the interaction Hamiltonian \hat{H}_{int} (5.2) implies the existence of an interaction energy E_{int}

$$E_{\text{int}} = \langle \psi | \hat{H}_{int} | \psi \rangle$$

$$= \frac{\mu_{\text{B}}}{\hbar} \langle \psi | \hat{\vec{L}} | \psi \rangle \cdot \vec{H}_{ex} + 2 \frac{\mu_{\text{B}}}{\hbar} \langle \psi | \hat{\vec{S}} | \psi \rangle \cdot \vec{H}_{ex} ,$$
(5.6)

which is experimentally confirmed very well by atomic spectroscopy. This says however that, when the magnetic interactions are correctly incorporated into the RST dynamics, the non-relativistic formula (5.6) for the external magnetic energy $E_{\rm int}$ must be also deducible from our RST results.

Indeed, such a deduction can easily be attained in the following way: restricting ourselves for a moment to a single particle (a = 1) with normalized four-current $k^{\mu}(\vec{r})$ (3.26), its external *magnetic* energy $(M_*^{(m)}c^2, \text{ say})$ is deduced either directly from the one-particle equation (3.4a) with $A^2_{\ \mu} = B_{\mu} \equiv 0$, or from the one-particle version of the energy functional $E_{\rm T}$ (3.39):

$$E_{\rm T} \Rightarrow M_* c^2 + E_{\rm es}^{(m)} . \tag{5.7}$$

Indeed, if one could omit here the external term $E_{\rm es}^{(m)}$ (2.48) (see below), so that the field energy $E_{\rm T}$ equals the mass energy M_*c^2 [21], both methods would yield the same result:

$$M_*^{(m)}c^2 = \hbar c \int d^3 \vec{r} \, \vec{k}(\vec{r}) \cdot \vec{A}_{ex}(\vec{r}) \,.$$
(5.8)

This is exactly the way in which the considered particle would also magnetically interact with the other (not considered) particle, cf. (4.7a)-(4.7b), so that there is actually no difference between external and internal magnetism. However the presumed omission of that external term $E_{es}^{(m)}$ in equation (5.7) is actually justified for the presence of a *homogeneous* external field \vec{H}_{ex} . The reason for this is that in this case the magnetic volume integral in equ. (2.48) may be converted to an (ill-defined) 2-surface integral at spatial infinity $(r \to \infty)$. This surface integral has to be conceived as the (infinite) energy content due to the (infinite) external source emitting the homogeneous field \vec{H}_{ex} ; and therefore that external term $E_{es}^{(m)}$ is to be omitted when one considers the energy of a localized particle.

Now the vector potential $\vec{A}_{ex}(\vec{r})$ (5.1) due to a constant magnetic field \vec{H}_{ex} is given (apart from a gauge transformation) by

$$\vec{A}_{ex}(\vec{r}) = -\frac{1}{2} \left(\vec{r} \times \vec{H}_{ex} \right) , \qquad (5.9)$$

and therefore the external magnetic energy (5.8) is found to be of the following form

$$M_*^{(m)}c^2 = -\vec{\mu}_{\rm J} \cdot \vec{H}_{ex}^{(ph)} .$$
(5.10)

Here the magnetic moment $\vec{\mu}_{\rm J}$ is given by

$$\vec{\mu}_{\rm J} = -\frac{1}{2} e_* \int \mathrm{d}^3 \vec{r} \left(\vec{r} \times \vec{k}(\vec{r}) \right) \,, \tag{5.11}$$

where e_* denotes the elementary charge unit and the physical field $\vec{H}_{ex}^{(ph)}$ is related to our geometric notation \vec{H}_{ex} through

$$\vec{H}_{ex} = \frac{e_*}{\hbar c} \vec{H}_{ex}^{(ph)} .$$
(5.12)

Next, recall that the Gordon decomposition of the four-current ${}^{(D)}j_{\mu}$ of a single Dirac particle reads [21]

$$k_{\mu} \equiv {}^{(D)}j_{\mu} = \bar{\psi}\gamma_{\mu}\psi = \frac{i\hbar}{2Mc} \left[\bar{\psi}\left(\partial_{\mu}\psi\right) - \left(\partial_{\mu}\bar{\psi}\right)\psi\right] + \frac{i\hbar}{Mc}\partial_{\nu}\left(\bar{\psi}\sigma_{\mu}{}^{\nu}\psi\right) , \qquad (5.13)$$

where the objects $\sigma_{\mu\nu}$ are the Spin(1,3) generators, i.e.

$$\sigma_{\mu\nu} = \frac{1}{4} \left[\gamma_{\mu}, \gamma_{\nu} \right] . \tag{5.14}$$

Observe here that we are allowed to resort to the Dirac current (5.13) of a *free* particle, because the magnetic energy (5.10) already contains the external field \vec{H}_{ex} linearly and we are satisfied with the deduction of that first-order approximation (5.6). Obviously, the Gordon decomposition (5.13) splits up the Dirac current ${}^{(D)}j_{\mu}$ into a sum of two parts, namely the drift part ${}^{(d)}k_{\mu}$

$${}^{(d)}k_{\mu} \doteq \frac{i\hbar}{2Mc} \left[\bar{\psi} \left(\partial_{\mu}\psi \right) - \left(\partial_{\mu}\bar{\psi} \right)\psi \right]$$
(5.15)

and the polarization part ${}^{(p)}k_{\mu}$

$${}^{(p)}k_{\mu} = \frac{i\hbar}{Mc} \partial_{\nu} \left(\bar{\psi} \sigma_{\mu}{}^{\nu} \psi \right) , \qquad (5.16)$$

where both parts obey a separate continuity equation, i.e. for the free particle:

$$\partial^{\mu(d)}k_{\mu} = 0 \tag{5.17a}$$

$$\partial^{\mu(p)}k_{\mu} = 0. \qquad (5.17b)$$

For the Gordon decomposition of a composite system see ref. [21]). Now insert the splitting

$$k_{\mu} = {}^{(d)}k_{\mu} + {}^{(p)}k_{\mu} \tag{5.18}$$

into the definition of the total magnetic moment $\vec{\mu}_{\rm J}$ (5.11) and find its analogous splitting as

$$\vec{\mu}_{\rm J} = \vec{\mu}_{\rm L} + \vec{\mu}_{\rm S} , \qquad (5.19)$$

with the orbital part $\vec{\mu}_{\rm L}$ being given by

$$\vec{\mu}_{\rm L} = -\frac{1}{2} e_* \int \mathrm{d}^3 \vec{r} \, \left(\vec{r} \times \vec{k}_d(\vec{r}) \right) \,, \tag{5.20}$$

and similarly for the spin part $\vec{\mu}_{\rm S}$

$$\vec{\mu}_{\rm S} = -\frac{1}{2} e_* \int \mathrm{d}^3 \vec{r} \left(\vec{r} \times \vec{k}_p(\vec{r}) \right)$$

$$\left(\vec{k}_d(\vec{r}) \doteqdot \left\{ {}^{(d)} k^j \right\}, \vec{k}_p(\vec{r}) \doteqdot \left\{ {}^{(p)} k^j \right\} \right) .$$
(5.21)

But clearly, the splitting of the total magnetic moment $\vec{\mu}_{J}$ (5.19) induces an analogous splitting of the magnetic energy (5.10), i.e.

$$M_*^{(m)}c^2 = E_{\rm L} + E_{\rm S} , \qquad (5.22)$$

with the orbital part $E_{\rm L}$ being found as

$$E_{\rm L} = -\vec{\mu}_{\rm L} \cdot \vec{H}_{ex} = \frac{1}{2} e_* \vec{H}_{ex} \cdot \int \mathrm{d}^3 \vec{r} \left(\vec{r} \times \vec{k}_d(\vec{r}) \right) , \qquad (5.23)$$

and similarly for the spin part $E_{\rm S}$

$$E_{\rm S} = -\vec{\mu}_{\rm S} \cdot \vec{H}_{ex} = \frac{1}{2} e_* \vec{H}_{ex} \int {\rm d}^3 \vec{r} \, \left(\vec{r} \times \vec{k}_p(\vec{r}) \right) \,. \tag{5.24}$$

Now it appears as a matter of course that when the RST correction $M_*^{(m)}c^2$ (5.22) is to be identified with the external magnetic energy E_{int} (5.6), the orbital part E_{L} (5.23) must be identified with the first part of E_{int} , i.e.

$$E_{\rm L} \Rightarrow \frac{\mu_{\rm B}}{\hbar} \vec{H}_{ex} \cdot \int d^3 \vec{r} \left(\bar{\psi}(\vec{r}) \cdot \hat{\vec{L}} \cdot \psi(\vec{r}) \right) , \qquad (5.25)$$

and similarly the spin part $E_{\rm S}$ (5.24) with the second part of $E_{\rm int}$:

$$E_{\rm S} \Rightarrow 2\frac{\mu_{\rm B}}{\hbar} \vec{H}_{ex} \cdot \int \mathrm{d}^3 \vec{r} \left(\bar{\psi}(\vec{r}) \cdot \hat{\vec{S}} \cdot \psi(\vec{r}) \right) \,. \tag{5.26}$$

Indeed, this claim (5.25)-(5.26) can easily be verified: first consider the orbital part $E_{\rm L}$ (5.23) and observe that the drift current $\vec{k}_d(\vec{r})$ reads in three - vector notation (to be deduced from its four - vector version (5.15))

$$\vec{k}_d(\vec{r}) = \frac{1}{2Mc} \left[\vec{\psi} \cdot \left(\frac{\hbar}{i} \vec{\nabla} \psi \right) - \left(\frac{\hbar}{i} \vec{\nabla} \bar{\psi} \right) \cdot \psi \right] \,. \tag{5.27}$$

Substituting this into the definition of the orbital magnetic moment $\vec{\mu}_{\rm L}$ (5.20) and observing the Hermiticity of the angular momentum operator $\hat{\vec{L}}$ yields immediately

$$\vec{\mu}_{\rm L} = -\frac{e_*}{2Mc} \int \mathrm{d}^3 \vec{r} \, \left(\bar{\psi} \cdot \hat{\vec{L}} \cdot \psi \right) \,. \tag{5.28}$$

But with this result, the orbital magnetic energy $E_{\rm L}$ (5.23) is easily seen to coincide with its expected form (5.25) and thus the orbital part of the magnetic interaction energy $E_{\rm int}$ (5.6) is in perfect agreement with RST.

By a similar reasoning it is also possible to verify the expected correspondence of the spin part (5.26), albeit by means of a somewhat more subtle argument. For this purpose, observe first that the space part $\vec{k_p}$ of the spin polarization current ${}^{(p)}k_{\mu}$ (5.16) reduces to a three-curl if the stationary field configurations (3.1a)-(3.1b) are considered:

$$\vec{k}_p(\vec{r}) = \vec{\nabla} \times \vec{P}_S(\vec{r}) . \tag{5.29}$$

Here the spin polarization density $\vec{P}_{S}(\vec{r})$ is defined by

$$\vec{P}_S(\vec{r}) = \frac{1}{Mc} \bar{\psi}(\vec{r}) \cdot \hat{\vec{S}} \cdot \psi(\vec{r}) , \qquad (5.30)$$

where the spin-operator $\hat{\vec{S}}$ is introduced through

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$$

$$\left(\vec{\sigma} = \left\{\sigma^{j}\right\} ; \sigma^{jk} = \frac{i}{2}\varepsilon^{kj}{}_{l}\sigma^{l}\right) .$$
(5.31)

Now, when that polarization current $\vec{k}_p(\vec{r})$ (5.29) is inserted into the definition of the spin magnetic moment $\vec{\mu}_S$ (5.21), one ends up with the following final form (by means of integrating by parts):

$$\vec{\mu}_{\rm S} = -\frac{e_*}{Mc} \int \mathrm{d}^3 \vec{r} \, \bar{\psi} \cdot \hat{\vec{S}} \cdot \psi \,. \tag{5.32}$$

But clearly, with this result the RST spin magnetic energy $E_{\rm S}$ (5.24) is again identified with the spin magnetic part of the interaction energy $E_{\rm int}$ (5.6) and this verifies the claimed correspondence (5.26).

Summarizing, the RST proposal (5.8) for the external magnetic interaction energy is found to be in full agreement with the non-relativistic description of the (experimentally well-established) Zeeman effects; and this in turn may be taken as confirmation that also the internal magnetic interparticle interactions are correctly taken into account by the RST energy functional $E_{\rm T}$ (3.39).

VI. GROUND-STATE INTERACTION ENERGY

For a demonstration of the results obtained so far, one may choose the simplest member of the para-system, which is the two-electron ground-state in the Coulomb field (3.9). Clearly, the arguments do apply also to all those excited states which own the same symmetry as the two particle ground-state, i.e. the highest possible symmetry. The simplification originates here from the fact that for the electrostatic approximation the mass eigenvalues \tilde{M}_a and the spatial parts of the wave functions ${}^{(a)}R_{\pm}(r)$ become identical, see equations $(3.15a) \cdot (3.18)$; and furthermore the time components $B_0(\vec{r})$ and $h_0(\vec{r})$ of the exchange vector potential B_{μ} and exchange current h_{μ} do vanish, which is consistent with the Poisson equation (3.7b). Furthermore, the identity of both mass eigenvalues implies that the exchange length $a_{\rm M}$ (4.20) becomes infinite which then annihilates the "electric" exchange field $\vec{X}(\vec{r})$ (4.21a) ($\rightsquigarrow \vec{X}(\vec{r}) \equiv 0$). As a consequence there are no "electric" exchange corrections ($M_a^{(h)} = 0$, see equation (4.24)), and thus the "magnetic" exchange corrections (4.25) simplify to

$$\hat{z}_1 \cdot M_1^{(g)} c^2 = \hat{z}_2 \cdot M_2^{(g)} c^2 = \frac{\hbar c}{4\pi\alpha_{\rm S}} \int \mathrm{d}^3 \vec{r} \, \vec{Y}^*(\vec{r}) \cdot \vec{Y}(\vec{r}) \,. \tag{6.1}$$

However, since the "magnetic" exchange field $\vec{Y}(\vec{r})$ and its vector potential $\vec{B}(\vec{r})$ (4.21b) do persist, they give rise to the emergence of exchange corrections for the para-system which are beyond the electrostatic approximation. The exchange vector potential $\vec{B}(\vec{r})$ obeys now the ordinary Poisson equation (cf. equ. (4.33))

$$\Delta \vec{B}(\vec{r}) = -4\pi\alpha_{\rm S}\vec{h}^*(\vec{r}) . \qquad (6.2)$$

with the corresponding simplification of the solution (4.36)

$$\vec{B}(\vec{r}) = \alpha_{\rm S} \int d^3 \vec{r}' \frac{\vec{h}^*(\vec{r}')}{|\vec{r} - \vec{r}'|} \,. \tag{6.3}$$

Observe also that the exchange three-current $\vec{h}(\vec{r})$ must become sourceless on behalf of the vanishing of its time component $h_0(\vec{r})$ (4.34)

$$\vec{\nabla} \cdot \vec{h}^*(\vec{r}) = 0. \tag{6.4}$$

Thus the Maxwellian exchange system (4.37a) - (4.37d) becomes truncated to its "magnetic" part

$$\vec{\nabla} \cdot \vec{Y}(\vec{r}) = 0 \tag{6.5a}$$

$$\vec{\nabla} \times \vec{Y}(\vec{r}) = 4\pi\alpha_{\rm S} \cdot \vec{h}^*(\vec{r}) . \qquad (6.5b)$$

Summarizing, the lowest-order corrections beyond the electrostatic approximation are described by the magnetostatic fields $\vec{H}_a(\vec{r})$ (4.18) and the "magnetic" exchange field $\vec{Y}(\vec{r})$ (6.1) so that the total correction $\Delta E_{\rm T}^{(mg)}$ (4.52) becomes for the ground-state of the parasystem

$$\Delta E_{\rm T}^{(mg)} \implies \frac{1}{2} \sum_{a=1}^{2} \hat{z}_a \cdot M_a^{(mg)} c^2 \qquad (6.6)$$
$$= \frac{1}{2} \hat{z}_1 \cdot \left(M_1^{(m)} + M_1^{(g)} \right) c^2 + \frac{1}{2} \hat{z}_2 \cdot \left(M_2^{(m)} + M_2^{(g)} \right) c^2 .$$

Since this energy correction consists of the proper magnetostatic contributions ($M_a^{(m)}$) and the "magnetic" exchange contributions ($M_a^{(g)}$), both subsystems have to be inspected now separately.

A. Exchange Corrections $M_a^{(g)}$

Obviously, it is merely a technical question whether one prefers to compute the exchange masses $M_a^{(g)}$ in their original form (4.9c)-(4.9d), being based upon the simultaneous use of the exchange current $\vec{h}(\vec{r})$ and the exchange vector potential $\vec{B}(\vec{r})$, or in the form (4.42a)-(4.42b) which makes use of the currents alone (a = 1, 2)

$$\hat{z}_a \cdot M_a^{(g)} c^2 = e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \, \frac{\vec{h}(\vec{r}) \cdot \vec{h}^*(\vec{r}')}{|\vec{r} - \vec{r}'|} \,, \tag{6.7}$$

or whether one prefers to deal with the above form (6.1) relying exclusively upon the "magnetic" exchange field strenghts $\vec{Y}(\vec{r})$. In any case one must know explicitly the functional form of the exchange current $\vec{h}(\vec{r})$. Therefore one first inserts the general stationary form of the wave functions $\psi_a(\vec{r},t)$ (3.1a) - (3.1b) into the general definition of the exchange current $h_{\mu}(\vec{r})$ (3.8b) and then finds the following form for the three-current $\vec{h}(\vec{r})$:

$$\vec{h}(\vec{r}) = \frac{i}{4\pi} \mathbb{R}_{+}(r) \cdot \vec{W}_{p}^{*}(\vartheta, \varphi) .$$
(6.8)

Here the three-vector \vec{W}_p depends on the spherical polar coordinates ϑ, φ in the following way

$$\vec{W}_p(\vartheta,\varphi) = -\cos\vartheta \left(\vec{e}_x + i\vec{e}_y\right) + e^{i\varphi}\sin\vartheta \,\vec{e}_z \,\,, \tag{6.9}$$

where $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are the basis vectors due to a Cartesian parametrization (x, y, z) of Euclidian three-space. Futhermore, the radial function \mathbb{R}_+ has been defined in terms of the ansatz functions ${}^{(a)}R_{\pm}(r)$ for the Pauli spinors ${}^{(a)}\phi_{\pm}$ as shown by equation (3.31). Since we are satisfied for the moment with the lowest-order approximation for the exchange masses $M_a^{(g)}$, we can resort to the non-relativistic approximation for the function \mathbb{R}_+ (3.32).

With the exchange current $\vec{h}(\vec{r})$ (6.8) being at hand now, one can in the next step look for the solution $\vec{B}(\vec{r})$ (6.3) of the Poisson equation (6.2). Clearly, the desired vector potential $\vec{B}(\vec{r})$ will have the same symmetry as its source $\vec{h}^*(\vec{r})$, i.e. one tries the product ansatz

$$\vec{B}(\vec{r}) = i \cdot r B(r) \cdot \vec{W}_p(\vartheta, \varphi) , \qquad (6.10)$$

and then one deduces the following differential equation for the radial ansatz function B(r)from the Poisson equation (6.2):

$$\frac{\mathrm{d}^2 B(r)}{\mathrm{d}r^2} + \frac{4}{r} \frac{\mathrm{d}B(r)}{\mathrm{d}r} = \alpha_{\mathrm{S}} \cdot \frac{\mathbb{R}_+}{r} . \qquad (6.11)$$

The solution of this equation is easily worked out as

$$B(r) = \frac{B_*}{3} \left(\frac{z_{ex}}{a_{\rm B}}\right)^3 \left\{ 2 \left[\left(\frac{z_{ex}r}{a_{\rm B}}\right)^{-1} + \left(\frac{z_{ex}r}{a_{\rm B}}\right)^{-2} \right] \cdot \exp\left(-2\frac{z_{ex}r}{a_{\rm B}}\right) + \left(\frac{z_{ex}r}{a_{\rm B}}\right)^{-3} \cdot \left[\exp\left(-2\frac{z_{ex}r}{a_{\rm B}}\right) - 1 \right] \right\}.$$
(6.12)

Obviously, this solution decays as r^{-3} at infinity $(r \to \infty)$ but is regular at the origin (r = 0)

$$B(r) = \frac{B_*}{3} \left(\frac{z_{ex}}{a_{\rm B}}\right)^3 \cdot \left\{-\frac{4}{3} + 2\frac{z_{ex}r}{a_{\rm B}} + \cdots\right\} .$$
(6.13)

The constant B_* is related to the normalization constant \mathbb{N}_* (3.27) through (non-relativistic limit)

$$B_* = \frac{3}{8} \left(\frac{a_{\rm B}}{z_{ex}}\right)^4 \cdot z_{\rm ex} \alpha_{\rm S}^2 \left(2MN_*^2\right) = \frac{3}{2} \alpha_{\rm S}^2 a_{\rm B} .$$
(6.14)

But once the exchange vector field $\vec{B}(\vec{r})$ is known, one can consider its curl $\vec{Y}(\vec{r})$ (4.21b), which appears in the following form

$$\vec{Y}(\vec{r}) = -\tilde{B}(r)\left[\vec{e}_x + i\vec{e}_y\right] + \frac{x + iy}{r}\frac{\mathrm{d}B(r)}{\mathrm{d}r} \cdot \vec{r}, \qquad (6.15)$$

with the radial function $\tilde{B}(r)$ being given by

$$\tilde{B}(r) \doteqdot r \frac{\mathrm{d}B(r)}{\mathrm{d}r} + 2B(r) . \qquad (6.16)$$

When this result is used in order to compute the exchange corrections $M_a^{(g)}$ (6.1), one is left after the the angular integration with the following radial problem (a = 1, 2):

$$\hat{z}_a \cdot M_a^{(g)} c^2 = \frac{4}{3} \frac{\hbar c}{\alpha_{\rm S}} \int_0^\infty \mathrm{d}r \ r^2 \left[\tilde{B}(r)^2 + 2B(r)^2 \right] \ . \tag{6.17}$$

However, observing here the specific functional form of B(r) (6.16) and repeatedly integrating by parts yields

$$\hat{z}_a \cdot M_a^{(g)} c^2 = -\frac{4}{3} \hbar c \int_0^\infty \mathrm{d}r \ r^3 B(r) \cdot \mathbb{R}_+(r) \ , \tag{6.18}$$

where the differential equation for B(r) (6.11) has also been used. Now it is just this latter form (6.18) for the exhange corrections $M_a^{(g)}$ which can also be recovered by starting from their original form (4.9c)-(4.9d) and using hereby the previously found results for the vector potential $\vec{B}(\vec{r})$ (6.10) and the exchange current $\vec{h}(\vec{r})$ (6.8).

Clearly, the general equivalence of both forms (4.9c) - (4.9d) and (4.25) for the exchange corrections $M_a^{(g)}$ has thus been exemplified merely in the lowest order (beyond the electrostatic approximation). But this can just be taken as a successful consistency test of the applied approximation technique, which consists in a combination of *linearizing* the magnetic interactions and additionally taking the *non-relativistic limit*. Indeed, the remaining radial integration of equation (6.18) can easily be done by use of the known functional forms of B(r) (6.12) and $\mathbb{R}_+(r)$ (3.32) which yields

$$\hat{z}_1 \cdot M_1^{(g)} c^2 = \hat{z}_2 \cdot M_2^{(g)} c^2 = \frac{1}{6} \left(z_{ex} \alpha_{\rm S} \right)^2 \cdot \frac{z_{ex} e^2}{a_{\rm B}} = \frac{4}{15} \left(z_{ex} \alpha_{\rm S} \right)^2 \mathring{E}_R^{(e)} .$$
(6.19)

This result verifies the expectation that the "magnetic" interactions are typically smaller than the "electric" ones (3.50) by a factor $(z_{\rm ex}\alpha_{\rm S})^2$ and therefore may be approximated here by their non-relativistic limit. But observe on the other hand that the magnetic corrections (6.19) vary as $z_{\rm ex}^3$ and therefore will become more important for the heavy atoms in comparison to their electric counterparts $\hat{E}_R^{(e)}$ (3.50) raising only linearly with $z_{\rm ex}$.

B. Relativistic Normalization

It must be remarked that the value of the renormalized "charges" \hat{z}_a (3.35) do not enter explicitly the result (6.19) so that the values of the corresponding renormalization integrals (3.35) are not explicitly needed. But clearly, the renormalized charges \hat{z}_a do enter the results in an implicit manner, namely through the proper relativistic normalization conditions [20, 21]. However, it is easy to demonstrate that for the presumed linearization (4.10a) - (4.10b) the renormalization of the charges is trivial, i.e. one can identify: $\hat{z}_a = 1$. The reason for this is that the deviation of \hat{z}_a from unity is induced by the integral of the entanglement vector G_{μ} [11, 20, 21]

$$G_{\mu} = \frac{i}{4\pi\alpha_{\rm S}} \left[B^{\nu} G^{*}_{\nu\mu} - B^{*\nu} G_{\nu\mu} \right]$$
(6.20)

over the time slices t = const which gives for the stationary field configurations

$$z_{1} - \hat{z}_{2} = -(z_{2} - \hat{z}_{2}) = \int_{t=\text{const}} G_{\mu} dS^{\mu} \Longrightarrow \int d^{3}\vec{r} \quad G_{0}(\vec{r}) \qquad (6.21)$$
$$= -\frac{1}{2\pi\alpha_{S}a_{M}} \int d^{3}\vec{r} \, \vec{B}^{*}(\vec{r}) \cdot \vec{B}(\vec{r}) \, .$$

However, for the ground state the exchange length $a_{\rm M}$ becomes infinite $(a_{\rm M} \to \infty)$ and thus the difference (6.21) between z_a and \hat{z}_a vanishes: $z_a = \hat{z}_a = 1$, (a = 1, 2).

This coincidence of the charges z_a and \hat{z}_a is not only a byproduct of the applied linearization, but for the ground state it has a deeper origin. Indeed, it is closely related to the symmetry of the two-particle ground-state. Observe here again that, when the normalization integral is done over a time slice (t = const) of the Minkowskian space-time, it is only the time component G_0 of the entanglement vector G_{μ} (6.20) which is relevant:

$$G_0(\vec{r}) = \frac{i}{4\pi\alpha_{\rm S}} \left[\vec{B}^*(\vec{r}) \cdot \vec{X}(\vec{r}) - \vec{B}(\vec{r}) \cdot \vec{X}^*(\vec{r}) \right] \,. \tag{6.22}$$

This component however vanishes, because the "electric" exchange vector $\vec{X}(\vec{r})$ must be put to zero when its source $h_0(\vec{r})$ vanishes, see equation (4.37a)-(4.37b). The latter fact, however, is a consequence of the symmetry of the ground-state ansatz (3.1a)-(3.3f).

C. Magnetic Corrections $M_a^{(m)}$

In order to complete the magnetic part $\Delta E_{\rm T}^{(mg)}$ (6.6) of the interaction energy, we have to consider the magnetostatic mass corrections $M_a^{(m)}$ (4.7a)-(4.7b). This will be done by explicit computation of the magnetostatic vector potentials $\vec{A}_a(\vec{r})$ (4.29b), however through solving directly the associated Poisson equation (4.27b), instead of computing the integral (4.29b). The reason for this is that both magnetostatic potentials $\vec{A}_a(\vec{r})$ (a = 1, 2) are of a similar functional form as the "magnetic" exchange potential $\vec{B}(\vec{r})$ (6.10), so that one can take over the functional form of the corresponding solution for the desired potentials $\vec{A}_a(\vec{r})$. Clearly, this similarity of the vector potentials $\vec{B}(\vec{r})$ and $\vec{A}_a(\vec{r})$ has a deeper geometric meaning to be discussed readily.

The close relationship between all three potentials $\vec{B}(\vec{r})$, $\vec{A}_a(\vec{r})$ (a = 1, 2) is recognized most immediatly by demonstrating that the Poisson equations for the magnetic potentials $\vec{A}_a(\vec{r})$ (4.27b) are efficitvely the same as for the exchange potential $\vec{B}(\vec{r})$ (6.2), i.e. the right - hand sides of both Poisson equations effectively do agree. This does not mean that the currents $\vec{k}_a(\vec{r})$ are identical to the exchange current $\vec{h}(\vec{r})$ (6.8), but when one computes the three - currents $\vec{k}_a(\vec{r})$ (3.8a) by means of the stationary field configurations (3.1a) - (3.1b), one finds the following form:

$$\vec{k}_a(\vec{r}) = k_a(r) \cdot \vec{V}_p(\vartheta, \varphi) .$$
(6.23)

Here the vector field \vec{V}_p is given by

$$\vec{V}_p(\vartheta,\varphi) = \sin\vartheta \cdot \left[-\sin\varphi \cdot \vec{e}_x + \cos\varphi \cdot \vec{e}_y\right] \doteq \sin\vartheta \cdot \vec{e}_\varphi \tag{6.24}$$

and thus is to be conceived as the magnetic analogue of its exchange counterpart $\vec{W}_p(\vartheta, \varphi)$ (6.9). Furthermore, the scalar prefactors $k_a(r)$ in equation (6.23) are found to be of the following form

$$k_1(r) = \frac{1}{2\pi} {}^{(1)}R_+(r) \cdot {}^{(1)}R_-(r)$$
(6.25a)

$$k_2(r) = -\frac{1}{2\pi} {}^{(2)}R_+(r) \cdot {}^{(2)}R_-(r) . \qquad (6.25b)$$

However, since the radial functions of both particles do coincide for the ground-state (i.e. ${}^{(1)}R_+(r) = {}^{(2)}R_+(r)$, ${}^{(1)}R_-(r) = {}^{(2)}R_-(r)$) one has

$$k_1(r) = -k_2(r) \equiv \frac{1}{4\pi} \mathbb{R}_+(r) ,$$
 (6.26)

and therefore both single-particle currents $\vec{k}_a(\vec{r})$ (6.23) flow in opposite directions around the z-axis (see fig.1)

$$\vec{k}_1(\vec{r}) = -\vec{k}_2(\vec{r}) \doteq \vec{k}(\vec{r}) .$$
 (6.27)

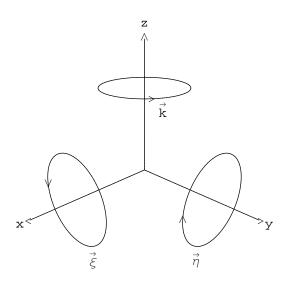


FIG. 1: Isotropy of the Ground - State

Each coordinate axis is encircled by the same kind of circular flow \vec{k} (6.27), $\vec{\xi}$ (6.28b) and $\vec{\eta}$ (6.28c) which yields the equipartition of the magnetic interaction energy $\Delta E_T^{(mg)}$ (6.38) into the identical contributions $\Delta E_T^{(z)}$ (6.39), $\Delta E_T^{(x)}$ (6.41a) and $\Delta E_T^{(y)}$ (6.41b).

Now recall the expectation that the ground - state symmetry has to be the highest possible one, so that the z - axis cannot be singled out in any way as compared to the x - and y - axis. Indeed, this is really the case if one takes also into account the exchange current $\vec{h}(\vec{r})$ (6.8) which splits up into its real and imaginary parts $\vec{\xi}(\vec{r})$ and $\vec{\eta}(\vec{r})$ as follows:

$$\vec{h}(\vec{r}) = \vec{\xi}(\vec{r}) + i\vec{\eta}(\vec{r})$$
 (6.28a)

$$\vec{\xi}(\vec{r}) = \frac{\mathbb{R}_{+}(r)}{4\pi} \left[-\cos\vartheta \cdot \vec{e}_{y} + \sin\vartheta \sin\varphi \cdot \vec{e}_{z} \right]$$
(6.28b)

$$\vec{\eta}(\vec{r}) = \frac{\mathbb{R}_+(r)}{4\pi} \left[-\cos\vartheta \cdot \vec{e}_x + \sin\vartheta \cos\varphi \cdot \vec{e}_z \right] \,. \tag{6.28c}$$

Evidently, this result says now that all three axes are encircled by the same type of flow, namely the x-axis by $\vec{\xi}(\vec{r})$ (6.28b), the y-axis by $\vec{\eta}(\vec{r})$ (6.28c) and the z-axis by $\vec{k}(\vec{r})$ (6.27); see fig.1. Here it must be stressed that this isotropic geometry for the ground-state could only be attained by completing the magnetostatic potentials $\vec{A}_a(\vec{r})$ by the exchange potential $\vec{B}(\vec{r})$ into a triplet of vector potentials. Since $\vec{B}(\vec{r})$ owes its existence to the use of the non-abelian group U(2) instead of the abelian U(1), which is generally used in the conventional approach to the electromagnetic interactions (classical and quantum), the "magnetic" exchange interactions are now seen to equip the RST with a truly *non-abelian* character!

For the explicit construction of the magnetostatic potentials $\vec{A}_a(\vec{r})$, one tries an ansatz which has the same symmetry as their sources $\vec{k}_a(\vec{r})$ (6.23), quite similarly as it was done for the exchange potential $\vec{B}(\vec{r})$ (6.10), i.e. one puts

$$\vec{A}_1(\vec{r}) = rA_1(r) \cdot \vec{V}_p(\vartheta, \varphi) \tag{6.29a}$$

$$\vec{A}_2(\vec{r}) = rA_2(r) \cdot \vec{V}_p(\vartheta, \varphi) .$$
(6.29b)

Inserting this into the Poisson equations (4.27b) yields the following differential equations for the radial ansatz functions $A_a(r)$:

$$\frac{\mathrm{d}^2 A_1(r)}{\mathrm{d}r^2} + \frac{4}{r} \frac{\mathrm{d}A_1(r)}{\mathrm{d}r} = 2\alpha_{\mathrm{S}} \frac{{}^{(1)}R_+(r) \cdot {}^{(1)}R_-(r)}{r} \equiv 4\pi k_1(r)$$
(6.30a)

$$\frac{\mathrm{d}^2 A_2(r)}{\mathrm{d}r^2} + \frac{4}{r} \frac{\mathrm{d}A_2(r)}{\mathrm{d}r} = 2\alpha_{\mathrm{S}} \frac{{}^{(2)}R_+(r) \cdot {}^{(2)}R_-(r)}{r} \equiv 4\pi k_2(r) .$$
(6.30b)

Since for the ground - state symmetry the radial parts of both wave functions $\psi_a(\vec{r})$ do agree (i.e. ${}^{(1)}R_+(r) = {}^{(2)}R_+(r)$, ${}^{(1)}R_-(r) = {}^{(2)}R_-(r)$), one arrives at

$$\frac{\mathrm{d}^2 A_1(r)}{\mathrm{d}r^2} + \frac{4}{r} \frac{\mathrm{d}A_1(r)}{\mathrm{d}r} = \alpha_{\mathrm{S}} \frac{\mathbb{R}_+(r)}{r}$$
(6.31a)

$$\frac{\mathrm{d}^2 A_2(r)}{\mathrm{d}r^2} + \frac{4}{r} \frac{\mathrm{d}A_2(r)}{\mathrm{d}r} = -\alpha_{\mathrm{S}} \frac{\mathbb{R}_+(r)}{r} , \qquad (6.31\mathrm{b})$$

which is effectively the same equation as for the radial exchange function B(r) (6.11). Therefore, one can directly take over the solutions as

$$A_1(r) = -A_2(r) \equiv B(r) , \qquad (6.32)$$

where the radial function B(r) has already been specified by equation (6.12).

Clearly, the identification (6.32) of all radial functions $A_a(\vec{r})$ and $B(\vec{r})$ is a further indication of the ground-state isotropy and it is instructive to elucidate this effect also from another point of view. For this purpose, reconsider the source equations (2.40a)-(2.40d) and insert therein the general shapes of the currents $\vec{k}_a(\vec{r})$ (6.23) and $\vec{h}(\vec{r})$ (6.8), together with the vector potentials $\vec{A}_a(\vec{r})$ (6.29a)-(6.29b) and $\vec{B}(\vec{r})$ (6.10). The right-hand sides of the first two equations (2.40a)-(2.40b) then turn out to be zero

$$B^{\mu}h_{\mu} - B^{\mu*}h_{\mu}^* = 0 , \qquad (6.33)$$

and therefore the currents $\vec{k}_a(\vec{r})$ are found to be sourceless

$$\vec{\nabla} \cdot \vec{k}_a(\vec{r}) = 0. \qquad (6.34)$$

Observe that this property is also shared by the exchange current $\vec{h}(\vec{r})$, cf. (6.4). The remaining two equations (2.40c)-(2.40d) yield the relation

$$B(r) = \frac{A_1(r) - A_2(r)}{k_1(r) - k_2(r)} \cdot \frac{\mathbb{R}_+(r)}{4\pi} .$$
(6.35)

Now recall here that, because of the ground-state symmetry, the denominator becomes on account of equation (6.26)

$$k_1(r) - k_2(r) = \frac{\mathbb{R}_+(r)}{2\pi}$$
(6.36)

and furthermore the numerator appears by virtue of equation (6.32) as

$$A_1(r) - A_2(r) = 2B(r) \tag{6.37}$$

so that the relation (6.35) degenerates to an identity. Therefore the source equations (2.40a) - (2.40d) are automatically satisfied when all RST objects $\left\{\vec{k}_a(\vec{r}), \vec{h}(\vec{r}), \vec{A}_a(\vec{r}), \vec{B}(\vec{r})\right\}$ own the symmetry properties of the ground - state, irrespective of the precise functional form of the radial parts ${}^{(a)}R_{\pm}(r)$ of the wave functions!

With these prerequisites at hand, we are now able to test a further consequence of the ground-state isotropy: if it is true that, on account of the isotropic geometry, all three spatial directions are equivalent, then the corresponding currents $\vec{k}(\vec{r})$ (6.27), $\vec{\xi}(\vec{r})$ (6.28b) and $\vec{\eta}(\vec{r})$ (6.28c) have to contribute equal parts to the magnetic energy $\Delta E_{\rm T}^{(mg)}$ (6.6), see fig. 1:

$$\Delta E_{\rm T}^{(mg)} = \Delta E_{\rm T}^{(x)} + E_{\rm T}^{(y)} + E_{\rm T}^{(z)} . \qquad (6.38)$$

Now the contribution of the z-direction is given by the currents $\vec{k}_a(\vec{r})$, cf. (4.39)

$$\Delta E_{\rm T}^{(z)} \doteq \frac{1}{2} \left(\hat{z}_1 \cdot M_1^{(m)} c^2 + \hat{z}_2 \cdot M_2^{(m)} c^2 \right) = e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \frac{\vec{k}(\vec{r}) \cdot \vec{k}(\vec{r}')}{|\vec{r} - \vec{r}'|} \,. \tag{6.39}$$

Next, the contributions of the x- and y-directions are given by the "magnetic" exchange energy $\hat{z}_a \cdot M_a^{(g)} c^2$, see equations (4.42a)-(4.42b) for infinite exchange length ($a_{\rm M} \to \infty$):

$$\Delta E_{\rm T}^{(x)} + \Delta E_{\rm T}^{(y)} \doteq \frac{1}{2} \left(\hat{z}_1 \cdot M_1^{(g)} c^2 + \hat{z}_2 \cdot M_2^{(g)} c^2 \right) = e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \, \frac{\vec{h}(\vec{r}) \cdot \vec{h}^*(\vec{r}')}{|\vec{r} - \vec{r}'|} \,, \qquad (6.40)$$

i.e. for either direction separately

$$\Delta E_{\rm T}^{(x)} = e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \, \frac{\vec{\xi}(\vec{r}) \cdot \vec{\xi}(\vec{r}')}{|\vec{r} - \vec{r}'|} \tag{6.41a}$$

$$\Delta E_{\rm T}^{(y)} = e^2 \iint d^3 \vec{r} \, d^3 \vec{r}' \, \frac{\vec{\eta}(\vec{r}) \cdot \vec{\eta}(\vec{r}')}{|\vec{r} - \vec{r}'|} \,. \tag{6.41b}$$

But since the three currents $\vec{k}(\vec{r}), \vec{\xi}(\vec{r}), \vec{\eta}(\vec{r})$ differ only by their spatial orientation, not by their intrinsic pattern of flux lines and strength, all three contributions are identical, cf. (6.19)

$$\Delta E_T^{(\text{mg})} = \Delta E_T^{(x)} + \Delta E_T^{(y)} + \Delta E_T^{(z)} = \frac{1}{4} \left(z_{ex} \alpha_{\rm S} \right)^2 \cdot \frac{z_{ex} e^2}{a_{\rm B}} = \frac{2}{5} \left(z_{ex} \alpha_{\rm S} \right)^2 \cdot \overset{\circ}{E}_R^{(e)} . \tag{6.42}$$

This is the definitive result for the magnetic corrections; and can now to be used in order to check our preliminary estimate of the magnitude of these magnetic contributions which has been made within the framework of the electrostatic approximation (see end of sect. III). Namely for small values of the coupling constant ($z_{ex}\alpha_S \ll 1$), the preliminary picture of the magnetic interactions did identify the electrostatic RST prediction $\Delta E_{RST}^{(e)}$ with the non-relativistic interaction energy $\dot{E}_R^{(e)}$, see equation (3.49). On the other hand the discrepancy between these RST predictions $\Delta E_{RST}^{(e)}$ and the experimental data ΔE_{exp} was attributed to the magnetic energy $\Delta E_T^{(mg)}$, see equation (3.57). Therefore the limit value f_0^2 of the geometric factor f_*^2 (3.58) for ($z_{ex}\alpha_S$)² \ll 1 appears now as

$$f_0^2 = \frac{1}{(z_{\rm ex}\alpha_{\rm S})^2} \cdot \frac{\Delta E_T^{\rm (mg)}}{\overset{\circ}{E_R^{(e)}}} = \frac{2}{5}$$
(6.43)

which appears to be in acceptable agreement with the values of f_*^2 as presented in table I. Clearly, for a more precise test of the limit values ε_0 (3.52) and f_0^2 (6.43) one would have to extend table I to smaller values of the coupling parameter $z_{\text{ex}}\alpha_{\text{S}}$, possibly up to neutral helium ($z_{\text{ex}} = 2$, see a separate paper).

D. Theory vs. Experiment

Apart from such an internal consistency test for the RST predictions, it is of course highly interesting to compare the complete RST predictions $\Delta E_{\text{RST}}^{(\text{emg})}$, i.e the sum of electric and magnetic contributions

$$\Delta E_{\rm RST}^{\rm (emg)} \doteq \Delta E_{\rm RST}^{\rm (e)} + \Delta E_T^{\rm (mg)} , \qquad (6.44)$$

Element	$\Delta E_{\rm exp}$	$\Delta E_{\rm RST}^{\rm (emg)}$	$\frac{\Delta E_{\rm exp} - \Delta E_{\rm RST}^{\rm (emg)}}{\Delta E_{\rm exp}}$	Relativistic MBPT	Rel. all order	MCDF	Unified
$(z_{\rm ex})$	[12]	(6.44)		[17, 18]	MBPT [19]	[14]-[17]	[13]
Ge (32)	$562,5{\pm}1,6$	564,9	-0,42%	561,9	562,1	562,1	562,1
Xe (54)	$1027,2{\pm}3,5$	1031	-0,37%	1028,1	1028,4	1028,2	1028,8
Dy (66)	$1341,6\pm 4,3$	1336	$0,\!41\%$	$1336,\! 6$	1337,2	1336,5	1338,2
W (74)	1568 ± 15	1570	-0,11%	$1574,\! 6$	$1574,\!8$	$1574,\! 6$	$1576,\!6$
Bi (83)	1876 ± 14	1868	$0,\!43\%$	1882,7	_	1880,8	1886,3

TABLE II: Comparison of the theoretical predictions for the ground-state interaction energy with the experimental data [12]. All energies are measured in [eV]. Already after the inclusion of the magnetic interactions in the lowest-order approximation, the RST predictions $\Delta E_{\rm RST}^{\rm (emg)}$ (third column) meet with the corresponding predictions of the other theoretical approaches (right half of the table).

to both the experimental data and to other theoretical predictions, such as the relativistic 1/Z expansion [13], the multiconfiguration Dirac - Fock method (MCDF) [14]- [17] and relativistic many - body pertubation theory (MBPT) [17]-[18], or the all - order technique for relativistic MBPT [19]. The predictions of these four theoretical approaches are collected in table II together with the experimental data [12] in order to oppose them to the present RST predictions $\Delta E_{\rm RST}^{(\rm emg)}$ (6.44). As it can easily be seen from a comparison of table I and table II, the RST predictions appear now to be very satisfying: after the magnetic interactions have been included into the RST results to yield the interaction energy $\Delta E_{\rm RST}^{(\rm emg)}$ (6.44), third column of table II, the deviation from the experimental data $\Delta E_{\rm exp}$ (second column of table 2) has decreased to less than a half percent which does no longer depend upon the nuclear charge $z_{\rm ex}$. This means that the discrepancy between the experimental values $\Delta E_{\rm exp}$ and the electrostatic RST predictions $\Delta E_{\rm RST}^{(e)}$, ranging from 1,7% for germanium(32) up to 11,5% for bismuth (83) (see table I), is actually caused by the magnetic interactions with a remaining uncertainty of roughly 0,4%. This supports the RST picture of the simultaneous action of electric and magnetic forces in the electronic orbits of an atom.

A further, very satisfying feature of the RST results $\Delta E_{\text{RST}}^{(\text{emg})}$ (6.44) refers to the comparison to the other four theoretical approaches, see the last four columns of table II. Here the RST predictions appear to be of the same order of precision as the corresponding predictions of the other four theoretical approaches, see table III of ref. [12]. Summarizing, one can therefore judge that the RST predictions can compete succesfully with these other approaches, even when the magnetic interactions are treated in the lowest-order approximation! Clearly, in the next step one will study the higher-order approximations of the magnetic RST interactions, which may be expected to lead even closer to the experimental data.

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