

# Finite-size anyons and perturbation theory

Stefan Mashkevich\*

*Institute for Theoretical Physics, 252143 Kiev, Ukraine*

*and*

*Centre for Advanced Study, Drammensveien 78, N-0271 Oslo, Norway*

(March 22, 2018)

## Abstract

We address the problem of finite-size anyons, i.e., composites of charges and finite radius magnetic flux tubes. Making perturbative calculations in this problem meets certain difficulties reminiscent of those in the problem of pointlike anyons. We show how to circumvent these difficulties for anyons of arbitrary spin. The case of spin  $1/2$  is special because it allows for a direct application of perturbation theory, while for any other spin, a redefinition of the wave function is necessary. We apply the perturbative algorithm to the  $N$ -body problem, derive the first-order equation of state and discuss some examples.

PACS number(s): 03.65.Ge, 05.30.-d, 05.70.Ce, 71.10.Pm

Typeset using REVTeX

---

\*Email: mash@phys.unit.no

## I. INTRODUCTION

The topology of two-dimensional space allows for existence of anyons [1,2], particles with statistics intermediate between bosonic and fermionic. However, the real world being three-dimensional, no particles with inherent anyonic statistics can exist; in reality such statistics can only arise effectively by means of an interaction. Remarkably, a composite made of a charge  $e$  and a magnetic flux  $\Phi$  is an anyon [2], because interchanging two such composites multiplies their wave function by  $\exp(i\pi\alpha)$ , where  $\alpha = e\Phi/2\pi$ , in the spirit of the Aharonov-Bohm effect. Such composites arise naturally in Chern-Simons field theory, where, the magnetic field being proportional to the charge density, a point charge is at the same time a point flux. Quasiparticle excitations in the fractional quantum Hall effect are anyons [3] just due to the charge-flux interaction.

Since anyons are not fundamental particles but composites or quasiparticle excitations, they should have finite size. For example, the size of the FQHE excitations is of the order of the magnetic length [4]. Also, in field theory, if the gauge field Lagrangian is the Chern-Simons term *plus* some other term, a point charge generates a magnetic field smeared over a finite region. The simplest example is Maxwell-Chern-Simons theory [5,6]. Clearly, if the radius of the flux tube, which is essentially the inverse photon mass, is much smaller than any other distance scale (mean interparticle distance and/or thermal wavelength), then the particles are effectively ideal (pointlike) anyons. In the opposite limiting case, one has particles in a magnetic field without any change of statistics. Hence there is “distance dependent statistics” [7]: If the particles themselves are, say, bosons, they behave like bosons when being close together, but like anyons when being far apart.

One more parameter, the size of the anyons, being present makes quantum mechanical problems more complicated. The two-body problem was considered in Refs. [6–9], where also some general results and conjectures for the  $N$ -body problem can be found. Since even fewer results can be derived exactly than for ideal anyons, it is natural to try using perturbation theory for small values of the statistical parameter. For ideal anyons, however, perturbation

theory gives senseless infinite results in the  $s$ -wave sector close to bosonic statistics, due to the singular nature of the anyonic interaction. There are several methods to improve the situation, which are in fact equivalent to each other [10] and may be reduced to the following: Modify the original Hamiltonian  $H_N$  by adding a repulsive contact interaction, i.e., consider  $H'_N = H_N + c \sum_{j < k}^{1,N} \delta^2(\mathbf{r}_{jk})$ ; this does not affect the exact solutions, but for a special value of  $c$  all the divergences cancel, leading to the correct finite result [11–13]. This value, as was demonstrated in [14], corresponds to assigning to the particles a spin  $1/2$ , interacting with the magnetic field inside the flux tube (cf. [15]). It is also possible (but not necessary) to redefine the wave function in such a way that the terms leading to divergences cancel already in the Hamiltonian (in addition, also the three-body interaction terms of  $H_N$  then cancel).

Consider the situation in more detail. Since the nature of the difficulty is connected with two-body interaction, it is in fact sufficient to restrict oneself to the two-body problem. In the  $L = 0$  sector of the relative Hamiltonian of two ideal anyons there are two linearly independent solutions, one regular and the other one singular at  $r = 0$  ( $r$  is the interparticle distance) but still normalizable. A generic wave function will behave for small  $r$  as  $r^\alpha + \kappa r^{-\alpha}$ , where  $\kappa$  has to be fixed as a boundary condition at  $r = 0$ , which corresponds to choosing one of the possible self-adjoint extensions of the Hamiltonian [16]. Adding a repulsive contact term excludes the singular solution, i.e., fixes  $\kappa = 0$ . However, this holds for any value of  $c$ , while perturbation theory works for one special value only. On the other hand, if the singular flux tube is understood as a limiting case of a finite radius one, then normally only the regular solution will survive, even with no contact term at all. In fact, the contact term has to be regularized itself—and thereby given a physical meaning—and the result (even in the singular limit) may depend on the regularization chosen. A simple and natural way is to ascribe to the particles a magnetic moment, or spin, which couples to the magnetic field inside the flux tube. The solution will then exhibit a continuous dependence on the value of the spin, but in the singular limit the problem becomes scale invariant, and therefore the dimensionful parameter  $\kappa$  can only tend to  $\infty$  or to  $0$  (unless a distance scale is introduced

by hand [16], which will not be considered here). That is, the solution can only tend to the purely singular one (as we will see, this happens for spin  $+1/2$ , where  $+$  means attractive, i.e., parallel to the flux tube) or to the purely regular one (for any other spin).

Our present goal is to reproduce these results within (first-order) perturbation theory, in order to make possible its application to the  $N$ -body problem. It works best for spin  $\pm 1/2$ , considered in detail in Ref. [14], but does not work directly for any other spin. Analyzing the two-body problem, we show how the wave function has to be redefined in order that the first-order result be in agreement with the exact solution. We then move on to the  $N$ -body problem. Here again, spin  $\pm 1/2$  is most simple in the sense that it is apparently the only case allowing for a cancellation of the three-body terms; however, our algorithm always works in the sense that it allows to get rid of the singularities. We apply it to derive the first-order perturbative equation of state, using the second-quantized formalism. The limiting cases of small and large flux tube radius reproduce the ideal anyon and the mean-field equations of state, respectively, and an example illustrates the interpolation between the two.

## II. THE TWO-BODY PROBLEM

Consider the problem of two finite-size anyons. Let  $m$  be their mass,  $e$  the charge and  $\sigma$  the spin (more precisely, the projection of the spin on the  $z$  axis, which can be positive or negative),  $\alpha$  the statistics parameter at infinite distance and  $\varepsilon(r)$  the function describing the flux profile (which we assume to be radially symmetric), so that

$$\Phi(r) = \frac{2\pi\alpha}{e}\varepsilon(r), \quad B(r) = \frac{\alpha}{er}\varepsilon'(r) \quad (1)$$

are the flux through the circle of radius  $r$  and the magnetic field at a distance  $r$ , respectively; one has  $\varepsilon(\infty) = 1$ , and we will assume in the sequel that  $\varepsilon(0) = 0$ , i.e., there is no singular flux at the center (even if the magnetic field still may be singular). As well as in Ref. [14], spin does not appear from a relativistic formulation but is introduced by hand. The magnetic moment, which couples to the magnetic field, is  $\mu = \sigma \frac{e}{m}$  [17]. Then the radial part of the

relative Hamiltonian, with a harmonic potential added (the latter, as usually, serves only for discretizing the spectrum and is irrelevant for the essence of the matter), is

$$\mathcal{H} = \frac{1}{m} \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{[L - \alpha\varepsilon(r)]^2}{r^2} - \frac{2\sigma\alpha\varepsilon'(r)}{r} \right) + \frac{1}{4}m\omega^2 r^2; \quad (2)$$

$L$  is the angular momentum, and we assume the “bare” particles to be bosons, whence  $L$  must be even. We now wish to treat the  $\alpha$  dependent terms as a perturbation. The difficulties arise for  $L = 0$ , when the unperturbed wave functions do not vanish at the origin. In fact, it is enough to consider the ground state. For  $\alpha = 0$ , its wave function is

$$\psi_0(r) = \sqrt{m\omega} \exp\left(-\frac{1}{4}m\omega r^2\right) \quad (3)$$

and its energy is  $\omega$ . Assume, for simplicity, that  $\varepsilon(r) = 1$  for  $r > R$  (that is, there is no magnetic field outside the circle of radius  $R$ ), where  $R$  is the size of the anyons. Then the perturbative correction from the third term of (2) is

$$\begin{aligned} \Delta E &= \langle \psi_0 | \frac{\alpha^2 \varepsilon^2(r)}{mr^2} | \psi_0 \rangle \\ &= \int_0^\infty \frac{\alpha^2 \varepsilon^2(r)}{mr^2} m\omega \exp\left(-\frac{1}{2}m\omega^2 r^2\right) r dr \\ &= \int_0^R + \alpha^2 \omega \int_R^\infty \frac{1}{r} \exp\left(-\frac{1}{2}m\omega^2 r^2\right) dr \\ &\xrightarrow{R \rightarrow 0} \alpha^2 \omega (-\ln R + \text{finite terms}), \end{aligned} \quad (4)$$

diverging in the singular limit  $R \rightarrow 0$ . Therefore, except possibly for some special values of  $\sigma$ , the ground state of  $\mathcal{H}$  for small enough  $R$  cannot be reproduced perturbatively in a straightforward way. (In fact, for  $R = 0$  the exact result, as we will see, is  $\Delta E = |\alpha|\omega$  for any  $\sigma \neq +1/2$ .)

### III. EXACT SOLUTION

To be able to check the perturbative results, let us first find the exact ground state energy of  $\mathcal{H}$ . Symmetry with respect to a change  $(\alpha, \sigma) \rightarrow (-\alpha, -\sigma)$  is present, so it is enough to consider  $\alpha \geq 0$ , which is what we will assume, unless otherwise specified. For  $r > R$ , the wave function with the correct behavior at infinity is

$$\psi_{>}(r) = r^\alpha U\left(\frac{1}{2}\left(1 + \alpha - \frac{E}{\omega}\right), 1 + \alpha; \frac{1}{2}m\omega r^2\right) \exp\left(-\frac{1}{4}m\omega r^2\right), \quad (5)$$

where  $U$  is the confluent hypergeometric function. The energy  $E$  can be determined by writing the boundary condition at  $r = R$ ,

$$\frac{\psi'(R)}{\psi(R)} = \frac{\lambda\alpha}{R} \quad (6)$$

and substituting first  $\psi_{>}(r)$  and then  $\psi_{<}(r)$ , the wave function for  $r < R$ , thereby obtaining two equations connecting  $E$  and  $\lambda$ . To make the point clear, let us introduce two restrictions: (i)  $\alpha \ll 1$ , since we are interested in perturbation theory; (ii)  $q \equiv m\omega R^2/2 \ll 1$ , since we are now concerned about the singular limit, in which  $q \rightarrow 0$ . In this approximation<sup>1</sup>, the first equation in question reads

$$\frac{E}{\omega} = 1 + \alpha \frac{1 + \lambda - (1 - \lambda)q^\alpha}{1 + \lambda + (1 - \lambda)q^\alpha}. \quad (7)$$

Note that substituting the asymptotic form  $\psi(r) = r^\alpha + \kappa r^{-\alpha}$  into (6) yields the relation  $\kappa = R^{2\alpha}(1 - \lambda)/(1 + \lambda)$ . Therefore in the singular limit there are only two possibilities: If  $\lambda \rightarrow -1$  (and  $1 + \lambda$  tends to zero faster than  $R^{2\alpha}$ , which will be true for small enough  $\alpha$ ), then the purely singular solution survives and  $E \rightarrow (1 - \alpha)\omega$ , in any other case the regular solution survives and  $E \rightarrow (1 + \alpha)\omega$ . On the other hand, for perturbation theory it is the factor  $q^\alpha$  in (7) that is troublesome, because its perturbative expansion  $q^\alpha = 1 + \alpha \ln q + \frac{\alpha^2 \ln^2 q}{2} + \dots$  needs more and more terms as  $q$  tends to 0. This factor cancels out for  $\lambda \rightarrow \pm 1$ , and one expects just these two situations to cause no difficulties for perturbation theory.

To proceed with the exact solution, one has to supply a function  $\varepsilon(r)$  for  $r < R$ . The simplest choice here is  $\varepsilon(r) = r^2/R^2$ , corresponding to the magnetic field being uniform inside the circle of radius  $R$  [8,9,14]. Then (2) becomes

$$\mathcal{H}_{<} = \frac{1}{m} \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) + \frac{1}{4}m\tilde{\omega}^2 r^2 + E_0, \quad (8)$$

---

<sup>1</sup>We will, however, continue to refer to the result as “exact”, just to stress that it is obtained directly from the Schrödinger equation.

where

$$\tilde{\omega}^2 = \omega^2 + \frac{\alpha^2}{4m^2R^4}, \quad E_0 = -\frac{4\sigma\alpha}{mR^2}. \quad (9)$$

The corresponding solution—regular at the origin, since there is no singular flux tube—is

$$\psi_{<}(r) = {}_1F_1\left(\frac{1}{2}\left(1 - \frac{E - E_0}{\tilde{\omega}}\right), 1; \frac{1}{2}m\tilde{\omega}r^2\right) \exp\left(-\frac{1}{4}m\tilde{\omega}r^2\right), \quad (10)$$

and substitution into (6) yields

$$\lambda = \frac{1}{\alpha} \frac{qE/\omega + 2\sigma\alpha}{qE/2\omega - 1}. \quad (11)$$

For small enough  $q$  one may put  $\lambda = -2\sigma$ , because  $q$  tends to zero faster than  $q^\alpha$ . Consequently, the singular solution arises for  $\sigma = +1/2$  (i.e., attractive), the regular one does for any other  $\sigma$ . Perturbation theory experiences difficulties for any  $\sigma \neq \pm 1/2$ . In particular, for the spinless case  $\sigma = 0$  one gets  $E/\omega = 1 + \alpha(1 - q^\alpha)/(1 + q^\alpha)$ , i.e.,

$$\Delta E = -\alpha\omega \tanh\left(\frac{\alpha}{2} \ln q\right). \quad (12)$$

For  $q \rightarrow 0$ , this tends to  $\Delta E = \alpha\omega$ , but to see this, one has to take into account *all* orders in  $\alpha$ .

#### IV. PERTURBATION THEORY

Let us now come back to perturbation theory. Following the idea of [11,12], in order to get rid of the singularities we redefine the wave function as

$$\psi(r) = f(r)\tilde{\psi}(r), \quad (13)$$

where  $f(r)$  is to be fixed at our convenience. Then the Hamiltonian acting on  $\tilde{\psi}(r)$  is

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1 + \tilde{\mathcal{H}}_2, \quad (14)$$

where

$$\tilde{\mathcal{H}}_0 = \frac{1}{m} \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) + \frac{1}{4} m \omega^2 r^2, \quad (15)$$

$$\tilde{\mathcal{H}}_1 = -\frac{2}{m} \frac{f'(r)}{f(r)} \frac{d}{dr}, \quad (16)$$

$$\tilde{\mathcal{H}}_2 = \frac{1}{m} \left( -\frac{f''(r)}{f(r)} - \frac{1}{r} \frac{f'(r)}{f(r)} + \frac{\alpha^2 \varepsilon^2(r)}{r^2} - \frac{2\sigma\alpha\varepsilon'(r)}{r} \right). \quad (17)$$

The purpose of the redefinition is to have the dangerous  $1/r^2$  term canceled, by an appropriate choice of  $f(r)$ . Asymptotically for  $r \rightarrow \infty$ , when  $\varepsilon(r) = 1$ , demanding it to be canceled by any linear combination of the first two terms of  $\tilde{\mathcal{H}}_2$  will make  $f(r)$  be a power function, and then each of these two terms will itself be proportional to  $1/r^2$  and therefore should be canceled as well. As we will see, the correct way is to demand that all four terms be canceled, i.e., that  $\tilde{\mathcal{H}}_2 \equiv 0$ , or

$$r^2 f''(r) + r f'(r) - [\alpha^2 \varepsilon^2(r) - 2\sigma\alpha\varepsilon'(r)] f(r) = 0. \quad (18)$$

This is the essence of the perturbative algorithm for our problem: Find  $f(r)$  from (18), choosing the solution which is nonsingular at the origin, and then regard  $\tilde{\mathcal{H}}_1$  as a perturbation. This algorithm will yield the correct result in the singular limit.

Remarkably, for  $\sigma = \mp 1/2$  there is a universal solution to (18),

$$f(r) = \exp \left[ \pm \alpha \int_0^r \frac{\varepsilon(r')}{r'} dr' \right], \quad (19)$$

which is precisely the ansatz considered in [14]. Then one obtains  $\mathcal{H}_1 = \mp \frac{2}{m} \frac{\alpha\varepsilon(r)}{r} \frac{d}{dr}$ , which yields the correct first-order result; in the singular limit it is  $\Delta E = \pm\alpha\omega$ . In fact, here this redefinition is not necessary; acting directly with the original problem (2) is more complicated but still possible, because the divergent first-order contribution from the  $1/r^2$  term turns out to be canceled by the second-order contribution from the spin term, while the first-order contribution from the latter gives the correct answer.

For arbitrary spin, there is apparently no universal solution. However, use can be made of the fact that  $\alpha \ll 1$ , to construct an approximate one. Coming back to the uniform magnetic field model, one has



$$\begin{cases} r^2 f''(r) + r f'(r) - \left[ \frac{\alpha^2 r^4}{R^4} - \frac{4\sigma\alpha r^2}{R^2} \right] f(r) = 0, & r < R, \\ r^2 f''(r) + r f'(r) - \alpha^2 f(r) = 0, & r > R. \end{cases} \quad (20)$$

It is easy to see that  $f_<(r) = 1 - \sigma\alpha \frac{r^2}{R^2}$  is the nonsingular solution to the first equation, neglecting terms of the order  $\alpha^2$ . The general solution to the second one is  $f_>(r) = C_1 r^\alpha + C_2 r^{-\alpha}$ , and matching  $f$  and  $df/dr$  at  $r = R$  yields

$$f(r) = \begin{cases} 1 - \sigma\alpha \frac{r^2}{R^2}, & r < R, \\ \frac{1-\sigma(2+\alpha)}{2} \left(\frac{r}{R}\right)^\alpha + \frac{1+\sigma(2-\alpha)}{2} \left(\frac{R}{r}\right)^\alpha, & r > R. \end{cases} \quad (21)$$

Now the first-order correction from  $\mathcal{H}_1$  is

$$\begin{aligned} \Delta E &= \langle \psi_0 | \tilde{\mathcal{H}}_1 | \psi_0 \rangle \\ &= 2\sigma\alpha\omega \frac{qe^{-q} + e^{-q} - 1}{q} + \alpha\omega \int_q^\infty \frac{(1-2\sigma)x^\alpha - (1+2\sigma)q^\alpha}{(1-2\sigma)x^\alpha + (1+2\sigma)q^\alpha} e^{-x} dx. \end{aligned} \quad (22)$$

For  $q \ll 1$ , the first term can be neglected, and in the second term one may replace  $x^\alpha$  by 1 (this is legal unless  $|\ln x| \gtrsim 1/\alpha$ , but the main contribution to the integral is given by the values  $x \sim 1$ ), and then the lower limit of integration can be replaced by 0, which yields the exact result (7).

It is worthwhile to note that the “straightforward” result (4), for the spinless case, is correct in a sense; indeed, if in (12) one fixes  $q$  and goes to the limit  $\alpha \rightarrow 0$ , one does get (4). This is natural: There being no singularity, the straightforward perturbation theory works for small enough  $\alpha$ ; but its range of validity shrinks to zero in the limit  $q \rightarrow 0$ , and to get the expression which remains valid in this limit, one has to redefine the wave function as described above.

Concerning the excited states, for the ones with  $L = 0$  and the radial quantum number  $n \neq 0$  one has to apply the same algorithm, because its sense is to take care of the short-distance behavior of the wave function, which is independent of  $n$ . For  $L \neq 0$ , perturbation theory is directly applicable, because already the unperturbed wave functions vanish at the origin, although making the redefinition will do no harm. Thus, the algorithm can be applied for all the states, which is essential for the second-quantized formalism.

## V. THE $N$ -BODY PROBLEM AND THE PERTURBATIVE EQUATION OF STATE

The main goal of perturbation theory is its application to the  $N$ -body problem. The Hamiltonian in our case reads

$$H_N = \frac{1}{2m} \sum_{j=1}^N \left[ \left( -i \frac{\partial}{\partial \mathbf{r}_j} - e \sum_{k \neq j} \mathbf{A}(\mathbf{r}_{jk}) \right)^2 - 2\sigma e \sum_{k \neq j} B(r_{jk}) \right] + V, \quad (23)$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{\alpha \mathbf{e}_z \times \mathbf{r}}{e r^2} \varepsilon(r) \quad (24)$$

is the vector potential and  $B(r)$  is the magnetic field as in (1);  $\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k$ , and  $r_{jk} = |\mathbf{r}_{jk}|$ .

The wave function now is to be transformed as

$$\psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \left( \prod_{j < k}^{1, N} f(r_{jk}) \right) \tilde{\psi}(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (25)$$

with  $f(r)$  determined as above. Then the Hamiltonian acting on  $\tilde{\psi}$  is

$$\begin{aligned} \tilde{H}_N = \frac{1}{2m} \sum_{j=1}^N \left[ -\frac{\partial^2}{\partial \mathbf{r}_j^2} + 2i\alpha \sum_{k \neq j} \frac{\mathbf{e}_z \times \mathbf{r}_{jk}}{r_{jk}} \varepsilon(r_{jk}) \frac{\partial}{\partial \mathbf{r}_j} - 2 \sum_{k \neq j} \frac{f'(r_{jk}) \mathbf{r}_{jk}}{f(r_{jk}) r_{jk}} \frac{\partial}{\partial \mathbf{r}_j} \right. \\ \left. + \sum_{\substack{k, l \neq j \\ k \neq l}} \left( -\frac{f'(r_{jk}) f'(r_{jl})}{f(r_{jk}) f(r_{jl})} + \alpha^2 \frac{\varepsilon(r_{jk}) \varepsilon(r_{jl})}{r_{jk} r_{jl}} \right) \frac{\mathbf{r}_{jk} \mathbf{r}_{jl}}{r_{jk} r_{jl}} \right] + V, \quad (26) \end{aligned}$$

where Eq. (18) has been taken into account. This contains two-body as well as three-body interaction terms. If  $\sigma = \pm 1/2$  then, because of Eq. (19), the three-body terms cancel, otherwise they do not. This certainly makes the problem more complicated, however these terms are of second order in  $\alpha$  and they do not produce any singularities. Therefore in the first order one omits them and considers the second and the third terms as perturbation. The first-order contribution of the second term, in fact, vanishes, and by virtue of the symmetry of the wave function the contribution of the third term can be represented as

$$\Delta E = \frac{N(N-1)}{2} \langle \psi_{\text{sym}} | \tilde{\mathcal{H}}_1(\mathbf{r}_1, \mathbf{r}_2) | \psi_{\text{sym}} \rangle, \quad (27)$$

where

$$\tilde{\mathcal{H}}_1(\mathbf{r}_1, \mathbf{r}_2) = -\frac{2}{m} \frac{f'(r_{12})}{f(r_{12})} \frac{\mathbf{r}_{12}}{r_{12}} \frac{\partial}{\partial \mathbf{r}_1} \quad (28)$$

[cf. (16)].

It is now possible to derive the (first-order) perturbative equation of state for finite-size anyons near both Bose and Fermi statistics. (In the second case, as well as for bosons outside the  $s$ -wave sector, the redefinition of the wave function is not necessary but does not change the result.) The simplest way is to use the second quantized formalism. The starting point is the expression for the first-order correction to the thermodynamic potential [20]

$$\begin{aligned} \Omega_1 = & \int_0^\beta d\beta_1 \int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \tilde{\mathcal{H}}_1(\mathbf{r}_1, \mathbf{r}_2) \left[ \left\{ \psi^\dagger(\mathbf{r}_1, \beta_1) \psi(\mathbf{r}_1, \beta_1) \right\} \left\{ \psi^\dagger(\mathbf{r}_2, \beta_1) \psi(\mathbf{r}_2, \beta_1) \right\} \right. \\ & \left. \pm \left\{ \psi^\dagger(\mathbf{r}_1, \beta_1) \psi(\mathbf{r}_2, \beta_1) \right\} \left\{ \psi^\dagger(\mathbf{r}_2, \beta_1) \psi(\mathbf{r}_1, \beta_1) \right\} \right], \end{aligned} \quad (29)$$

where  $\left\{ \psi^\dagger(\mathbf{r}_1, \beta_1) \psi(\mathbf{r}_2, \beta_2) \right\}$  is the one-particle thermal Green function, and the upper/lower sign refers to bosons/fermions. Taking into account the explicit form of  $\tilde{\mathcal{H}}_1$  (note that the derivative acts on  $\psi^\dagger$  only), it is possible to do the spatial integration by parts, and the surface term will vanish. There is an expression (see [20] for details)

$$\left\{ \psi^\dagger(\mathbf{r}_1, \beta_1) \psi(\mathbf{r}_2, \beta_1) \right\} = -\sum_{s=1}^{\infty} (\pm z)^s G(\mathbf{r}_1, \mathbf{r}_2; s\beta), \quad (30)$$

where  $z$  is the fugacity and

$$G(\mathbf{r}_1, \mathbf{r}_2; \beta) = \frac{1}{\lambda^2} \exp \left[ -\frac{\pi(\mathbf{r}_1 - \mathbf{r}_2)^2}{\lambda^2} \right] \quad (31)$$

is the one-particle plane wave thermal Green function in the thermodynamic limit,  $\lambda = \sqrt{2\pi\beta/m}$  being the thermal wavelength. Equation (29) then turns into

$$\Omega_1 = \alpha \frac{V}{2\lambda^2} \sum_{n=1}^{\infty} c_n (\pm z)^n \quad (32)$$

with

$$c_n = \frac{1}{2} \sum_{s=1}^{n-1} \frac{I_{s, n-s}}{s(n-s)}, \quad (33)$$

$$I_{st} = \frac{1}{\alpha} \int_0^\infty dr \frac{d}{dr} \left[ r \frac{f'(r)}{f(r)} \right] \left[ 1 \pm \exp \left( -\frac{s+t}{st} \frac{\pi r^2}{\lambda^2} \right) \right]. \quad (34)$$

According to Eq. (18), one has

$$\begin{aligned} f(r) &\xrightarrow{r \rightarrow 0} 1, \\ f(r) &\xrightarrow{r \rightarrow \infty} C_1 r^\alpha + C_2 r^{-\alpha}, \end{aligned} \quad (35)$$

therefore the integral above is in fact proportional to  $\alpha$ , and  $c_n \propto 1$ . Now, one has for the pressure

$$\begin{aligned} P\beta &= -(\Omega_0 + \Omega_1)/V \\ &= \frac{1}{\lambda^2} \sum_{n=1}^{\infty} \left( \pm \frac{1}{n^2} - \alpha c_n \right) (\pm z)^n \end{aligned} \quad (36)$$

[ $\Omega_0 = \mp(V/\lambda^2) \sum_{n=1}^{\infty} (\pm z)^n/n^2$  being the unperturbed thermodynamic potential] and for the density

$$\rho = z \frac{\partial}{\partial z} (P\beta) \quad (37)$$

$$= \frac{1}{\lambda^2} \left[ \mp \ln(1 \mp z) - \alpha \sum_{n=1}^{\infty} n c_n (\pm z)^n \right]; \quad (38)$$

neglecting  $\alpha^2$ , the solution for  $z$  is

$$z = z_0 + \alpha(1 \mp z_0) \sum_{n=1}^{\infty} n c_n (\pm z_0)^n, \quad (39)$$

where

$$z_0 = \pm [1 - \exp(\mp \lambda^2 \rho)] \quad (40)$$

is the unperturbed fugacity. Equations (36) and (39) explicitly give  $P$  as a function of  $\rho$ , i.e., the equation of state, which is thus obtained for any flux profile in terms of integrals.

Introducing the virial expansion,

$$P\beta = \rho [1 + A_2(\rho\lambda^2) + A_3(\rho\lambda^2)^2 + \dots], \quad (41)$$

the first few virial coefficients are

$$\begin{aligned} A_2 &= \mp \frac{1}{4} + \alpha \left( -\frac{1}{2} c_1 + c_2 \right), \\ A_3 &= \frac{1}{36} \pm \alpha \left( \frac{1}{3} c_1 - 2c_2 + 2c_3 \right), \\ A_4 &= \alpha \left( -\frac{1}{8} c_1 + \frac{7}{4} c_2 - \frac{9}{2} c_3 + 3c_4 \right), \\ A_5 &= -\frac{1}{3600} \pm \alpha \left( \frac{1}{30} c_1 - c_2 + 5c_3 - 8c_4 + 4c_5 \right). \end{aligned} \quad (42)$$

## VI. EXAMPLES

Consider some particular cases. We will now drop the condition  $\alpha > 0$ . According to (35), the main term of  $f(r)$  at  $r \rightarrow \infty$  is proportional to  $r^{|\alpha|}$  unless the corresponding  $C$  vanishes, whence it is proportional to  $r^{-|\alpha|}$  [in particular, it is so when  $\sigma \frac{\alpha}{|\alpha|} = +1/2$  and  $\varepsilon(r) = \eta(r)$ ]. In the first case, it is easy to see from (34) that

$$I_{st} = \begin{cases} (1 \pm 1) \frac{\alpha}{|\alpha|} & \text{for } R \ll \lambda, \\ \frac{\alpha}{|\alpha|} & \text{for } R \gg \lambda. \end{cases} \quad (43)$$

This leads to simple results:

$$R \ll \lambda: \quad A_2 = \mp 1/4 + \frac{1 \pm 1}{2} |\alpha|, \quad (44)$$

$$R \gg \lambda: \quad A_2 = \mp 1/4 + \frac{1}{2} |\alpha|, \quad (45)$$

and the higher virial coefficients are unaffected. The first equation is the well-known first-order result for ideal anyons [20], the second one is a mean-field result [21]—indeed, in that limit the magnetic field gets smeared over the whole volume. In the second case—that is, essentially for attractive spin 1/2—there is an additional minus sign in the correction terms, so that the second virial coefficient can become lower than the bosonic one (cf. [22]).

The transition between the two limiting regimes can be observed on a simple model where an explicit expression for  $c_n$  is within reach. Let  $\sigma = -1/2$  and  $\varepsilon(r) = 1 - \exp(-r^2/R^2)$ , so that the magnetic field  $B(r) = (2\alpha/eR^2) \exp(-r^2/R^2)$  is Gaussian. Then, by virtue of (19),  $rf'(r)/f(r) = \varepsilon(r)$ , and one gets

$$I_{st} = 1 \pm \frac{1}{1 + \xi^2 \frac{s+t}{st}}, \quad (46)$$

$$c_n = \frac{\gamma + \psi(n)}{n} \pm \frac{1}{2\sqrt{n^2 + 4n\xi^2}} \left[ \psi\left(\frac{n + \sqrt{n^2 + 4n\xi^2}}{2}\right) - \psi\left(\frac{2 - n + \sqrt{n^2 + 4n\xi^2}}{2}\right) - \psi\left(\frac{n - \sqrt{n^2 + 4n\xi^2}}{2}\right) + \psi\left(\frac{2 - n - \sqrt{n^2 + 4n\xi^2}}{2}\right) \right], \quad (47)$$

where

$$\xi = \sqrt{\pi} \frac{R}{\lambda}, \quad (48)$$

$\gamma$  is Euler's constant,  $\psi(x) = \Gamma'(x)/\Gamma(x)$ . [The same expressions will of course be valid for any other  $\sigma$  and  $\varepsilon(r)$  that lead to the same  $f(r)$ , through Eq. (18).]

At the lowest order in  $\xi$  and  $1/\xi$ , respectively, the virial coefficients are

$$\begin{aligned} A_2 &= \mp \frac{1}{4} + \alpha \left( \frac{1 \pm 1}{2} \mp \xi^2 \right), \\ A_3 &= \frac{1}{36} + \alpha \frac{\xi^2}{2}, \\ A_4 &= \mp \alpha \frac{\xi^2}{12}, \\ A_5 &= -\frac{1}{3600}, \end{aligned} \quad (49)$$

$$\begin{aligned} A_2 &= \mp \frac{1}{4} + \alpha \left( \frac{1}{2} \pm \frac{1}{4\xi^2} \right), \\ A_3 &= \frac{1}{36} + \alpha \frac{1}{6\xi^2}, \\ A_4 &= \pm \alpha \frac{1}{16\xi^2}, \\ A_5 &= -\frac{1}{3600} + \alpha \frac{1}{60\xi^2}. \end{aligned} \quad (50)$$

The Bose and Fermi  $\xi$  dependent terms are equal to each other for odd coefficients and have opposite signs for even ones, and for  $\xi \ll 1$  they in fact vanish for all odd coefficients but the third. Plots of the first five virial coefficients as functions of  $\xi$  near Bose statistics, showing the intermediate behavior, are displayed in Fig. 1.

Note that there is no symmetry with respect to  $\alpha \rightarrow -\alpha$  here; indeed, the spin is repulsive for  $\alpha > 0$  but attractive for  $\alpha < 0$ .

## VII. DISCUSSION

Let us summarize the main points of our reasoning. In the problem of finite-size anyons that was considered, there are no singularities and therefore all results, exact or perturbative, are finite. Therefore, had it been possible to calculate the perturbative corrections to all

orders (and provided the series converged), the exact result for the problem at hand could be obtained without any special treatment. What is achieved by the redefinition of the wave function—in a way that actually takes into account the short-distance behavior of the exact solution—is the result being obtained at lower order of perturbation theory than it would be without this redefinition. For spin  $1/2$ , it is obtained at first order instead of second; for any other spin, at first order instead of infinite.

There is a certain subtlety concerning the transformation of the Hamiltonian. The correct singular limit is  $\varepsilon(r) = \eta(r)$  (the step function) rather than  $\varepsilon(r) = 1$  [14], so that the spin term in (2) becomes a contact term  $-\frac{4\pi\sigma\alpha}{m}\delta^2(\mathbf{r})$ . For  $\sigma = \mp 1/2$ , the function  $f(r)$  defined by (19) tends in the singular limit to  $f_0(r) = r^{\pm\alpha}$ , and it is the equality  $\Delta \ln f_0(r) = \pm 2\pi\alpha\delta^2(\mathbf{r})$  that ensures the cancellation of the contact term when the Hamiltonian is redefined [10,14]. Now, for arbitrary  $\sigma$ , the function  $f(r)$  defined by (21) also tends to  $r^\alpha$ , but only *pointwise*; an elementary calculation shows that  $\Delta \ln f(r) \rightarrow -4\pi\sigma\alpha\delta^2(\mathbf{r})$ , thereby ensuring the correct transformation of the Hamiltonian. Again, spin  $1/2$  is singled out, in the sense that it is in this case only that the limiting transition under the operator  $\Delta$  is legal.

The encoding of the short-distance behavior of the two-body wave function in perturbation theory allows for a perturbative treatment of the  $N$ -body problem. Again, this turns out to be most simple for spin  $1/2$  (because the three-body terms cancel), but still possible, in principle, for arbitrary spin. The perturbative equation of state is obtained in an explicit form and shows smooth interpolation between the two limiting cases, the ideal anyon one and the mean field one.

Numerous stimulating discussions with Stéphane Ouvry are gratefully acknowledged. I am also thankful to Giovanni Amelino-Camelia for useful remarks. Parts of this work were done at the theory division of the Institut de Physique Nucléaire, Orsay and at the Centre for Advanced Study at the Norwegian Academy of Science and Letters, Oslo, thanks to which are due for kind hospitality and support.

## REFERENCES

- [1] J.M. Leinaas, J. Myrheim, Nuovo Cimento B **37**, 1 (1977).
- [2] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982), *ibid.* **49**, 957 (1982).
- [3] R.B. Laughlin, Phys. Rev. Lett. **50**, 1953 (1983); B. Halperin, *ibid.* **52**, 1583 (1984).
- [4] D. Arovas, J.R. Schrieffer, F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
- [5] K. Shizuya, H. Tamura, Phys. Lett. B **252**, 412 (1990).
- [6] G. Zinoviev, S. Mashkevich, H. Sato, JETP **78**, 105 (1994) [Russian original: Zh. Eksp. Teor. Fiz. **105**, 198 (1994)].
- [7] S. Mashkevich, Phys. Rev. D **48**, 5953 (1993).
- [8] C.A. Trugenberger, Phys. Lett. B **288**, 121 (1992).
- [9] S. Mashkevich, G. Zinoviev, JETP **82**, 813 (1996). [Russian original: Zh. Eksp. Teor. Fiz. **109**, 1512 (1996).]
- [10] S. Ouvry, Phys. Rev. D **50**, 5296 (1994).
- [11] J. McCabe, S. Ouvry, Phys. Lett. B **260**, 113 (1991); A. Comtet, J. McCabe, S. Ouvry, Phys. Lett. B **260**, 372 (1991).
- [12] D. Sen, Nucl. Phys. B **360**, 397 (1991).
- [13] G. Amelino-Camelia, Phys. Rev. D **51**, 2000 (1995).
- [14] A. Comtet, S. Mashkevich, S. Ouvry, Phys. Rev. D **52**, 2594 (1995).
- [15] C.R. Hagen, Phys. Rev. D **52**, 2466 (1995).
- [16] C. Manuel, R. Tarrach, Phys. Lett. B **268**, 222 (1991); M. Bourdeau, R.D. Sorkin, Phys. Rev. D **45**, 687 (1992); G. Amelino-Camelia, hep-th/9502105.
- [17] In fact, we are tacitly assuming that the gyromagnetic ratio for the spin is  $g = 2$



(while  $2\sigma$  might in principle be fractional) [18]. More generally,  $\sigma$  in our formulas would of course be replaced by  $g\sigma/2$ , with the conclusions changing correspondingly if  $g \neq 2$ . In particular, for relativistic *fundamental* particles (not composites) of integer or half-integer spin  $\sigma \neq 0$  there is the result  $g = 1/\sigma$  [19], so that in this case  $\sigma = \pm 1, \pm 3/2, \pm 2, \dots$  are all equivalent to  $\sigma = \pm 1/2$ . I thank C.R. Hagen for pointing this out.

[18] C. Chou, V.P. Nair, A.P. Polychronakos, Phys. Lett. B **304**, 105 (1993); G. Gat, R. Ray, Phys. Lett. B **340**, 162 (1994).

[19] C.R. Hagen, W.J. Hurley, Phys. Rev. Lett. **24**, 1381 (1970).

[20] A. Dasnières de Veigy, S. Ouvry, Phys. Lett. B **291**, 130 (1992).

[21] S. Viefers, Cand. Sci. thesis (Oslo, 1993).

[22] T. Blum, C.R. Hagen, S. Ramaswamy, Phys. Rev. Lett. **64**, 709 (1990).

## Figure caption

Fig. 1a–d. The virial coefficients  $a_2$  through  $a_5$  as functions of  $\xi$  in the Gaussian model, for  $\alpha = 0.1$ .

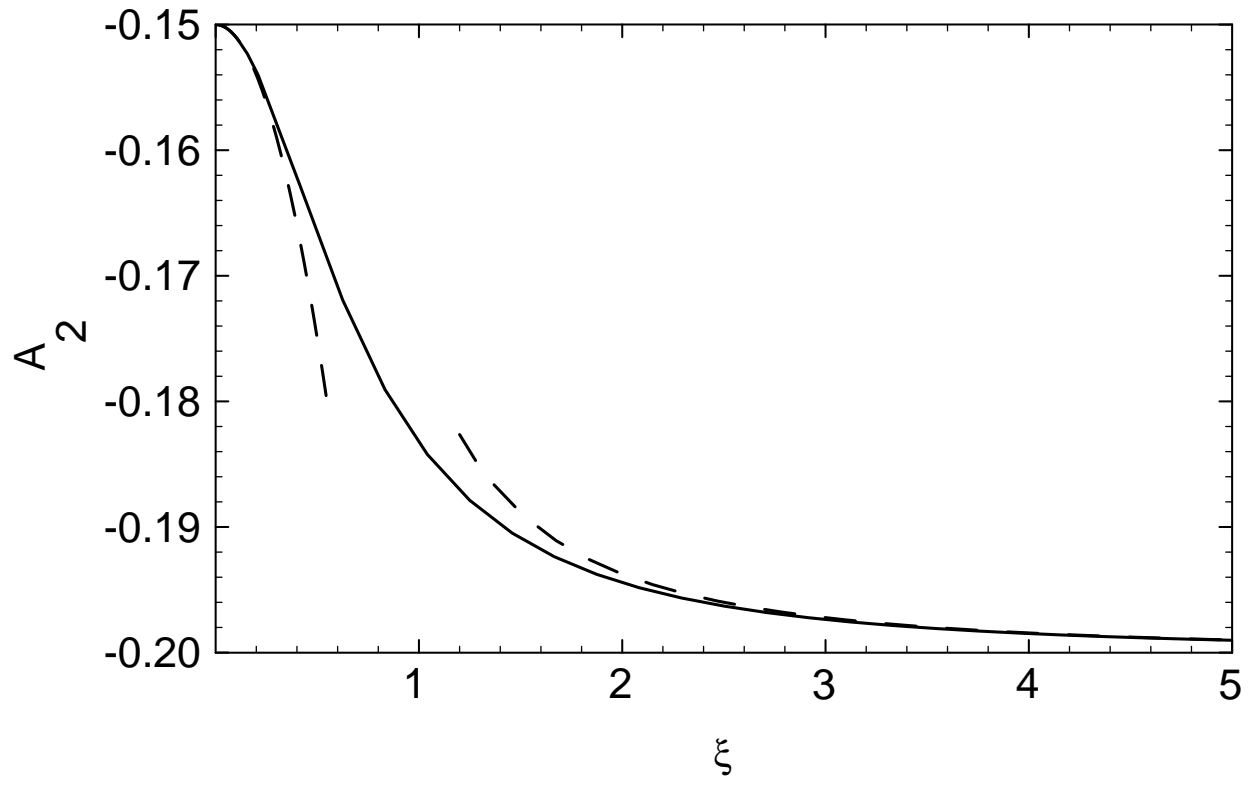


Fig. 1a

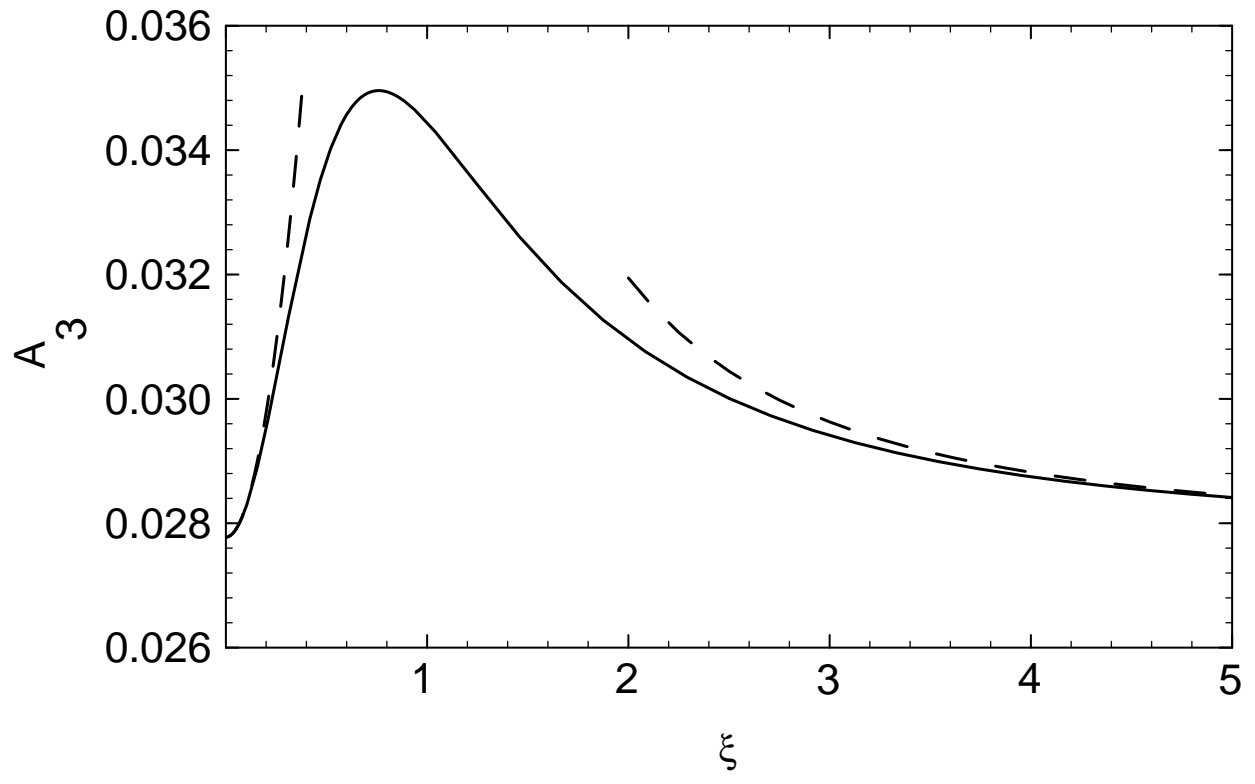


Fig. 1b

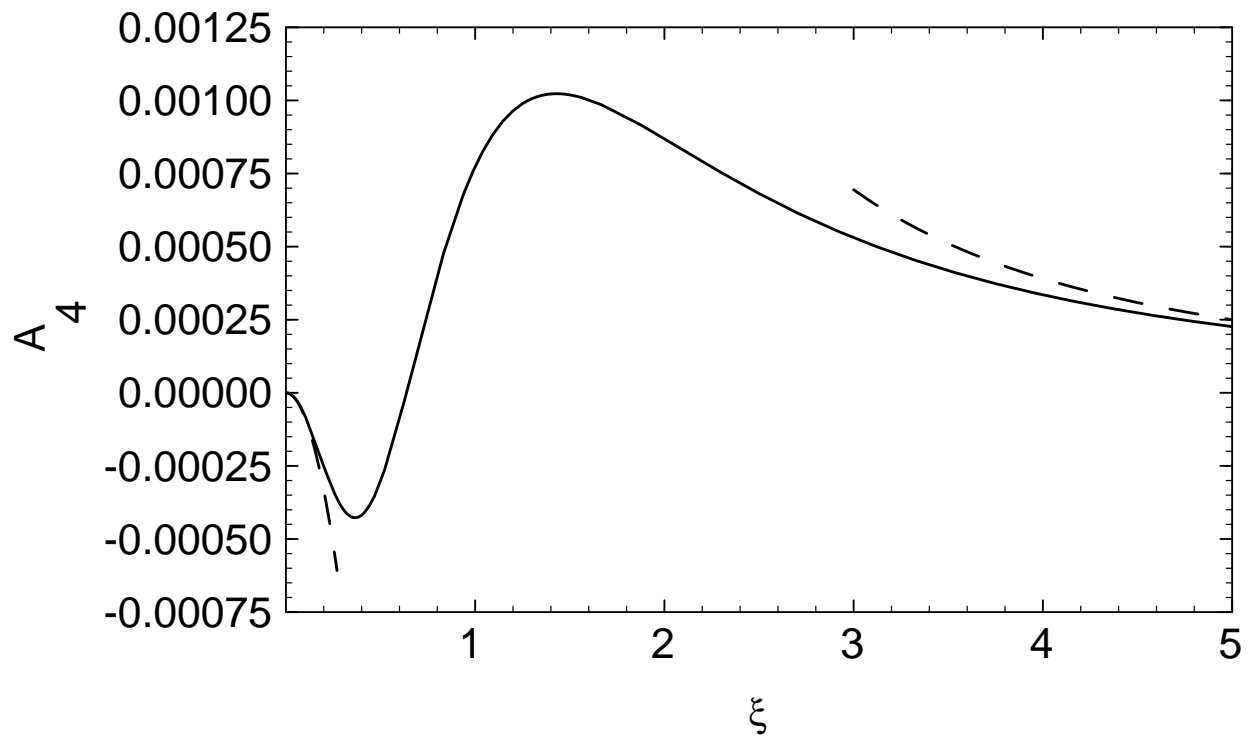


Fig. 1c

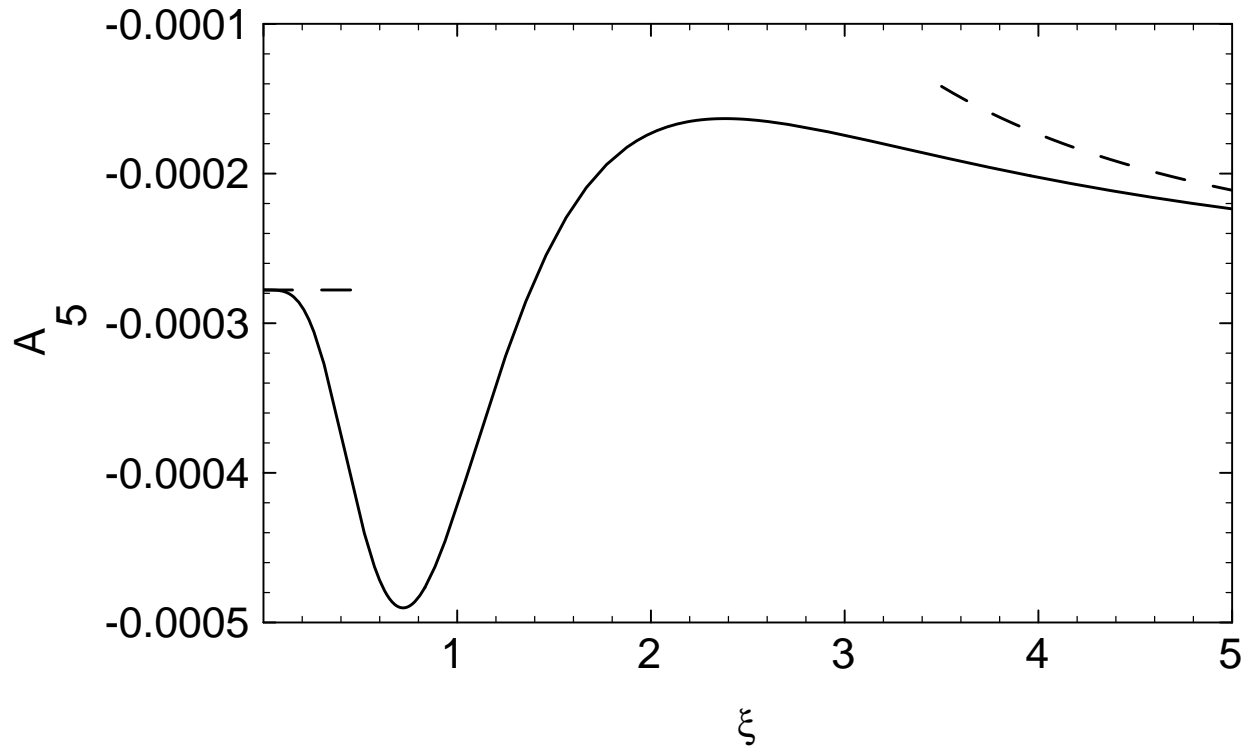


Fig. 1d