BOSONIZATION AND DUALITY
IN CONDENSED MATTER SYSTEMS

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Abstract

We show that abelian bosonization of 1+1 dimensional fermion systems can be interpreted as duality transformation and, as a conseguence, it can be generalized to arbitrary dimensions in terms of gauge forms of rank $d - 1$, where d is the dimension of the space. This permit to treat condensed matter systems in $d > 1$ as gauge theories. Furthermore we show that in the "scaling" limit the bosonized action is quadratic in a wide class of condensed matter systems.

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1. Introduction

The procedure of writing $1 + 1$ dimensional fermionic systems in terms of boson fields (bosonization) has now a long history [1]. (Few years ago a somewhat different procedure of bosonization have been discovered for $2 + 1$ dimensional systems, involving the introduction of Chern–Simons gauge fields and generalising Jordan–Wigner transformation [2]; here we are not dealing with it).

Only recently it has been realized [3,4] that the abelian bosonization of one–dimensional systems is a special case of a more general (and now obiquitous!) transformation:duality, without restriction on dimensions. It then follows that one can generalize the abelian bosonization to arbitrary dimensions (although in general it is less powerful) in terms of gauge forms (antisymmetric tensor gauge fields) of rank $d - 1$, where d is the space dimension, playing the role of the scalar field in $d = 1$.

One can then apply the bosonization in particular to condensed matter systems [4]. This permits to treat non–relativistic Fermi systems with positive density at $T \sim 0$ as gauge theories ($d > 1$) and to apply to them methods developed in the analysis of gauge theories in high–energy physics. As an application we will briefly discuss the Wilson criterion for the existence of the charge operator.

Furthermore, for a large class of systems (free electron gas, insulators, Hall fluids, B.C.S. superconductors...) one can prove that the bosonic action is quadratic in a suitably defined "scaling limit". It also follows from general properties of bosonization that density–density or current–current (two–body) perturbations are exactly gaussian in the bosonic field, this lead to the conjecture that it is possible a classification of large–scale charge properties of condensed matter systems in universality classes, using vacuum polarization tensor.

Some applications of these ideas are sketched and the relation of this bosonization procedure with Luther–Haldane bosonization of Fermi liquids is exhibited.

2. Bosonization

Bosonization corresponds roughly to the following statement: in $d = 1$ a quantum theory of a fermion field $\hat{\psi}$ with linear dispersion relation can be written in terms of a quantum scalar field $\hat{\phi}$ with quadratic dispersion relation, describing fluctuations of fermion–antifermion pairs. [In condensed matter systems, fermions with linear dispersion are obtained linearizing the dispersion re-

lation of non–relativistic fermions around the two points of the Fermi surface, a procedure legitimate if we are interested in large scale properties. In high–energy physics $\hat{\psi}$ is just the massless Dirac field]. More precisely, setting the Fermi velocity $v_F = 1$ in condensed matter systems, and the velocity of light $c = 1$ in relativistic systems, let $\psi, \bar{\psi}$ denote two–component Grassman fields and ϕ a complex field describing in the euclidean path–integral formalism a massless Dirac field $\hat{\psi}$ and a neutral scalar field $\hat{\phi}$, respectively. Bosonization can be stated as follows: the (euclidean) correlation functions corresponding to the lagrangian $\mathcal{L}_F = \bar{\psi}\slashed{\partial}\psi$ of the (euclidean) fields : $\bar{\psi}\gamma_\mu\psi$: $(x) \equiv \mathcal{J}_\mu(x)$ (the 2–current), : $\bar{\psi}\psi$: $(x), \psi_R(x) \equiv (\frac{1+\gamma_5}{2})\psi(x),\ \psi_L(x) \equiv (\frac{1-\gamma_5}{2})$, are identical to the correlation functions corresponding to the lagrangian $\mathcal{L}_B = \frac{1}{8\pi}(\partial_\mu\phi)^2$ of the fields $\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\phi(x)$, : $\cos\phi$: (x), : $e^{+\frac{i}{2}\phi(x)}$: $D(x,1)$,: $e^{-\frac{i}{2}\phi(x)}$: $D(x,1)$, ... where : : denotes normal ordering (and from now on it will be omitted) and $D(x,1)$ is a disorder field [4,5] creating a vortex of unit vorticity at $x \in \mathbf{R}^2$.

It has been realized in [3] (and independently in a preliminary version of [4]) that this bosonization formulas are just a special version of the <u>duality transformation</u> in $d = 1$.

3. Duality

We now outline the general structure of duality.

<u>Remark on notations</u> To avoid topological complications we work in $\mathbf{R}^{d+1}$, furthermore to avoid the cumbersome use of multiindices we use the language of forms: given an antisymmetric tensor field $F_{\mu_1,..\mu_k}$ we define a k-form $F \equiv \frac{1}{k!}F_{\mu_1...\mu_k}dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$, where $\wedge$ is the wedge ($\equiv$ antisymmetric tensor) product. We denote by $\Lambda^k(\mathbf{R}^{d+1})$ the group of k-forms on $\mathbf{R}^{d+1}$ under pointwise addition, by d the exterior differential $d : \Lambda^k \to \Lambda^{k+1}$, with $dF = \frac{1}{k!}\partial_\mu F_{\mu_1...\mu_k}dx^\mu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$, by * the Hodge–star $^* : \Lambda^k \to \Lambda^{d+1-k}$, with $^*F = \frac{1}{k!}\frac{1}{(d+1-k)!}F^{\mu_1...\mu_k}\varepsilon_{\mu_1...\mu_k...\mu_{d+1}}dx^{\mu_{k+1}} \wedge ... \wedge dx^{\mu_{d+1}}$ and by (,) the inner product in Λ^k: for $F, F' \in \Lambda^k(\mathbf{R}^{d+1})$

$$(F, F') = \frac{1}{k!}\int d^{d+1}x\, F_{\mu_1...\mu_k}F^{\mu_1\cdots\mu_k}(x) = \int F \wedge^* F'.$$

To discuss duality we need two basic facts

i) <u>Poincarè lemma</u>: let $F \in \Lambda^k(\mathbf{R}^{d+1})$ be closed, i.e. $dF = 0$, then there exists $A \in \Lambda^{k-1}(\mathbf{R}^{d+1})$ such that $F = dA$

ii) Denote by $\Lambda^k/d\Lambda^{k-1}$ the quotient group of equivalence classes $[F] =$

$\{F' \in \Lambda^k | F' - F = d\zeta, \zeta \in \Lambda^{k-1}\}$, then d establishes a group isomorphism between $\Lambda^k/d\Lambda^{k-1}$ and the image of d in Λ^{k+1}, the group of closed $(k+1)$-forms.

Basic formula

Suppose we can formally write the euclidean partition function of a quantum field theory in terms of a k-form F in $\mathbf{R}^{d+1}$ as

$$Z = \int \mathcal{D}F e^{-S(F)} \delta(dF). \tag{1}$$

Then we have a "dual formulation" of such a theory in terms of a $(d-k)$-form B, invariant under the gauge transformation

$$B \to B + d\zeta, \zeta \in \Lambda^{d-k-1}(\mathbf{R}^{d+1})$$

or, alternatively, in terms of a $(d-k+1)$-form H, satisfying $dH = 0$.

To find, (heuristically) this dual formulation we first express the constraint $dF = 0$ in (1) by a Fourier representation of the δ–functional:

$$\delta(dF) = \int \mathcal{D}[B] e^{i \int F \wedge dB}$$

where $\mathcal{D}[B]$ denotes the normalized measure on the gauge equivalence classes

$$[B] = \{B' \in \Lambda^{d-k}(\mathbf{R}^{d+1}) | B' - B = d\zeta, \zeta \in \Lambda^{d-k}(\mathbf{R}^{d+1})\}.$$

Alternatively one can use the gauge–fixing + Faddev–Popov ghost procedure to properly define a BRS invariant measure for B [6]. Define $\tilde{S}(dB)$ through the functional integral Fourier transform

$$e^{-\tilde{S}(dB)} \equiv \int \mathcal{D}F e^{-S(F)} e^{i \int F \wedge dB}. \tag{2}$$

Then

$$Z = \int \mathcal{D}F e^{-S(F)} \delta(dF) = \int \mathcal{D}F e^{-S(F)} \int \mathcal{D}[B] e^{i \int F \wedge dB} = \tag{3}$$

$$\int \mathcal{D}[B] e^{-\tilde{S}(dB)} = \int \mathcal{D}H e^{-\tilde{S}(H)} \delta(dH),$$

where in the last equality we used the previously defined properties i) and ii).

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<u>Examples</u>: a) <u>Abelian gauge theories</u>

Consider a quantum field theory described in the euclidean formulation in terms of a $(k-1)$-form A and whose action S is invariant under the gauge transformation

$$A \to A + d\zeta, \zeta \in \Lambda^{k-2}.$$

Then, using the isomorphism established by d, one can change variable in the path–integral representation of the partition function from A to a k-form F, constrained by $dF = 0$:

$$Z = \int \mathcal{D}[A]e^{-S(dA)} = \int \mathcal{D}Fe^{-S(F)}\delta(dF).$$

[For $k = 1$, $d\zeta$ is replaced by a closed $0-$form, i.e. a constant]. The corresponding duality is widely known as <u>Wegner – t'Hooft duality</u> [7]. In the lattice version, in $d = 1$ for $\mathbf{Z}_2$-valued 0-forms, it has already been introduced by <u>Kramers and Wannier</u> [8] in 1941 for the Ising model.

b) <u>Theories with global abelian gauge invariance</u>

Consider a quantum field theory expressed in euclidean formalism in terms of "charged" fields χ, χ^* whose action $S(\chi, \chi^*)$ is invariant under an abelian (e.g. $U(1)$) global gauge transformation

$$\chi(x) \to e^{i\alpha}\chi(x), \ \chi^*(x) \to e^{-i\alpha}\chi^*(x).$$

We promote the global gauge invariance to a local gauge invariance introducing a minimal coupling between χ, χ^* and a $U(1)$-gauge field A. Integrating over A and setting $dA = 0$ one recovers the original theory. In formulas, for the partition function we have:

$$Z = \int \mathcal{D}\chi\mathcal{D}\chi^*e^{-S(\chi,\chi^*)} = \int \mathcal{D}\chi\mathcal{D}\chi^*\mathcal{D}Ae^{-S(\chi,\chi^*,A)}\delta(dA) = \int \mathcal{D}Ae^{-S(A)}\delta(dA),$$

where $S(\chi, \chi^*, A)$ is gauge invariant and $S(A)$ is the effective action obtained integrating out χ, χ^*. A suitable version of duality for models of class b) gives the <u>abelian T–duality</u> [9] and as we shall see, bosonization is just duality in case b when χ is the Fermi field ψ.

4. General features of duality

Let us outline some general properties of duality following simply from the defini-tion.

1) From the property that the square of a Fourier transformation is parity it follows that:

$$\tilde{\tilde{S}}(F) = S(-F)$$

2) Correlation functions at non–coinciding arguments of $-i(\frac{\delta S}{\delta F})_{\mu_1...\mu_k}$ are given in the dual theory by correlation functions of $(^*dB)_{\mu_1...\mu_k}$ (or $(^*H)_{\mu_1...\mu_k}$).

In fact, denoting by $\langle\ \rangle$ the expectation value in the original (F) theory and by $\langle\ \tilde{}\ \rangle$ the expectation values in the dual $(B$ or $H)$ theory, and omitting all indices, we have that

$$\Big\langle\prod_j\Big(-i\frac{\delta S}{\delta F(x_j)}\Big)\Big\rangle = Z^{-1}\int \mathcal{D}[B]\int\mathcal{D}F\prod_j\Big(-i\frac{\delta}{\delta F(x_j)}\Big)e^{-S(F)}e^{i\int F\wedge dB} =$$

$$Z^{-1}\int\mathcal{D}[B]\int\mathcal{D}Fe^{-S(F)}\prod_j\Big(-i\frac{\delta}{\delta F(x_j)}\Big)e^{i\int F\wedge dB} =$$

$$= \Big\langle\prod_j(^*dB)(x_j)\tilde{\Big\rangle} = \Big\langle\prod_j(^*H)(x_j)\tilde{\Big\rangle}, \tag{4}$$

where in the second equality integration by parts has been used.

For models in class a) the equation of motion of the F theory are written as $d^*\frac{\delta S}{\delta F} = 0$. They are mapped by duality to the Bianchi identities $dH = 0$ and conversely the Bianchi identities given by $dF = 0$ are mapped to $d^*\frac{\delta\tilde{S}}{\delta H} = 0$. Hence, duality interchanges equations of motions and Bianchi identities.

<u>Remark</u> In $d = 3$, for $k = 2$, also H is a two-form and we denote it by $\tilde{F}$. Under duality

$$\begin{pmatrix}-i\frac{\delta S}{\delta F}\\ ^*F\end{pmatrix} \to \begin{pmatrix}^*\tilde{F}\\ i\frac{\delta S}{\delta F}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-i\frac{\delta\tilde{S}}{\delta\tilde{F}}\\ ^*\tilde{F}\end{pmatrix}. \tag{5}$$

Furthemore for such values of d, k one can add to the action the θ term $\frac{\theta}{2\pi}\int F\wedge F$ and the theory is invariant under $\theta \to \theta + 2\pi$. Under this transformation

$$\begin{pmatrix} -i\frac{\delta S}{\delta F} \\ {}^*F \end{pmatrix} \rightarrow \begin{pmatrix} -i\frac{\delta S}{\delta F} + {}^*F \\ {}^*F \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i\frac{\delta S}{\delta F} \\ {}^*F \end{pmatrix}. \tag{6}$$

One recognizes the 2×2 matrices in (5) (6) as the S and T generators of $SL(2, \mathbf{Z})$, hence one can construct a full $SL(2, \mathbf{Z})$ group of equivalent descriptions of the theory. An $N = 2$ supersymmetric version of these transformation is a building block of Seiberg-Witten discussion of low-energy $N = 2$ Super-Yang Mills, with gauge group $SU(2)$ [10].

For models in class b), $-i\frac{\delta S}{\delta A_\mu}(x) = \mathcal{J}_\mu(x)$, the current associated to the global $U(1)$ symmetry, hence current correlation functions are expressed in the dual theory as $^*dB$–correlation functions and the analogue of the equation of motion in models of class a) is just current conservation:

$$d^*(-i\frac{\delta S}{\delta A}) = d^*\mathcal{J} = 0$$

3) <u>order–disorder duality</u>

Let Σ_p be a p-dimensional surface and denote by $\tilde{\Sigma}_p$ its Poincarè dual $(d + 1 - p)$-current, so that for $F \in \Lambda^p(\mathbf{R}^{d+1})$ we have:

$$\int_{\Sigma_p} F = \int F \wedge \tilde{\Sigma}_p.$$

In a theory of gauge forms F of rank k the "<u>Wilson loop</u>" order field $W_\alpha(\Sigma_k)$, $\alpha \in \mathbf{R}$, is defined by

$$W_\alpha(\Sigma_k) = e^{i\alpha \int_{\Sigma_k} F} = e^{i\alpha \int F \wedge \tilde{\Sigma}_k} \tag{7}$$

and it measures the "magnetic flux" through Σ_k.

The <u>"Wegner– t'Hooft" disorder field</u> $D_\alpha(\Sigma_{d+1-k})$ in the same theory is obtained instead shifting F in the action by $\alpha\tilde{\Sigma}_{d+1-k}$, i.e.

$$\langle D_\alpha\left(\Sigma_{d+1-k}\right)\rangle = \langle e^{-[S(F-\alpha\tilde{\Sigma}_{d+1-k})-S(F)]}\rangle \tag{8}$$

and it measures the "electric flux" through Σ_{d+1-k} (Normalisation factors are omitted in (7) (8), see [4]).

<u>Duality exchanges Wegner – t'Hooft disorder field and Wilson loop order field</u>, in fact

$$\langle W_\alpha(\Sigma_k) \rangle = \int \mathcal{D}F\mathcal{D}[B]e^{-S(F)}e^{i\int F\wedge dB}e^{i\alpha \int F\wedge\tilde{\Sigma}_k}$$

$$= \int \mathcal{D}F\mathcal{D}[B]e^{-S(F-\alpha\tilde{\Sigma}_k)}e^{i\int F\wedge dB} = \langle D_\alpha(\Sigma_k)\tilde\rangle,$$

where in the second equality we use the change of variable $F \to F + \alpha\tilde{\Sigma}_k$.

5. Bosonization in condensed matter system

It has been proved in [3,4], that abelian bosonization is duality for a model in class b) with $\chi \equiv \psi$ the massles Dirac field in $d = 1$.
The proof for the partition function is immediate [3] using the old result by Schwinger

$$\int \mathcal{D}\bar\psi\mathcal{D}\psi e^{-\int \bar\psi(\slashed\partial - \slashed A)\psi} = e^{-\frac{1}{2\pi}(dA, \Delta^{-1}dA)}$$

where Δ is the two–dimensional laplacian. In fact, with $B \in \Lambda^0(\mathbf{R}^2)$,

$$Z = \int \mathcal{D}\bar\psi\mathcal{D}\psi e^{-\int \bar\psi\slashed\partial\psi} = \int \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}[A]\mathcal{D}Be^{-\int \bar\psi(\slashed\partial - \slashed A)\psi}e^{i\int A\wedge dB} =$$

$$\int \mathcal{D}[A]e^{-\frac{1}{2\pi}(dA, \Delta^{-1}dA)}e^{i\int A\wedge dB} = \int \mathcal{D}Be^{-\frac{\pi}{2}(B,\Delta B)} = \int \mathcal{D}\phi e^{-\frac{1}{8\pi}(\partial_\mu\phi)^2},$$

where we identify $B \equiv \frac{\phi}{2\pi}$. The proof for current correlation functions [3] follows from property 2) in sect 4 at non–coinciding arguments and can be extended also to coinciding points, using gauge invariance [4]. The proof for fermion correlation functions is slightly more involved, see [4].

A basic message we learn from this identification is the possibility to extend bosonization to arbitrary Fermi systems replacing ϕ by a $(d-1)$-gauge form B and in particular one can obtain a bosonized (dual) action $\tilde{S}(dB)$ for condensed matter systems in arbitrary dimensions.

However, the problem we are faced on, is that even if bosonization as duality is always in principle applicable, it becomes useful only if $\tilde{S}(dB)$ has a tractable form at least for some "reference systems". This is not true in general, of course; in this respect Schwinger result for massless Dirac fields in $d = 1$ is very special! However, one can hope that $\tilde{S}(dB)$ simplify at large scales. To discuss large-scale properties of $T \sim 0$ systems we proceed as follows: we confine our Fermi system

in cubes $\Omega_\lambda = \{\lambda x | x \in \Omega\}, \Omega$ being a fixed cube in $\mathbf{R}^d$ and $\lambda, 1 \leq \lambda < \infty$, a scale parameter. We keeps the particle density constant and couple the fermions to a $U(1)$-gauge field $A^\lambda(\lambda x) \equiv \lambda^{-1} A(x)$ where A is an arbitrary but λ-independent gauge potential. Let $S^{\Omega_\lambda}(A^\lambda)$ denote the corresponding gauge-invariant action and expand it in Laurent series around $\lambda = \infty$:

$$S^{\Omega_\lambda}(A^\lambda) \underset{\lambda \nearrow \infty}{\sim} \lambda^k \sum_{n=0}^{\infty} \lambda^{-n} S^{(n)}(A). \tag{9}$$

We call the leading term in this expansion the "scaling limit" of the effective action $S(A)$ of our system and we denote it by $S^\star(A)$. It is expected to give a good description of large scale properties of $S(A)$. The dual action is denoted by $\tilde{S}^\star(dB)$. Somewhat remarkably, one can prove [4,11,12] that $S^\star(A)$, and hence $S^\star(dB)$, is quadratic for insulators (I), Hall fluids (H), free electron gas (F), B.C.S. superconductors (S). [The proof does not use small - A arguments nor in general follows from dimensional analysis, furthermore an analogous statement is false for an analogous treatment of the spin degrees of freedom, where A is non–abelian. Let us outline the basic ideas of the proof in cases I,H,F.] The proof is easy if the spectrum is gapful (I,H). In fact, as a consequence, the connected current correlation functions $\langle \prod_j \mathcal{J}^{\mu_j}(x_j) \rangle^c$ decay exponentially as $|x_i - x_j| \to \infty$, so that in the scaling limit they become distributions with point-like support, given by linear combinations of δ-functions and a finite number of derivative of δ. In turn, one easily realizes that

$$\prod_j \frac{\delta}{\delta A_{\mu_j}(x_j)} S(A) = \langle \prod_j \mathcal{J}^{\mu_j}(x_j) \rangle^c,$$

so that connected current correlation functions are just the coefficients of a series expansion of $S(A)$ in power of A. As a result $S^\star(A)$ is local and its form can then be determined by using dimensional arguments and symmetries.

For example for parity preserving rotation symmetric insulators one finds [11]

$$S^\star(A) = \int d^{d+1}x \{c_1 F_{ij}(A) F^{ij}(A) + c_2 F_{0i}(A) F^{0i}(A)\}(x) \tag{10}$$

where $F^{\mu\nu}(A) = \partial^{[\mu} A^{\nu]}$, so that $S^\star(A)$ is Maxwell–like; for Laughlin fluids (Hall fluids at Laughlin plateaux, where only a $U(1)$ symmetry appears) one finds, as a consequence of broken parity [11] the Chern-Simons action

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$$S^\star(A) = c \int A \wedge dA. \tag{11}$$

The proof [4,12] is less easy for electron gas and superconductors where the absence of gap forbids any argument of locality. Let us outline the idea for the electron gas, it will turn out that the result follows, roughly speaking, treating a d-dimensional Fermi surface as the union of 1-dimensional Fermi surfaces corresponding to its rays. We start noticing that at large scale only regions close to the Fermi surface contribute to the fermion propagator, which can be approximately written as

$$\langle \psi^*(\lambda x)\psi(\lambda y)\rangle \underset{\lambda \nearrow \infty}{\sim} \frac{1}{\lambda} \int_{S^{d-1}} d\boldsymbol{\omega} e^{ik_F \lambda \boldsymbol{\omega}\cdot(\mathbf{x}-\mathbf{y})}(\frac{k_F}{2\pi})^{d-1} G_{\boldsymbol{\omega}}\Big(x_0 - y_0, \boldsymbol{\omega}\cdot(\mathbf{x}\cdot\mathbf{y})\Big), \tag{12}$$

with

$$G_{\boldsymbol{\omega}}(x_0, \boldsymbol{\omega}\cdot\mathbf{x}) = \int \frac{dk_0}{2\pi}\frac{dk_1}{2\pi}\frac{e^{-i(k_0 x_0 + k_1 \boldsymbol{\omega}\cdot\mathbf{x})}}{ik_0 - v_F k_1} \tag{13}$$

where $d\boldsymbol{\omega}$ is the uniform measure on the $d-1$–dimensional unit sphere S^{d-1}, k_F is the Fermi momentum and v_F the Fermi velocity and from now on we set $v_F = 1$. Let us introduce a field $\psi_{\boldsymbol{\omega}}$ for each point indexed by $\boldsymbol{\omega}$ of the Fermi surface and using a "relativistic notation" set $\psi_{[\boldsymbol{\omega}]} = \binom{\psi_{\boldsymbol{\omega}}}{\psi_{-\boldsymbol{\omega}}}$, where $[\boldsymbol{\omega}] \equiv \{\boldsymbol{\omega}, -\boldsymbol{\omega}\}$. Then, the approximate formula (12) is recovered identifying

$$\psi(\lambda x) \underset{\lambda \nearrow \infty}{\sim} \int d\boldsymbol{\omega} e^{-ik_F \lambda \, \boldsymbol{\omega}\cdot\mathbf{x}}\psi_{\boldsymbol{\omega}}(\lambda x^0, \lambda \, \boldsymbol{\omega}\cdot\mathbf{x})$$

and replacing the free electron action in the scaling limit by the integral of one–dimensional massless Dirac action:

$$(\frac{k_F}{2\pi})^{d-1} \int_{S^{d-1}} d[\boldsymbol{\omega}] \int d^{d+1}x \bar{\psi}_{[\boldsymbol{\omega}]}\partial\!\!\!/_{\boldsymbol{\omega}}\psi_{[\boldsymbol{\omega}]}(x) \equiv S_0(\psi_{[\boldsymbol{\omega}]}, \bar{\psi}_{[\boldsymbol{\omega}]}) \tag{14}$$

where $\partial_{\boldsymbol{\omega}}^\mu = (\partial_0, \boldsymbol{\omega}\cdot\boldsymbol{\nabla})$. The possibility of expressing the action in the limit of $\lambda \nearrow \infty$ as an integral over one-dimensional actions persists if we couple the free fermions to a gauge field A, in fact

$$S(\psi, \psi^*, A^\lambda) \underset{\lambda \nearrow \infty}{\sim} S_0(\psi_{[\boldsymbol{\omega}]}, \bar{\psi}_{[\boldsymbol{\omega}]}) + \lambda^d i \int d[\boldsymbol{\omega}] \int d^{d+1}x \, A_\mu^{\boldsymbol{\omega}}(x)j_{[\boldsymbol{\omega}]}^\mu(x; \lambda) \tag{15}$$

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where $A_\mu^{\boldsymbol{\omega}} = (A_0, \boldsymbol{\omega} \cdot \mathbf{A})$ and

$$j_{[\boldsymbol{\omega}]}^\mu(x; \lambda) = \frac{1}{\lambda} \int d[\boldsymbol{\omega}'] e^{-i\lambda k_F (\boldsymbol{\omega}-\boldsymbol{\omega}') \cdot \mathbf{x}} \bar{\psi}_{[\boldsymbol{\omega}']}(x^0, \mathbf{x} \cdot \boldsymbol{\omega}') \gamma^\mu \psi_{[\boldsymbol{\omega}]}(x^0, \mathbf{x} \cdot \boldsymbol{\omega}).$$

<u>Remark</u> Formally, in the limit $\lambda \nearrow \infty$

$$j_{[\boldsymbol{\omega}]}^\mu(x; \lambda) \to \frac{1}{\lambda^d} \delta(\mathbf{x} \wedge \boldsymbol{\omega}) \bar{\psi}_{[\boldsymbol{\omega}]} \gamma_\mu \psi_{[\boldsymbol{\omega}]}(x^0, \mathbf{x} \cdot \boldsymbol{\omega}),$$

however perturbation by A and $\lambda \nearrow \infty$ limit do <u>not</u> commute! [12].

Since for every ray $[\boldsymbol{\omega}]$ in (15), the action of $\psi_{[\boldsymbol{\omega}]}$ is 1-dimensional, the effective action is quadratic in A^0 and as a consequence the full effective action is also quadratic in A, being integral of quadratic actions. One can easily verify that denoting by $(\Pi_F^\star)^{\mu\nu}$ the scaling limit of the free electron vacuum polarization tensor,

$$S^\star(A) = \int d^{d+1}x \, d^{d+1}y \, A_\mu(x) (\Pi_F^\star)^{\mu\nu}(x-y) A(y) \equiv (A, \Pi_F^\star A). \qquad (16)$$

<u>Remark</u> <u>Relation with Luther–Haldane bosonization [12]</u>

Since $\psi_{[\boldsymbol{\omega}]}$ is a 1+1 Dirac massless Fermi field, one can directly bosonize (15) in terms of a scalar real field $\varphi_{[\boldsymbol{\omega}]}$, and one obtains

$$j_{[\boldsymbol{\omega}]}^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_{\boldsymbol{\omega}\nu} \varphi_{[\boldsymbol{\omega}]}$$

This procedure gives the (euclidean version of the) Luther-Haldane bosonization [13]. The relation of $\varphi_{[\boldsymbol{\omega}]}$ with the dual field B is given by

$$\mathcal{J}^0 = (^*dB)^0 = \frac{1}{2\pi} \int d[\boldsymbol{\omega}] \boldsymbol{\omega} \cdot \boldsymbol{\nabla} \varphi_{[\boldsymbol{\omega}]}$$

$$\mathcal{J}^k = (^*dB)^k = \frac{1}{2\pi} \int d[\boldsymbol{\omega}] \omega^k \partial_0 \varphi_{[\boldsymbol{\omega}]}$$

6. Adding perturbations

According to property 2) of duality, density-density or current-current pertur-
bations in the dual theory are quadratic in B. Hence, if we have a "reference"
system with scaling limit effective action $S_0^\star(A) = (A, \Pi^\star A)$ and, as a conseguence,
bosonized action $S_0^\star(dB) = ({}^*dB, (\Pi^\star)^{-1}\, {}^*dB)$, one would be tempted to say that
the perturbed system have a quadratic (!) scaling limit bosonized action given by
$S^\star(dB) = ({}^*dB, ((\Pi^\star)^{-1} + V^\star)^* dB)$, where V is the perturbation kernel.

However this holds only if the following <u>perturbative assumption</u> (P) is
satisfied: perturbation and scaling limit commute.

<u>Remark</u>: What could happen is that the perturbation drive the reference system
away from its fixed point in the scaling limit. A typical example is obtained
choosing S_0 as the action of free fermions and V a Cooper interaction: the scaling
limit of the perturbed system is known to describe a superconductor!. Assumption
P can be argued to hold for perturbed free systems if V is long range and the
Cooper channel is tunnel off [12,14].

If assumption P holds, then, in the scaling limit of the perturbed system,
the two–point current correlation function is given by

$$\langle \mathcal{J}^\mu(x)\mathcal{J}^\nu(y)\rangle^\star = \langle ({}^*dB)^\mu(x)({}^*dB)^\nu(y)\rangle^\star = [((\Pi^\star)^{-1} + V^\star)^{-1}]^{\mu\nu}(x, y), \quad (17)$$

Equation (17) is exactly the result of R.P.A.! Hence, assumption P implies exact-
ness of R.P.A. in the scaling limit. This explains e.g. why in a free electron system
perturbed by a Coulomb potential the plasmon gap obtained by R.P.A. coincides
with the non–perturbative exact result obtained by Morchio and Strocchi [15] by
a "generalized Goldstone theorem".

To summarize, bosonization combined with assumtpion P gives a method
to treat non–relativistic $T \sim 0$ systems in the scaling limit as gauge theories for
$d > 1$. One can then apply to them the techniques elaborated in the analysis of
gauge theories. As an application we discuss the Wilson criterion for the existence
of the charge operator.

7. Existence of the charge operator

As remarked before, by duality a Wilson loop $W_\alpha(\Sigma_d)$ measures the charge
contained in a d-dimensional surface Σ_d in the dual (B) theory. One can prove

that, if it exists, the charge operator $\mathcal{Q}$ can be defined through the (weak) limit

$$e^{i\alpha\mathcal{Q}} = \lim_{R\nearrow\infty} \frac{W_\alpha(\Sigma_d^R)}{\langle W_\alpha(\Sigma_d^R)\rangle}, \tag{18}$$

where Σ_d^R is a ball of radius R in the time 0 (hyper-)plane. The normalization ensures that if $\mathcal{Q}$ exists it annihilates the vacuum. For the existence of the limit (18), one can use the Wilson criterion, proved to be correct for many lattice gauge theories: the limit exists if for $R\nearrow\infty$

$$\langle W_\alpha(\Sigma_d^R)\rangle \geq e^{-c|\partial\Sigma_d^R|},$$

where $|\partial\Sigma|$ denote the volume of the boundary of Σ, i.e. if the Wilson loop has "perimeter decay", and the limit does not exist if it has a faster decay, e.g. as $R\nearrow\infty$

$$\langle W_\alpha(\Sigma_d^R)\rangle \leq e^{-c|\partial\Sigma_d^R|\ln R}.$$

In the B–theory one can easily compute

$$\langle W_\alpha(\Sigma_d^R)\rangle \underset{R\nearrow\infty}{\sim} \begin{cases} e^{-c|\partial\Sigma_d^R|} & \text{I} \\ 1 & \text{H} \\ e^{-c|\partial\Sigma_d^R|\ln R} & \text{F} \\ e^{-c|\partial\Sigma_d^R|\ln R} & \text{S.} \end{cases}$$

This implies existence of the charge operator $\mathcal{Q}$ for insulators and Hall fluids, so that in these systems $\mathcal{Q}$ defines a superselection rule. Viceversa, $\mathcal{Q}$ does not exists for the free electron gas and for superconductors, signalizing that charge fluctuations diverge in the thermodynamic limit. Under assumption P it follows that for a long-range repulsive density-density perturbation we obtain perimeter decay also for perturbed free systems and superconductors: the long range perturbation depresses charge fluctuations and $\mathcal{Q}$ is again well defined.

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