March 13, 2018 PAR-LPTHE 95/49

hep-th/yynnmmm

BILAYERS IN FOUR DIMENSIONS AND SUPERSYMMETRY

Romain ATTAL [1]

Laboratoire de Physique Théorique et Hautes Energies, [2]
Universités Pierre et Marie Curie (Paris 6) et Denis Diderot (Paris 7);
Unité de recherche associée au CNRS (D0 280).

Abstract: I build $N = 1$ superstrings in $\mathbb{R}^{\not 4}$ out of purely geometric bosonic data. The world-sheet is a bilayer of uniform thickness and the $2D$ supercharge vanishes in a natural way.

[1] E-mail: attal@lpthe.jussieu.fr.

[2] LPTHE tour 16 / 1^{er} étage, Université P. et M. Curie, BP 126, 4 place Jussieu, F 75252 PARIS CEDEX 05 (France).

1 Introduction

The usual approach to superstrings [1] uses anticommuting variables which are not very intuitive objects. In order to understand them better, I have sought for a more pictorial description. The basic idea is to use standard bosonization techniques [2] and to interpret geometrically the compactified bosonic field as kinks in the normal bundle. This is only possible when the space-time is a four-manifold. The resulting model is the following: I consider a bilayer with a uniform thickness living in a four dimensional, flat Euclidean space and choose an action proportional to the total area A of this bilayer. I show that this is a σ-model, taking values in the projectified normal bundle, which can be fermionized into a worldsheet Dirac fermion coupled to the normal connection [3]. For a particular value of the thickness, related to the string tension, this model is equivalent to a free four-vector Majorana fermion with the orthogonality constraint of a spinning string (the massless Dirac-Ramond equation) [4].

2 Action

Our bilayers are described by:

⋆ a smooth closed orientable $2D$ surface Σ, with p marked points $S_1 \cdots S_p$;

⋆ an immersion $X : \Sigma \to \mathbb{R}^4$;

⋆ a smooth section of the projectified normal bundle induced by X on Σ ($Y \in \Gamma(PN_X\Sigma)$ can be singular at the punctures $S_1 \cdots S_p$) ;

⋆ a thickness $2\delta > 0$.

The S_i's are the limits of infinitesimal circles mapped to twisted strings. If $y(P)$ is a unit vector in the line $Y(P)$ ($\forall P \in \Sigma$), the area of the bilayer $(X \pm \delta y)(\Sigma)$ is:

$$A = \int_\Sigma d\xi^1 \wedge d\xi^2 \left\{ (\mathrm{det}[\partial_\mathrm{a}(\mathrm{X}+\delta \mathrm{y}).\partial_\mathrm{b}(\mathrm{X}+\delta\mathrm{y})])^{1/2} + (\mathrm{det}[\partial_\mathrm{a}(\mathrm{X}-\delta \mathrm{y}).\partial_\mathrm{b}(\mathrm{X}-\delta\mathrm{y})])^{1/2} \right\} \tag{1}$$

which I expand in powers of δ:

$$A = 2 \int_\Sigma d\xi^1 \wedge d\xi^2 \; g^{1/2} \left(1 + \frac{\delta^2}{2} \; g^{ab} \; \partial_a y^\perp . \partial_b y^\perp + \delta^2 \; \mathcal{R} + \mathcal{O}(\delta^4) \right) . \tag{2}$$

Here, $\xi = (\xi^1;\xi^2)$ is a local coordinate system on Σ, the dot denotes the standard inner product in $\mathbb{R}^4$, $\partial_a y^\perp$ is the normal part of $\partial_a y$, $g_{ab} = \partial_a X.\partial_b X$, $g = \mathrm{det}[\mathrm{g_{ab}}]$, and $\mathcal{R}$ is Ricci's scalar curvature. The $\mathcal{O}(\delta^4)$ terms, containing more derivatives, are irrelevant, and I drop the topological term $\int_\Sigma d\xi^1 \wedge d\xi^2 \; g^{1/2} \; \mathcal{R} = 8\pi(1-\mathrm{genus}(\Sigma))$. The second term in (2) can be rewritten as follows. Pick a generic $N \in \Gamma(N_X\Sigma)$ with isolated zeros $Z_1 \cdots Z_q$ of indices $\iota_1 \cdots \iota_q$. The normal $n = N/\|N\|$ and binormal b define a right handed orthonormal frame in $N_X\Sigma$ over $\Sigma_Z = \Sigma \setminus \{Z_1 \cdots Z_q\}$, where the normal connection $\nabla^\perp$ is represented by the matrix $\begin{pmatrix} d & -T \\ T & d \end{pmatrix}$ with $d = d\xi^1\partial_1 + d\xi^2\partial_2$ and $T = b.dn$. If $\theta : \Sigma_Z \to \mathbb{R}/\pi\mathbb{Z}$ is the angle from $\pm n$ to Y, we have:

$$\pm \, y \;\; = \;\; \cos\theta \; n + \sin\theta \; b \; ,$$

$$
\begin{aligned}
dy^{\perp} &= \pm(d\theta + T)\ (\cos\theta\ b - \sin\theta\ n)\ , \\
A &= 2\int_{\Sigma} d\xi^1 \wedge d\xi^2\ g^{1/2} + \delta^2 \int_{\Sigma} \omega \wedge *\omega\ ,
\end{aligned} \tag{3}
$$

where $\omega = *(d\theta + T)\ (= (\partial_1\theta + T_1)d\xi^2 - (\partial_2\theta + T_2)d\xi^1 \ \text{ if } \ g_{ab} = e^{\phi}\delta_{ab})$.
I take the action to be $S = \mu A$, μ being the string tension of one layer. In the partition function $\mathcal{Z}(\mathcal{X}) = \int \mathcal{D}\theta\ \rceil^{-\mu\delta^{\in}\int_{\pm}\omega\wedge*\omega}$, we sum over the θ's which satisfy $\oint_{Z_j}\omega = 0$, since Y is regular at these points, and $\oint_{S_i}\omega = n_i\pi$ $(n_i \in \mathbb{Z})$ (the boundary strings can be twisted). Among these functions, the classical configurations are the solutions of the equation of motion $d\omega = 0$ and are parametrized by $H_1(\Sigma;\mathbb{Z})$.

3 Fermions

Since $PN_X\Sigma$ is a circle bundle, this system admits kinks and a fermionic representation by holonomies [5]. If $\gamma : [0;1] \rightarrow \Sigma$ is a path joining P_0 to P, we define:

$$
\begin{aligned}
b &= exp\ (k\int_{\gamma} id\theta - \omega) \qquad c = exp\ (-k\int_{\gamma} id\theta - \omega) \\
\bar{b} &= exp\ (k\int_{\gamma} id\theta + \omega) \qquad \bar{c} = exp\ (-k\int_{\gamma} id\theta + \omega)\ .
\end{aligned} \tag{4}
$$

Due to the equation of motion $(d\omega = 0)$, their correlators only depend on $[\gamma] \in H_1(\Sigma, P - P_0;\mathbb{Z})$. In order to recover

$$
\frac{1}{\mathcal{Z}(\mathcal{X})}\int D\theta\ e^{-\mu\delta^2\int_{\Sigma}\omega\wedge*\omega}\ b(z)c(0) = \langle b(z)c(0)\rangle \sim z^{-1}\ , \tag{5}
$$

on $\mathbb{C}$ and without the gauge field T, we must fix $k = (2\pi\mu\delta^2)^{1/2}$, as can be seen after a Gaussian integration. Moreover, for the special value $k = 1$, i.e. $\delta = (2\pi\mu)^{-1/2} = \delta_0$, there is no quartic term in the fermionic action [6] and $\psi = \begin{pmatrix} c \\ \bar{b} \end{pmatrix}$ satisfies the following equation of motion:

$$
\begin{pmatrix} 0 & 2\partial + i(T_1 + iT_2) \\ 2\bar{\partial} + i(T_1 - iT_2) & 0 \end{pmatrix} \begin{pmatrix} c \\ \bar{b} \end{pmatrix} = (\partial\!\!\!/ + iT\!\!\!/)\psi = 0\ . \tag{6}
$$

This shows that ψ is a $2D$ Dirac spinor and a vector in $N_X\Sigma$:

$$
\psi \in \Gamma(K^{1/2} \otimes_{\mathbb{C}} N_X\Sigma) \oplus \Gamma(K^{-1/2} \otimes_{\mathbb{C}} N_X\Sigma)\ . \tag{7}
$$

Here, $N_X\Sigma$ is viewed as a complex line bundle on Σ , K denotes the canonical line bundle of holomorphic $(1,0)$-forms on Σ , $K^{1/2}$ is one of the $2^{2\text{genus}(\Sigma)+\text{p}}$ spin structures on Σ [2] , K^* is the dual bundle of K and $K^{-1/2} = K^{1/2} \otimes_{\mathbb{C}} K^*$. Since the normal connection $\nabla^{\perp}$ is the projection on $N_X\Sigma$ of the trivial connection ∇ acting on sections of the total bundle $X^*(T\mathbb{R}^4) = T\Sigma \oplus_{\mathbb{R}}^{\perp} N_X\Sigma$, we can replace ψ by a free four-vector Majorana fermion

$$
\Psi \in \Gamma(K^{1/2} \otimes_{\mathbb{R}} X^*(T\mathbb{R}^4)) \oplus \Gamma(K^{-1/2} \otimes_{\mathbb{R}} X^*(T\mathbb{R}^4)) \ \text{ and } \ \partial\!\!\!/\Psi = 0\ , \tag{8}
$$

with the orthogonality constraint $\Psi.dX = 0$ to be applied on the Hilbert space in order to recover the same number of degrees of freedom in (7) and (8). We thus obtain three equivalent descriptions of a fermionic string satisfying the (massless) Dirac-Ramond equation:

⋆ a σ-model in $PN_X\Sigma$;

⋆ $\psi \in \Gamma(K^{1/2} \otimes_{\mathbb{C}} N_X\Sigma) \oplus \Gamma(K^{1/2} \otimes_{\mathbb{C}} N_X\Sigma)$ and $(\partial\!\!\!/ + iT\!\!\!/)\psi = 0$;

⋆ $\Psi \in \Gamma(K^{1/2} \otimes_{\mathbb{R}} X^*(T\mathbb{R}^4)) \oplus \Gamma(K^{-1/2} \otimes_{\mathbb{R}} X^*(T\mathbb{R}^4))$, Ψ is real , $\partial\!\!\!/\Psi = 0$ and $\Psi.dX = 0$

.

4 Conclusion

The previous computations suggest a simple picture for superstrings in four dimensions: they are double covers of bosonic strings and the two nearby world-sheets must be separated by $2\delta_0$ in order to have free fields. This suggests that one interpret the tachyonic instability of bosonic strings as a phase transition to a fermionic vacuum.

Acknowledgements: I thank Jean-Benoît Bost and Krzysztof Gawedzki for helpful discussions.

References

[1] M.B. Green, J.H. Schwarz and E. Witten: "Superstring Theory". (Cambridge University Press, 1987).

[2] L. Alvarez-Gaumé, J.-B. Bost, G. Moore, P. Nelson and C. Vafa: "Bosonization on higher genus Riemann surfaces". Commun. Math. Phys. **112**, 503-552 (1987).

[3] M.D. Spivak: "A comprehensive introduction to differential geometry". (Publish or Perish, 1979).

[4] P. Ramond: "Dual theory for free fermions". Phys. Rev. **D3**, 2415-2418 (1971).

[5] T.H.R. Skyrme: "Kinks and the Dirac equation". J. Math. Phys. **12** (8), 1735-1743 (1971).

[6] E.Charpentier and K.Gawedzki: "Bosonization in background of metric and gauge". J. Math. Phys. **34** (2), 381-436 (1993).