Extravariables in the BRST Quantization of Second-Class Constrained Systems. Existence Theorems

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Abstract

In this paper we show how the BRST quantization can be applied to systems possessing only second-class constraints through their conversion to some first-class ones starting with our method exposed in [Nucl.Phys. B456 (1995)473]. Thus, it is proved that i) for a certain class of second-class systems there exists a standard coupling between the variables of the original phase-space and some extravariables such that we can transform the original system into a one-parameter family of first-class systems; ii) the BRST quantization of this family in a standard gauge leads to the same path integral as that of the original system. The analysis is accomplished in both reducible and irreducible cases. In the same time, there is obtained the Lagrangian action of the first-class family and its provenience is clarified. In this context, the Wess-Zumino action is also derived. The results from the theoretical part of the paper are exemplified in detail for the massive Yang-Mills theory and for the massive abelian three-form gauge fields.

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1 Introduction

During the last years, the BRST method imposed itself as the only covariant quantization method for gauge theories. It is well-known that at the Hamiltonian level, to gauge theories correspond first (and eventually second)-class constraints. The canonical quantization of the theories with both first and second-class constraints has been accomplished in [1] and [2], while the BRST quantization of such theories is presented in [3]. A natural tendency is that of also quantizing the systems possessing only second-class constraints in the BRST formalism. This cannot be done directly because these theories do not possess gauge invariances. This is why it is necessary to implement in the theory some gauge invariances. This can be achieved by transforming the original second-class system into a first-class one in the original phase-space [4] or into a larger one obtained from the original phase-space by introducing some extra variables [5]-[6]. The BRST quantization of those second-class systems whose constraint matrix does not depend on the canonical variables is shown in [4] and is realized through implementing some gauge invariances in the original phase-space. Many authors [7]-[16] have applied the methods from [5]-[6] and succeeded in quantizing (in the BV, BRST or other methods) various models. The BRST quantization of second-class systems in a larger phase-space has not been gained in a general manner up to present. This is actually the purpose of our work. Namely, in this paper it will be shown how to realize in general the BRST quantization in a larger phase-space for systems subject only to second-class constraints. More precisely, starting with an original second-class system, we shall implement the following steps: i) we shall transform this system into a first-class one in the original phase-space [4]; ii) from this last system we shall build a one-parameter family of first-class systems in a larger phase-space in the case of irreducible original second-class constraints, as well as in the case where these initial constraints preserve somehow the trace of reducibility of a certain first-class system; iii) we shall quantize the first-class family in the light of the BRST formalism, obtaining in the end that its path integral is identical with the one of the original system. This is the meaning of applying the BRST quantization to second-class systems. We mention that our method of turning the original second-class system into a first-class family to be employed in step ii) is different from that exposed in [5]-[6]. In this paper we use for the sake of simplicity the notations of finite-dimensional analytical mechanics, but the analysis can be straightforwardly extended to field theory. Related to the BRST quantization, we follow the same lines as in [17].

The paper is organized into seven sections. In Sec.2 we shall briefly review the BRST quantization of second-class constrained systems in the original phase-space. Sec.3 is devoted to the construction of the one-parameter family of first-class systems. There it will be proved the existence of the Hamiltonian of the first-class family and it will be obtained its concrete form. In Sec.4 we shall quantize in the antifield BRST formalism the first-class family and prove that its path integral coincides with the one of the original system. Sec.5 focuses on the Lagrangian approach of the first-class family. Here it will be inferred the Lagrangian form of the path integral for the first-class family under some simple assumptions and it will be clarified the origin of this family. The Wess-Zumino action [18] associated with the introduction of extravariables is also emphasised. In Sec.6 there will be exposed two examples illustrating the results derived in the theoretical part of the paper. Sec.7 outlines some conclusions.

2 The BRST quantization of second-class systems in the original phase-space

We follow the presentation of Ref. [4], to which we refer for details and proofs. Our starting point is represented by a system with the canonical Hamiltonian H, described by N canonical pairs (q^i, p_i), and subject to the second-class constraints $\chi_\alpha = 0$, where $\chi_\alpha = (G_a, C_a)$ such that the constraint functions G_a to satisfy

$$[G_a, G_b] = C_{ab}{}^{c} G_c. \tag{1}$$

The symbol $[,]$ denotes the Poisson bracket. Because the constraint functions χ_α are second-class, it results simply from (1) that

$$\det C_{\alpha\beta} = (\det(\Delta_{ab}))^2 \neq 0, \tag{2}$$

where $C_{\alpha\beta} = [\chi_\alpha, \chi_\beta]$ and $\Delta_{ab} = [C_a, G_b]$. We treat only the case where Δ_{ab}'s do not depend on the canonical variables.

The first step in our quantization procedure consists in the construction of a first-class Hamiltonian with respect to the functions G_a. Related to this matter, the next theorem holds.

Theorem 1 *Let H be the canonical Hamiltonian of the system subject to the second-class constraints, $\chi_\alpha = 0$. Then, there exists a function $\overline{H} = H +$ "extraterms in q's and p's" such that $\overline{H}$ is first-class with respect to the constraints $G_a = 0$*

$$\left[\overline{H}, G_a\right] = f^b_{\ a} G_b, \tag{3}$$

with $f^b_{\ a}$ some functions of q's and p's.

Proof. The proof is given in [4].□

The concrete form of $\overline{H}$ reads [4]

$$\begin{aligned}\overline{H} \;=\; & H + \sum_{k=0}^{\infty} \frac{(-)^{k+1}}{(k+1)!} \left[\ldots \left[\left[H, G_{m_{k+1}} \right] \Delta^{m_{k+1} a_{k+1}}, G_{m_k} \right] \Delta^{m_k a_1}, \ldots, G_{m_1} \right] \cdot \\ & \Delta^{m_1 a_k} C_{a_1} C_{a_2} \ldots C_{a_{k+1}} + \lambda^a G_a, \end{aligned} \tag{4}$$

where Δ^{ab} is the inverse of Δ_{ab}, and λ^a's are some functions taken such that $f^b_{\ a} = \lambda^c C_{ca}^{\ \ \ b} + (-)^{\epsilon_a \epsilon_b} \left[\lambda^b, G_a\right]$. In the last formula, ϵ_a denotes the Grassmann parity of the function G_a. Making a co-ordinate transformation of the type [4]

$$\left(q^i, p_i\right) \rightarrow \left(Q^a, P_a, z^\Delta, \overline{p}_\Delta\right), \tag{5}$$

such that $P_a = G_a$, $Q^a = \Delta^{ab} C_b$, $\left[z^\Delta, P_a\right] = \left[\overline{p}_\Delta, P_a\right] = 0$, and $\left(z^\Delta, \overline{p}_\Delta\right)$ to be canonical pairs, we associate to the original system described by the action

$$S_0\left[q^i, p_i, \mu^\alpha\right] = \int dt \left(\dot{q}^i p_i - H - \mu^\alpha \chi_\alpha\right), \tag{6}$$

a first-class system with the action

$$S_0\left[Q^a, P_a, z^\Delta, \overline{p}_\Delta, v^a\right] = \int dt \left(\dot{Q}^a P_a + \dot{z}^\Delta \overline{p}_\Delta - \tilde{H} - v^a P_a\right). \tag{7}$$

In (7), $\tilde{H}\left(Q^a, z^\Delta, \overline{p}_\Delta\right) = \overline{H} - \lambda^a G_a = H\left(0, z^\Delta, \overline{p}_\Delta\right) \equiv h\left(z^\Delta, \overline{p}_\Delta\right)$, as deduced in [4]. Action (7) is invariant under the gauge transformations $\delta_\epsilon Q^a = \epsilon^a$, $\delta_\epsilon v^a = \dot{\epsilon}^a$, $\delta_\epsilon z^\Delta = \delta_\epsilon \overline{p}_\Delta = \delta_\epsilon P_a = 0$. Transformation (5) is not canonical in general, but its Jacobian is equal to unity, as indicated in [4].

Let's pass now to the antifield BRST quantization of action (7). More precisely, we shall show that the path integral associated to action (6) is the same with the one corresponding to (7) after its BRST quantization in a gauge-fixing fermion implementing the canonical gauge conditions $C_a = 0$. These gauge conditions are equivalent to the conditions $Q^a = 0$. The next theorem is helpful in finding the correct form of the above mentioned gauge-fixing fermion.

Theorem 2 *There exists a set of functions $f^a(Q)$ such that*

$$\frac{1}{2}\Delta_{ab} f^a(Q) f^b(Q) = \sum_{k=0}^{\infty} \frac{(-)^{k+1}}{(k+1)!} \frac{\overrightarrow{\partial}^{k+1} H}{\partial Q^{a_1} \ldots \partial Q^{a_{k+1}}} Q^{a_1} \ldots Q^{a_{k+1}}. \tag{8}$$

Proof. The proof is given in [4].□

The form of the functions $f^a(Q)$ reads [4]

$$\begin{aligned} f^a(Q) &= Q^a - \frac{1}{3!}\Delta^{(ab)} \left.\frac{\overrightarrow{\partial}^3 H}{\partial Q^b \partial Q^{b_1} \partial Q^{b_2}}\right|_{Q=0} Q^{b_1} Q^{b_2} + \\ & \sum_{k=3}^{\infty} \frac{1}{k!} \left.\frac{\overrightarrow{\partial}^k f^c(Q)}{\partial Q^{c_1} \ldots \partial Q^{c_k}}\right|_{Q=0} Q^{c_1} \ldots Q^{c_k}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \frac{1}{k!} \left.\frac{\overrightarrow{\partial}^k f^c(Q)}{\partial Q^{c_1} \ldots \partial Q^{c_k}}\right|_{Q=0} &= -\Delta^{(cm)} \left(\frac{1}{(k+1)!} \left.\frac{\overrightarrow{\partial}^{k+1} H}{\partial Q^m \partial Q^{c_1} \ldots \partial Q^{c_k}}\right|_{Q=0} + \right. \\ & \left. \frac{1}{2!}\Delta_{ab} \sum_{j=2}^{k-1} \frac{1}{j!\,(k-j+1)!} \left.\frac{\overrightarrow{\partial}^j f^a(Q)}{\partial Q^m \partial Q^{c_1} \ldots \partial Q^{c_{j-1}}}\right|_{Q=0} \left.\frac{\overrightarrow{\partial}^{k-j+1} f^b(Q)}{\partial Q^{c_j} \ldots \partial Q^{c_k}}\right|_{Q=0} \right). \end{aligned}$$

In the last formulas, $\Delta^{(ab)}$ is the inverse of the matrix $\Delta_{(ab)} = \frac{1}{2}(\Delta_{ab} + \Delta_{ba})$. It appears clearly from (9) that $f^a(Q) = 0$ implies $Q^a = 0$. Taking the gauge-fixing fermion of the form $\Psi = -\int dt \left(\overline{\eta}^a \Delta_{ab} f^b(Q)\right)$, we obtain

$$Z_\Psi = Z, \tag{10}$$

where Z_Ψ is the path integral of the first-class system in the gauge Ψ, and Z is the path integral of the original system and is given by [17]

$$Z = \int \mathcal{D}q\, \mathcal{D}p\, \mathcal{D}\mu\, \left(\det C_{\alpha\beta}\right)^{1/2} \exp\left(iS_0\left[q^i, p_i, \mu^\alpha\right]\right). \tag{11}$$

At the level of independent variables $\left(z^{\Delta},\overline{p}_{\Delta}\right)$, formula (11) becomes

$$Z=\int\mathcal{D}z^{\Delta}\,\mathcal{D}\,\overline{p}_{\Delta}\exp\left(i\int dt\left(\dot{z}^{\Delta}\overline{p}_{\Delta}-h\left(z^{\Delta},\overline{p}_{\Delta}\right)\right)\right). \tag{12}$$

3 The construction of the one-parameter family of first-class systems

Within this section we shall extend the original phase-space and shall construct in the new phase-space a one-parameter family of first-class systems associated to the original theory. In the sequel we shall consider only those purely bosonic systems with the primary constraints $G_a=0$ and the secondary ones $C_a=0$, and whose phase-space is described by the real functions (q^i,p_i). The case of the systems having only primary second-class constraints is treated in [19]. We make, without affecting the generality, the assumption that the functions C_a can be written under the form

$$C_a=C_a^0+C_a^1, \tag{13}$$

such that $[C_a^0,C_b^0]=[G_a,C_b^0]=0$ strongly, and $[C_a^1,G_b]=\Delta_{ab}$. Indeed, the form (13) is not an additional restriction because if we make the transformation (5) we can always take $C_a^0=m_a\left(\overline{p}_{\Delta}\right)$ and $C_a^1\equiv C_a-m_a\left(\overline{p}_{\Delta}\right)$, for some functions m_a. In order to build the first-class family invoked above, the next theorem is crucial in order to couple the original variables with the ones to be added below.

Theorem 3 *Let H be the canonical Hamiltonian of a system possessing the primary second-class constraints $P_a=0$, and the secondary ones $Q_a=0$. Then,*

i) the sole real solution of the system $f^a(Q)=0$, where $f^a(Q)$ fulfill (8), is $Q^a=0$;

ii) $\det\left(\frac{\partial f^a(Q)}{\partial Q^b}\right)\neq 0$, *for every Q^a real.*

Proof. i) From (9) it results directly that $Q^a=0$ is solution for $f^a(Q)=0$. It remains to prove that this is the only real solution. Representing the canonical Hamiltonian as a series of powers in Q^a's, [4]

$$H\left(\left(Q^a, z^\Delta, \overline{p}_\Delta\right)\right) = H\left(0, z^\Delta, \overline{p}_\Delta\right) + \sum_{j=2}^{\infty} \frac{1}{j!} \left. \frac{\partial^j H\left(Q^a, z^\Delta, \overline{p}_\Delta\right)}{\partial Q^{a_1} \ldots \partial Q^{a_j}} \right|_{Q=0} Q^{a_1} \ldots Q^{a_j}, \tag{14}$$

and introducing (14) in (8), we obtain

$$\frac{1}{2} \Delta_{ab} f^a(Q) f^b(Q) = H\left(0, z^\Delta, \overline{p}_\Delta\right) - H\left(Q^a, z^\Delta, \overline{p}_\Delta\right). \tag{15}$$

Differentiating (15) with respect to Q^c, it follows

$$\frac{1}{2} \Delta_{(ab)} \frac{\partial f^a(Q)}{\partial Q^c} f^b(Q) = -\frac{\partial H\left(Q^a, z^\Delta, \overline{p}_\Delta\right)}{\partial Q^c}. \tag{16}$$

On the other hand, as $Q^a = 0$ are a consequence of the constraints $P_a = 0$, it results

$$\dot{P}_c = -\frac{\partial H\left(Q^a, z^\Delta, \overline{p}_\Delta\right)}{\partial Q^c} = 0 \Longrightarrow Q^c = 0. \tag{17}$$

Comparing (16) and (17), it is clear that if there exists an other real solution of $f^a(Q) = 0$ than $Q^a = 0$, e.g. $Q_0^a \neq 0$, then the system will also have the secondary constraints $Q_0^a = 0$, which contradicts the hypotheses. Thus, i) is proved.

ii) From (9) we get $\det\left(\frac{\partial f^a(Q)}{\partial Q^b}\right)\Big|_{Q=0} = 1$, so the last determinant is non-vanishing. In this way, it remains to be proved that

$$\det\left(\frac{\partial f^a(Q)}{\partial Q^b}\right)\Bigg|_{Q\neq 0} \neq 0, \qquad \text{for every } Q^a \text{ real}. \tag{18}$$

Using the result from i), it follows that we can represent $f^a(Q)$ under the form

$$f^a(Q) = V^a{}_b Q^b, \tag{19}$$

where $V^a{}_b$ is an invertible matrix depending on Q^a's, z^Δ's and $\overline{p}_\Delta$'s. With the aid of the fact that $Q^a = 0$ are the only secondary constraints, we have

$$-\frac{\partial H\left(Q^a, z^\Delta, \overline{p}_\Delta\right)}{\partial Q^c} = \overline{V}_{cb} Q^b, \tag{20}$$

with $\overline{V}_{cb}$ an invertible matrix depending on the same variables as $V^a{}_b$. Introducing (19) and (20) in (16), we infer

$$\frac{1}{2}\Delta_{(ab)}V^b{}_c\frac{\partial f^c(Q)}{\partial Q^d}=\overline{V}_{cd},\qquad \mathit{for\ every\ real}\ Q^a\neq 0. \tag{21}$$

Taking the determinant in both hands of (21) and taking into account that the $V^b{}_c$'s and $\overline{V}_{cd}$ are both invertible, it results immediately (18). This proves ii).□

It is easy to see that ii) implies det $\left(\frac{\partial f^a(C)}{\partial C_b}\right)\neq 0$, for every real functions C_b. In the last relation, $f^a(C)$ is obtained from $f^a(Q)$ using $Q^a=\Delta^{ab}C_b$. The importance of the last theorem resides in the fact that we can implement in a simple manner some secondary first-class constraints $\gamma_a=0$ for the first-class family we intend to construct through the term $\gamma_a f^a(C)$ which we shall introduce in the Hamiltonian of this family. It is precisely this term which will couple the original variables with the new ones. This way of coupling is one of the main points in our approach and reveals the main difference between our conversion method and the one presented in [5]-[6]. Indeed, choosing γ_a such that $[G_a,\gamma_b]=0$ strongly, we infer that $\left[G_a,\gamma_b f^b(C)\right]=-\Delta_{ca}\frac{\partial f^b(C)}{\partial C_c}\gamma_b$. If we succeed in finding a Hamiltonian H^* for the first-class family satisfying $[H^*,G_a]=\left[\gamma_b f^b(C),G_a\right]$, then the consistency of the primary constraints $G_a=0$ will imply the secondary constraints $\gamma_a=0$ as det $\left(\frac{\partial f^b(C)}{\partial C_c}\right)\neq 0$. This problem will be treated in the next two subsections. The splitting (13) of the functions C_a will evidence two important cases, namely the irreducible case where the functions C_a^0 are all independent, and the reducible case where these functions are reducible. These cases will be treated separately.

3.1 The irreducible case

In this subsection we shall consider the case of the functions C_a^0 being all independent. Then, the construction of the first-class family goes as follows. For every pair (G_a,C_a) we introduce a canonical pair $(z^a,\overline{p}_a)$, so the new phase-space will have the local co-ordinates $(q^i,p_i,z^a,\overline{p}_a)$. We construct the Hamiltonian H^* of the first-class family such that the gauge algebra to be

$$[G_a,H^*]=-\Delta_{ca}\frac{\partial f^b(C)}{\partial C_c}\gamma_b, \tag{22}$$

$$\left[\gamma_a, H^*\right] = \left[C_a^0, f^b\left(C\right)\right]\gamma_b, \tag{23}$$

$$\left[G_a, \gamma_b\right] = 0, \tag{24}$$

with the choice

$$\gamma_a \equiv C_a^0 + \lambda \overline{p}_a, \tag{25}$$

λ being the non-vanishing parameter of the first-class family. We take the Hamiltonian H^* of the form

$$H^* = H' - \frac{\lambda^2}{2}\Delta^{ab}\overline{p}_a\overline{p}_b + \gamma_a f^a\left(C\right) + g\left(q^i, p_i, z^a, \overline{p}_a\right), \tag{26}$$

where

$$H' = \tilde{H} + \frac{1}{2}\Delta^{ab}C_a^0 C_b^0, \tag{27}$$

and $g\left(q^i, p_i, z^a, \overline{p}_a\right)$ is a function to be further derived. We notice that the first piece of H^* is the Hamiltonian of the first-class system in the original phase-space associated to the original theory. In fact, H' and $\tilde{H}$ are in the same class of gauge invariant functions with respect to the G_a's. In principle, the second term in the right hand of (27) can be taken any function of C_a^0's. As it will be seen, the necessity of taking this term quadratic in C_a^0's is directly connected with the choice of the quadratic term in the momenta $\overline{p}_a$'s from (26). This last quadratic term is motivated by a simpler passing to the Lagrangian formalism (see Sec.5). Replacing (26) in (22) and (23), we get the equations

$$\left[G_a, g\left(q^i, p_i, z^a, \overline{p}_a\right)\right] = 0, \tag{28}$$

$$\left[C_a^0, H'\right] + \lambda\left[\overline{p}_a, g\left(q^i, p_i, z^a, \overline{p}_a\right)\right] = 0. \tag{29}$$

Now, it appears more obviously the reason of choosing the gauge algebra of the form (22-24), the secondary first-class constraints as in (25) and the Hamiltonian H^* like in (26). With these choices, the first-class family is determined from the original system up to the function $g\left(q^i, p_i, z^a, \overline{p}_a\right)$, which has to fulfill (28-29). The next theorem shows that this function can be also completely gained from the original system.

Theorem 4 *There exists a function $g\left(q^i, p_i, z^a, \overline{p}_a\right)$ satisfying (28-29).*

Proof. The proof is intended to be constructive, finally obtaining the concrete form of $g\left(q^i,p_i,z^a,\overline{p}_a\right)$. We represent this function as a series of powers in z^a's with coefficients depending on $\left(q^i,p_i,\overline{p}_a\right)$

$$g\left(q^i,p_i,z^a,\overline{p}_a\right)=\sum_{k=1}^{\infty}\overset{(k)}{g}_{a_1\ldots a_k}\left(q^i,p_i,\overline{p}_a\right)z^{a_1}\ldots z^{a_k}. \tag{30}$$

Inserting (30) in (29) and identifying the coefficients of the same powers in z^a's, we find the following tower of equations

$$\lambda\overset{(1)}{g}_a=\left[C_a^0,H'\right], \tag{31}$$

$$2\lambda\overset{(2)}{g}_{a_1a_2}=\left[C_{a_1}^0,\overset{(1)}{g}_{a_2}\right], \tag{32}$$

$$\vdots$$

$$k\lambda\overset{(k)}{g}_{a_1\ldots a_k}=\left[C_{a_1}^0,\overset{(k-1)}{g}_{a_2\ldots a_k}\right], \tag{33}$$

$$\vdots$$

Using (31-33), we deduce in a simple manner

$$\overset{(k)}{g}_{a_1\ldots a_k}=\frac{1}{k!\lambda^k}\left[C_{a_1}^0,\left[C_{a_2}^0,\ldots,\left[C_{a_k}^0,H'\right]\ldots\right]\right]. \tag{34}$$

In this way, we proved that (30) with the coefficients (34) is the solution of (29). Using the Jacobi identity we get immediately

$$\left[G_a,\overset{(k)}{g}_{a_1\ldots a_k}\right]=0, \qquad \mathit{for\ every}\ k, \tag{35}$$

so (28) is also fulfilled. This ends the proof.□

Because $\left[C_a^0,H'\right]=\left[C_a^0,\tilde{H}\right]$, we finally obtain

$$g\left(q^i,p_i,z^a,\overline{p}_a\right)=\sum_{k=1}^{\infty}\frac{1}{k!\lambda^k}\left[C_{a_1}^0,\left[C_{a_2}^0,\ldots,\left[C_{a_k}^0,\tilde{H}\right]\ldots\right]\right]z^{a_1}\ldots z^{a_k}. \tag{36}$$

In this way, we associated to the original system depicted by action (6) a one-parameter family of first-class systems described by the action

$$S_0\left[q^i, p_i, z^a, \overline{p}_a, v^a, u^a\right] = \int dt \left(\dot{q}^i p_i + \dot{z}^a p_a - H^* - v^a G_a - u^a \gamma_a\right). \tag{37}$$

Action (37) is invariant under the gauge transformations:

$$\delta_\epsilon q^i = \left[q^i, G_a\right]\epsilon_1^a + \left[q^i, C_a^0\right]\epsilon_2^a, \tag{38}$$

$$\delta_\epsilon p_i = \left[p_i, G_a\right]\epsilon_1^a + \left[p_i, C_a^0\right]\epsilon_2^a, \tag{39}$$

$$\delta_\epsilon z^a = \lambda \epsilon_2^a, \tag{40}$$

$$\delta_\epsilon \overline{p}_a = 0, \tag{41}$$

$$\delta_\epsilon v^a = \dot{\epsilon}_1^a - C_{bc}{}^{a} v^b \epsilon_1^c, \tag{42}$$

$$\delta_\epsilon u^a = \dot{\epsilon}_2^a + \left[C_b^0, f^a(C)\right]\epsilon_2^b - \Delta_{cb}\frac{\partial f^a(C)}{\partial C_c}\epsilon_1^b. \tag{43}$$

These gauge transformations will be used in Sec.4 to the BRST quantization of the irreducible first-class family.

3.2 The reducible case

Within this subsection, we shall examine the case where the functions C_a^0 are not all independent. This means that there exist some functions on q^i's and p_i's denoted by $Z^a_{\ a_1}$, not all vanishing, such that

$$Z^a_{\ a_1} C_a^0 = 0. \tag{44}$$

We note that relations (44) represent some identities holding for all q^i 's and p_i's. Taking the Poisson brackets of G_a's and C_a^0's with both hands of (44), we get the following identities

$$\left[G_b, Z^a_{\ a_1}\right] = 0, \tag{45}$$

$$\left[C_b^0, Z^a_{\ a_1}\right] = 0. \tag{46}$$

With the aid of (46) we deduce, supposing that (44) are the sole reducibility relations for C_a^0's, the identities

$$\left[Z^a_{\ a_1}, Z^b_{\ b_1}\right] = 0. \tag{47}$$

At this point we are able to build consistently the first-class family in the reducible case. This construction goes as follows. For every pair (G_a, C_a) we introduce a new canonical pair $(z^a, \overline{p}_a)$ such that the consistency of the G_a's to imply the secondary constraints $\gamma_a = 0$, with γ^a given by (25). From (44) it results simply

$$Z^a_{\ a_1}\gamma_a = \lambda Z^a_{\ a_1}\overline{p}_a = 0. \tag{48}$$

For every relation (48) we add a new canonical pair (y^{a_1}, π_{a_1}) together with the constraint

$$\gamma_{a_1} \equiv \pi_{a_1} = 0, \tag{49}$$

such that the consistency of the last constraints to imply the secondary ones of the form

$$\overline{\gamma}_{a_1} \equiv -Z^a_{\ a_1}\overline{p}_a = 0, \tag{50}$$

which are precisely (48) up to a factor. Through this mechanism we cannot generate new constraints even if the reducibility functions $Z^a_{\ a_1}$ are not all independent. In general, we can assume that there exist some non-vanishing functions of (q^i, p_i), denoted by $Z^{a_1}_{\ \ a_2}, \ldots, Z^{a_{k-1}}_{\ \ a_k}$, such that the next identities to hold

$$Z^a_{\ a_1}Z^{a_1}_{\ \ a_2} = 0, \tag{51}$$

$$\vdots$$

$$Z^{a_{k-2}}_{\ \ a_{k-1}}Z^{a_{k-1}}_{\ \ a_k} = 0. \tag{52}$$

From (51) we draw that

$$Z^{a_1}_{\ \ a_2}\overline{\gamma}_{a_1} = 0. \tag{53}$$

If we repeated now the procedure between formulas (48-50), we would introduce some new canonical pairs (x^{a_2}, Π_{a_2}) together with the constraints $\Pi_{a_2} = 0$, such that their consistency to induce the "tertiary constraints"

$$\tilde{\gamma}_{a_2} \equiv Z^{a_1}_{\ \ a_2}\overline{\gamma}_{a_1} = 0. \tag{54}$$

Relations (54) are not constraints but identities due to (51). Thus, the maximal set of constraints we can generate through the above procedure is given by $G_a = 0$, $\gamma_a = 0$ and (49-50). From (45-47) it follows that all the previous constraints are first-class. A new feature of these constraints is that they become reducible. The reducibility relations read

$$Z^a_{\ a_1}\gamma_a + \lambda\overline{\gamma}_{a_1} = 0. \tag{55}$$

In the sequel, we shall build the Hamiltonian $\tilde{H}^*$ of the reducible first-class family to satisfy

$$\left[G_a, \tilde{H}^*\right] = -\Delta_{ca} \frac{\partial f^b (C)}{\partial C_c} \gamma_b, \tag{56}$$

$$\left[\gamma_{a_1}, \tilde{H}^*\right] = \overline{\gamma}_{a_1}, \tag{57}$$

$$\left[\gamma_a, \tilde{H}^*\right] = \left[C_a^0, f^b (C)\right] \gamma_b, \tag{58}$$

$$\left[\overline{\gamma}_{a_1}, \tilde{H}^*\right] = M_{a_1}^a \gamma_a, \tag{59}$$

with $M_{a_1}^a$ some functions to be subsequently determined. In the reducible case, we take the Hamiltonian $\tilde{H}^*$ of the form

$$\tilde{H}^* = H' - \frac{\lambda^2}{2} \Delta^{ab} \overline{p}_a \overline{p}_b + \gamma_a f^a (C) - y^{a_1} \overline{\gamma}_{a_1} + \tilde{g} \left(q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}\right), \tag{60}$$

with $\tilde{g} (q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1})$ a function to be further obtained. Inserting (60) in (56-58) we achieve the following equations

$$\left[G_a, \tilde{g} \left(q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}\right)\right] = 0, \tag{61}$$

$$\left[\pi_{a_1}, \tilde{g} \left(q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}\right)\right] = 0, \tag{62}$$

$$\left[C_a^0, H'\right] + \lambda \left[\overline{p}_a, \tilde{g} \left(q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}\right)\right] = 0. \tag{63}$$

It is simply to observe that the function $g (q^i, p_i, z^a, \overline{p}_a)$ given by (36) verifies automatically (61-63). Thus, it is left to be shown that $g (q^i, p_i, z^a, \overline{p}_a)$ verifies also (59). This is the aim of the next theorem.

Theorem 5 *Let f be a solution of equations (61-63). Then, f satisfies (59), where $\tilde{H}^*$ is given by (60) with $\tilde{g} = f$.*

Proof. Taking the Poisson bracket of both hands in (55) with $\tilde{H}^*$ we get

$$\left[Z_{a_1}^a \gamma_a, \tilde{H}^*\right] + \lambda \left[\overline{\gamma}_{a_1}, \tilde{H}^*\right] = 0. \tag{64}$$

From (58) (which is verified if f is solution for (61-63)) and (64) we infer

$$\left[\overline{\gamma}_{a_1}, \tilde{H}^*\right] = -\frac{1}{\lambda} \left(Z_{a_1}^b \left[C_b^0, f^a (C)\right] + \left[Z_{a_1}^a, \tilde{H}^*\right]\right) \gamma_a. \tag{65}$$

The last relations are nothing but (59), with

$$M^{a}_{\ a_1} = -\frac{1}{\lambda}\left(Z^{b}_{\ a_1}\left[C^{0}_{b}, f^{a}\left(C\right)\right] + \left[Z^{a}_{\ a_1}, \tilde{H}^{*}\right]\right). \tag{66}$$

This completes the proof.□

Now it is clear the reason for choosing $\tilde{H}^{*}$ to fulfill (56-59). Indeed, accordingly the above theorem equations (56-59) are compatible with

$$\tilde{g}\left(q^{i}, p_{i}, z^{a}, \overline{p}_{a}, y^{a_1}, \pi_{a_1}\right) = g\left(q^{i}, p_{i}, z^{a}, \overline{p}_{a}\right), \tag{67}$$

with g given by (36).

To conclude with, we associated to the original theory (6) a reducible first-class family described by the action

$$S'_{0}\left[q^{i}, p_{i}, z^{a}, \overline{p}_{a}, y^{a_1}, \pi_{a_1}, v^{a}, v^{a_1}, u^{a}, u^{a_1}\right] = \int dt\left(\dot{q}^{i}p_{i} + \right.$$
$$\left.\dot{z}^{a}\overline{p}_{a} + \dot{y}^{a_1}\pi_{a_1} - \tilde{H}^{*} - v^{a}G_{a} - v^{a_1}\gamma_{a_1} - u^{a}\gamma_{a} - u^{a_1}\overline{\gamma}_{a_1}\right). \tag{68}$$

The gauge invariances of action (68) are deduced to be

$$\delta_{\epsilon}q^{i} = \left[q^{i}, G_{a}\right]\epsilon^{a}_{1} + \left[q^{i}, C^{0}_{a}\right]\epsilon^{a}_{2} - \left[q^{i}, Z^{a}_{\ a_1}\right]\overline{p}_{a}\epsilon^{a_1}_{4}, \tag{69}$$

$$\delta_{\epsilon}p_{i} = \left[p_{i}, G_{a}\right]\epsilon^{a}_{1} + \left[p_{i}, C^{0}_{a}\right]\epsilon^{a}_{2} - \left[p_{i}, Z^{a}_{\ a_1}\right]\overline{p}_{a}\epsilon^{a_1}_{4}, \tag{70}$$

$$\delta_{\epsilon}z^{a} = \lambda\epsilon^{a}_{2} - Z^{a}_{\ a_1}\epsilon^{a_1}_{4}, \tag{71}$$

$$\delta_{\epsilon}\overline{p}_{a} = 0, \tag{72}$$

$$\delta_{\epsilon}y^{a_1} = \epsilon^{a_1}_{3}, \tag{73}$$

$$\delta_{\epsilon}\pi_{a_1} = 0, \tag{74}$$

$$\delta_{\epsilon}v^{a} = \dot{\epsilon}^{a}_{1} - C_{bc}^{\ \ a}v^{b}\epsilon^{c}_{1}, \tag{75}$$

$$\delta_{\epsilon}v^{a_1} = \dot{\epsilon}^{a}_{3}, \tag{76}$$

$$\delta_{\epsilon}u^{a} = \dot{\epsilon}^{a}_{2} + \left[C^{0}_{b}, f^{a}\left(C\right)\right]\epsilon^{b}_{2} - \Delta_{cb}\frac{\partial f^{a}\left(C\right)}{\partial C_{c}}\epsilon^{b}_{1} -$$
$$\frac{1}{\lambda}\left(Z^{b}_{\ a_1}\left[C^{0}_{b}, f^{a}\left(C\right)\right] + \left[Z^{a}_{\ a_1}, \tilde{H}^{*}\right]\right)\epsilon^{a_1}_{4} + Z^{a}_{\ a_1}\epsilon^{a_1}_{5}, \tag{77}$$

$$\delta_\epsilon u^{a_1} = \epsilon_3^{a_1} + \dot{\epsilon}_4^a + \lambda \epsilon_5^{a_1}. \tag{78}$$

The gauge parameters $\epsilon_5^{a_1}$ appear due to the reducibility relations (55) which allow us to introduce the additional gauge invariances [17], [20]

$$\delta_\epsilon u^A = Z^A_{\ b_1} \epsilon_5^{b_1}, \tag{79}$$

where $u^A = (u^a, u^{a_1})$, and $Z^A_{\ b_1} = \left(Z^a_{\ b_1}, \lambda \delta^{a_1}_{\ b_1} \right)$ are the reducibility functions from (55). The gauge invariances (69-78) will be employed within the BRST quantization of the reducible first-class family.

4 The antifield BRST quantization of the first-class family

In this section we shall quantize the first-class families constructed earlier in the context of the antifield BRST formalism based on path integrals. As there appear major differences between the reducible and irreducible cases, we shall treat them separately.

4.1 The quantization in the irreducible case

The starting point is given by action (37) together with the gauge transformations (38-43). Because the constraints are irreducible, the minimal ghost spectrum [17] will contain only the ghosts (η_1^a, η_2^a) correspondent to the gauge parameters $(\epsilon_1^a, \epsilon_2^a)$. The Grassmann parities and ghost numbers of the above ghosts are all equal to one. For all the variables $\Phi^I = (q^i, p_i, z^a, \overline{p}_a, v^a, u^a)$ we introduce the antifields [17]

$$\Phi_I^* = \left(q_i^*, p^{*i}, z_a^*, \overline{p}^{*a}, v_a^*, u_a^* \right),$$

all of Grassmann parity one and ghost number minus one. The non-minimal sector is taken to contain the variables

$$(B_1^a, B_{1a}^*, B_2^a, B_{2a}^*, \overline{\eta}_{1a}^*, \overline{\eta}_1^a, \overline{\eta}_{2a}^*, \overline{\eta}_2^a).$$

Then, the non-minimal solution of the master equation reads

$$S = S_0 \left[q^i, p_i, z^a, \overline{p}_a, v^a, u^a \right] + \int dt \left(q_i^* \left(\frac{\partial G_a}{\partial p_i} \eta_1^a + \frac{\partial C_a^0}{\partial p_i} \eta_2^a \right) - \right.$$

$$p^{*i}\left(\frac{\partial G_a}{\partial q^i}\eta_1^a+\frac{\partial C_a^0}{\partial q^i}\eta_2^a\right)+\lambda z_a^*\eta_2^a+v_a^*\left(\dot{\eta}_1^a-C_{bc}{}^{\;\;a}v^b\eta_1^c\right)+$$
$$u_a^*\left(\dot{\eta}_2^a+\left[C_b^0,f^a\left(C\right)\right]\eta_2^b-\Delta_{cb}\frac{\partial f^a\left(C\right)}{\partial C_c}\eta_1^b\right)+$$
$$\overline{\eta}_{1a}^*B_1^a+\overline{\eta}_{2a}^*B_2^a+\ldots)\,, \tag{80}$$

where "..." signify the terms of antighost numbers greater than one. These terms are not essential because of the special form of the gauge-fixing fermion to be outlined below. The standard gauge-fixing fermion in our methods reads

$$\Psi'=-\int dt\left(\overline{\eta}_1^a\Delta_{ab}f^b\left(\rho_c\right)-\lambda\overline{\eta}_2^a\Delta_{ab}z^b\right), \tag{81}$$

where

$$\rho_c=C_c-\frac{1}{\lambda}\left[C_c,C_b^0\right]z^b. \tag{82}$$

We observe that Ψ' reduces to Ψ (given in Sec.2) in the absence of the extravariables ($z^a=0$). The gauge-fixing fermion (81) implements the canonical gauge conditions $C_a=0$ and $z^a=0$. Eliminating in the usual manner the antifields from (80), we derive the next gauge-fixed action

$$\begin{aligned}S_{\Psi'} &= S_0\left[q^i,p_i,z^a,\overline{p}_a,v^a,u^a\right]+\int dt\left(\lambda^2\overline{\eta}_2^a\Delta_{ab}\eta_2^b+\lambda\Delta_{ab}B_2^az^b-\right.\\ &\quad \overline{\eta}_1^a\Delta_{ab}\frac{\partial f^b\left(\rho\right)}{\partial\rho_c}\left(\Delta_{cd}\eta_1^d-\frac{1}{\lambda}z^d\left[\left[C_c,C_d^0\right],C_e^0\right]\eta_2^e\right)-\\ &\quad \left.\Delta_{ab}B_1^af^b\left(\rho\right)\right).\end{aligned} \tag{83}$$

Employing repeatedly the Jacobi identity together with the fact that the term $\frac{\partial^2 f^a(\rho)}{\partial\rho_b\partial\rho_c}$ is symmetric in b and c, it is simply to see that (83) is invariant under the following BRST transformations

$$sq^i=\left[q^i,G_a\right]\eta_1^a+\left[q^i,C_a^0\right]\eta_2^a, \tag{84}$$

$$sp_i=\left[p_i,G_a\right]\eta_1^a+\left[p_i,C_a^0\right]\eta_2^a, \tag{85}$$

$$sz^a=\lambda\eta_2^a, \tag{86}$$

$$s\overline{p}_a=0, \tag{87}$$

$$sv^a=\dot{\eta}_1^a-C_{bc}{}^{\;\;a}v^b\eta_1^c, \tag{88}$$

$$su^a = \dot{\eta}_2^a + \left[C_b^0, f^a(C)\right]\eta_2^b - \Delta_{cb}\frac{\partial f^a(C)}{\partial C_c}\eta_1^b, \tag{89}$$

$$s\eta_1^a = s\eta_2^a = 0, \tag{90}$$

$$s\overline{\eta}_1^a = B_1^a, \tag{91}$$

$$s\overline{\eta}_2^a = -B_2^a, \tag{92}$$

$$sB_1^a = sB_2^a = 0. \tag{93}$$

The path integral correspondent to action (83) takes the form

$$Z_{\Psi'} = \int \mathcal{D}q^i\,\mathcal{D}p_i\,\mathcal{D}z^a\,\mathcal{D}\overline{p}_a\,\mathcal{D}\overline{\eta}_1^a\,\mathcal{D}\eta_1^a\,\mathcal{D}\overline{\eta}_2^a\,\mathcal{D}\eta_2^a\,\mathcal{D}v^a\,\mathcal{D}u^a\,\mathcal{D}B_1^a\,\mathcal{D}B_2^a\,\cdot \exp\left(iS_{\Psi'}\right). \tag{94}$$

Integrating in (94) over all the variables excepting the q^i's and p_i's we derive the following form of the path integral for the irreducible first-class family

$$Z_{\Psi'} = \int \mathcal{D}q^i\,\mathcal{D}p_i\,\det\left(\Delta_{cd}\right)\prod_a \delta\left(G_a\right)\prod_b \delta\left(C_b\right)\exp\left(i\int dt\left(\dot{q}^i p_i - \tilde{H}\right)\right). \tag{95}$$

If we integrate in (11) over μ^α's, it follows

$$Z_{\Psi'} = Z. \tag{96}$$

After performing the above integration, the exponents of the path integrals (11) and (95) differ through a term which vanishes when $C_a = 0$, but this term is not important because of the factors $\prod_b \delta\left(C_b\right)$ in the measure from (95). It is not hard to see that if we make the transformation (5) in (95) and further integrate over (Q^a, P_a), we get that $Z_{\Psi'}$ will be given by (12). We notify that (95) is identical to the path integral derived in [3] in the case of purely second-class systems. Formula (96) represents the main result of this subsection and one of the major results in this paper. It states that the path integral of the irreducible first-class family is the same with the one of the original second-class system in our standard gauge (81). This is the meaning of applying the BRST quantization to second-class systems.

4.2 The quantization in the reducible case

In this subsection we start with action (68) and its gauge invariances (69-78). For the sake of generality, we presume that the first-class family is k-th order reducible, with $k \geq 2$. Then, the ghost spectrum [17] contains the ghosts $(\eta_1^a, \eta_2^a, \eta_3^{a_1}, \eta_4^{a_1}, \eta_5^{a_1})$, all with the Grassmann parities and ghost numbers equal to one, as well as the ghosts η^{a_k}, these ones with the Grassmann parities $k\ (mod\, 2)$ and ghost numbers k. The antifield spectrum contains the antifields $\left(q_i^*, p^{*i}, z_a^*, \overline{p}^{*a}, y_{a_1}^*, \pi^{*a_1}, v_a^*, v_{a_1}^*, u_a^*, u_{a_1}^*\right)$, all with the Grassmann parities equal to one and ghost numbers equal to minus one, as well as the antifields $\eta_{a_{k-1}}^*$ with the Grassmann parities $k\ (mod\, 2)$ and ghost numbers $(-k)$. Then, the minimal solution of the master equation is expressed by

$$S' = S_0'\left[q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}, v^a, v^{a_1}, u^a, u^{a_1}\right] + \int dt \left\{ q_i^* \left(\frac{\partial G_a}{\partial p_i}\eta_1^a + \frac{\partial C_a^0}{\partial p_i}\eta_2^a - \frac{\partial Z^a_{\ a_1}}{\partial p_i}\overline{p}_a \eta_4^{a_1}\right) - p^{*i}\left(\frac{\partial G_a}{\partial q^i}\eta_1^a + \frac{\partial C_a^0}{\partial q^i}\eta_2^a - \frac{\partial Z^a_{\ a_1}}{\partial q^i}\overline{p}_a \eta_4^{a_1}\right) + z_a^*\left(\lambda \eta_2^a - Z^a_{\ a_1}\eta_4^{a_1}\right) + y_{a_1}^*\eta_3^{a_1} + v_a^*\left(\dot{\eta}_1^a - C_{bc}^{\ \ a} v^b \eta_1^c\right) + v_{a_1}^*\dot{\eta}_3^{a_1} + u_{a_1}^*\left(\dot{\eta}_4^{a_1} + \eta_3^{a_1} + \lambda\eta_5^{a_1}\right) + u_a^*\left[\dot{\eta}_2^a + \left[C_b^0, f^a(C)\right]\eta_2^b - \Delta_{cb}\frac{\partial f^a(C)}{\partial C_c}\eta_1^b - \frac{1}{\lambda}\left(Z^b_{\ a_1}\left[C_b^0, f^a(C)\right] + \left[Z^a_{\ a_1}, \tilde{H}^*\right]\right)\eta_4^{a_1} + Z^a_{\ a_1}\eta_5^{a_1}\right] + \sum_{j=2}^{k}\eta_{a_{j-1}}^* Z^{a_{j-1}}_{\ a_j}\eta^{a_j} + \ldots \right\}, \tag{97}$$

where "..." signify other terms with antighost numbers greater than one, which are not important due to the concrete form of the gauge-fixing fermion to be given below. We introduce a non-minimal sector [17] such that the non-minimal solution of the master equation to become

$$S'' = S' + \int dt \left(\overline{\eta}_{1a}^* B_1^a + \overline{\eta}_{2a}^* B_2^a + \overline{\eta}_1^{*a_1} B_{1a_1} + \overline{\eta}_2^{*a_1} B_{2a_1} + \overline{\eta}_3^{*a_1} B_{3a_1} + \sum_{j=2}^{k}\overline{\eta}^{*a_j} B_{a_j}\right). \tag{98}$$

The gauge-fixing fermion in the reducible case has the form

$$\Psi'' = \Psi' + \int dt \left(-\overline{\eta}_{1a_1} y^{a_1} - \overline{\eta}_{2a_1}\eta_4^{a_1} - \overline{\eta}_{3a_1} u^{a_1} + \sum_{j=2}^{k}\overline{\eta}_{a_j}\eta^{a_j}\right), \tag{99}$$

where Ψ' is given in (81).The gauge-fixing fermion (99) implements the canonical gauge conditions $C_a = 0$, $z^a = 0$, $y^{a_1} = 0$, $u^{a_1} = 0$. Eliminating as usually the antifields from (98), we infer the following gauge-fixed action

$$S''_{\Psi''} = S_{\Psi'} + \int dt \left(\overline{\eta}_1^a \Delta_{ab} \frac{\partial f^b(\rho)}{\partial \rho_c} \left[\rho_c, Z^d_{\ a_1} \right] \overline{p}_d \eta_4^{a_1} - \overline{\eta}_2^a \Delta_{ab} Z^b_{\ a_1} \eta_4^{a_1} - \right.$$
$$\overline{\eta}_{1a_1} \eta_3^{a_1} - \frac{1}{\lambda} \overline{\eta}_{3a_1} \left(\dot{\eta}_4^{a_1} + \eta_3^{a_1} + \lambda \eta_5^{a_1} \right) - y^{a_1} B_{1a_1} - \eta_4^{a_1} B_{2a_1} -$$
$$\left. \frac{1}{\lambda} u^{a_1} B_{3a_1} + \sum_{j=2}^{k} \eta^{a_j} B_{a_j} \right), \tag{100}$$

with $S_{\Psi'}$ as in (83). Integrating now in the path integral of the reducible first-class family, $Z_{\Psi''}$ (corresponding to (100)) over all the variables excepting q^i's and p_i's, we find

$$Z_{\Psi''} = Z. \tag{101}$$

Formula (101) is the basic result of this subsection and, actually, of this paper. It expresses the fact that in the reducible case the path integral of the first-class family is the same with the one of the original second-class theory. We are able now to explain in what sense the original second-class system maintains the trace of reducibility of a certain first-class system. At the classical level, we obtain from (68) putting all the extravariables equal to zero the Hamiltonian action of the original system. Thus, at the classical level the second-class system comes from the reducible first-class family (68). At the path integral level, formula (101) shows that the path integral of the original system is coming from the BRST quantization of the reducible first-class family. So, the original system can be regarded at both levels as coming from the reducible first-class family. This is the meaning of the original system preserving the relic of the reducibility of the first-class family.

5 The Lagrangian approach of the first-class family

In this section we shall derive under some simplifying assumptions the Lagrangian form of the path integrals deduced in the previous section and clarify the physical origin of the first-class family in both reducible and irreducible cases. Related to the physical origin, we shall emphasise the Wess-Zumino action in these cases. A different way of deriving the Lagrangian form of the path integral is presented in [22]. There, the linear part in the Lagrange multipliers associated to the secondary, tertiary, ... constraints are eliminated through a canonical transformation.Further, the integration in the path integral over the momenta and the Lagrange multipliers of the primary constraints leads to a Lagrangian, while the integration over the remaining multipliers (by stationary-phase method) gives the Lagrangian measure in the path integral. In the sequel, we expose an alternative method under special hypotheses. Again, we shall consider separately the two cases.

5.1 The Lagrangian approach in the irreducible case

If in (83) we make the transformations $f^a(\rho) \longrightarrow f^a(\rho) + \frac{1}{2}B_1^a$, which do not affect its BRST invariances (as $sB_1^a = 0$), and integrate in the corresponding path integral over all the variables excepting (q^i, p_i), we find

$$\begin{aligned} Z_{\Psi'} &= \int \mathcal{D}q^i\, \mathcal{D}p_i \prod_e \delta(G_e) \det\left(\Delta_{ab} \frac{\partial f^b(C)}{\partial C_c} \Delta_{cd}\right) \cdot \\ &\quad \exp\left(i \int dt \left(\dot{q}^i p_i - H\right)\right). \end{aligned} \tag{102}$$

If the C_a's depend only on the q^i's, the integration over the p_i's in (102) leads us to the following form of the Lagrangian path integral

$$Z_{\Psi'} = \int \mathcal{D}q^i\ \det\left(\Delta_{ab} \frac{\partial f^b(C)}{\partial C_c} \Delta_{cd}\right) \exp\left(i \int dt\, L_0\left(q^i, \dot{q}^i\right)\right), \tag{103}$$

where $L_0(q^i, \dot{q}^i)$ is the Lagrangian of the original second-class system. From (103) it results that if the original canonical Hamiltonian is more than quadratic in the functions C_a (see (9)) in the Lagrangian path integral it will appear the non-trivial local measure

$$\mu = \det\left(\Delta_{ab} \frac{\partial f^b(C)}{\partial C_c} \Delta_{cd}\right). \tag{104}$$

In the case of H at most quadratic in the C_a's, the measure (104) reduces to $\mu = \det(\Delta_{ad})$, so the Lagrangian path integral takes the simple form

$$Z_{\Psi'} = \int \mathcal{D}q^i\ (\det C_{\alpha\beta})^{1/2} \exp\left(i \int dt\, L_0\left(q^i, \dot{q}^i\right)\right). \tag{105}$$

The last formula remains also valid when the C_a's depend on p_i's because in this case $\left.\frac{\partial^k H}{\partial C_{a_1}\ldots\partial C_{a_k}}\right|_{C_a=0} = 0$ for any $k > 2$.

In the sequel we shall make clear the physical provenance of the irreducible first-class family. In this end, we consider for simplicity $C_a^0 \equiv 0$. Under this circumstance, action (37) (which describes the irreducible first-class family) reduces to

$$S_0\left[q^i, p_i, z^a, \overline{p}_a, v^a, u^a\right] = \int dt\left(\dot{q}^i p_i + \dot{z}^a \overline{p}_a - \overline{H}^* - v^a G_a - \lambda u^a \overline{p}_a\right), \quad (106)$$

where

$$\overline{H}^* = \tilde{H} - \frac{\lambda^2}{2}\Delta^{ab}\overline{p}_a\overline{p}_b + \lambda\overline{p}_a f^a(C). \quad (107)$$

Action (106) takes into account the primary, as well as the secondary constraints. Passing from this extended action to the total one [17] (taking $u^a = 0$) and making in the resulting action the transformation (5), we infer [4]

$$S_0\left[Q^a, P_a, z^\Delta, \overline{p}_\Delta, z^a, \overline{p}_a, v^a\right] = \int dt\left(\dot{Q}^a P_a + \dot{z}^\Delta\overline{p}_\Delta + \dot{z}^a\overline{p}_a - h^* - v^a P_a\right), \quad (108)$$

with

$$h^* = h\left(z^\Delta, \overline{p}_\Delta\right) - \frac{\lambda^2}{2}\Delta^{ab}\overline{p}_a\overline{p}_b + \lambda\overline{p}_a f^a(Q), \quad (109)$$

and $h\left(z^\Delta, \overline{p}_\Delta\right) = \tilde{H}\left(Q^a, z^\Delta, \overline{p}_\Delta\right) = H\left(0, z^\Delta, \overline{p}_\Delta\right)$ [4]. Eliminating from (108) all the momenta and Lagrangian multipliers on their equations of motion [21], we get the Lagrangian action of the irreducible first-class family under the form

$$S_0^L\left[Q^a, z^\Delta, z^a\right] = \int dt\left(l\left(z^\Delta, \dot{z}^\Delta\right) - \frac{1}{2\lambda^2}\Delta_{ab}\left(\lambda f^a(Q) - \dot{z}^a\right)\left(\lambda f^b(Q) - \dot{z}^b\right)\right). \quad (110)$$

In (110) $l\left(z^\Delta, \dot{z}^\Delta\right)$ is the Lagrangian corresponding to $h\left(z^\Delta, \overline{p}_\Delta\right)$. It is clear that for $z^a = 0$ action (110) reduces to the original Lagrangian action. The gauge invariances of (110) are as follows

$$\delta_\epsilon Q^a = R^a_b(Q)\,\dot{\epsilon}^b, \quad (111)$$

$$\delta_\epsilon z^a = \lambda \epsilon^a, \tag{112}$$

$$\delta_\epsilon z^\Delta = 0, \tag{113}$$

where $R^a_{\ b}(Q)$ is the inverse of the matrix $\frac{\partial f^a(Q)}{\partial Q^b}$ (from Theorem 3 it is obvious that the inverse exists). The gauge transformations (111-113) result from (38-43) via $\delta_\epsilon u^a = 0$ [21].

In order to reveal the origin of the first-class family we consider the Lagrangian action

$$\overline{S}_0 [z^a] = \int dt \left(-\frac{1}{2\lambda^2} \Delta_{ab} \dot{z}^a \dot{z}^b \right). \tag{114}$$

This action is invariant under the rigid (Noether) transformations

$$\delta_\epsilon z^a = \lambda \epsilon^a, \tag{115}$$

with all ϵ^a constant. Gauging now symmetries (115) (i.e. ϵ^a are arbitrary functions of time), action (114) is no more gauge invariant. Thus, it is necessary to introduce in (114) some additional variables in order to obtain a gauge-invariant action. Under this observation, it results clearly that action (110) comes from the gauging of the rigid symmetries (115) through the introduction of the variables $\left(Q^a, z^\Delta\right)$ which transform accordingly (111), (113). The action of the first-class family contains some mixing-component terms of the type "current-current", with the "currents"

$$j_a = \frac{1}{\lambda} \Delta_{ab} \left(\lambda f^b (Q) - \dot{z}^b \right). \tag{116}$$

These "currents" are conservative and gauge-invariant and come from the rigid invariances (115) of the action (110) via Noether's theorem.

The Wess-Zumino action in the irreducible case is defined by

$$\begin{aligned} S_0^{WZ} \left[Q^a, z^\Delta, z^a\right] &= S_0^L \left[Q^a, z^\Delta, z^a\right] - S_0^L \left[Q^a, z^\Delta, z^a = 0\right] = \\ & \int dt \left(-\frac{1}{2\lambda^2} \Delta_{ab} \dot{z}^a \dot{z}^b + \frac{1}{\lambda} \Delta_{(ab)} f^a (Q) \dot{z}^b \right), \end{aligned} \tag{117}$$

and obvious vanishes when $z^a = 0$. The Wess-Zumino action was introduced for the first time in the context of anomalous field theories [18]. For the chiral Schwinger model this action was discovered by Fadeev and Shatashvili in the framework of the canonical quantization of this model. In the case of our formalism the Wess-Zumino action is necessary in order to make gauge-invariant the original second-class system, such that to apply subsequently the BRST formalism. We remark that a piece in the Wess-Zumino action is precisely action (114). From (110), we observe that the "Wess-Zumino variables" z^a are introduced in order to compensate in a certain sense the unphysical variables from the original theory, Q^a. Indeed, in (110) the z^a's are coupled only to the unphysical variables Q^a through the "current-current" terms. The earlier separation in physical and unphysical variables is a consequence of the fact that we took $C_a^0 \equiv 0$. In the case of field theory it is exactly the presence of C_a^0 non-identically vanishing in γ_a which ensures the Lorentz covariance of the Lagrangian action of the first-class family due to the fact that the function g given in (36) is non-vanishing. The proof of this last conclusion is technically difficult in general, and this is why we shall exemplify it on the models exposed in Sec.6.

5.2 The Lagrangian approach in the reducible case

The Lagrangian path integral of the reducible first-class family is obtained analogously with the irreducible case. Making in (100) the transformation $f^a(\rho) \longrightarrow f^a(\rho) + \frac{1}{2}B_1^a$ and integrating in the correspondent path integral over all the variables excepting (q^i, p_i) we get that $Z_{\Psi''}$ is also given by formula (102). The procedure of passing from (102) to (103-105) is identical with the one from the irreducible case, finally deriving the same results.

Next, we shall analyze the origin of the reducible first-class family. We shall consider the case $C_a^0 = 0$, too. Now, action (68) takes the form

$$S_0'\left[q^i, p_i, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}, v^a, v^{a_1}, u^a, u^{a_1}\right] = \int dt \left(\dot{q}^i p_i + \right.$$
$$\left. \dot{z}^a \overline{p}_a + \dot{y}^{a_1}\pi_{a_1} - \widehat{H^*} - v^a G_a - v^{a_1}\gamma_{a_1} - \lambda u^a \overline{p}_a - u^{a_1}\overline{\gamma}_{a_1}\right), \quad (118)$$

where $\widehat{H^*} = \overline{H}^* - y^{a_1}\overline{\gamma}_{a_1}$. The passing from the extended action (118) to its correspondent total action is gained putting $u^a = u^{a_1} = 0$. Making in this total action the transformation (5), we get

$$S_0'\left[Q^a, P_a, z^\Delta, \overline{p}_\Delta, z^a, \overline{p}_a, y^{a_1}, \pi_{a_1}, v^a, v^{a_1}\right] = \int dt \left(\dot{Q}^a P_a + \right.$$
$$\left. \dot{z}^\Delta \overline{p}_\Delta + \dot{z}^a \overline{p}_a + \dot{y}^{a_1}\pi_{a_1} - h^* + y^{a_1}\overline{\gamma}_{a_1} - v^a P_a - v^{a_1}\gamma_{a_1}\right). \quad (119)$$

Eliminating from action (119) the momenta and the Lagrangian multipliers on their equations of motion, we deduce the Lagrangian action of the reducible first-class family as

$$S_0'^L\left[Q^a, z^\Delta, z^a, y^{a_1}\right] = \int dt \left(l\left(z^\Delta, \dot{z}^\Delta\right) - \right.$$
$$\left.\frac{1}{2\lambda^2}\Delta_{ab}\left(\lambda f^a(Q) + Z^a_{\ a_1} y^{a_1} - \dot{z}^a\right)\left(\lambda f^b(Q) + Z^b_{\ b_1} y^{b_1} - \dot{z}^b\right)\right). \quad (120)$$

In order to obtain action (120) we presumed that $Z^a_{\ a_1}$'s do not depend on the momenta. Action (120) is invariant under the gauge transformations

$$\delta_\epsilon Q^a = R^a_{\ b}(Q)\left(\dot{\epsilon}^b + Z^b_{\ b_1}\bar{\epsilon}^{b_1} - \dot{Z}^b_{\ b_1}\epsilon^{b_1}\right), \quad (121)$$

$$\delta_\epsilon z^a = \lambda\epsilon^a - Z^a_{\ a_1}\epsilon^{a_1}, \quad (122)$$

$$\delta_\epsilon y^{a_1} = -\dot{\epsilon}^{a_1} - \lambda\bar{\epsilon}^{a_1}, \quad (123)$$

$$\delta_\epsilon z^\Delta = 0. \quad (124)$$

The gauge transformations (121-124) result from (69-78) via $\delta_\epsilon u^a = \delta_\epsilon u^{a_1} = 0$.Due to (45) it follows that $Z^a_{\ a_1}$ do not depend on the Q^a's, such that, from (124) we have $\delta_\epsilon Z^a_{\ a_1} = 0$. As expected, (121-124) represent a set of reducible gauge transformations. If the functions C^0_a are k-th order reducible, then (121-124) possess the same reducibility order. Indeed, denoting $X^\alpha = \left(Q^a, z^\Delta, z^a, y^{a_1}\right)$ we have the reducibility relations

$$Z^a_{\ \alpha}\delta_\epsilon X^\alpha = 0, \quad (125)$$

with

$$Z^a_{\ \alpha} = \left(\lambda\delta^a_{\ b}, 0, \dot{R}^a_{\ b}, R^a_{\ b}Z^b_{\ a_1}\right). \quad (126)$$

Obviously, the reducibility relations (125) are written in De Witt notations. Because of (51-52) we further find the reducibility relations

$$Z^{A_1}_{\ a}Z^a_{\ \alpha} = 0, \quad (127)$$

$$Z^{A_2}_{\ A_1}Z^{A_1}_{\ a} = 0, \quad (128)$$

$$\vdots$$

$$Z^{A_{k-2}}_{\ A_{k-1}}Z^{A_{k-1}}_{\ A_k} = 0, \quad (129)$$

where $Z^{A_1}_{\ a} = \left(0, 0, 0, Z^{a_1}_{\ a_2}\right)$, and $Z^{A_{k-j}}_{\ A_{k-j+1}} = \left(0, 0, 0, Z^{a_{k-j}}_{\ a_{k-j+1}}\right)$.

In the reducible case, too, action (120) results from the gauging of some rigid symmetries. In this case, there appear two important cases.

i) Firstly, we consider

$$Z^{a}_{\ a_1}\Theta^{a_1} \neq 0, \tag{130}$$

for all Θ^{a_1}'s constant, (i.e. the $Z^{a}_{\ a_1}$'s do not contain derivative terms to act upon the Θ^{a_1}'s). Next, we shall show that action (120) also comes from the gauging of some rigid symmetries of action (114). Action (114) is invariant under the rigid transformations (with two sets of constant parameters)

$$\delta_\epsilon z^a = \lambda \epsilon^a - Z^{a}_{\ a_1}\epsilon^{a_1}. \tag{131}$$

The gauging of the last symmetries implies the necessity of introducing some new variables in order to obtain from (114) a gauge-invariant action. As $\delta_\epsilon \overline{S}_0 = \int dt \left(-\frac{1}{\lambda^2}\Delta_{(ab)}\dot{z}^b \left(\lambda\dot{\epsilon}^a - Z^{a}_{\ a_1}\dot{\epsilon}^{a_1} - \dot{Z}^{a}_{\ a_1}\epsilon^{a_1}\right)\right)$ it follows that it is necessary to introduce the variables $\left(Q^a, z^\Delta, y^{a_1}\right)$ having the gauge transformations

$$\delta_\epsilon Q^a = R^{a}_{\ b}(Q)\left(\dot{\epsilon}^b - \dot{Z}^{b}_{\ b_1}\epsilon^{b_1}\right), \tag{132}$$

$$\delta_\epsilon y^{a_1} = -\dot{\epsilon}^{a_1}, \tag{133}$$

$$\delta_\epsilon z^\Delta = 0, \tag{134}$$

such that the gauge-invariant action deriving from $\overline{S}_0$ to have precisely the form (120). We notice that the introduction of the terms $\lambda f^a(Q) + Z^{a}_{\ a_1}y^{a_1}$ in $\overline{S}_0$ (in order to get (120)) allows the additional gauge invariances of this term of the form

$$\delta_\epsilon Q^a = R^{a}_{\ b}(Q)\, Z^{b}_{\ b_1}\overline{\epsilon}^{b_1}, \tag{135}$$

$$\delta_\epsilon y^{a_1} = -\lambda\overline{\epsilon}^{a_1}, \tag{136}$$

which are due to the manifest reducibility of the first-class family. The gauge invariances (132-136) are nothing but (121-124). In this way we evidenced that action (120) comes from the gauging of the rigid symmetries (131) of action (114). At the same time, action (120) is invariant under the rigid transformations (131). Then, there result from Noether's theorem the conserved "currents"

$$j_a = \frac{1}{\lambda}\Delta_{ab}\left(\lambda f^b(Q) + Z^{b}_{\ b_1}y^{b_1} - \dot{z}^b\right), \tag{137}$$

corresponding to the rigid parameters ϵ^a, and

$$j_{a_1} = -\frac{1}{\lambda^2} Z^a_{\ a_1} \Delta_{ab} \left(\lambda f^b\left(Q\right) + Z^b_{\ b_1} y^{b_1} - \dot{z}^b\right) \equiv -\lambda Z^a_{\ a_1} j_a, \tag{138}$$

associated to the rigid parameters ϵ^{a_1}. These "currents" present an interesting feature, namely they are k-th order reducible. Using (137-138) we have the reducibility relations

$$Z^a_{\ a_1} j_a + \lambda j_{a_1} \equiv Z^\Lambda_{\ a_1} j_\Lambda = 0, \tag{139}$$

where $Z^\Lambda_{\ a_1} = \left(Z^a_{\ a_1}, \lambda \delta^{b_1}_{a_1}\right)$ and $j_\Lambda = (j_a, j_{b_1})$. From (51-52) we further find the reducibility relations

$$Z^{a_1}_{\ \Lambda_1} Z^\Lambda_{\ a_1} = 0, \tag{140}$$

$$Z^{\Lambda_1}_{\ \Lambda_2} Z^{a_1}_{\ \Lambda_1} = 0, \tag{141}$$

$$\vdots$$

$$Z^{\Lambda_{k-1}}_{\ \Lambda_k} Z^{\Lambda_{k-2}}_{\ \Lambda_{k-1}} = 0, \tag{142}$$

with $Z^{a_1}_{\ \Lambda_1} = \left(Z^{a_1}_{\ a_2}, 0\right)$ and $Z^{\Lambda_{k-j-1}}_{\ \Lambda_{k-j}} = \left(Z^{a_{k-j-1}}_{\ a_{k-j}}, 0\right)$. The reducible "currents" (137-138) are gauge-invariant under the gauge transformations (121-124). Thus, the action of the reducible first-class family contains some mixing-component terms of the type "current-current" $-\frac{1}{2}\Delta^{ab} j_a j_b$, with j_a given by (137).

ii) Secondly, we consider

$$Z^a_{\ a_1} \Theta^{a_1} = 0, \tag{143}$$

only for all Θ^{a_1}'s constant, (i.e. the $Z^a_{\ a_1}$'s contain derivative terms to act upon the Θ^{a_1}'s). Now we prove that action (120) also results from the gauging of some rigid symmetries, but not for action (114). We start with the action

$$\tilde{S}_0\left[z^a, y^{a_1}\right] = -\int dt \left(\frac{1}{2\lambda^2} \Delta_{ab} \left(Z^a_{\ a_1} y^{a_1} - \dot{z}^a\right)\left(Z^b_{\ b_1} y^{b_1} - \dot{z}^b\right)\right), \tag{144}$$

which is invariant under the rigid transformations

$$\delta_\epsilon z^a = \lambda \epsilon^a, \tag{145}$$

$$\delta_\epsilon y^{a_1} = -\lambda \bar{\epsilon}^{a_1}. \tag{146}$$

Gauging these symmetries, we infer

$$\delta_\epsilon \tilde{S}_0 = \int dt \frac{1}{\lambda} \Delta_{(ab)} \left(Z^a_{\ a_1} \bar{\epsilon}^{a_1} + \dot{\epsilon}^a \right) \left(Z^b_{\ b_1} y^{b_1} - \dot{z}^b \right),$$

so that it is necessary to introduce the new variables $\left(Q^a, z^\Delta\right)$ with the gauge transformations

$$\delta_\epsilon Q^a = R^a_{\ b}(Q) \left(\dot{\epsilon}^b + Z^b_{\ b_1} \bar{\epsilon}^{b_1} \right), \tag{147}$$

and (124) for the z^Δ's, further resulting the gauge-invariant action derived from $\tilde{S}_0$ precisely of the form (120). We observe that action (144) possesses the additional gauge invariances

$$\delta_\epsilon Q^a = -\frac{1}{\lambda} R^a_{\ b}(Q) \dot{Z}^b_{\ b_1} \epsilon^{b_1}, \tag{148}$$

$$\delta_\epsilon z^a = -Z^a_{\ a_1} \epsilon^{a_1}, \tag{149}$$

$$\delta_\epsilon y^{a_1} = -\dot{\epsilon}^{a_1}, \tag{150}$$

such that $\delta_\epsilon \left(\lambda f^a(Q) + Z^a_{\ a_1} y^{a_1} - \dot{z}^a \right) = 0$ under the prior transformations. Thus, the last transformations are independent of the non-invariant form of (144) under the gauge version of (145-146). In the case of $\dot{Z}^b_{\ b_1} = 0$, the invariances (148-150) reduce to (149-150), the last ones representing some gauge symmetries characteristic to the terms $\left(Z^a_{\ a_1} y^{a_1} - \dot{z}^a \right)$ containing only extravariables. To conclude with, action (120) comes from the gauging of the rigid symmetries (145-146) of action (144) and, in the same time, possesses some supplementary gauge invariances because (143) do not hold if Θ^{a_1}'s are functions of time. Obviously, action (120) is also invariant under (145-146) so that we obtain via Noether's theorem the gauge-invariant "currents" (137) (the rigid symmetries (146) lead to some trivial "currents") which are no longer reducible.

In the end of this subsection we emphasise the Wess-Zumino action corresponding to the reducible case

$$\begin{aligned} S_0'^{WZ}\left[Q^a, z^\Delta, z^a, y^{a_1}\right] &= S_0'\left[Q^a, z^\Delta, z^a, y^{a_1}\right] - S_0'\left[Q^a, z^\Delta, z^a = 0, y^{a_1} = 0\right] \equiv \\ & -\int dt \left(\frac{1}{2\lambda^2} \Delta_{ab} \left(Z^a_{\ a_1} y^{a_1} - \dot{z}^a \right) \left(Z^b_{\ b_1} y^{b_1} - \dot{z}^b \right) + \right. \\ & \left. \frac{1}{\lambda} \Delta_{(ab)} f^a(Q) \left(Z^b_{\ b_1} y^{b_1} - \dot{z}^b \right) \right). \end{aligned} \tag{151}$$

We remark that in the reducible case the Wess-Zumino action contains action (144).

6 Examples

In this section we illustrate the general theory presented in this paper on two representative models.

6.1 The Massive Yang-Mills theory

The Lagrangian action describing the massive Yang-Mills theory reads

$$S_0^L[A] = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} - \frac{1}{2} M^2 A^a_\mu A_a^\mu \right), \tag{152}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^a_{\ bc} A^b_\mu A^c_\nu$. The canonical analysis of this model furnishes the canonical Hamiltonian

$$H = \int d^3x \left(\frac{1}{2} \pi_{ia} \pi_i^a + \frac{1}{4} F^a_{ij} F_a^{ij} - A^a_0 D_i \pi^i_a + \frac{1}{2} M^2 A^a_\mu A^\mu_a \right), \tag{153}$$

together with the primary, respectively secondary constraints

$$G_a \equiv \pi^0_a = 0, \tag{154}$$

$$C_a \equiv -D_i \pi^i_a + M^2 A^0_a = 0, \tag{155}$$

where $D_i \pi^i_a = \partial_i \pi^i_a - f^c_{\ ab} \pi^i_c A^b_i$ and π^μ_a denote the canonical momenta of A^a_μ. It is easy to see that the above constraints are second-class as the matrix $[C_a, G_b] = M^2 \delta_{ab} \equiv \Delta_{ab}$ has a non-vanishing determinant. We observe that $[G_a, G_b] = 0$ strongly. We choose

$$C^0_a = -\partial_i \pi^i_a, \tag{156}$$

and

$$C^1_a = f^c_{\ ab} \pi^i_c A^b_i + M^2 A^0_a, \tag{157}$$

such that $[C^0_a, C^0_b] = [C^0_a, G_b] = 0$, and $[C^1_a, G_b] = \Delta_{ab}$. It follows that the massive Yang-Mills theory verifies the hypotheses of our methods. The

Hamiltonian H' for our model is given by

$$H' = \int d^3x \left(\frac{1}{2}\pi_{ia}\pi_i^a + \frac{1}{4}F_{ij}^a F_a^{ij} + \frac{1}{2}M^2 A_i^a A_a^i - \right.$$
$$\left. \frac{1}{2M^2} f^c{}_{ba} f^a{}_{de} \pi_c^i A_i^b \pi_j^d A^{je}\right), \tag{158}$$

such that $[H', G_a] = 0$. The functions C_a^0 being irreducible, we introduce the additional canonical pairs (φ^a, Π_a) in number equal to the number of pairs (G_a, C_a). The above canonical pairs play the role of the pairs $(z^a, \overline{p}_a)$ from the general theory. The functions $f^a(C)$ for our model have the form

$$f^a(C) = \Delta^{ab} C_b. \tag{159}$$

The Hamiltonian of the first class family is

$$H^* = H' + \int d^3x \left(-\frac{\lambda^2}{2M^2}\Pi_a \Pi^a - \right.$$
$$\left. \frac{1}{M^2}\left(\lambda \Pi_a - \partial_i \pi_a^i\right)\left(f^a{}_{bc}\pi_j^b A^{jc} - M^2 A_0^a\right) + g\right), \tag{160}$$

where

$$g(A, \pi, \varphi, \Pi) =$$
$$\int d^3x \left(\frac{1}{M^2} f^a{}_{bc}\Pi_a \pi_i^b \partial^i \varphi^c + \frac{1}{\lambda}\left(f_{abc}\partial_i A_j^a \left(A^{ib}\partial^j\varphi^c - A^{jb}\partial^i\varphi^c\right) - \right.\right.$$
$$\left. f^a{}_{mn} f_{abc} A_i^m A_j^n A^{ib}\partial^j \varphi^c + \frac{1}{M^2} f^a{}_{bc} f_{amn}\pi^{ic}\pi_j^m A_i^b \partial^j \varphi^n - M^2 A_i^a \partial^i \varphi_a\right) +$$
$$\frac{1}{\lambda^2}\left(\frac{1}{2}M^2 \partial^i\varphi^a \partial_i \varphi_a - \frac{1}{2M^2} f_{abc} f^c{}_{mn}\pi_i^a \pi_j^m \partial^i \varphi^b \partial^j \varphi^n - f_{abc}\partial_i A_j^a \partial^i \varphi^b \partial^j \varphi^c + \right.$$
$$\left. \frac{1}{2} f^a{}_{bc} f_{amn} A_i^b \partial_j \varphi^c \left(A^{im}\partial^j \varphi^n - A^{jm}\partial^i \varphi^n\right) + \frac{1}{2} f^a{}_{mn} f_{abc} A_i^m A_j^n \partial^i \varphi^b \partial^j \varphi^c\right) -$$
$$\left. \frac{1}{\lambda^3} f^a{}_{bc} f_{amn} A_i^m \partial^i \varphi^b \partial_j \varphi^c \partial^j \varphi^n + \frac{1}{4\lambda^4} f^a{}_{bc} f_{amn}\partial^i \varphi^b \partial^j \varphi^c \partial_i \varphi^m \partial_j \varphi^n\right). \tag{161}$$

The first-class constraints of the first-class family are $G_a \equiv \pi_a^0 = 0$ and $\gamma_a \equiv \lambda \Pi_a - \partial_i \pi_a^i = 0$, such that the gauge algebra of the first-class family reads

$$[G_a, G_b] = [G_a, \gamma_b] = [\gamma_a, \gamma_b] = 0, \tag{162}$$

$$[G_a, H^*] = -\gamma_a, \tag{163}$$

$$[\gamma_a, H^*] = 0. \tag{164}$$

The gauge invariances of the extended action are in this case: $\delta_\epsilon A_0^a = \epsilon_1^a$, $\delta_\epsilon A_i^a = \partial_i \epsilon_2^a$, $\delta_\epsilon \pi_a^\mu = 0$, $\delta_\epsilon \varphi^a = \lambda \epsilon_2^a$, $\delta_\epsilon \Pi_a = 0$, $\delta_\epsilon v^a = \dot{\epsilon}_1^a$, $\delta_\epsilon u^a = \dot{\epsilon}_2^a - \epsilon_1^a$. The gauge-fixing fermion (81) for our model reads

$$\Psi' = -\int d^4x \left(\overline{\eta}_1^a \left(M^2 A_a^0 - \frac{1}{\lambda} f^b_{\ ac} \pi_b^i \tilde{A}_i^c \right) + \lambda M^2 \overline{\eta}_2^a \varphi^a \right), \tag{165}$$

with $\tilde{A}_\mu^a = A_\mu^a - \frac{1}{\lambda} \partial_\mu \varphi^a$. The path integral (95) for the massive Yang-Mills theory in the gauge (165) after integration over (A_0^a, π_a^0) is given by

$$Z_{\Psi'} = \int \mathcal{D}A_i^a \, \mathcal{D}\pi_a^i \, \exp\left(i\overline{S} \right), \tag{166}$$

where

$$\overline{S} = \int d^4x \left(\dot{A}_i^a \pi_a^i - \frac{1}{2} \pi_{ia} \pi_i^a - \frac{1}{4} F_{ij}^a F_a^{ij} - \right.$$
$$\left. \frac{1}{2} M^2 A_i^a A_a^i + \frac{1}{2M^2} \left(D_i \pi_a^i \right)^2 \right). \tag{167}$$

The results obtained in (166-167) are identical to the ones derived in [2], [4] through other methods.

The gauge invariances of the total action for our model are inferred from the ones of the extended action taking $\delta_\epsilon u^a = 0$, which further implies $\dot{\epsilon}_2^a = \epsilon_1^a$. Then, the gauge invariances of the total action are given by: $\delta_\epsilon A_\mu^a = \partial_\mu \epsilon_2^a$, $\delta_\epsilon \pi_a^\mu = 0$, $\delta_\epsilon \varphi^a = \lambda \epsilon_2^a$, $\delta_\epsilon \Pi_a = 0$, $\delta_\epsilon v^a = \ddot{\epsilon}_2^a$. These gauge transformations are written now under a manifestly covariant form. This is because of the non-trivial term $-\partial_i \pi_a^i \equiv C_a^0$ in the secondary constraints γ_a. It is precisely this term which induces $\delta_\epsilon A_i^a = \partial_i \epsilon_2^a \neq 0$ and so further implies the above covariance of the gauge transformations and also $g\left(A, \pi, \varphi, \Pi\right) \neq 0$. We shall see below that $g\left(A, \pi, \varphi, \Pi\right)$ of the form (161) ensures the manifestly covariance of the Lagrangian action for the first-class family. We can reach this action eliminating from the total action the momenta and Lagrange multipliers v^a on their equations of motion, namely

$$\pi_a^i = -\tilde{F}_a^{0i}, \tag{168}$$

$$\Pi_a = \frac{1}{\lambda}\left(M^2\tilde{A}^0_a + f_{abc}\tilde{F}^b_{0i}\tilde{A}^{ic}\right), \tag{169}$$

$$\pi^0_a = 0, \tag{170}$$

with $\tilde{F}^a_{\mu\nu} = \partial_\mu\tilde{A}^a_\nu - \partial_\nu\tilde{A}^a_\mu - f^a_{\ bc}\tilde{A}^b_\mu\tilde{A}^c_\nu$. Then, the Lagrangian action of the first-class family reads

$$S^L_0\left[A,\varphi\right] = \int d^4x\left(-\frac{1}{4}\tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu}_a - \frac{1}{2}M^2\tilde{A}^a_\mu\tilde{A}^\mu_a\right). \tag{171}$$

This action has the gauge invariances $\delta_\epsilon A^a_\mu = \partial_\mu\epsilon^a_2$, $\delta_\epsilon\varphi^a = \lambda\epsilon^a_2$ and comes from the gauging of the action

$$S_0\left[\varphi\right] = -\int d^4x\,\frac{M^2}{2\lambda^2}\partial_\mu\varphi^a\partial^\mu\varphi_a, \tag{172}$$

which allows the rigid symmetries $\delta_\epsilon\varphi^a = \lambda\epsilon^a_2$, with ϵ^a_2 all constant. The conserved gauge-invariant currents corresponding to the last rigid invariance, but for action (171) are

$$j^\mu_a = \frac{1}{\lambda}\left(f^b_{\ ac}\tilde{A}^c_\nu\tilde{F}^{\mu\nu}_b - M^2\tilde{A}^\mu_a\right). \tag{173}$$

Action (171) together with the currents (173) coincide in the abelian limit ($f^a_{\ bc} = 0$) with our results derived in [24]and also with the one resulting from Stueckelberg's formalism [25]. The Wess-Zumino action in this case takes the form

$$S^{WZ}_0\left[A,\varphi\right] = -\frac{1}{2\lambda}\int d^4x\left(f^a_{\ bc}\left(A^b_\mu\partial_\nu\varphi^c + A^c_\nu\partial_\mu\varphi^b - \frac{1}{\lambda}\partial_\mu\varphi^b\partial_\nu\varphi^c\right)\cdot\right.$$
$$\left(F^{\mu\nu}_a + \frac{1}{2\lambda}g^{\mu\alpha}g^{\nu\beta}f_{ade}\left(A^d_\alpha\partial_\beta\varphi^e + A^e_\beta\partial_\alpha\varphi^d - \frac{1}{\lambda}\partial_\alpha\varphi^d\partial_\beta\varphi^e\right)\right) +$$
$$\left.M^2\left(\frac{1}{\lambda}\partial_\mu\varphi^a\partial^\mu\varphi_a - 2A^a_\mu\partial^\mu\varphi_a\right)\right). \tag{174}$$

This ends the analysis of the model under consideration.

6.2 Massive abelian three-form gauge fields

This model is an example of reducible theory. We are starting with the Lagrangian action [26]

$$S^L_0\left[A\right] = \int d^4x\left(-\frac{1}{2\cdot 4!}F_{\alpha\beta\gamma\rho}F^{\alpha\beta\gamma\rho} - \frac{M^2}{2\cdot 3!}A_{\alpha\beta\gamma}A^{\alpha\beta\gamma}\right), \tag{175}$$

where $F^{\alpha\beta\gamma\rho} = \partial^{[\alpha}A^{\beta\gamma\rho]} \equiv \partial^\alpha A^{\beta\gamma\rho} - \partial^\beta A^{\alpha\gamma\rho} + \partial^\gamma A^{\rho\alpha\beta} - \partial^\rho A^{\gamma\alpha\beta}$, with $A^{\alpha\beta\gamma}$'s antisymmetric in all indices. The canonical analysis of this model provides the canonical Hamiltonian

$$H = \int d^3x \left(3\pi_{ijk}\pi_{ijk} - 3A_{0jk}\partial_i\pi^{ijk} + \frac{M^2}{2\cdot 3!}A_{\alpha\beta\gamma}A^{\alpha\beta\gamma} \right), \tag{176}$$

as well as the primary, respectively secondary constraints

$$G^{ij} \equiv \pi^{0ij} = 0, \tag{177}$$

$$C^{ij} \equiv -3\partial_k\pi^{kij} + \frac{M^2}{2}A^{0ij} = 0, \tag{178}$$

where π^{kij}'s are the canonical momenta of the A_{kij}'s. It is simple to check that the above constraints are second-class, the matrix $[G_{ij}, C_{kl}] \equiv \Delta_{ij;kl} = -\frac{M^2}{4}(g_{ik}g_{jl} - g_{il}g_{jk})$ having a non-vanishing determinant. In the last relation g_{ij}'s denote the spatial part of the Minkowskian metric. It is clear that $[G_{ij}, G_{kl}] = 0$ strongly. We take the analogous of C_a^0's, respectively C_a^1's of the form

$$C_{ij}^0 \equiv -3\partial^k\pi_{kij}, \tag{179}$$

$$C_{ij}^1 = \frac{M^2}{2}A_{0ij}, \tag{180}$$

so the model satisfies the conditions from our methods. In this case, the functions C_{ij}^0 are second-order reducible, the reducibility relations being $\partial^i C_{ij}^0 = 0$ and $\partial^j\partial^i C_{ij}^0 = 0$. The Hamiltonian H' is in the present case

$$H' = \int d^3x \left(3\pi_{ijk}\pi_{ijk} + \frac{M^2}{2\cdot 3!}A_{ijk}A^{ijk} \right). \tag{181}$$

Because we are in the reducible case, we extend the original phase-space as follows. For every pair (G_{ij}, C_{ij}) we introduce a bosonic canonical pair (A^{ij}, Π_{ij}), with the new fields antisymmetric in their indices, such that the new secondary constraints to be

$$\gamma_{ij} \equiv \lambda\Pi_{ij} - 3\partial^k\pi_{kij} = 0. \tag{182}$$

It is simply to check that

$$\partial^i\gamma_{ij} = \lambda\partial^i\Pi_{ij} = 0. \tag{183}$$

For every relation (183) we introduce the new canonical pair (A^{0i}, Π_{0i}) and the constraint

$$\gamma_i \equiv \Pi_{0i} = 0, \tag{184}$$

such that its consistency to imply the constraint

$$\overline{\gamma}_i = -\partial^j \Pi_{ji} = 0. \tag{185}$$

In this way we associated to the original system a one-parameter family of first-class systems with the first class constraints (177), (182), (184-185). Now, the first-class constraints become reducible

$$\lambda \overline{\gamma}_j + \partial^i \gamma_{ij} = 0. \tag{186}$$

The first-class Hamiltonian of the reducible first-class family reads

$$H^* = H' + \int d^3x \left(-\lambda^2 \Pi_{ij} \Pi^{ij} + A^{0ij} \gamma_{ij} - 2A^{0j} \partial^i \Pi_{ij} + \right.$$

$$\left. + \frac{M^2}{3!\lambda^2} \left(\frac{1}{2} \partial^{[i} A^{jk]} \partial_{[i} A_{jk]} - A^{ijk} \partial_{[i} A_{jk]} \right) \right), \tag{187}$$

where

$$g = \int d^3x + \frac{M^2}{3!\lambda^2} \left(\frac{1}{2} \partial^{[i} A^{jk]} \partial_{[i} A_{jk]} - A^{ijk} \partial_{[i} A_{jk]} \right). \tag{188}$$

The gauge invariances of the extended action for this model are $\delta_\epsilon A^{0ij} = \epsilon_1^{ij}$, $\delta_\epsilon A^{ijk} = \partial^{[i} \epsilon_2^{jk]}$, $\delta_\epsilon A^{0i} = \epsilon_1^{0i}$, $\delta_\epsilon A^{ij} = \lambda \epsilon_2^{ij} + \frac{1}{2} \partial^{[i} \epsilon_2^{0j]}$, $\delta_\epsilon \pi_{0ij} = \delta_\epsilon \pi_{ijk} = \delta_\epsilon \Pi_{0i} = \delta_\epsilon \Pi_{ij} = 0$, $\delta_\epsilon v^{ij} = \dot{\epsilon}_1^{ij}$, $\delta_\epsilon v^i = \dot{\epsilon}_1^{0i}$, $\delta_\epsilon u^{ij} = \dot{\epsilon}_2^{ij} - \epsilon_1^{ij} - \frac{1}{2} \partial^{[i} \epsilon^{j]}$, $\delta_\epsilon u^j = \dot{\epsilon}_2^{0j} - 2\epsilon_1^{0j} + \lambda \epsilon^j$, with ϵ^j due to the reducibility relations (186) (they play the role of $\epsilon_5^{a_1}$ in the general theory).

The gauge-fixing fermion (99) for our model is

$$\Psi'' = -\int d^4x \left(\overline{\eta}_1^{ij} C_{ij} + \overline{\eta}_2^i A_{0i} + \lambda \frac{M^2}{2} \overline{\eta}_2^{ij} A_{ij} + \overline{\eta}_1^i \eta_{2_{0i}} + \frac{1}{\lambda} u^i \overline{\eta}_i \right), \tag{189}$$

where the bar variables belong to the non-minimal sector, while the ghosts $\eta_{2_{0i}}$ correspond to the gauge parameters ϵ_2^{0i}. The path integral in this case will read

$$Z_{\Psi''} = \int \mathcal{D}A^{ijk} \, \mathcal{D}\pi_{ijk} \, \exp\left(i\overline{S}' \right), \tag{190}$$

where

$$\overline{S}' = \int d^4x \left(\dot{A}^{ijk}\pi_{ijk} - 3\pi_{ijk}\pi_{ijk} - \frac{M^2}{2\cdot 3!}A^{ijk}A_{ijk} + \frac{9}{M^2}\left(\partial^i\pi_{ijk}\right)^2 \right). \quad (191)$$

The gauge invariances of the total action in this case are obtained from the extended ones making $\delta_\epsilon u^{ij} = \delta_\epsilon u^j = 0$. They take the form $\delta_\epsilon A^{\alpha\beta\gamma} = \partial^{[\alpha}\epsilon_2^{\beta\gamma]}$, $\delta_\epsilon A^{\alpha\beta} = \lambda\epsilon_2^{\alpha\beta} + \frac{1}{2}\partial^{[\alpha}\epsilon_2^{0\beta]}$, $\delta_\epsilon\pi_{0ij} = \delta_\epsilon\pi_{ijk} = \delta_\epsilon\Pi_{0i} = \delta_\epsilon\Pi_{ij} = 0$, $\delta_\epsilon v^{ij} = \ddot{\epsilon}_2^{ij} - \frac{1}{2}\partial^{[i}\dot{\epsilon}^{j]}$, $\delta_\epsilon v^i = \frac{1}{2}\left(\ddot{\epsilon}_2^{0i} + \lambda\dot{\epsilon}^j\right)$. In this case, the gauge transformations of the fields $A^{\alpha\beta\gamma}$'s and $A^{\alpha\beta}$'s are manifestly covariant too, due on one hand to the presence in (182) of the non-vanishing functions C_{ij}^0 and on the other to the constraints (185). The Lagrangian action of the reducible first-class family reads

$$S_0'^L\left[A^{\alpha\beta\gamma}, A^{\alpha\beta}\right] = \int d^4x \left(-\frac{1}{2\cdot 4!}F_{\alpha\beta\gamma\rho}F^{\alpha\beta\gamma\rho}\right) - \int d^4x \, \frac{M^2}{2\cdot 3!}\left(A^{\alpha\beta\gamma} - \frac{1}{\lambda}F^{\alpha\beta\gamma}\right)\left(A_{\alpha\beta\gamma} - \frac{1}{\lambda}F_{\alpha\beta\gamma}\right), \quad (192)$$

where $F^{\alpha\beta\gamma} = \partial^{[\alpha}A^{\beta\gamma]} \equiv \partial^\alpha A^{\beta\gamma} + \partial^\gamma A^{\alpha\beta} + \partial^\beta A^{\gamma\alpha}$. Action (192) allows the gauge invariances $\delta_\epsilon A^{\alpha\beta\gamma} = \partial^{[\alpha}\epsilon^{\beta\gamma]}$, $\delta_\epsilon A^{\alpha\beta} = \lambda\epsilon^{\alpha\beta} + \partial^{[\alpha}\epsilon^{\beta]}$. This action comes from the gauging of the rigid symmetries $\delta_\epsilon A^{\alpha\beta} = \lambda\epsilon^{\alpha\beta}$ (here, $\epsilon^{\alpha\beta}$ are all constant) of the action

$$\tilde{S}_0\left[A^{\alpha\beta}\right] = -\int d^4x \, \frac{M^2}{2\cdot 3!}\frac{1}{\lambda^2}F^{\alpha\beta\gamma}F_{\alpha\beta\gamma}. \quad (193)$$

We are under the conditions of Sec. 5.1, case ii) because $Z^\alpha\epsilon^\beta = 0$, for ϵ^β's constant, with $Z^\alpha \equiv \partial^\alpha$. When ϵ^β's depend on x action (193) possesses some gauge symmetries independent of the presence of the fields $A^{\alpha\beta\gamma}$, namely

$$\delta_\epsilon A^{\alpha\beta} = \partial^{[\alpha}\epsilon^{\beta]}. \quad (194)$$

Formula (194) is the analogous of (149-150) from the general theory, in the case $\dot{Z}^a_{a_1} = 0$. The conserved gauge-invariant currents correspondent to the rigid symmetries $\delta_\epsilon A^{\alpha\beta} = \lambda\epsilon^{\alpha\beta}$ for action (192) take the form

$$j^\gamma{}_{\alpha\beta} = \frac{M^2}{3!}\left(A^\gamma{}_{\alpha\beta} - \frac{1}{\lambda}F^\gamma{}_{\alpha\beta}\right). \quad (195)$$

Action (192) describes a field theory with abelian two and three-form gauge fields coupled through a mixing-component term of the type current-current, with the gauge-invariant current (195).

The Wess-Zumino action in this case is precisely

$$S_0^{\prime WZ}\left[A^{\alpha\beta\gamma},A^{\alpha\beta}\right]=-\frac{M^2}{12\lambda}\int d^4x\,F^{\alpha\beta\gamma}\left(\frac{1}{\lambda}F_{\alpha\beta\gamma}-2A_{\alpha\beta\gamma}\right). \tag{196}$$

More on abelian p-form gauge fields can be found in [27]. This completes our analysis.

7 Conclusion

In this paper it was shown in detail the way of quantizing the systems with only second-class constraints in the BRST formalism by converting the original second-class constraints into some first-class ones in a larger phase-space. The main advantage of our method consists in the fact that it is standard. Thus, the existence of the functions we are working with is fully proved and, in addition, their concrete form is output. The way of implementing the secondary first-class constraints exposed in this paper emphasises the main difference between our conversion method and the BFT method [5]-[6]. Indeed, the presence of the term $\gamma_a f^a\left(C\right)$ in the Hamiltonian of the first-class family is decisive in underlining this difference as the functions $f^a\left(C\right)$ are a characteristic of our method and do not appear in the BFT method. In addition, we expose a conversion method for the reducible case, too.

At the same time, it is clarified the provenance of the first-class family in the reducible, as well as irreducible case. As was exhibited, the first-class family results from the gauging of some rigid symmetries of a certain action. In the context of building up this family, the Wess-Zumino action appears natuarally, its concrete form being computed in both cases.

The two examples illustrating the theoretical part of the paper also evidence that our method lead to a manifestly covariant form of the Lagrangian action corresponding to the first-class family. The presence of the non-identically vanishing functions C_a^0 in (25), and implicitly in (36) is crucial in order to establish the covariance.

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