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# Spectral invariants for the Dirac equation on the d-ball with various boundary conditions

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Abstract

The mode properties for spectral and mixed boundary conditions for massless spin-half fields are derived for the $d$–ball. The corresponding functional determinants and traced heat-kernel coefficients are presented, the latter as polynomials in $d$.

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## 1. Introduction.

In putative supersymmetric field theories on manifolds with boundary, the question arises of the boundary conditions satisfied by the higher-spin fields. These problems are encountered for example in quantum cosmology [1–9] and have become more pressing recently, particularly in gauge theories [10–12].

It is generally assumed that the conditions should be such as to make the relevant operators self-adjoint (see *e.g.* [13]). One possibility is the spectral condition introduced by Atiyah, Patodi and Singer [14] in their extension of the spin-index theorem to the non-empty boundary case. Although it seems that spectral conditions are not suitable for supersymmetry, they are of undoubted interest beyond this particular purpose.

In this paper we report on the calculation of important quantities in the spectral geometry of spin-1/2 fields on the $d$-ball with these nonlocal boundary conditions, namely the integrated heat-kernel asymptotic expansion coefficients and the functional determinants. For comparison, we also treat the case of mixed (local) conditions, which are possibly of more supersymmetric significance.

Although the $d$-ball is a very particular manifold, it turns out in the corresponding scalar Dirichlet and Neumann cases that the results are surprisingly restrictive of the *general* form of the heat-kernel expansion [15]. One of the motivations for the present calculation is to prepare the way for a similar discussion with spinors.

## 2. Spinor modes on the d-ball. Spectral conditions.

The eigenvalue Dirac equation on the Euclidean d-ball is

$$-i\Gamma^\mu\nabla_\mu\psi_\pm = \pm k\psi_\pm, \quad \Gamma^{(\mu}\Gamma^{\nu)} = g^{\mu\nu}, \tag{1}$$

and the nonzero modes are separated in polar coordinates, $ds^2 = dr^2 + r^2 d\Omega^2$, in standard fashion to be regular at the origin, ($A$ is a radial normalisation factor),

$$\begin{aligned}\psi_\pm^{(+)} &= \frac{A}{r^{(d-2)/2}}\begin{pmatrix} iJ_{n+d/2}(kr)\, Z_+^{(n)}(\Omega) \\ \pm J_{n+d/2-1}(kr)\, Z_+^{(n)}(\Omega)\end{pmatrix} \\ \psi_\pm^{(-)} &= \frac{A}{r^{(d-2)/2}}\begin{pmatrix} \pm J_{n+d/2-1}(kr)\, Z_-^{(n)}(\Omega) \\ iJ_{n+d/2}(kr)\, Z_-^{(n)}(\Omega)\end{pmatrix}.\end{aligned} \tag{2}$$

Here the $Z_\pm^{(n)}(\Omega)$ are the well-known spinor modes on the unit $(d-1)$–sphere (some modern references are [16–18]) satisfying the intrinsic equation

$$-i\gamma^j\widetilde{\nabla}_j Z_\pm^{(n)} = \pm\lambda_n Z_\pm^{(n)}, \tag{3}$$

where

$$\lambda_n = \big(n + \frac{d-1}{2}\big), \quad n = 0, 1, \ldots\ .$$

Each eigenvalue is greater than 1/2 and has degeneracy

$$\frac{1}{2} d_s \binom{d+n-2}{n}.$$

The dimension, $d_s$, of $\psi$–spinor space is $2^{d/2}$ if $d$ is even. For odd $d$ it is $2^{(d+1)/2}$ and has been doubled in order to implement the boundary conditions. Appendix A contains a more systematic discussion of $\gamma$-matrices and spinors.

The projected $\gamma$-matrices are given by

$$\Gamma^r = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \Gamma^j = \begin{pmatrix} \mathbf{0} & i\gamma^j \\ -i\gamma^j & \mathbf{0} \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \tag{4}$$

Spectral boundary conditions are applied, effectively as in D'Eath and Esposito [3], by setting the negative (positive) $Z$-modes of the positive (negative) chirality parts of $\psi$, to zero at $r = 1$, the other modes remaining free. This leads to the condition $J_{n+d/2-1}(k) = 0$.

Roughly speaking, spectral conditions amount to requiring that zero-modes of (1) should be square-integrable on the elongated manifold obtained from the ball by extending the narrow collar (of approximate, product metric $dr^2 + d\Omega^2$) just inside the surface to values of $r$ ranging from 1 to $\infty$. This will be so if the modes of $A = \Gamma^r\Gamma^j\nabla_j$ with *negative* eigenvalues are suppressed at the boundary, (*e.g.* [14,19–26] ). At $r = 0$ the modes vanish except that with $n = 0$ which has the opposite handedness.

From (3) and (4), the boundary operator, $A_0$, is $A_0 = \Gamma^r\Gamma^j\nabla_j\big|_{r=1} = \big(\Gamma^5 \otimes -i\gamma^j\big)\big(\mathbf{1} \otimes \widetilde{\nabla}_j\big) = \Gamma^5 \otimes \big(-i\gamma^j\widetilde{\nabla}_j\big)$ and its eigenstates are

$$A_0\begin{pmatrix} Z_+^{(n)} \\ Z_-^{(n)} \end{pmatrix} = \lambda_n \begin{pmatrix} Z_+^{(n)} \\ Z_-^{(n)} \end{pmatrix}, \quad A_0\begin{pmatrix} Z_-^{(n)} \\ Z_+^{(n)} \end{pmatrix} = -\lambda_n \begin{pmatrix} Z_-^{(n)} \\ Z_+^{(n)} \end{pmatrix}. \tag{5}$$

Then, from (2), we see that the negative modes of $A_0$ are associated with the radial factor $J_{n+d/2-1}(kr)$, hence the condition quoted above.

We put $p = n + d/2 - 1$ making the implicit eigenvalue equation,

$$J_p(k) = 0 \tag{6}$$

with degeneracies

$$N_p^{(d)} = \frac{d_s}{(d-2)!}\big(p - \frac{d}{2} + 2\big)\big(p - \frac{d}{2} + 3\big)\ldots\big(p + \frac{d}{2} - 1\big) \tag{7}$$

where $p \geq d/2-1$ and is integral for even $d$ but half odd-integral for odd $d$. The form of the degeneracies shows that $p$ can start at 1 if $d$ is even and at 1/2 if $d$ is odd. For $d=4$, we obtain agreement with D'Eath and Esposito [3]. The normalisation in (2) is $A = \big(J_{n+d/2}(k)\big)^{-1}$.

The case of the disc, $d=2$, needs special treatment. The implicit equation is still (6), with $p = 1, 2, \ldots$, but the degeneracy is just 2.

## 3. Mixed boundary conditions.

For mixed boundary conditions, [6,26–30], we apply $P_+\psi = 0$ at $r=1$ where the projection is

$$P_+ = \frac{1}{2}\big(\mathbf{1} - i\Gamma^5\Gamma^\mu\, n_\mu\big) \tag{8}$$

in terms of the inward normal $n_\mu$.

For the geometry of the ball

$$P_+ = \frac{1}{2}\begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ -i\mathbf{1} & \mathbf{1} \end{pmatrix}$$

and so for $\psi_\pm^{(+)}$,

$$J_{n+d/2}(k) = \mp J_{n+d/2-1}(k)$$

and for $\psi_\pm^{(-)}$,

$$J_{n+d/2-1}(k) = \mp J_{n+d/2}(k), \quad n = 0, 1, 2, \ldots.$$

Thus, taking $p = n + d/2$, the implicit eigenvalue equation is as in [2],

$$J_p^2(k) - J_{p-1}^2(k) = 0 \tag{9}$$

while the degeneracies are

$$N_p^{(d)} = \frac{d_s}{2(d-2)!}\big(p - \frac{d}{2} + 1\big)\big(p - \frac{d}{2} + 2\big)\ldots\big(p + \frac{d}{2} - 2\big) \tag{10}$$

where $p \geq d/2$ and is integral for even $d$ but half odd-integral for odd $d$. The form of the degeneracies shows again that $p$ can start at 1 if $d$ is even and at 1/2 if $d$ is odd. In two dimensions, the degeneracy is unity.

For both conditions, the $\zeta$–function is $\zeta_d(s) = \sum_p \sum_{k_p} N_p^{(d)}(k_p)^{-2s}$, $k_p$ being the positive roots of (6) or of (9). The functional determinant is $\exp\big(-\zeta_d'(0)\big)$ and the traced heat-kernel expansion, $K(\tau) = \sum_{n=0,1/2,\ldots} B_n\, \tau^{n-d/2}$.

## 4. Polynomial form of traced heat-kernel coefficients.

Specific, integral forms exist for the first few *local* coefficients, [28,27,15], which, computed on the $d$-ball, give

$$
\begin{aligned}
B_0^{(L)}(d) &= \frac{2^{-d-1}d_s}{\Gamma(1+d/2)},\\
B_{1/2}^{(L)}(d) &= 0,\\
B_1^{(L)}(d) &= -\frac{2^{-d}d_s}{6\,\Gamma(d/2)}\,(d-1),\\
B_{3/2}^{(L)}(d) &= \frac{2^{-d}d_s\sqrt{\pi}}{64\,\Gamma(d/2)}\,(d-1)\,(d-3),\\
B_2^{(L)}(d) &= \frac{2^{-d}\,d_s}{3780\,\Gamma(d/2)}\,(d-1)\,(d+3)\,(17d-46).
\end{aligned}
\tag{11}
$$

For particular $d$'s, (11) is consistent with the results obtained from (9) and (10) using the method described in Bordag *et al* [31]. The individual values following from this calculation are not displayed here since they are better used to construct the polynomial content of the coefficients, some higher examples of which are exhibited in Appendix B. There is no difficulty in finding any coefficient.

Turning to the spectral case, although there appears to be no known general forms corresponding to those for local coefficients, polynomial expressions can be obtained in the present geometry. These are written conjecturally as

$$
\begin{aligned}
B_n^{(S)}(d) &= 2^{-d}d_s\bigg(\frac{\overline{F}_n(d)}{\Gamma\big((d+1)/2\big)}+\sqrt{\pi}\,\frac{\overline{G}_n(d)}{\Gamma\big(d/2\big)}\bigg), \quad n=1/2,3/2,\ldots\\
&= 2^{-d}d_s\bigg(\frac{\overline{F}_n(d)}{\Gamma\big(d/2\big)}+\sqrt{\pi}\,\frac{\overline{G}_n(d)}{\Gamma\big((d+1)/2\big)}\bigg), \quad n=0,1,\ldots
\end{aligned}
\tag{12}
$$

where $\overline{F}_n$ and $\overline{G}_n$ are polynomials of degree $2n-1$. For $n\geq 1$ a factor of $d-1$ is extracted, $\overline{F}_n=(d-1)F_n$, $\overline{G}_n=(d-1)G_n$ and the $F$ and $G$ fitted using specifically evaluated coefficients over several dimensions. This yields

$$
\begin{aligned}
&\overline{F}_0(d)=\frac{1}{d}, \quad \overline{G}_0(d)=0,\\
&\overline{F}_{1/2}(d)=\frac{1}{2}, \quad \overline{G}_{1/2}(d)=-\frac{1}{2},\\
&F_1(d)=\frac{1}{3}\,, \quad G_1(d)=-\frac{1}{4}\,,\\
&F_{3/2}(d)=\frac{1}{24}\,(4d-11)\,, \quad G_{3/2}(d)=-\frac{1}{192}\,(7d-17)\,,\\
&F_2(d)=\frac{1}{945}\,(d-6)\,(5d-13)\,, \quad G_2(d)=-\frac{1}{384}\,(d-6)\,(7d-20)\,.
\end{aligned}
\tag{13}
$$

Further polynomials are given in Appendix C. The forms have been checked to $d=19$. The coefficients for $d=4$ were also given earlier by Kirsten and Cognola [32].

We remark on the circumstance that alternate spectral coefficients (depending on the dimension) are comprised of two parts, one proportional to $\sqrt{\pi}$ and the other to $1/\sqrt{\pi}$. By contrast, for local (mixed) boundary conditions there are no $1/\sqrt{\pi}$ terms and this would be the expected behaviour.

A similar structure to (12) is encountered in the case of local conditions for physical components of *higher* spin fields in four dimensions, [32].

When, as here, the manifold is not product near the boundary, the spectral asymptotic expansion has been established by Grubb [33] and by Grubb and Seeley [34]. In the product, cylindrical case Grubb and Seeley give a construction of the $\zeta$–function in terms of the $\zeta$–functions on the doubled manifold and on the boundary which would yield a structure for the heat-kernel somewhat akin to (12).

It should also be remarked that, in the general case, if $d$ is even there can be logarithmic terms in the heat-kernel expansion, equivalent to double poles in the $\zeta$–function. These are absent here, the reason possibly being that the heat-kernel expansion for a massless Dirac field on the odd dimensional boundary, $\mathrm{S}^{d-1}$, terminates with the $\tau^{-1/2}$ term, a well known fact. This mechanism is explicit for the even $d$-hemisphere using Grubb and Seeley's product construction.

The values (13) show, in particular, that the massless spin-1/2 scaling behaviour is governed in the spectral case by the numbers,

$$\begin{aligned}
&\zeta_2(0)=-\frac{1}{12}, \quad \zeta_3(0)=0, \quad \zeta_4(0)=\frac{11}{360}, \quad \zeta_5(0)=0,\\
&\zeta_6(0)=-\frac{191}{15120}, \quad \zeta_7(0)=0, \quad \zeta_8(0)=\frac{2497}{453600}, \quad \textit{etc.}
\end{aligned} \tag{14}$$

which equal those for local (or mixed) boundary conditions as was noted by D'Eath and Esposito [3] in four dimensions. The mixed values also follow from those on the $d$-hemisphere by conformal invariance which may account for the equality since the Grubb-Seeley formula shows that on the hemisphere $\zeta_d^{(S)}(0)=\zeta_d^{(L)}(0)$, each being half the full sphere value.

In addition we note the result,

$$B^{(S)}_{d/2-1}(d)=0, \quad d \text{ even} \tag{15}$$

which corresponds to the vanishing residue of the pole of the spectral $\zeta$–function at $s=1$.

**5. Spectral functional determinants.**

Application of the techniques fully described in our earlier works [35–37] leads straightforwardly to

$$\begin{aligned}
\zeta_2'(0) &= 2\zeta_R'(-1) + \frac{2}{3}\ln 2 + \frac{5}{12},\\
\zeta_3'(0) &= -\frac{3}{2}\zeta_R'(-2) + \frac{1}{6}\ln 2 + \frac{11}{48},\\
\zeta_4'(0) &= \frac{2}{3}\big(\zeta_R'(-3) - \zeta_R'(-1)\big) + \frac{1}{45}\ln 2 - \frac{2489}{30240},\\
\zeta_5'(0) &= \frac{5}{8}\zeta_R'(-2) - \frac{5}{16}\zeta_R'(-4) - \frac{59}{720}\ln 2 - \frac{17497}{241920},\\
\zeta_6'(0) &= \frac{4}{15}\zeta_R'(-1) - \frac{1}{3}\zeta_R'(-3) + \frac{1}{15}\zeta_R'(-5) - \frac{1}{189}\ln 2 + \frac{6466519}{207567360},\\
\zeta_7'(0) &= -\frac{259}{960}\zeta_R'(-2) + \frac{35}{192}\zeta_R'(-4) - \frac{7}{320}\zeta_R'(-6) + \frac{2179}{60480}\ln 2 + \frac{59792179}{2075673600},\\
\zeta_8'(0) &= -\frac{4}{35}\zeta_R'(-1) + \frac{7}{45}\zeta_R'(-3) - \frac{2}{45}\zeta_R'(-5)\\
&\qquad + \frac{1}{315}\zeta_R'(-7) + \frac{23}{14175}\ln 2 - \frac{183927381289}{14079294028800}.
\end{aligned} \tag{16}$$

The four dimensional result is that already computed in [32,37].

**6. Mixed functional determinants.**

The mixed determinants are likewise found to be given in terms of

$$\begin{aligned}
\zeta_2'(0) &= 2\zeta_R'(-1) + \frac{1}{6}\ln 2 - \frac{1}{12},\\
\zeta_3'(0) &= -\frac{3}{2}\zeta_R'(-2) + \frac{1}{4}\ln 2 + \frac{1}{16},\\
\zeta_4'(0) &= \frac{251}{15120} - \frac{11}{180}\ln 2 + \frac{2}{3}\big(\zeta_R'(-3) - \zeta_R'(-1)\big),\\
\zeta_5'(0) &= -\frac{91}{3840} - \frac{3}{32}\ln 2 - \frac{5}{16}\zeta_R'(-4) + \frac{5}{8}\zeta_R'(-2),\\
\zeta_6'(0) &= -\frac{28417}{4989600} + \frac{191}{7560}\ln 2 + \frac{1}{15}\zeta_R'(-5) - \frac{1}{3}\zeta_R'(-3) + \frac{4}{15}\zeta_R'(-1),\\
\zeta_7'(0) &= \frac{47941}{4838400} + \frac{5}{128}\ln 2 - \frac{7}{320}\zeta_R'(-6) + \frac{35}{192}\zeta_R'(-4) - \frac{259}{960}\zeta_R'(-2),\\
\zeta_8'(0) &= \frac{14493407}{6399679104} - \frac{2497}{226800}\ln 2 + \frac{1}{315}\zeta_R'(-7) - \frac{2}{45}\zeta_R'(-5)\\
&\qquad + \frac{7}{45}\zeta_R'(-3) - \frac{4}{35}\zeta_R'(-1).
\end{aligned} \tag{17}$$

It is worth noting that the two- three- and four-dimensional results agree with those found by one of us (JSA) using a conformal transformation method [38]. Again, the four dimensional result is that given in [32,37].

## 7. Conclusion.

As noted earlier, the specific expressions obtained here may be of use in tying down the general form of the heat-kernel coefficients in the spectral case, if there is one. Grubb and Seeley's [34] formal results on the expansion have already been alluded to. The work of Gilkey [19] is mostly concerned with that combination of coefficients relevant for the spin index.

More might be said for the mixed coefficients. For example, the general form of the mixed $B_{5/2}$ could be written down following [27]. Then, specialising to a flat ambient manifold, precise values for some coefficients, and for combinations of others, could be obtained in the manner of van den Berg (reported in [15]) who used the Dirichlet scalar polynomials computed by Levitin [39]. This programme will be pursued elsewhere. Unfortunately the procedure will not be as informative as the corresponding scalar one, where one has the extra control provided by the Robin multiplier. For example, using the polynomials derived by Levitin, the following Neumann coefficients in lemma 5.1 of [15] are very easily obtained,

$$d_{30} = 2160, \quad d_{31} = 1080, \quad d_{32} = 360, \quad d_{33} = \frac{885}{4}, \quad d_{34} = \frac{315}{2}$$

$$d_{35} = 150,\ d_{36} = \frac{2041}{128},\ d_{37} = \frac{417}{32},\ d_{38} + d_{39} = \frac{1175}{32},\ d_{40} = \frac{231}{8}.$$

## 8. Acknowledgments.

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**Appendix A. $\gamma$–matrices and spinors.**

In a $d$–dimensional space, we denote by $\gamma^a_{(d)}$ , $a = 1, 2, \dots d$, the $\gamma$–matrices projected along some $d$–bein system. If $d$ is even, the $\gamma$'s are defined inductively by

$$
\begin{aligned}
&\gamma^j_{(d)} = \begin{pmatrix} \mathbf{0} & i\gamma^j_{(d-2)} \\ -i\gamma^j_{(d-2)} & \mathbf{0} \end{pmatrix}, \quad j = 1, 2, \dots d-1, \\
&\gamma^d_{(d)} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \gamma^{d+1}_{(d)} = \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}
\end{aligned}
\tag{18}
$$

starting from the Pauli matrices

$$
\gamma^1_{(2)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^2_{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The matrices (18) satisfy the Dirac anti-commutation formula

$$
\gamma^a_{(d)}\, \gamma^b_{(d)} + \gamma^b_{(d)}\, \gamma^a_{(d)} = 2\delta^{ab}.
$$

In the body of this paper on the $d$-ball, $\gamma^{d+1}_{(d)}$ is denoted by $\Gamma^5$ and $\gamma^d_{(d)}$ by $\Gamma^r$, the (outward) radial matrix. For example, the mixed projector (8) is written here as

$$
P_+\psi = \frac{1}{2}\big(\mathbf{1} - i\gamma^{d+1}_{(d)}\gamma^a_{(d)} e^\mu_a\, n_\mu\big)\psi = 0
$$

where $e^\mu_a$ is the $d$–bein.

For spectral conditions, in the terminology of [19], $\widetilde{\gamma}^j_{(d-1)} = i\gamma^d_{(d)}\, \gamma^j_{(d)}$ is the *induced tangential* Clifford module structure on the boundary confined spinor bundle and satisfies

$$
\widetilde{\gamma}^i_{(d-1)}\, \widetilde{\gamma}^j_{(d-1)} + \widetilde{\gamma}^j_{(d-1)}\, \widetilde{\gamma}^i_{(d-1)} = 2\delta^{ij}.
$$

In the present work, the matrices for odd $d$ are defined in terms of those for even $d$ in the following way,

$$
\begin{aligned}
&\gamma^j_{(d)} = \gamma^j_{(d+1)}, \quad j = 1, 2, \dots d-1, \\
&\gamma^d_{(d)} = \gamma^{d+1}_{(d+1)}, \qquad \gamma^{d+1}_{(d)} = \gamma^{d+2}_{(d+1)},
\end{aligned}
\tag{19}
$$

and $\gamma^d_{(d+1)}$ is not used. Again, $\gamma^d_{(d)}$ is the radial matrix and $\gamma^{d+1}_{(d)}$ is '$\Gamma^5$'. This particular choice has the advantage of giving the same mode structure in both odd and even dimensions.

In effect, we are defining spinors on odd $\mathcal{M}$ through those on even $\mathbb{R}\times\mathcal{M}$ by ignoring the added dimension, *e.g.* by taking fields uniform on the $\mathbb{R}$.

Of course this is what physicists have done automatically from the first when separating variables for the Dirac equation in, say, polar coordinates. A pertinant case is the Casimir energy in a spatial 3-sphere in Minkowski space-time *e.g.* [40–42].

The use of doubled $\gamma$-matrices for odd dimensions in the present paper was motivated originally by the desire to implement mixed boundary conditions, for which $d$ matrices are needed to contract into the normal plus one further matrix that anti-commutes with these. Since there are not enough matrices in the usual irreducible representation (of dimension $2^{(d-1)/2}$) of the Clifford-Dirac algebra to accomplish this, the dimension was doubled and $\gamma$-matrices of one *higher* dimension used, with a single redundancy. This means, for example, that 2-spinors can be defined on the 3-sphere, but not on the 3-hemisphere. These doubled-up matrices also allow one to discuss spectral conditions for odd $d$, as in the text. Another approach, using pin manifolds, is discussed by Gilkey, [19] section 9.

Trautman [43] refers to spinors in $\mathbb{R}^{2n}$ as *Dirac* spinors, which, when restricted to a hypersurface, become *Cartan* spinors. There seems to be no reason why this terminology cannot be extended to curved spaces.

**Appendix B. Mixed coefficient polynomials.**

The mixed coefficients have the structure,

$$\begin{aligned} B_n^{(L)}(d) &= \frac{2^{-d} d_s}{\Gamma(d/2)} \sqrt{\pi}\,(d-1)\,P_n(d), \quad n = 1/2, 3/2, \ldots \\ &= \frac{2^{-d} d_s}{\Gamma(d/2)}\,(d-1)\,P_n(d), \quad n = 1, 2, \ldots, \end{aligned} \tag{20}$$

with the polynomials

$$\begin{aligned}
P_{5/2}^{(L)}(d) &= \frac{1}{122880}\,(d+1)\,(d-5)\,(89\,d-263)\,, \\
P_{3}^{(L)}(d) &= -\frac{1}{1247400}\,\left(15600+11426\,d-9169\,d^2+1006\,d^3+61\,d^4\right), \\
P_{7/2}^{(L)}(d) &= \frac{1}{495452160}\,(d-7) \\
&\quad \left(393039+368952\,d-147742\,d^2-33848\,d^3+9167\,d^4\right), \\
P_{4}^{(L)}(d) &= -\frac{1}{219988969200}\,\left(1908965520+1529812932\,d-808656824\,d^2\right. \\
&\quad \left.-197908917\,d^3+105046309\,d^4-10068831\,d^5+83899\,d^6\right), \\
P_{9/2}^{(L)}(d) &= \frac{1}{20927899238400}\,(d+1)\,(d-9) \\
&\quad \left(10887720195-916876245\,d-2084061206\,d^2+333544346\,d^3\right. \\
&\quad \left.+40853459\,d^4-6852869\,d^5\right), \\
P_{5}^{(L)}(d) &= -\frac{1}{6830657493660000} \\
&\quad \left(57920260204800+47074221218160\,d-20614444675524\,d^2\right. \\
&\quad -7939793557052\,d^3+2539767459817\,d^4+254749941880\,d^5 \\
&\quad \left.-118154075186\,d^6+9525728692\,d^7-170628227\,d^8\right).
\end{aligned}$$

**Appendix C. Spectral coefficient polynomials.**

$$F_{5/2}(d) = -\frac{1}{60480}\,(46809 - 27899\,d + 4536\,d^2 - 160\,d^3),$$

$$G_{5/2}(d) = \frac{1}{368640}\,(9927 - 5129\,d + 369\,d^2 + 65\,d^3),$$

$$F_3(d) = -\frac{1}{405405}\,(d-8)\,\big(1542 - 385\,d - 171\,d^2 + 40\,d^3\big),$$

$$G_3(d) = \frac{1}{737280}\,(d-8)\,\big(63600 - 33668\,d + 3924\,d^2 + 65\,d^3\big),$$

$$F_{7/2}(d) = -\frac{1}{103783680}\,\big(221818311 - 156858900\,d + 35468617\,d^2 - 2592500\,d^3 - 24928\,d^4 + 5120\,d^5\big),$$

$$G_{7/2}(d) = \frac{1}{371589120}\,(4501359 - 827409\,d - 1050058\,d^2 + 372374\,d^3 - 35141\,d^4 + 475\,d^5),$$

$$F_4(d) = -\frac{1}{13749310575}\,(d-10)\,\big(23041368 + 2531082\,d - 8288995\,d^2 + 1680941\,d^3 + 24355\,d^4 - 16775\,d^5\big)\,,$$

$$G_4(d) = \frac{1}{743178240}\,(d-10)\,\big(170021376 - 111709248\,d + 21793760\,d^2 - 1009228\,d^3 - 50096\,d^4 + 475\,d^5\big)\,,$$

$$F_{9/2}(d) = -\frac{1}{56317176115200}\,(464260690378485 - 373244849131275\,d + 106164603742547\,d^2 - 12690585476317\,d^3 + 504841197392\,d^4 + 7017579968\,d^5 - 26306560\,d^6 - 34355200\,d^7),$$

$$G_{9/2}(d) = \frac{1}{62783697715200}\,(509445573615 + 91281582927\,d - 236987179165\,d^2 + 54541934915\,d^3 + 2269235885\,d^4 - 1736240947\,d^5 + 155559425\,d^6 - 3457375\,d^7),$$

$$F_5(d) = -\frac{1}{8538321867075}\,\big(d-12\big)\,\big(9567536832 + 3811378020\,d - 4614231340\,d^2 + 405754883\,d^3 + 230777003\,d^4 - 42837163\,d^5 + 1236545\,d^6 + 84980\,d^7\big),$$

$$G_5(d) = \frac{1}{125567395430400}\,(d-12)\,\big(105020227952640 - 80282869575168\,d + 20966354815040\,d^2 - 2131292479600\,d^3 + 52372511840\,d^4 + 1363355768\,d^5 + 125123300\,d^6 - 3457375\,d^7\big)\,.$$

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