

## The Exact $S$ -matrix of the Deformed $c = 1$ Matrix Model

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### ABSTRACT

We consider the  $c = 1$  matrix model deformed by the operator  $\frac{1}{2}M \text{Tr} \Phi^{-2}$ , which was conjectured by Jevicki and Yoneya to describe a two-dimensional black hole of mass  $M$ . We calculate the exact non-perturbative  $S$ -matrix and show that all the amplitudes involving an odd number of particles vanish at least to all orders of perturbation theory. We conjecture that these amplitudes vanish non-perturbatively and prove this for the  $2n \rightarrow 1$  scattering. For the 2- and 4-particle amplitudes we give some leading terms of the perturbative expansion.

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There has been a considerable amount of speculation on the relation between the  $c = 1$  matrix model and two-dimensional stringy black holes [1,2,3,4]. Recently Jevicki and Yoneya [5] made an interesting proposal that a stationary black hole of mass  $M$  is described by the large- $N$  Hermitian matrix quantum mechanics with potential  $U(\Phi) = \frac{1}{2} \text{Tr}(-\Phi^2 + M\Phi^{-2})$ . The matrix eigenvalues act as free fermions, and their Fermi level  $\mu$  is set to zero. The deformation of the  $c = 1$  matrix model by the operator  $\text{Tr} \Phi^{-2}$  is uniquely determined by the requirement that it preserve the  $w_\infty$  symmetry structure [6]. There are further arguments why operators with negative powers of  $\Phi$  should be identified with “wrongly dressed” Liouville theory operators, of which the black hole mass perturbation is the leading example [5,7–10]. In Ref. [5] some calculations were performed in the deformed matrix model with a number of intriguing results. It was found that  $1/\sqrt{M}$  plays the role of the string coupling constant  $g_{\text{st}}$ , in agreement with string theory in the two-dimensional black hole background. The tree level odd-point functions were found to vanish, which provided one more argument in favor of the black hole analogy. Further studies of the deformed model, including some loop corrections, were performed in [9,10].

In this Letter we calculate the exact non-perturbative  $S$ -matrix of the fermion density perturbations in the deformed matrix model. We find that all the odd-point functions vanish at least to all orders in  $g_{\text{st}}$ . Furthermore, we show that the  $2k \rightarrow 1$  amplitudes vanish non-perturbatively and conjecture that this is true for odd-point functions with other kinematical structures. For the 2- and 4-point functions we give a few leading terms of the loop expansion.

Our exact solution of the deformed matrix model is based on the powerful method of Moore, Plesser and Ramgoolam [11], who constructed the  $S$ -matrix of the  $c = 1$  matrix model in terms of the single-fermion reflection coefficient. Remarkably, in the deformed model the reflection coefficient can also be calculated exactly. The crucial observation is that the single fermion wave function with energy  $(-\epsilon)$ , which satisfies the Schrödinger equation

$$\left( \frac{d^2}{dx^2} + x^2 - \frac{M}{x^2} - 2\epsilon \right) \psi_\epsilon(x) = 0, \quad (1)$$

is explicitly given by

$$\psi_\epsilon(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{i\pi}{2}(\alpha+\frac{1}{2})} e^{-\epsilon\pi/4} \frac{|\Gamma(\frac{1}{2} + \frac{i\epsilon}{2} + \alpha)|}{\Gamma(2\alpha + 1)} M_{i\epsilon/2, \alpha}(ix^2), \quad (2)$$

where  $\alpha = \frac{1}{4}\sqrt{1+4M}$  and  $M_{i\epsilon/2,\alpha}$  is the Whittaker function. The wave function is properly normalized and satisfies the correct boundary condition  $\psi_\epsilon(0) = 0$ . The scattering phase shift can be read off from the asymptotic formula,

$$\psi_\epsilon(x \rightarrow \infty) = \frac{1}{\sqrt{2\pi x}} \left( e^{-ix^2/2} e^{i\epsilon \ln x} S + e^{ix^2/2} e^{-i\epsilon \ln x} S^* \right), \quad (3)$$

where

$$S \equiv e^{i\pi(2\alpha+1)/4} \sqrt{\frac{\Gamma(\frac{1}{2} - \frac{i\epsilon}{2} + \alpha)}{\Gamma(\frac{1}{2} + \frac{i\epsilon}{2} + \alpha)}}.$$

Now we can calculate the asymptotic behavior of the resolvent  $I(x_1, x_2) = \langle x_1 | \frac{1}{H - \mu - iq} | x_2 \rangle$ . Introducing the classical time  $\tau$  through  $x^2(\tau) = \mu + \sqrt{M + \mu^2} \cosh(2\tau)$ , we find

$$I(x_1, x_2; q > 0) \underset{x_1, x_2 \rightarrow \infty}{=} \frac{i}{\sqrt{x_1 x_2}} \left\{ e^{i|G(\tau_1) - G(\tau_2)|} e^{-q|\tau_1 - \tau_2|} + R_q e^{i(G(\tau_1) + G(\tau_2))} e^{-q(\tau_1 + \tau_2)} \right\}, \quad (4)$$

where  $G(\tau) = -\frac{1}{4}\sqrt{M + \mu^2} e^{2\tau} + \mu\tau + \pi/4 + \mathcal{O}(e^{-2\tau})$  is the WKB phase factor for large  $\tau$ . The reflection coefficient is

$$R_q = \left( \frac{4}{M + \mu^2} \right)^{|q|/2} \frac{\Gamma(\frac{1}{2} - \frac{i\mu}{2} + \frac{|q|}{2} + \alpha)}{\Gamma(\frac{1}{2} + \frac{i\mu}{2} - \frac{|q|}{2} + \alpha)} e^{i[\frac{1}{2}\mu \log[(M + \mu^2)/4] - \mu + \pi\alpha]}. \quad (5)$$

As shown in Ref. [11], any scattering amplitude of the fermion density perturbations can be written in terms of integrals of products of reflection coefficients. Schematically, the relation is [11]

$$A(q_i) = \sum \int \prod (R_Q R_Q^*). \quad (6)$$

The  $l \rightarrow m$  amplitude is  $A_{l \rightarrow m}(q_1, \dots, q_l; -q_{l+1}, \dots, -q_{l+m})$ , where all  $q_i$  are taken to be positive. For now we work in the Euclidean domain and later continue to the Minkowski

signature. The explicit formula for the  $n \rightarrow 1$  amplitude reads

$$\begin{aligned}
A_{n \rightarrow 1}(q_1, \dots, q_n; -q) = & i^{n+1} \left\{ \sum_{\{i_1\}} \int_{q_{i_1}}^q dx R_{q-x} R_x^* - \sum_{\{i_1, i_2\}} \int_{q_{i_1+q_{i_2}}}^q dx R_{q-x} R_x^* + \right. \\
& \left. \dots + (-1)^{n-1} \sum_{\{i_1, \dots, i_{n-1}\}} \int_{q_{i_1+\dots+q_{i_{n-1}}}}^q dx R_{q-x} R_x^* - \int_0^q dx R_{q-x} R_x^* \right\}, \tag{7}
\end{aligned}$$

where  $\{i_1, i_2, \dots, i_k\}$  is a subset of  $\{1, 2, \dots, n\}$ . Similarly the  $2 \rightarrow 2$  amplitude in the kinematic region  $q_1 = \max\{q_i\}$  is given by

$$\begin{aligned}
A_{2 \rightarrow 2}(q_1, q_2; -q_3, -q_4) = & - \int_{q_1}^{q_1+q_2} dx R_{q_1+q_2-x} R_x^* - \int_0^{q_2} dx R_{q_1+q_2-x} R_x^* \\
& + \frac{1}{2} \left\{ \int_0^{q_2} dx R_{q_3-x} R_{q_2-x} R_{x+q_1-q_3}^* R_x^* + \int_{q_3-q_2}^{q_3} dx R_{q_3-x} R_{q_1-x} R_{x+q_2-q_3}^* R_x^* + (q_3 \rightarrow q_4) \right\}. \tag{8}
\end{aligned}$$

Our goal is to generate the asymptotic expansions of correlation functions in powers of  $g_{\text{st}} = 1/\sqrt{M}$ . In the following we set the Fermi level  $\mu$  to zero, according to the proposal of Ref. [5]. Our methods work equally well for  $\mu \neq 0$ , and we will report those results in a later publication. Let us first find the asymptotic expansion of the reflection coefficient. Introducing  $r_q \equiv e^{-i\pi\alpha} R_q$ , we have

$$\begin{aligned}
r_q &= \left(1 + \frac{1}{4M}\right)^{|q|/2} F(\alpha, q), \\
F(\alpha, q) &= \alpha^{-|q|} \frac{\Gamma(\frac{1}{2} + \frac{|q|}{2} + \alpha)}{\Gamma(\frac{1}{2} - \frac{|q|}{2} + \alpha)}. \tag{9}
\end{aligned}$$

It is easy to show that

$$F(-\alpha, q) = F(\alpha, q) \frac{1 + e^{-2\pi i\alpha} e^{-\pi i|q|}}{1 + e^{-2\pi i\alpha} e^{\pi i|q|}}. \tag{10}$$

The fraction on the right-hand side is equal to 1, up to terms that are invisible in the asymptotic expansion in powers of  $1/\alpha$ . Therefore, the odd powers are absent from the

asymptotic expansion,

$$F(\alpha, q) = 1 + \sum_{k=1}^{\infty} d_k(q) \alpha^{-2k} . \quad (11)$$

It follows that there are no odd powers of  $1/\sqrt{M}$  in the asymptotic expansion of the reflection coefficient,

$$r_q(M) = 1 + \sum_{k=1}^{\infty} c_k(q) M^{-k} . \quad (12)$$

The first few coefficients are given by

$$\begin{aligned} c_1(q) &= \frac{1}{24} q(7 - 4q^2) , \\ c_2(q) &= \frac{1}{5760} q(q-2)(501 + 128q - 536q^2 - 128q^3 + 80q^4) , \\ c_3(q) &= \frac{1}{2903040} q(q-2)(q-4)(115173 + 67968q - 137060q^2 \\ &\quad - 78720q^3 + 25072q^4 + 10752q^5 - 2240q^6) . \end{aligned} \quad (13)$$

Simple scaling arguments indicate that all the odd-point functions are expanded in odd powers of  $g_{st} = 1/\sqrt{M}$ . However, these powers are missing from Eq. (12) and, therefore, from Eq. (6) for the correlation functions. It follows immediately that *all the odd-point functions vanish to all orders in  $g_{st}$* . In fact, all the  $2k \rightarrow 1$  amplitudes vanish non-perturbatively. To prove this, consider formula (7) for a general  $n \rightarrow 1$  amplitude and perform the substitution  $x \rightarrow q - x$  in each of the integrals. Since the integrand  $r_{q-x} r_x$  is symmetric under this substitution, it easily follows that

$$A_{n \rightarrow 1} = (-1)^{n+1} A_{n \rightarrow 1} , \quad (14)$$

and therefore  $A_{2k \rightarrow 1} = 0$ . This result depends on  $R_q$  being real, up to a  $q$ -independent overall phase. For  $\mu \neq 0$  this property is lost, so that the odd-point functions no longer vanish non-perturbatively [9]. For  $\mu = 0$ , on the other hand, we expect that the odd-point functions with all kinematical structures vanish non-perturbatively.

Contrary to the odd-point functions, the even-point functions do not vanish and have non-trivial loop expansions. Using Eqs. (7) and (12), we find for the 2-point function

$$\begin{aligned}
A_{1 \rightarrow 1}(q, -q) &= \int_0^q dx r_x r_{q-x} = q + \frac{1}{24M} q^2 (7 - 2q^2) \\
&+ \frac{1}{5760M^2} q^2 (q-2) (501 + 128q - 236q^2 - 48q^3 + 24q^4) \\
&+ \frac{1}{2903040M^3} q^2 (q-2) (q-4) (115173 + 67968q - 49490q^2 \\
&\quad - 29328q^3 + 6088q^4 + 2688q^5 - 464q^6) + \dots
\end{aligned} \tag{15}$$

The one-loop result agrees with the collective field theory calculation [9]. The form of the higher-loop corrections is so intricate, however, that they would be virtually impossible to obtain in the bosonized formalism. Our results, on the other hand, give the entire non-perturbative answer in one compact formula. We also used Eq. (7) to find the expansion of the  $3 \rightarrow 1$  amplitude,

$$\begin{aligned}
A_{3 \rightarrow 1}(q_1, q_2, q_3; -q) &= 2 \int_0^q dx r_x r_{q-x} - 2 \sum_{i=1}^3 \int_0^{q_i} dx r_x r_{q-x} \\
&= \frac{1}{M} q_1 q_2 q_3 q \left[ 1 + \frac{1}{24M} (q-2) \left( 15 + 4q - q^2 - 3(q_1^2 + q_2^2 + q_3^2) \right) + \dots \right].
\end{aligned} \tag{16}$$

For the  $2 \rightarrow 2$  kinematic structure we find, using Eq. (8),

$$\begin{aligned}
A_{2 \rightarrow 2}(q_1, q_2; -q_3, -q_4) &= \frac{1}{M} q_1 q_2 q_3 q_4 \left[ 1 + \frac{1}{24M} \left( -30 + 7(q_1 + q_2) \right. \right. \\
&\quad \left. \left. + 12(q_1 + q_2)^2 - 12(q_1 q_2 + q_3 q_4) - 2(q_1 + q_2)^3 - 2q_1^3 + 6q_1 q_3 q_4 \right) + \dots \right].
\end{aligned} \tag{17}$$

Here the tree level answers agree with the collective field theory calculations of Ref. [5], but the loop corrections are new.

The correlation functions we derived constitute the Euclidean continuation of the  $S$ -matrix elements of the collective field theory. In order to continue back to the Minkowski signature, we have to take  $|q_i| \rightarrow -i\omega_i$  [12, 11]. For instance, the  $3 \rightarrow 1$  amplitude becomes

$$\mathcal{A}_{3 \rightarrow 1}(\omega_1, \omega_2, \omega_3; \omega) = \frac{1}{M} \omega_1 \omega_2 \omega_3 \omega \left[ 1 - \frac{1}{24M} (2 + i\omega) \left( 15 - 4i\omega + \omega^2 + 3(\omega_1^2 + \omega_2^2 + \omega_3^2) \right) + \dots \right],$$

where all energies are assumed positive. This continuation takes  $R_q$  into a pure phase and,

according to the arguments of Ref. [11], the non-perturbative  $S$ -matrix is unitary. This is, of course, related to the total reflection from the potential which approaches  $\infty$  as  $x \rightarrow 0$ .

For the Euclidean signature, the correlation functions of the ‘‘tachyon operators’’ of string theory are simply related to  $A(q_i)$  [13,14],

$$\langle T_{q_1} T_{q_2} \dots T_{q_n} \rangle = A(q_1, q_2, \dots, q_n) \prod_{i=1}^n L(q_i) . \quad (18)$$

The external leg factor  $L(q)$  can be calculated from a matrix model representation of the tachyon operator [11],

$$T_q \sim f(|q|) \int dt e^{iqt} \text{Tr} e^{-l\Phi^2(t)} , \quad (19)$$

where  $f(|q|)$  is a smooth function which determines the normalization. One finds that

$$L(q) \sim \int_{-\infty}^{\infty} d\tau e^{-l\sqrt{M} \cosh 2\tau} e^{-|q|\tau} \sim (l\sqrt{M})^{|q|/2} \Gamma(-|q|/2) . \quad (20)$$

We may chose the operator normalization  $f(|q|)$  so that

$$L(q) = M^{|q|/4} \frac{\Gamma(-|q|/2)}{\Gamma(|q|/2)} . \quad (21)$$

Now  $L(i\omega)$  is a pure phase, as needed for the unitarity of the Minkowski signature  $S$ -matrix. Note that  $L(q)$  has poles for  $|q| = 2n$ ,  $n > 0$ , while for the conventional  $c = 1$  model the poles occur for  $|q| = n > 0$ . This agrees with an argument for the position of the poles based on energy sum rules in the black hole conformal field theory [5,8]. Reproducing our exact correlation functions in the context of conformal field theory poses an interesting challenge.

In this Letter we calculated the exact  $S$ -matrix of the deformed  $c = 1$  matrix model for  $M > 0$ , which has been conjectured to describe the stationary black hole background of two-dimensional string theory. Even if the black hole analogy fails, this model is interesting in its own right because it leads to a new non-perturbatively calculable unitary  $S$ -matrix. There are many interesting extensions of this work. For example, one may consider the case of  $M < 0$ , which has been conjectured to describe a ‘‘naked singularity’’ [5]. We hope to return to these problems in a future paper.

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