hep–th/9511058
NSF–ITP–95–141
BONN–TH–95–20

CONIFOLD TRANSITIONS AND MIRROR SYMMETRIES

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ABSTRACT

Recent work initiated by Strominger has lead to a consistent physical interpretation of certain types of transitions between different string vacua. These transitions, discovered several years ago, involve singular conifold configurations which connect distinct Calabi-Yau manifolds. In this paper we discuss a number of aspects of conifold transitions pertinent to both worldsheet and spacetime mirror symmetry. It is shown that the mirror transform based on fractional transformations allows an extension of the mirror map to conifold boundary points of the moduli space of weighted Calabi-Yau manifolds. The conifold points encountered in the mirror context are not amenable to an analysis via the original splitting constructions. We describe the first examples of such nonsplitting conifold transitions, which turn out to connect the known web of Calabi-Yau spaces to new regions of the collective moduli space. We then generalize the splitting conifold transition to weighted manifolds and describe a class of connections between the webs of ordinary and weighted projective Calabi-Yau spaces. Combining these two constructions we find evidence for a dual analog of conifold transitions in heterotic N=2 compactifications on K3×T^2 and in particular describe the first conifold transition of a Calabi-Yau manifold whose heterotic dual has been identified by Kachru and Vafa. We furthermore present a special type of conifold transition which, when applied to certain classes of Calabi-Yau K3 fibrations, preserves the fiber structure.

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1. Introduction

The fact that singular varieties of conifold type describe transition points between Calabi-Yau manifolds with different Hodge numbers has been realized several years ago in the framework of complete intersection Calabi-Yau manifolds embedded in products of projective spaces [1]. Only recently however has a physical interpretation of such transitions been found through work initiated by Strominger [2]. The picture which emerges is that the singular configurations of [1] correspond, in the low–energy effective action, to divergences that arise from integrating out massive modes which become massless at the singularity. The question whether the string can propagate consistently on conifold configurations is important not only because of the possibility of phase transitions between Calabi-Yau manifolds in the early universe, but also because it is of relevance for problems in mirror symmetry [3], and heterotic $K3\times T^2$-type II Calabi-Yau duality [4, 5], so–called worldsheet and spacetime mirror symmetry, or first and second quantized mirror symmetry [5]. Indeed, it was remarked in [6] that the link provided by the splitting process between the individual moduli spaces of many Calabi-Yau manifolds should make it possible to extend mirror symmetry to the global moduli space of all Calabi-Yau spaces. It was furthermore emphasized in [4, 5] that the conjectured heterotic-type II duality inevitably leads to the problem of singular configurations because the moduli space of $K3\times T^2$ contains points of enhanced symmetries where the massless spectrum changes.

Conifold transitions between Calabi-Yau spaces were introduced in [1] in the context of complete intersection manifolds embedded in products of ordinary projective space, and subsequent discussions [7, 8] have focused exclusively on such manifolds. In the intervening years however this class has been found to be wanting. Both, worldsheet and spacetime mirror symmetry, necessitate a more general framework which, in first approximation, is provided by weighted projective varieties[1]. One of the virtues of the class of weighted complete intersection Calabi-Yau manifolds lies in the fact that for a large subclass of these spaces we have at our disposal a physical construction of mirror pairs[9], based on fractional transformations. After reviewing and extending the discussion of the mirror transform of [9] in Section 2 we show in Section 3 that via this construction it is possible to explicitly mirror map appropriate submanifolds of the moduli space of Calabi-Yau spaces. These submanifolds of moduli space generically contain conifold points and therefore the fractional mirror transform allows us to trace Calabi-Yau configurations to the boundary of moduli space, thereby providing a concrete realization of the scenario envisioned in [6]. The singular varieties encountered in these regions of moduli space are, however, not of the type originally discussed in [1], and therefore might be expected to have novel features. In

[1]We will not discuss manifolds embedded in toric varieties.

Section 4 we describe the first examples of such conifold transitions relevant for mirror symmetry and find that indeed the resolved manifolds open paths into new regions of the collective moduli space of all Calabi-Yau manifolds.

Section 5 leads us from world sheet mirror symmetry to heterotic-type II duality. We describe a construction, based again on fractional transformations, which provides connections between various subwebs of Calabi-Yau moduli spaces. The classes of manifolds involved turn out to consist of spaces which are all K3 fibrations, and therefore are of relevance for a second unification problem - spacetime mirror symmetry. Part of our later discussion will therefore focus on these classes of spaces. Even though the particular types of conifold transitions described in [1], so–called splitting and contraction, are not the most general kind of transition involving nodes, as we emphasize in Section 4, they do have an important advantage. Whereas in general it can be quite difficult to decide whether the resolved manifold does in fact define a Calabi-Yau manifold, rather than some more general space, the splitting construction not only automatically guarantees that the resolved manifold is of Calabi-Yau type, it also provides an explicit algebraic representation of the two Calabi-Yau manifolds connected through some common conifold configuration. This simplifies the analysis of the whole process considerably. Because, as just mentioned, both types of mirror symmetry demand the class of weighted manifolds as a sort of 'minimal' framework, it is clearly of importance to generalize the discussion of [1] to this more general context. Perhaps the most intriguing problem is the possibility of finding a heterotic analog of the Calabi-Yau conifold transition.

In the remaining part of the paper we initiate the analysis of conifold transitions of splitting type in the context of weighted complete intersection Calabi-Yau manifolds. The generalization of the constructions of [1] to the weighted framework introduces some new twists which we discuss in the two final Sections. The first problem which has to be circumvented concerns the question of transversality of the split configuration. As in the case of weighted hypersurfaces it is not always possible to find a quasismooth manifold for a given combination of weights. We deal with this question in Section 6 and describe a class of weighted splits which do connect quasismooth varieties. In our discussion of such weighted conifold transitions we will find support for the speculation in [4] that a heterotic analog of conifold transitions indeed exists. In particular we describe the first conifold transition of a manifold whose heterotic dual has previously been suggested by Kachru and Vafa [4][2]. The generalization of the splitting construction of [1] to the weighted framework also provides support for the notion of a universal moduli space of Calabi-Yau manifolds, generalizing to the weighted category observations made in [1, 7] in the context

[2]A different type of transition has been discussed in [10].

of ordinary complete intersection Calabi-Yau spaces.

Finally, we describe the behavior of the K3 fibration of Calabi-Yau manifolds, introduced in Section 5, under conifold transitions. The fact that K3 fibrations are central to the problem of heterotic-type II duality has been recognized in [11] and further discussed in [12]. We introduce a particular type of splitting type conifold transition which preserves the K3 fibration structure, thereby showing that the property of K3 fibrations for Calabi-Yau threefolds extends to the class of general complete intersection manifolds of arbitrary codimension, and therefore is much more general than hitherto expected. We end by describing in Section 7 the new phenomenon of 'colliding singularities' which occurs in conifold transitions between weighted manifolds.

2. The Fractional Transformation Mirror Transform

Our main tool in tracing the mirror map along certain directions to the boundary of moduli space is the mirror transform based on fractional transformations. This construction was introduced in [9] in order to establish explicitly the existence of mirror symmetry discovered in the first reference of [3] in the framework of weighted Calabi-Yau manifolds. In the following we briefly review the discussion of [9] and make it more precise[3].

The essential ingredient of the fractional transformation mirror construction is the basic isomorphism[4]

$$\begin{aligned} &\mathbb{C}_{\left(\frac{b}{g_{ab}},\frac{a}{g_{ab}}\right)}\left[\frac{ab}{g_{ab}}\right] \ni \left\{z_1^a + z_2^b = 0\right\} \Big/ \mathbb{Z}_b : [\,(b-1) \quad 1\,] \\ \sim\ &\mathbb{C}_{\left(\frac{b^2}{h_{ab}},\frac{a(b-1)-b}{h_{ab}}\right)}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \left\{y_1^{a(b-1)/b} + y_1 y_2^b = 0\right\} \Big/ \mathbb{Z}_{b-1} : [\,1 \quad (b-2)\,] \end{aligned} \tag{1}$$

induced by the fractional transformations

$$\begin{aligned} z_1 &= y_1^{1-\frac{1}{b}}, \qquad y_1 = z_1^{\frac{b}{b-1}} \\ z_2 &= y_1^{\frac{1}{b}} y_2, \qquad y_2 = \frac{z_2}{z_1^{\frac{1}{b-1}}} \end{aligned} \tag{2}$$

in the path integral of the theory. Here g_{ab} is the greatest common divisor of a and b and h_{ab} is the greatest common divisor of b^2 and $(ab-a-b)$. The action of a cyclic group $\mathbb{Z}_b$ of order b

[3]Due to some mishap this article has appeared twice. The paper published in Phys.Lett. **B268**(1991)47 is an identical copy of [9].

[4]In [9] the modding on the rhs of this relation was ignored because in all the applications discussed in that paper this additional orbifolding in the image theory turned out to be trivial simply because the action became part of the projective equivalence. In general, however, the action on the rhs can not be neglected.

denoted by $[m\ \ n]$ indicates that the symmetry acts like $(z_1, z_2) \mapsto (\alpha^m z_1, \alpha^n z_2)$ where α is the b^{th} root of unity.

The ideal of the first cover theory

$$\mathcal{J}_z = [\partial_1 p, \partial_2 p] = \left[z_1^{a-1}, z_2^{b-1}\right]. \tag{3}$$

generates the $\mu = (a-1)(b-1)$–dimensional ring

$$\mathcal{R}_z = \{z_1^p z_2^q \mid 0 \leq p \leq a-2, \quad 0 \leq q \leq b-2\} \tag{4}$$

whereas the ideal of the second cover theory

$$\mathcal{I}_y = [\partial_1 p, \partial_2 p] = \left[\frac{a(b-1)}{b} y_1^{\frac{a(b-1)}{b}-1} + y_2^b, y_1 y_2^{b-1}\right] \tag{5}$$

generates the $\frac{a}{b}(b-1)^2 + 1$–dimensional polynomial ring

$$\mathcal{R}_y = \{y_2^{b-1}\} \cup \{y_1^p y_2^q \mid 0 \leq p \leq \frac{a}{b}(b-1) - 1, 0 \leq q \leq (b-2)\} \tag{6}$$

We are interested in the states of the orbifold theories. First consider the invariant sectors:

$$\mathcal{R}_z^{inv} = \{z_1^p z_2^q \mid 0 \leq p \leq a-2, \quad 0 \leq q \leq b-2, \quad p(b-1) + q = 0 \mod b\} \tag{7}$$

i.e. $p - q = 0 \bmod b$ and therefore $p = q + nb$ for some integer $n \in \mathbb{N}$. Thus dim $\mathcal{R}_z^{inv} = a(b-1)/b$. Similarly

$$\mathcal{R}_y^{inv} = \{y_2^{b-1}\} \cup \{y_1^p y_2^q \mid 0 \leq p \leq \frac{a}{b}(b-1) - 1, 0 \leq q \leq (b-2), p + q(b-2) = 0 \mod (b-1)\} \tag{8}$$

and hence $p = q + n(b-1)$ and the dimension is dim $\mathcal{R}_y^{inv} = a(b-1)/b + 1$. Hence there is only one twisted state in the z–orbifold which is mapped by fractional transformations into a monomial of the y–theory.

It follows from the analysis in [9] that twisted states are of the form (z_2^p / z_1^q) with $p, q \in \mathbb{Z}$. The first constraint comes from invariance under the $\mathbb{Z}_b$ action, which leads to the relation $p = nb + q(b-1)$ for some integer $n \in \mathbb{Z}$. Thus these rational forms take the form $z_2^{nb+q(b-1)} / z_1^q \leftrightarrow y_1^n y_2^{nb+q(b-1)}$, for $n, q \in \mathbb{Z}$ with the unitarity constraint $q\,(1 - 1/a - 1/b) + n \geq 0$. For $n \geq 1$, $q \geq 0$ clearly all image states are in the ideal of the y–theory. Hence the states above with $q > 0$, any n, and $q < 0, nb < -q(b-1)$ are possible twisted states, which, for $n = 0, q > 0$ lead to monomials: $z_2^{q(b-1)} / z_1^q \longleftrightarrow y_2^{q(b-1)}$. For $q > 2$ the y–monomials belong to the ideal as well, leaving us with two twisted states $z_2^{b-1} / z_1 \leftrightarrow y_2^{b-1}$ and $z_2^{2(b-1)} / z_1^2 \leftrightarrow y_2^{2(b-1)}$. Both of these states are in the invariant sector of the image theory and thus survive the $\mathbb{Z}_{b-1}$–modding. The final reduction comes from realizing that the state $z_2^{2(b-1)}/z_1^2 \leftrightarrow y_2^{2(b-1)}$ is in fact equivalent to an invariant state: via the y–ideal the above state can be written as $y_2^b y_2^{b-2} = y_1^{\frac{a}{b}(b-1)-1} y_2^{b-2}$ which the fractional transformation map into $z_1^{a-2} z_2^{b-2}$ which is the top state in the $\mathcal{R}_z$ ring invariant with respect to the $\mathbb{Z}_b$–action. Thus the invariant sector of the y–orbifold theory is mapped into the invariant ring of the z–theory plus one twisted state. A simple application of this discussion to the isomorphism $\mathbb{P}_{(1,7,2,2,2)}[14] \sim \mathbb{P}_{(1,3,1,1,1)}[7]$ [9] allows an explicit relation between the purely polynomial chiral ring on the rhs and the chiral ring on the lhs which is supplemented by blow–up modes originating from the singular set described by a genus 15 curve. The blow–up modes thus acquire a representation as rational expressions in the coordinates.

The basic isomorphism itself provides the mirror of weighted spaces only in very few cases, such as $\mathbb{P}_{(3,8,33,66,88,132)}[264]^{(57,81)}/\mathbb{Z}_2 \sim \mathbb{P}_{(3,8,66,88,99)}[264]^{(81,57)}$. Much more powerful, however, is a simple iteration of the basic isomorphism as described in [9].

It is important to realize that even though the basic isomorphism maps a Fermat type orbifold into a tadpole orbifold the fractional transformation mirror transform is not restricted to Fermat type polynomials. Consider e.g. the manifold embedded in

$$\mathbb{P}_{(3,6,6,4,5)}[24]_{-48}^{(10,34)} \ni \{p = z_1^8 + z_2^4 + z_3^4 + z_4^6 + z_4 z_5^4 = 0\}. \tag{9}$$

Orbifolding this space with respect to a $\mathbb{Z}_4$ symmetry with the action $\mathbb{Z}_4 \; : \; [\;0\;0\;1\;0\;3\;]$ and using the appropriate fractional transformations as discussed above leads to the mirror manifold

$$\mathbb{P}_{(9,18,12,13,20)}[72]_{48}^{(34,10)} \ni \{p = z_1^8 + z_2^4 + z_3^6 + z_3 z_5^3 + z_5 z_4^4 = 0\}. \tag{10}$$

3. Mirror Mapping Moduli Spaces

We can now apply fractional transformations to map moduli spaces. A simple example which illustrates this phenomenon is furnished by the quintic which, at the exactly solvable point, takes the form

$$(3^5)_{A_4^5} \;\sim\; \mathbb{C}^*_{(1,1,1,1,1)}[5] \;\sim\; \mathbb{P}_4[5] \ni \left\{\sum_{i=1}^5 z_i^5 = 0\right\}. \tag{11}$$

Using the iteration of the basic isomorphism as described in Section 2 one finds that orbifolding the Landau–Ginzburg theory with respect to the cyclic group

$$\mathbb{Z}_5^3 : \quad \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix} \tag{12}$$

leads to the mirror spectrum. Using the fractional transformations

$$z_1 = y_1^{4/5},\ z_2 = y_1^{1/5} y_2^{4/5},\ z_3 = y_2^{1/5} y_3^{4/5}, z_4 = y_3^{1/5} y_4,\ z_5 = y_5 \tag{13}$$

that follow from the iteration of the basic isomorphism shows that this orbifold is isomorphic to the complete intersection

$$\mathbb{P}_{(80,60,65,51,64)}[320]_{200}^{101} \ni \{z_1^4 + z_1 z_2^4 + z_2 z_3^4 + z_3 z_4^5 + z_5^5 = 0\}, \tag{14}$$

where again no orbifolding is necessary on the mirror side of the isomorphism. The manifold (14) is indeed one of two spaces in the list of weighted hypersurfaces which has the appropriate mirror spectrum of the quintic (see the first ref. of [3] and [13]).

It is important to emphasize that the fractional mirror transform can be applied to the whole relevant part of the moduli space: since the complex sector of the mirror of the quintic is 1–dimensional, we need to establish a map that relates a 1–dimensional subspace of the 101–dimensional space of complex deformations to the 1–dimensional subspace of the mirror. This is achieved by considering

$$\mathbb{P}_4[5] \ni \left\{ \sum_{i=1}^{5} z_i^5 - 5\psi \prod_{i=1}^{5} z_i = 0 \right\}. \tag{15}$$

The crucial point here is that for each value of ψ the configuration features the mirror discrete group (12) as a symmetry group and we can mod out this action. Furthermore for each value of ψ we can apply our fractional transformation and map this configuration into

$$\mathbb{P}_{(80,60,65,51,64)}[320]_{200}^{101} \ni \{z_1^4 + z_1 z_2^4 + z_2 z_3^4 + z_3 z_4^5 + z_5^5 - 5\psi \prod_{i=1}^{5} z_i = 0\}. \tag{16}$$

The same holds for the second representation of the mirror which appears in the lists of [13] [5].

The above example of a map between moduli spaces is the simplest example of a vast class of manifolds, the moduli spaces of which feature the very same structure as the quintic in a one–dimensional subspace. In the class of weighted Calabi-Yau hypersurfaces of degree d in weighted projective space $\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d]$ with $d = \sum_{i=1}^{5} k_i$ there always exists a 1–dimensional family of manifolds

$$\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d] \ni \left\{ p_0(z_i) - d\psi \prod_i z_i = 0 \right\}. \tag{17}$$

[5] The second configuration with the mirror spectrum of the quintic, $\mathbb{P}_{(64,48,52,51,41)}[256]_{200}^{101}$ is isomorphic to the one just discussed because the additional $\mathbb{Z}_5$ modding via which it is obtained from the quintic is part of the projective equivalence.

Since any action preserving the holomorphic threeform also leaves invariant this canonical deformation and furthermore fractional transformations leave invariant this monomial it follows without any further check that the fractional mirror transform always applies to this 1–dimensional subspace. The important aspect of this class is that by moving along this canonical direction in moduli space one eventually runs into singularities, which generically are conifolds. Thus our mirror construction inevitably leads us to consider conifold configurations.

For most weighted hypersurfaces the 1–dimensional family just described is only part of a higher–dimensional moduli space. The fractional transformation mirror transform, of course, applies to mirror pairs involving larger moduli spaces as well. An example that has received attention recently is given by the family

$$\mathbb{P}_{(1,1,2,2,2)}[8] \ni \{p = z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 - 8\psi \prod_{i=1}^{5} z_i - 2\lambda z_1^4 z_2^4 = 0\}. \tag{18}$$

This theory has the spectrum $(h^{(1,1)}, h^{(2,1)}) = (2, 86)$ and the mirror can be obtained by applying the fractional transformations

$$z_1 = y_1^{7/8}, \quad z_2 = y_1^{1/8} y_2^{3/4}, \quad z_3 = y_2^{1/4} y_3^{3/4}, \quad z_4 = y_4^{1/4} y_5, \quad z_5 = y_5 \tag{19}$$

to the orbifold $\mathbb{P}_{(1,1,2,2,2)}[8]/\mathbb{Z}_8 \times \mathbb{Z}_3^2$ with respect to the action

$$\mathbb{Z}_8 \times \mathbb{Z}_3^2 : \quad \begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix}. \tag{20}$$

This leads to the polynomial

$$p = y_1^7 + y_1 y_2^6 + y_2 y_3^3 + y_3 y_4^4 + y_5^4 - 8\psi \prod_{i=1}^{5} y_i - 2\lambda y_1^4 y_2^3 \tag{21}$$

which lives in the configuration $\mathbb{P}_{(4,4,8,5,7)}[28]$ for which one finds the mirror spectrum $(h^{(1,1)}, h^{(2,1)}) = (86, 2)$, as expected.

Other examples which have been the focus of recent investigations of the conjectured heterotic-type II duality [4, 11, 14, 15, 16, 17, 18, 19] can be analyzed in the same manner. The simpler of the two most prominent members is the two–parameter family

$$\mathbb{P}_{(1,1,2,2,6)}[12]^{(2,128)} \ni \{p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi \prod_{i=1}^{5} z_i - 2\lambda z_1^6 z_2^6 = 0\} \tag{22}$$

which is mapped via the $\mathbb{Z}_6^2 \times \mathbb{Z}_2$ fractional transformations $z_1 = y_1, z_2 = y_2^{5/6}, z_3 = y_2^{1/6} y_3^{5/6}$, $z_4 = y_4^{1/6} y_5^{1/2}$ into the mirror family

$$\mathbb{P}_{(25,30,54,82,109)}[300] \ni \{p = y_1^{12} + y_2^{10} + y_2 y_3^6 + y_3 y_4^3 + y_4 y_5^2 - 12\psi \prod_{i=1}^{5} y_i - 2\lambda y_1^6 y_2^5 = 0\}, \qquad (23)$$

which describes the type IIB dual of a heterotic string vacuum with 129 hypermultiplets and 3 vector multiplets. Finally, the three–parameter family

$$\mathbb{P}_{(1,1,2,8,12)}[24] \ni \{p = z_1^{24} + z_2^{24} + z_3^{12} + z_4^3 + z_5^2 - 12\psi \prod_{i=1}^{5} z_i - 2\lambda z_1^6 z_2^6 z_3^6 - \sigma z_1^{12} z_2^{12} = 0\} \qquad (24)$$

and its mirror, obtained via the $\mathbb{Z}_{24} \times \mathbb{Z}_3 \times \mathbb{Z}_2$ fractional transformations $z_1 = y_1^{23/24}$, $z_2 = y_1^{1/24} y_2^{1/2}$, $z_3 = y_3^{2/3}$, $z_4 = y_3^{1/3} y_4$, $z_5 = y_2^{1/2} y_5$ into the mirror family

$$\mathbb{P}_{(24,44,69,161,254)}[552] \ni \{p = y_1^{23} + y_1 y_2^{12} + y_2 y_5^2 + y_3^8 + y_3 y_4^3 - 12\psi \prod_{i=1}^{5} y_i - 2\lambda y_1^6 y_2^3 y_3^4 - \sigma y_1^{12} y_2^6 = 0\} \quad (25)$$

describe the type IIA and IIB duals respectively of a heterotic vacuum with 244 hypermultiplets and 4 vector multiplets. Our construction clearly generalizes to ever more general moduli spaces.

4. New Directions in the Global Calabi-Yau Moduli Space via General Conifold Transitions

The splitting construction in either the ordinary projective class of Calabi-Yau manifolds [1], or in the weighted category, which we will discuss below, is particularly simple because it provides simple representations of the different smooth phases that are connected via a singular variety. In general such a simple description of the manifold 'on the other side' is not to be expected, even if one starts out with a complete intersection. A class of manifolds which illustrates the necessity of considering more general conifold transitions is provided by the 1–parameter families of (17). A particularly simple subclass is obtained by considering spaces of Brieskorn–Pham type

$$\mathbb{P}_{(k_1,\ldots,k_5)}[d] \ni \left\{ \sum_{i=1}^{5} k_i z_i^{d/k_i} - d\psi \prod_{i=1}^{5} z_i = 0 \right\}. \qquad (26)$$

These varieties acquire singularities at $\psi^d = 1$, the singular points are nodes, and there are $d^3 / \prod_i k_i$ of them.

The natural question arises what the manifolds are that are found after traversal of the conifold. In the present case the splitting construction does not provide insight and one has to take recourse to more general considerations concerning the resolution of singularities in Calabi-Yau manifolds. The general theory is rather more involved because it is not automatically guaranteed that the resolved variety is projective [20], in contrast to the splitting construction. Once this question has been answered, however, it is not difficult to compute the Hodge numbers. The nodes at the conifold point are resolved by introducing a sphere $\mathbb{P}_1 \sim S^2$, in contrast to blowing up. Thus the surgery involves replacing a three-sphere S^3 by a projective curve, thereby changing the Euler number by +2. Hence the Euler number of the resolved manifold becomes[6]

$$\chi(\tilde{M}) = \chi(M) + 2N \tag{27}$$

if N is the number of nodes. In a Calabi-Yau manifold this can only be achieved by increasing $h^{(1,1)}$ by unity or decreasing $h^{(2,1)}$ by unity. Thus

$$h^{(1,1)}(\tilde{M}) = h^{(1,1)}(M) + \delta, \qquad h^{(2,1)}(\tilde{M}) = h^{(2,1)}(M) - (N - \delta) \tag{28}$$

where δ is the number of linearly dependent vanishing cycles.

Consider e.g. the simplest space of Brieskorn–Pham type, the family of quintics (15) at $\psi^5 = 1$. The starting point here is a family of smooth manifolds with $(h^{(1,1)}, h^{(2,1)}) = (1, 101)$ which acquires $d^3/\prod k_i = 125$ nodes at the conifold. Thus the Euler number of the resolved manifold is $\chi(\tilde{M}) = -200 + 2 \cdot 125 = 50$ and the Hodge numbers are $(h^{(1,1)}(\tilde{M}), h^{(2,1)}(\tilde{M})) = (1+\delta, 101-(125-\delta))$. If we wish to fix the configuration of the nodes then we expect the resolved manifold to have fewer complex deformations since the resolution only introduces $\mathbb{P}_1$s and we lose the complex deformations which would kill the nodes. The quintic is rather special since the number of nodes it acquires at the conifold is larger than the number of complex deformations one starts out with. Therefore it leads to a resolved space which is rigid. Because there are 24 more nodes than there are complex deformations one finds $\delta = 24$ and the resolved space in fact has the Hodge numbers $(h^{(1,1)}(\tilde{M}), h^{(2,1)}(\tilde{M})) = (25, 0)$. It has been checked in [21] that the manifold is indeed Calabi-Yau.

A further example of a Brieskorn–Pham type variety whose conifold transition leads to a rigid manifold as well is the one–parameter family of hypersurfaces

$$\mathbb{P}_{(1,1,1,1,2)}[6] \ni \left\{ \sum_{i=1}^{4} z_i^6 + 2z_5^3 - 6\psi \prod_{i=1}^{5} z_i = 0 \right\}, \tag{29}$$

which acquires a conifold configuration at $\psi = 1$ with 108 nodes. The resolution of these nodes leads to a smooth rigid manifold with $\chi = 12$.

[6] For general weighted Calabi-Yau manifold this result is not correct as we will discuss in the last Section.

The fact that the cohomology of this example is produced neither by the class of all complete intersection Calabi-Yau manifolds [1] nor by the class of all weighted hypersurfaces or, more generally, the complete class of Landau–Ginzburg theories, shows that general resolutions allow us to explore new, yet uncharted, territory of the global moduli space of all Calabi-Yau manifolds.

5. Calabi-Yau Isomorphisms: Connecting Collective Webs and new K3 fibrations.

In this Section we discuss two further applications of fractional transformations which will turn out to be useful in the following parts of the paper. The first is that they lead to a particularly simple class of intersection points between the moduli spaces of different types of Calabi-Yau spaces, whereas the second shows how insight into the fiber structure of certain Calabi-Yau manifolds can be gained from fractional transformations.

It was shown in [22] that the moduli space of Calabi-Yau manifolds embedded in weighted projective spaces is connected to the moduli space of manifolds embedded in products of ordinary projective space. This arose simply because there exist isomorphisms between weighted hypersurfaces and ordinary complete intersections of higher codimension, the simplest example being the relation

$$\mathbb{P}_{(1,1,2,2,2)}[8]^{(2,86)} \sim \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}. \tag{30}$$

Fractional transformations in fact lead to an explanation simpler and more general than the analysis of [22], providing a great many of such identifications. Consider the following class of manifolds of Brieskorn–Pham type

$$\mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}[2k] \tag{31}$$

with $k = (2k_1 + k_2 + k_3 + k_4 - 1)$ and $2k/(2k_1 - 1) \in 2\mathbb{N}$. Viewing these string vacua as a Landau–Ginzburg theory we can add trivial factors y_i^2 without changing the model. Adding two such factors and applying the basic isomorphism (1) to the two parts $(x_i^{2k/(2k_i-1)} + y_i^2)$ in the resulting representation of the theory changes the configuration to

$$\mathbb{C}_{(2(2k_1-1),2(2k_1-1),2k_2,2k_3,2k_4,(k_2+k_3+k_4),(k_2+k_3+k_4))}[2k]. \tag{32}$$

If the $\mathbb{Z}_2$'s happen to act trivially we can use the construction of [23] to derive the corresponding manifold of codimension 2, arriving at the relations

$$\mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}[2k] \sim \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{((2k_1-1),(2k_1-1),k_2,k_3,k_4)} \end{matrix} \begin{bmatrix} 2 & 0 \\ (2k_1-1) & k \end{bmatrix}. \tag{33}$$

For $k_1 = k_2 = k_3 = k_4 = 1$ we recover the example above, discussed in detail in [22]. This class of spaces thus provides a great many of identifications between hypersurfaces and Calabi-Yau manifolds of higher codimension.

A second reason why this class of manifolds is of interest comes from the fact that the relations (33) are also useful for explorations of spacetime mirror symmetry. It has been recognized early on [11] that important for the heterotic-type II duality [4, 5] is the fact that the Calabi-Yau manifolds involved are K3–fibrations. It is therefore of some importance to gain insight into the nature of such manifolds in order to obtain further examples of dual pairs beyond the few which have been the focus of most of the discussions so far. Following the analysis of [24] it can readily be seen that all manifolds of the type (31) are in fact K3 fibrations. Defining a divisor $D_\lambda \in \mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}[(2k_1-1)\ 2k]$ via $(z_1 - \lambda z_2) = 0$, and applying the (1–1) coordinate transformation $y_1 = z_1^2$, shows that the fibers are described by the K3 configurations

$$\mathbb{P}_{(2k_1-1,k_2,k_3,k_4)}[k]. \tag{34}$$

This class thus provides a pool of K3 fibrations which considerably extends the list of examples of K3 fibrations enumerated in [11]. For convenience we provide the complete set of models in the Appendix.

An important aspect of the class of fibrations (31) is that the equivalences (33) trivially allow the identification of the (possible) type II image of the heterotic dilaton. This is because for N=2 heterotic vacua the dilaton couples to the rest of the moduli in such a way [25][7] that the intersection numbers of the corresponding modes on the type II Calabi-Yau dual, denoted by s and m_i, $i = 1, ..., n$, take the form

$$\kappa_{sss} = 0 = \kappa_{ssi}, \quad \kappa_{sij} = \mathrm{diag}(1, n). \tag{35}$$

This condition merely indicates that the corresponding Calabi-Yau dual is a fibered manifold and leaves open a number of different ways to fiber the manifold [27]. The condition derived from the heterotic theory which identifies the fibers as K3 varieties is the fact that the dual Calabi-Yau manifolds also have to satisfy $\int c_2(M) h_s = 24$, where h_s is the element in $\mathrm{H}^2(M)$ describing the dual image of the dilaton[8]. For manifolds with large Picard number b_2 it is quite involved to compute these couplings and identify the appropriate h_s. For our class of fibrations (31) however the equivalent representation as a codimension–two space allows for an immediate

[7]See [14, 26] for recent reviews of this subject as well as a more complete list of the original references.

[8]We are grateful to B.Hunt for correspondence on this point. A more detailed recent discussion of these facts can be found in [12].

identification – the image of the dilaton must be the Kähler form which descends down to the Calabi-Yau space from the ambient projective curve.

It should be noted that we have assumed the condition $(2k/(2k_1-1)) \in 2\mathbb{N}$ for convenience of presentation only. It is not necessary either for relations of the type we have discussed or for the manifold to be a K3 fibration. An example which illustrates this point is provided by the manifold

$$\mathbb{P}_{(3,3,4,4,14)}[28] \sim \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(3,3,2,2,7)} \end{matrix} \begin{bmatrix} 2 & 0 \\ 3 & 14 \end{bmatrix} \tag{36}$$

with K3 fiber $\mathbb{P}_{(3,2,2,7)}[14]$. In this more general class some of the new heterotic spectra found in [29] can be found and therefore it provides the 'missing' Calabi-Yau dual candidates of some known heterotic N=2 vacua.

A similar discussion applies to relations of the type

$$\mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4,2k_5)}[2a \;\; 2b] \sim \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{((2k_1-1),(2k_1-1),k_2,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ (2k_1-1) & a & b \end{bmatrix} \tag{37}$$

where $(a+b) = 2k_1 - 1 + \sum_{i=2}^{5} k_i$. This class describes K3 fibrations as well and generalizes the second type of Calabi-Yau spaces considered in [11].

6. Splitting and Contraction for Weighted CICYs and Spacetime Mirror Symmetry

Singularities are ubiquitous in the moduli space of Calabi-Yau spaces: no matter from which smooth point one starts, moving along a generic complex deformation will eventually lead to a singular configuration. What is not ubiquitous is knowledge about what happens 'on the other side' of the singularity, or whether it exists at all. The existence problem is far from obvious since the projectivity of the small resolution of singularities, obtained by deforming a family of smooth varieties V_t into a singular configuration V_0, is not easy to check in general.

In ref. [1] a certain type of conifold transition between Calabi-Yau spaces has been introduced which avoids this difficulty. The constructions of [1], called splitting and contraction, have the virtue that they describe conifold transitions of the family V_t of (quasi–)smooth varieties depending on some complex variable t

$$V_t \longrightarrow V_0 \longrightarrow \tilde{V} \tag{38}$$

which automatically provide relations between smooth *Calabi-Yau* manifolds (here $\tilde{V}$ denotes a small resolution Calabi-Yau manifold). We will show in this Section that the construction of [1] generalize to the weighted framework even though the story acquires some new twists.

There first new constraint that is specific to the class of weighted manifolds and has no counterpart in the ordinary projective class originates from the fact that for a given choice of weights there may not exist a quasismooth set of polynomials [9]. The problem is even more pronounced in the case of complete intersections with higher codimension than it is for hypersurfaces in weighted projective four–space, as discussed in the first reference of [3], and has in fact been one of the major stumbling blocks for the construction of the class of all Calabi-Yau manifolds embedded in products of weighted projective spaces. To illustrate the problem consider the following split

$$\mathbb{P}_{(k_1,k_1,k_2,k_3,k_4)}[d] \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_2,k_3,k_4)} \end{matrix} \begin{bmatrix} 1 & 1 \\ ak_1 & (d-ak_1) \end{bmatrix} \ni \left\{ \begin{array}{l} p_1 = x_1 Q(y_i) + x_2 R(y_i) \\ p_2 = x_1 S(y_i) + x_2 T(y_i) \end{array} \right\}, \tag{39}$$

where $d = 2k_1 + k_2 + k_3 + k_4$ and a is some postive integer. For $k_1 = k_2 = k_3 = k_4 = 1$ this reduces to the simplest type of split considered in [1], the rhs describing a $\mathbb{P}_1$–split $\tilde{V}$ of the determinantal variety

$$\mathbb{P}_{(k_1,k_1,k_2,k_3,k_4)}[d] \ni V_0 = \{p = QT - RS = 0\}, \tag{40}$$

which can be deformed into a smooth variety V_t (for favourable choices of weights).

Now if, for instance, the weights are such that the first polynomial involves only the first two coordinates of the weighted 4–space, then it is never possible to find transverse choices of polynomials. The equations that follow from the transversality condition, according to which $dp_1 \wedge dp_2 = 0$ may not have any solution on the manifold, lead to two branches. It suffices to discuss one of these. Assuming that indeed $Q = Q(y_1, y_2)$ and $R = R(y_1, y_2)$ leads to $0 \equiv Q\Big|_{(0,0,y_3,y_4,y_5)}$ and $0 \equiv R\Big|_{(0,0,y_3,y_4,y_5)}$, and therefore the equations restricted to the subvariety parametrized by $(0, 0, y_3, y_4, y_5)$ reduce to $0 = T$ and

$$\left(S\frac{\partial R}{\partial y_i}\right)\Big|_{(0,0,y_3,y_4,y_5)} = 0 = \left(\frac{\partial R}{\partial y_i}\frac{\partial T}{\partial y_j} - \frac{\partial R}{\partial y_j}\frac{\partial T}{\partial y_i}\right)\Big|_{(0,0,y_3,y_4,y_5)} \tag{41}$$

for all i and all $i < j$ respectively. If $a > 1$ then $\partial R/\partial y_i\Big|_{(0,0,y_3,y_4,y_5)} \equiv 0$ and the configuration is singular for all points on the curve $\mathbb{P}_{(k_2,k_3,k_4)}[d - ak_1]$. A codimension two Calabi-Yau configurations for which it is not possible to find quasismooth choice of polynomials is given by $(k_1, k_2, k_3, k_4) = (1, 3, 3, 3)$ with $a = 2$, for instance. Assuming, then, that $a = 1$ the analysis of the transversality equations reveals that quasismoothness can be obtained by requiring that both $(\partial S/\partial y_1) = 0$ and $(\partial T/\partial y_2) = 0$ and that both, S and T, depend on all but at most one

[9] We will not discuss possible generalizations, such as the one discussed in [28], in the present paper.

variables, and that they are of standard type in these variables. Furthermore, if S is independent of some variable then the polynomial T must be of Fermat type in this variable, and vice versa.

A simple example of a splittable configuration is provided by the quasismooth octic hypersurface

$$M = \mathbb{P}_{(1,1,2,2,2)}[8]^{(2,86)} \ni \left\{ \sum_i z_i^8 + \sum_j z_j^4 = 0 \right\}. \tag{42}$$

This manifold can be split into the codimension-two variety

$$M_{\rm split} = \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(2,2,1,1,2)} \end{matrix} \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix} \in \left\{ \begin{array}{l} p_1 = x_1 y_1 + x_2 y_2 \\ p_2 = x_1(y_2^3 + y_4^6 - y_5^3) + x_2(y_1^3 + y_3^6 + y_5^3) \end{array} \right\}, \tag{43}$$

which can be checked to be transverse. The determinantal variety leads to the singular octic

$$p_s = Q(y_i)T(y_i) - R(y_i)S(y_i) = y_1(y_1^3 + y_3^6 + y_5^3) - y_2(y_2^3 + y_4^6 - y_5^3) \tag{44}$$

fails to be transverse at $\mathbb{P}_{(2,2,1,1,2)}[2\ 2\ 6\ 6] = 18$ nodes. Hence the Euler number of the codimension two complete intersection is $\chi(M_{\rm split}) = \chi(M) + 2 \cdot 18 = -168 + 36 = -132$. Since $h^{(1,1)} = 3$, because of the additional $\mathbb{P}_1$, the complete massless spectrum that results is $(h^{(1,1)}, h^{(2,1)}) = (3, 69)$.

The conifold transition

$$\mathbb{P}_{(1,1,2,2,2)}[8]^{(2,86)} \longleftrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(2,2,1,1,2)} \end{matrix} \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}^{(3,69)} \tag{45}$$

is of some interest because it provides a possible ingredient of a sequence of string spectra discovered by Kachru and Vafa in their discussion of dual pairs of type II Calabi-Yau compactifications and heterotic $K3 \times T^2$ vacua. Starting with the $E_8 \times E_8$ heterotic string they considered a series of embeddings of SU(N) factors into one of the E_8s, thereby breaking this group down to E_7 (for N=2), E_6 (N=3), SO(10) (for N=4), or SU(5) (for N=5), respectively. The spectra obtained in this way are [4]

$$\mathrm{N}=2:\ (65,19), \quad \mathrm{N}=3:\ (84,18), \quad \mathrm{N}=4:\ (101,17), \quad \mathrm{N}=5:\ (116,16). \tag{46}$$

This sequence is intriguing: it has precisely the structure we expect from splitting transitions of the type discussed above [10]: because of the vanishing cycles the number of complex deformations is reduced in the transition $V_t \longrightarrow V_0$ and because of the properties of small resolutions new Kähler deformations are introduced in the smoothing process $V_0 \longrightarrow \tilde{V}$. However, because the

[10] Not of a general conifold transition however, as follows from our discussion in Section 4.

purported Hodge numbers $(h^{(1,1)}, h^{(2,1)})$ do not appear for any of the manifolds in the list of all CICYs [1] nor for any of the models in the list of all Landau–Ginzburg theories [13], it seems difficult at present to check for the possibility of a direct, simple analog of the splitting transition. Kachru and Vafa, however, made the intriguing observation that the transition from N=5 to N=4 is reminiscent of the splitting process of [1] applied to the codimension two Calabi-Yau manifold

$$\mathbb{P}_4[5]^{(1,101)} \longleftrightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^{(2,86)}, \tag{47}$$

provided an overall shift of 14 in the Hodge numbers is taken into account[11]. This shift of 14 results in the sequence of Euler numbers

$$\mathrm{N} = 2: \ -92, \quad \mathrm{N} = 3: \ -132, \quad \mathrm{N} = 4: \ -168, \quad \mathrm{N} = 5: \ -200. \tag{48}$$

The idea that there might indeed exist an analog of Calabi-Yau splitting in the context of N=2 heterotic string theory clearly would gain support if direct splits could be found for the remaining two embeddings of SU(N). We see that a candidate for the second element in the chain (48) is provided by the pair of spaces connected through the weighted split (45). To find the remaining elements of the sequence (48), recall from Section 5 that the octic $\mathbb{P}_{(1,1,2,2,2)}[8]$ has another representation as a codimension-two ordinary complete intersection Calabi-Yau manifold. Since we have just found the split of the quasismooth octic to a (3,69) manifold we might expect that an appropriate direct split of the second representation might exist as well. Indeed, using the ordinary splitting of [1] we find the sequence of splits

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}_{-168} \longleftrightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 3 \end{bmatrix}_{-132} \longleftrightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}_{-92}. \tag{49}$$

Jumping ahead we emphasize that all manifolds of (49),(43) are K3 fibrations. Thus we have established a direct split within the subclass of K3 fibered Calabi-Yau manifolds for each element in the sequence of SU(N)–embeddings discussed by Kachru and Vafa. The ability to find direct splits according to growing N may be interpreted as evidence that there might exist an alternative construction of Calabi-Yau splitting in the context of N=2 theories.

At present no conifold transition between Calabi-Yau manifolds with heterotic duals is known. As an initial step in this direction we present the first conifold transition of a Calabi-Yau manifold the dual of which has been identified by Kachru and Vafa[12]. The heterotic vacuum in question

[11]The origin of this shift remains obscure at present.

[12]This problem is also under consideration in the work of [30]

is constructed by starting with an eight–dimensional compactification with an enhanced gauge group $E_8 \times E_8 \times SU(3) \times U(1)^2$, obtained by going to a special point in the moduli space of the torus T^2. Choosing particular embeddings of the various relevant bundles into the gauge group factors one ends up with a theory with 102 hypermultiplets and 6 vector multiplets. Since there is only one space in the class (33) with the appropriate spectrum, given by $\mathbb{P}_{(1,1,2,4,4)}[12]^{(5,101)}$, it is very likely that this indeed describes the dual of the heterotic dual just described. This configuration can be split as follows

$$\mathbb{P}_{(1,1,2,4,4)}[12] \quad \longleftrightarrow \quad \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(4,4,1,1,2)} \end{matrix} \begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix}, \tag{50}$$

involving a conifold configuration with 32 nodes. The resulting cohomology for the split manifold is $(h^{(1,1)}, h^{(2,1)}) = (6, 70)$. We will show in the next Section that this conifold transition belongs to a whole class of weighted splits which connect Calabi-Yau manifolds that are all K3 fibrations. Thus our split (50) remains in the class of spaces relevant for heterotic-type II duality. Orbifolding this manifold by $\mathbb{Z}_6 \times \mathbb{Z}_3^2$ and iterating the basic isomorphism we find the mirror configuration to be $\mathbb{P}_{(20,24,49,54,93)}[240]^{(101,5)}$.

At this time only very few heterotic vacua along the lines of [4] have been constructed, with results [29] that are not too different from the corresponding heterotic spectrum $(n_H, n_V) = (71, 7)$. It should be expected that our splitting result will turn up as the number of heterotic vacua grows.

7. Conifold Transitions between K3 fibered Calabi-Yau manifolds.

As a second application we show how weighted splitting indicates that the deeper understanding recently obtained [11] of the appearance of the j–function in the context of spacetime mirror symmetry [4, 5] is far more general than initially thought. In order to do so, we recall that the underlying reason for the appearance of the j–function is to be found in the K3 fibration of the Calabi-Yau threefold. Because the web of Calabi-Yau manifolds can be traversed via conifold transitions, it is natural to ask what the behavior of K3 fibrations is under such transitions. Our discussion in the following will focus on the splitting construction, and for simplicity we discuss in detail one example, the quasismooth octic.

It is well–known that the octic (30) is a K3 fibration [24], i.e. the linear system L defined by the linear sections defines a family of K3–surfaces in the representation $\mathbb{P}_3[4]$. In more detail consider the divisor defined by the linear relation $z_2 = \theta z_1$, which leads to the family of hypersurfaces

$$\mathbb{P}_{(1,2,2,2)}[8] \ni \{p = (1+\theta^8 - 2\lambda\theta^4)z_1^8 + z_3^4 + z_4^4 + z_5^4 - 8\psi\theta z_1^2 \prod_{i=3}^{5} z_i = 0\}. \tag{51}$$

With $y_1 = z_1^2$ and $y_i = z_{i+1}, i = 2,3,4$, one arrives at the family of quartic K3–hypersurfaces of the form

$$\mathbb{P}_3[4] \ni \{p = (1+\theta^8 - 2\lambda\theta^4)y_1^4 + y_2^4 + y_3^4 + y_4^4 - 8\psi\theta \prod_{i=1}^{4} y_i = 0\}. \tag{52}$$

It is this structure of the K3 fibration of the Calabi-Yau threefold which explains [11] the appearance of the j–function [4].

Now, starting from the codimension two split of the octic as defined in (45), contraction leads to the determinantal variety

$$\mathbb{P}_{(1,1,2,2,2)}[8] \ni \{p_0 \equiv y_1(y_1^3 + y_3^6 - y_5^3) - y_2(y_2^3 + y_4^6 + y_5^3) = 0\} \tag{53}$$

which, via $y_4 = \theta y_3$, and the definitions $z_i = y_i, i = 1,2,\ z_3 = y_3^2,\ z_4 = y_5$, leads to the family of singular K3 surfaces

$$\mathbb{P}_3[4] \ni \{z_1^4 - z_2^4 + (z_1 - \theta z_2)z_3^3 + (z_1 - z_2)z_4^3 = 0\}. \tag{54}$$

Furthermore the codimension two variety describing the split contains the family of K3 surfaces

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_3\end{matrix}\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix} \ni \left\{\begin{array}{l} p_1 = x_1y_1 + x_2y_2 \\ p_2 = x_1(y_2^3 + \theta y_3^3 + y_4^3) + x_2(y_1^3 + y_3^3 - y_4^3) \end{array}\right\} \tag{55}$$

which can be seen to lead precisely to the determinantal K3 of (54). Thus we see that the splitting and contraction process not only relates K3–fibration but essentially takes place in the fiber, passing through singular K3 surfaces. The splitting conifold transitions therefore carry over the K3 fiber structure of the hypersurfaces to more complicated Calabi-Yau manifolds of higher codimension.

The above analysis clearly allows for generalizations. A simple class of splits is defined as follows

$$\mathbb{P}_{(2k-1,2k-1,2l,2l,2m)}[2(2k-1+2l+m)] \longleftrightarrow \begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(2l,2l,2k-1,2k-1,2m)}\end{matrix}\begin{bmatrix}1 & 1\\ 2l & 2(2k-1+l+m)\end{bmatrix}, \tag{56}$$

where the codimension one hypersurfaces, containing the K3 surfaces $\mathbb{P}_{(2k-1,l,l,m)}[2k-1+2l+m]$, split into codimension two manifolds containing codimension-two K3 manifolds

$$\mathbb{P}_{(2k-1,l,l,m)}[2k-1+2l+m] \longleftrightarrow \begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(l,l,m,2k-1)}\end{matrix}\begin{bmatrix}1 & 1\\ l & (l+m+2k-1)\end{bmatrix}. \tag{57}$$

As in the example above the quasismooth K3 hypersurfaces on the lhs are deformations of the determinantal K3s obtained by contracting the codimension-two K3 complete intersections of the split manifold. This construction is not restricted to this simple manifolds but can also be applied also to a variety of other classes, starting from more complicated spaces of higher codimension such as

$$\mathbb{P}_{(2k-1,2k-1,2l,2l,2m,2n)}[2a \;\; 2b] \;\longleftrightarrow\; \begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(2l,2l,2k-1,2k-1,2m,2n)}\end{matrix} \begin{bmatrix}1 & 1 & 0\\ 2l & 2(a-l) & 2b\end{bmatrix}, \tag{58}$$

with $(a+b) = 2k-1+2l+m+n$, or, more concretely,

$$\begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(1,1,1,1,2,2)}\end{matrix} \begin{bmatrix}1 & 1 & 0\\ 2 & 2 & 4\end{bmatrix} \;\longleftrightarrow\; \begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(1,1)}\\ \mathbb{P}_{(1,1,1,1,2,2)}\end{matrix} \begin{bmatrix}0 & 0 & 1 & 1\\ 1 & 1 & 0 & 0\\ 2 & 2 & 2 & 2\end{bmatrix}, \tag{59}$$

all of which are K3 fibrations.

The above splitting analysis provides strong evidence that the appearance of the j–function in the analysis of the heterotic-type II duality is not restricted to the simple Calabi-Yau spaces that have been considered in the literature but instead extends to the general class of weighted complete intersection Calabi-Yau manifolds of arbitrary codimension.

8. Colliding Singularities

Finally, we wish to point out a novel phenomenon that arises in conifold transitions between weighted Calabi-Yau manifolds. Namely, it can happen that a number N_i of the N hypersurface singularities sit on top of $\mathbb{Z}_{p_i}$ orbifold singularities of the weighted space. If such a situation occurs the results obtained for conifold transitions between manifolds embedded in products of ordinary projective spaces [1] are no longer correct.

Consider the manifold

$$M = \mathbb{P}_{(2,2,3,3,5)}[15]^{(7,43)} \ni \left\{z_1^6 z_3 + z_2^6 z_4 + z_3^5 + z_4^5 + z_5^3 = 0\right\} \tag{60}$$

with the split configuration

$$M_{\rm split} = \begin{matrix}\mathbb{P}_{(1,1)}\\ \mathbb{P}_{(3,3,2,2,5)}\end{matrix} \begin{bmatrix}1 & 1\\ 3 & 12\end{bmatrix}, \tag{61}$$

a quasismooth manifold of which is defined by the polynomials

$$\begin{aligned} 0 &= p_1 = x_1 y_1 + x_2 y_2 \\ 0 &= p_2 = x_1(y_2^4 + y_4^6 + y_4 y_5^2) + x_2(y_1^4 + y_3^6 + y_3 y_5^2). \end{aligned} \tag{62}$$

Because this configuration does not allow a Fermat type choice for the polynomials S and T enumerating the singularities involves the detailed structure of the defining polynomials. The number of nodes in this case is given by $N = \mathbb{P}_{(2,2,3,3,5)}[3\ 3\ 12\ 12] = 8$, and therefore we might have expected that the split manifold has Euler number -56. Computing the Euler number of the split manifold with the standard methods however leads to $\chi(M_{\rm split}) = -48$. The resolution of this discrepancy is found by noting that one of the nodes sits on top of a $\mathbb{Z}_5$ orbifold singularity. Hence the resolution

$$\chi(M_{\rm split}) = \chi(M) + 2N + \sum_i (p_i - 1)N_i \qquad (63)$$

leads to an additional contribution of $+8$ in the naive result, leading to agreement with the standard computation.

Acknowledgement

We are grateful to P. Berglund, S. Chaudhuri, S. Lang, J. Louis and F. Quevedo for discussions, and in particular B.Hunt for correspondence. This work was supported in part by NSF grant PHY–94–07194. M.L. thanks the Theory Group at the University of Bonn for hospitality during the course of part of this work.

Appendix: *The class of K3 fibrations of the type* $\mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}[2k]$, *with* $k = 2k_1 - 1 + k_2 + k_3 + k_4$ *and* $2k/(2k_1-1) \in 2\mathbb{N}$.

χ	$h^{(1,1)}$	$h^{(2,1)}$	Weights
108	60	6	(14,16,20,25,25)
96	59	11	(10,12,33,33,44)
96	55	7	(10,12,16,19,19)
84	54	12	(8,10,27,27,36)
84	54	12	(6,10,19,19,22)
84	50	8	(8,12,14,17,17)
72	49	13	(6,10,11,11,28)
72	49	13	(6,8,17,17,20)
72	44	8	(9,9,14,16,24)
64	43	11	(7,7,8,12,22)
60	44	14	(5,5,12,16,22)
48	35	11	(6,7,7,10,12)
48	43	19	(4,10,21,21,28)
48	41	17	(4,9,9,10,22)
48	39	15	(5,5,8,14,18)
48	39	15	(4,12,14,15,15)
48	39	15	(4,10,12,13,13)
36	38	20	(6,8,21,21,28)
36	38	20	(4,6,13,13,16)
36	34	16	(4,8,10,11,11)
32	33	17	(5,5,6,8,16)
24	32	20	(4,5,5,12,14)
24	29	17	(8,10,12,15,15)
24	27	15	(5,5,6,6,8)
24	33	21	(4,6,7,7,18)
12	36	30	(2,12,15,15,16)
12	36	30	(2,10,13,13,14)
16	31	23	(4,5,5,8,18)
0	35	35	(2,12,21,21,28)
0	27	27	(4,6,15,15,20)
0	34	34	(3,3,8,14,20)
0	23	23	(4,6,8,9,9)
0	23	23	(4,4,6,7,7)
0	31	31	(2,7,7,10,16)
0	31	31	(2,8,11,11,12)
-12	38	44	(3,3,8,20,26)
-12	30	36	(2,8,15,15,20)
-12	30	36	(2,6,11,11,14)
-12	25	31	(3,3,4,10,10)
-24	20	32	(3,3,4,4,10)
-24	29	41	(2,6,7,7,20)
-24	27	39	(3,3,4,10,16)
-24	23	35	(2,5,5,6,12)
-36	20	38	(2,4,7,7,8)
-36	20	38	(2,6,9,9,10)
-48	31	55	(3,3,4,16,22)
-48	15	39	(2,4,4,5,5)
-48	21	45	(2,4,5,5,14)
-48	19	43	(2,3,3,8,8)
-48	19	43	(2,4,9,9,12)

χ	$h^{(1,1)}$	$h^{(2,1)}$	Weights
-64	11	43	(6,7,7,8,28)
-72	21	57	(2,3,3,8,14)
-72	13	49	(5,5,8,12,30)
-72	10	46	(4,5,5,6,20)
-72	7	43	(4,5,5,6,10)
-84	12	54	(2,2,5,5,6)
-96	14	62	(3,3,8,10,24)
-96	11	59	(3,3,4,8,18)
-96	11	59	(2,2,3,3,8)
-96	7	55	(3,3,4,6,8)
-96	5	53	(2,3,3,4,6)
-108	6	60	(2,2,2,3,3)
-112	7	63	(2,5,5,8,20)
-120	25	85	(2,3,3,14,20)
-120	6	66	(2,3,3,4,12)
-132	7	73	(3,3,6,8,10)
-168	2	86	(1,1,2,2,2)
-192	11	107	(3,3,4,20,30)
-192	8	104	(1,1,4,4,6)
-192	5	101	(1,1,2,4,4)
-192	3	99	(1,1,2,2,4)
-204	14	116	(3,3,8,28,42)
-204	9	111	(1,1,4,6,6)
-232	9	125	(1,1,4,6,8)
-232	5	121	(1,1,2,4,6)
-240	11	131	(1,1,6,8,8)
-240	7	127	(2,3,3,16,24)
-252	2	128	(1,1,2,2,6)
-264	11	143	(1,1,6,8,10)
-272	7	143	(1,1,4,4,10)
-288	9	153	(1,1,4,8,10)
-288	4	148	(1,1,2,4,8)
-304	12	164	(1,1,8,10,12)
-312	11	167	(1,1,6,10,12)
-312	8	164	(1,1,4,6,12)
-312	5	161	(1,1,2,6,8)
-348	12	186	(1,1,8,12,14)
-368	10	194	(1,1,6,8,16)
-372	8	194	(1,1,4,8,14)
-372	4	190	(1,1,2,6,10)
-420	10	220	(1,1,6,10,18)
-432	13	229	(1,1,12,16,18)
-432	11	227	(1,1,8,10,20)
-480	11	251	(1,1,8,12,22)
-480	3	243	(1,1,2,8,12)
-528	7	271	(1,1,4,12,18)
-612	12	318	(1,1,12,16,30)
-624	9	321	(1,1,6,16,24)
-732	10	376	(1,1,8,20,30)
-960	11	491	(1,1,12,28,42)

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