Poisson Distributions for Sharp-Time Fields: Antidote for Triviality

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Abstract

Standard lattice-space formulations of quartic self-coupled Euclidean scalar quantum fields become trivial in the continuum limit for sufficiently high space-time dimensions, and in particular the moment generating functional for space-time smeared fields becomes a Gaussian appropriate to that of a (possibly generalized) free field. For sharp-time fields this fact implies that the ground-state expectation functional also becomes Gaussian in the continuum limit. To overcome these consequences of the central limit theorem, an auxiliary, nonclassical potential is appended to the original lattice form of the model and parameters are tuned so that a generalized Poisson field distribution emerges in the continuum limit for the ground-state probability distribution. As a consequence, the sharp-time expectation functional is infinitely divisible, but the Hamiltonian operator is such, in the general case, that the generating functional for the space-time smeared field is not infinitely divisible in Minkowski space. This feature permits the models in question to escape a manifestly trivial scattering matrix imposed on all infinitely divisible covariant Minkowski fields. Two sequentially related proposals for an alternative lattice formulation of interacting covariant models in four and more space-time dimensions are analyzed in some detail.

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1 Introduction and Overview

One of the more useful techniques to generate a Euclidean scalar quantum field theory is to take the continuum and infinite volume limit of a model theory defined on a finite space-time hypercubic lattice. Let the expression

$$\langle e^{\Sigma h_k \phi_k a^n} \rangle \equiv \int e^{[\Sigma h_k \phi_k a^n - \mathcal{W}(\phi)]} \, \Pi d\phi_k$$

schematically represent the generating functional for such a lattice theory. Here $\{h_k\}$ is an external field, $k = (k_1, k_2, \ldots, k_n)$, $k_j \in \mathbf{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}$, labels a lattice site, a denotes the lattice spacing, and n denotes the space-time dimension. The expression $\mathcal{W}(\phi)$ denotes the lattice action, which for a φ_n^4 model, for example, is conventionally taken to be

$$\mathcal{W}(\phi) = \tfrac{1}{2} Y(a) \Sigma (\phi_{k^*} - \phi_k)^2 a^{n-2} + \tfrac{1}{2} m_0^2(a) \Sigma \phi_k^2 a^n + g_0(a) \Sigma \phi_k^4 a^n,$$

where k^* denotes one of half of the nearest neighbors to k and the summations run over the sites and nearest neighbors as needed on a finite size lattice. The continuum limit includes an increase in the number of lattice sites as well as sending $a \to 0$. To facilitate convergence to potentially interesting results, cutoff-dependent coefficients (Y, m_0, and g_0) have been included. For $n = 2, 3$ the resultant theory that emerges in the continuum limit is nontrivial (non-Gaussian) and is in accord with that computed from a (renormalized) perturbation theory in the coupling constant g_0 [1]. When $n \geq 5$, and possibly for $n = 4$ as well, the theory that emerges in the continuum limit is trivial (Gaussian), to wit

$$\langle e^{\int h(x)\phi(x)\,d^n x} \rangle = e^{\frac{1}{2} \int h(x) C(x-y) h(y) \, d^n x \, d^n y} \ ,$$

which corresponds to a Minkowski theory that is a (possibly generalized) free field [2]. The manifest non-Gaussian character of the model on the lattice is subsumed into a possible influence on the correlation function C in the Gaussian limiting behavior.

The limiting behavior described above for $n \geq 5$, and possibly $n = 4$ as well, is characteristic of the general behavior that goes under the name of the Central Limit Theorem (CLT)[3]. Although the simplest illustrative examples of a CLT behavior are for independent identically distributed random variables, neither independence nor identity of the various distributions

are essential for its validity. If we accept for the further sake of argument that a CLT mechanism is behind the Gaussian limiting behavior for $n \geq 4$ (including $n = 4$ for convenience), then we are also drawn to consider Poisson distributions as they are the very kind that resist the vise grip of the CLT and lead to non-Gaussian and thereby potentially nontrivial continuum limits. By Poisson distributions we mean all those that go under the name Poisson, compound Poisson, or generalized Poisson distributions [4], which are described below as needed. It is the latter of these distributions that will be of physical interest in the present work.

Once the possibility of Poisson distributions is raised, there are (at least) two ways to proceed. From the point of view of Euclidean quantum field theory—where we presently find ourselves—a natural procedure, which we shall initially discuss, suggests itself, but we shall find it wanting in that although the Euclidean theory may be non-Gaussian the associated Minkowski theory is forced to have trivial scattering [5]. Our second construction of a Poisson distribution is qualitatively different than the first one, and nontrivial scattering in the Minkowski theory is not manifestly excluded. Although, in hindsight, the second approach may appear more natural, both standpoints are included for completeness. We now outline the two alternative approaches in more detail.

One may first be tempted to insist that the entire space-time dependent field distribution should be made into a generalized Poisson distribution in Euclidean space. This change of distribution is presumed to arise by the addition of an auxiliary, nonclasssical $[O(\hbar^2)]$ potential $\overline{P}$ to the conventional lattice action. Assuming for the moment we could find such a $\overline{P}$ let us examine its consequences for potential nontriviality. In particular, we suppose that both the Euclidean and Minkowski fields have the property of *infinite divisibility* [6][7]. For the Minkowski field this means that the generating functional for the smeared field operator

$$\varphi(h) \equiv \int h(x)\varphi(x)\,d^n x$$

has a form implicitly given by

$$\langle 0|e^{i\varphi(h)}|0\rangle \equiv \lim_{R\to\infty} \exp\{\, R\langle 0_R|\,[e^{i\varphi_R(h)} - 1]|0_R\rangle\},$$

where $|0_R\rangle$ is a normalized state and $\varphi_R(h)$ is a field operator for each R. As

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a consequence of this structure it follows, for example, that

$$\langle 0|\varphi(f_1)\varphi(f_2)\varphi(f_2)\varphi(f_1)|0\rangle^T \geq 0 \ .$$

Combined with Schwarz's inequality we have

$$
\begin{aligned}
0 \ &\leq \ |\langle 0|\varphi(g_1)\varphi(g_2)\varphi(f_2)\varphi(f_1)|0\rangle^T|^2 \\
&\leq \ \langle 0|\varphi(g_1)\varphi(g_2)\varphi(g_2)\varphi(g_1)|0\rangle^T \\
&\quad \times \langle 0|\varphi(f_1)\varphi(f_2)\varphi(f_2)\varphi(f_1)|0\rangle^T \ ,
\end{aligned}
$$

which for the asymptotic states of the Haag-Ruelle scattering theory implies that

$$
\begin{aligned}
0 \ &\leq \ |\langle 0|\varphi_{out}(g_1)\varphi_{out}(g_2)\varphi_{in}(f_2)\varphi_{in}(f_1)|0\rangle^T|^2 \\
&\leq \ \langle 0|\varphi_{out}(g_1)\varphi_{out}(g_2)\varphi_{out}(g_2)\varphi_{out}(g_1)|0\rangle^T \\
&\quad \times \langle 0|\varphi_{in}(f_1)\varphi_{in}(f_2)\varphi_{in}(f_2)\varphi_{in}(f_1)|0\rangle^T = 0 \ .
\end{aligned}
$$

The vanishing of this expression implies an absence of two-particle elastic scattering. Similar arguments may be extended to multi-particle scattering including particle production and annihilation [5]. Accepting the hypothesis that all asymptotic states are covered in this way leads to a unit S-matrix. Thus, insisting that the entire space-time dependent field distribution be a generalized Poisson distribution leads to trivial scattering, and so this particular hypothesis loses its appeal. However, there is also another route to follow.

Let us assume for the sake of argument that the sharp-time field is well defined (a discussion of lifting this assumption appears in the next Section). We observe that if the space-time smeared field has a Gaussian generating functional then this property is inherited by the sharp-time field as well. If we set

$$h(x) = g(\mathbf{x})\delta(t) \ , \quad \mathbf{x} \in \mathbb{R}^s \ , \quad s \equiv n - 1 \ ,$$

then, for $n \geq 4$,

$$\langle e^{\int g(\mathbf{x})\phi(\mathbf{x})\, d^s x} \rangle = e^{\frac{1}{2}\int g(\mathbf{x})C(\mathbf{x}-\mathbf{y})g(\mathbf{y})\, d^s x \, d^s y} \ .$$

At sharp time, e.g., $t = 0$, the Euclidean theory expectation coincides with the Minkowski theory expectation, and thus it follows that

$$\langle 0|e^{i\varphi(g)}|0\rangle = e^{-\frac{1}{2}\int g(\mathbf{x})C(\mathbf{x}-\mathbf{y})g(\mathbf{y})\, d^s x \, d^s y} \ ,$$

where we have changed g to ig for later convenience. We may diagonalize the field at $t = 0$ so that

$$\langle 0|e^{i\varphi(g)}|0\rangle = \int e^{i\phi(g)}|\Psi_G\{\phi\}|^2\,\delta\phi$$

where we have adopted a formal notation for a straightforward Gaussian functional integral. The expression $\Psi_G\{\phi\}$ represents the ground state functional, a Gaussian eigenstate of the Hamiltonian operator.

The Gaussian character of the continuum ground state contrasts with the *non*-Gaussian ground state of the lattice Hamiltonian operator prior to taking the continuum limit. The lattice Hamiltonian may be readily deduced from the lattice action, and it has the form

$$\begin{aligned}
\mathcal{H}_a &= -\tfrac{1}{2}Y^{-1}(a)a^{-s}\Sigma'\frac{\partial^2}{\partial\phi_k^2} + \tfrac{1}{2}Y(a)\Sigma'(\phi_{k^*} - \phi_k)^2 a^{s-2}\\
&\quad +\tfrac{1}{2}m_0^2(a)\Sigma'\phi_k^2 a^s + g_0(a)\Sigma'\phi_k^4 a^s + \text{const. },
\end{aligned}$$

where we assume that $g_0(a) > 0$ and that the prime on various sums indicates that only lattice sites with the same "time" value (equal k_1, say) are included. The "const." is adjusted so that the ground state $\Psi_a(\phi)$ of the lattice Hamiltonian satisfies

$$\mathcal{H}_a\Psi_a(\phi) = 0 \ .$$

In view of the nonvanishing quartic interaction, $\Psi_a(\phi)$ can not be a Gaussian, and as a consequence the characteristic functional

$$\int e^{i\Sigma'\phi_k g_k a^s}|\Psi_a(\phi)|^2\,\Pi'\,d\phi_k$$

is *non*-Gaussian although, as noted above, the continuum limit of this lattice characteristic functional is Gaussian. Thus, we may reasonably conclude that the distribution of field values at sharp time (e.g., $t = 0$) succumbs to the CLT, and in the continuum limit passes from a non-Gaussian to a Gaussian distribution.

Our strategy at this point is to add an (alternative) auxiliary, nonclassical $[O(\hbar^2)]$ potential P to the lattice Hamiltonian which has the effect of changing the ground state in such a way that in the continuum limit the characteristic functional for the sharp-time field corresponds to a generalized

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Poisson distribution. In symbols, in the first approach we adopt, we modify the lattice Hamiltonian so that its ground-state characteristic functional has the form given by

$$\int e^{i\Sigma'\phi_k g_k a^s} |\Psi_a(\phi)|^2 \, \Pi' d\phi_k = \exp\{-\int [1 - \cos(\Sigma'\phi_k g_k a^s)]|\psi_a(\phi)|^2 \, \Pi' d\phi_k\} \, ,$$

where

$$\int |\psi_a(\phi)|^2 \, \Pi' d\phi_k = \infty \, ,$$

and insist, moreover, in the continuum limit and written formally, that

$$\begin{aligned}
\langle 0|e^{i\varphi(g)}|0\rangle &= \int e^{i\phi(g)} |\Psi_P\{\phi\}|^2 \, \delta\phi \\
&= \exp(-\int \{1 - \cos[\phi(g)]\}|\psi\{\phi\}|^2 \, \delta\phi) \, ,
\end{aligned}$$

where

$$\int |\psi\{\phi\}|^2 \, \delta\phi = \infty \, .$$

In writing these expressions we have made explicit use of the expected symmetry $\psi_a(-\phi) = \psi_a(\phi)$ appropriate to the ground state in order to introduce the "cosine". The expression in the exponent in both of the last equations is often termed the *second characteristic* [6]; it also serves as the generator of the truncated equal-time correlation functions, when such moments exist. We suppose that the characteristic functional—and thus the second characteristic—is well defined for a wide class of test functions g, say $g \in \mathcal{S}(\mathbb{R}^s)$, the Schwartz space of C^∞ functions of rapid decrease. However, as already indicated, we will insist that $\int |\psi_a(\phi)|^2 \, \Pi' d\phi_k = \infty$ as well as $\int |\psi\{\phi\}|^2 \, \delta\phi = \infty$, conditions which characterize a *generalized* Poisson distribution for the lattice and in the continuum limit, respectively, as this divergence is necessary and sufficient in the present case for the smeared sharp-time field operator $\varphi(g)$, on the lattice and in the continuum, to have a purely absolutely continuous spectrum. The contrary situation for which $\int |\psi_a(\phi)|^2 \, \Pi' d\phi_k < \infty$, or especially $\int |\psi\{\phi\}|^2 \, \delta\phi < \infty$, conditions which describe a compound Poisson distribution, will be unsuitable.

Once having ensured that the sharp-time field has a generalized Poisson (and non-Gaussian) distribution, we choose the associated functional $\Psi_P\{\phi\}$ to be the ground state of the continuum Hamiltonian and, based on that choice, attempt to build multi-time correlation functions in a conventional

manner. Insofar as the sharp-time correlation functions are non-Gaussian, one may reasonably expect the multi-time correlation functions to be non-Gaussian as well. Since the present construction invokes an auxiliary interaction to generate a Poisson distribution only for the sharp-time fields, there is no reason, in general, to expect that the relativistic field will have a Poisson distribution for all space *and* time thereby avoiding manifest triviality of the S-matrix as discussed above.

Faced with the difficulty of analytically determining $\Psi_a(\phi)$ to carry out this program we introduce a second model designed so that the continuum limit of the time zero field, at least for several test cases, is identical to that based on the first model. In the second model we approximate $\Psi_a(\phi)$ directly thereby enabling us to recover the auxiliary nonclassical potential P directly. Given the auxiliary potential we propose a specific form of a lattice action that we believe embodies our proposals for an alternative and nontrivial quantization of self-coupled scalar fields such as φ_n^4 for $n \geq 4$ (and potentially an alternative quantization for $n = 2$ or 3 as well).

1.1 Illustrative Toy Model

It is instructive at this point to introduce a "toy" model to more easily visualize some aspects of the previous discussion. Let us consider the well-known Cauchy distribution in one dimension given by its characteristic function

$$
\begin{aligned}
C(p) \equiv e^{-b|p|} &= \int \frac{b\,e^{ipx}}{\pi(x^2 + b^2)}\,dx \\
&= \exp\{-b\int[1 - \cos(px)]\frac{dx}{\pi x^2}\} \ .
\end{aligned}
$$

Here $b > 0$ denotes an arbitrary scale parameter. The spectrum of x is absolutely continuous as signified by the probability density $b/[\pi(x^2 + b^2)]$. Observe the nonintegrable singularity in the second characteristic weight function, i.e., $\int dx/(\pi x^2) = \infty$. The two weight functions involved are, of course, related to one another. In particular,

$$
\begin{aligned}
|p| = \lim_{b \to 0} b^{-1}(1 - e^{-b|p|}) &= \lim_{b \to 0} \int (1 - e^{ipx}) \frac{dx}{\pi(x^2 + b^2)} \\
&= \int[1 - \cos(px)]\frac{dx}{\pi x^2} \ .
\end{aligned}
$$

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While analytically simple, this toy model is special because its moments do not exist. A closely related family of toy models is determined by the characteristic function

$$
\begin{aligned}
C_\rho(p) = e^{-b\sqrt{p^2+\rho^2}+b\rho} \quad &= \quad \int e^{ipx}\frac{b\rho}{\pi}\frac{K_1(\rho\sqrt{x^2+b^2}\,)}{\sqrt{x^2+b^2}}e^{b\rho}\,dx \\
&= \quad \exp\{-b\int[1-\cos(px)]\frac{\rho K_1(\rho|x|)}{\pi|x|}\,dx\}\;,
\end{aligned}
$$

where $\rho > 0$ is another parameter at our disposal [8]. The conventional modified Bessel function $K_1(z) \simeq \sqrt{\pi/2z}\,e^{-z}$ for $z \gg 1$, ensuring that all moments of the distribution exist. Moreover, since $K_1(z) \simeq 1/z$ for $0 < z \ll 1$, it follows that as $\rho \to 0$ these distributions converge to the Cauchy case. For any $\rho > 0$ observe that for *small* x the second characteristic weight function is effectively identical, i.e., $b/(\pi x^2)$, to that of the Cauchy distribution. Thus we are led to the notion of a set of different distributions all of which have the same singularity structure of the second characteristic weight function at small arguments but which differ from each other by their behavior for intermediate and large arguments. In the present case such a set includes, e.g., all distributions of the form

$$
\exp\{-b\int[1-\cos(px)]\frac{F(x)}{\pi x^2}\,dx\} \equiv \int e^{ipx}G(x)\,dx\;,
$$

where $F \in C^2$, $F(0) = 1$, $F \geq 0$, and $\int[F(x)/(1+x^2)]\,dx < \infty$. Although we can not analytically specify the L^1 nonnegative weight function G of the characteristic function in the general case, we know that it exists. The characterization of a distribution by means of its second characteristic will be an important tool in our study.

One aspect of our approach that does not have an analog in this toy model is the relation of the second model to the first, namely the fact that these two lattice field theories have different lattice starting points but equal continuum limits.

Before closing the discussion of our toy models it is worthwhile illustrating our suggested procedure for constructing multi-time correlation functions. For simplicity we confine ourselves to the Cauchy model (even though an entirely analogous procedure could be carried through for the toy model

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with finite moments). In particular, if we identify

$$\langle 0|e^{ipQ}|0\rangle \equiv e^{-b|p|} = \int e^{ipx} \frac{b\,dx}{\pi(x^2 + b^2)}$$

as the ground-state expectation function, then we are led to represent the Hamiltonian operator $\mathcal{H}$ by

$$
\begin{aligned}
\mathcal{H}_b &\equiv -\tfrac{1}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}(x^2 + b^2)^{1/2}\frac{\partial^2}{\partial x^2}(x^2 + b^2)^{-1/2} \\
&= -\tfrac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{2x^2 - b^2}{2(x^2 + b^2)^2} \ .
\end{aligned}
$$

For any $b > 0$ it is seen that this Hamiltonian has only one normalizable eigenstate, the ground state with zero energy; this paucity of normalizable eigenstates is related to the absence of moments for this model. It follows that

$$\langle 0|e^{ip'Q}e^{-i\mathcal{H}t}e^{ipQ}|0\rangle \equiv \frac{b}{\pi}\int \frac{e^{ip'x}}{\sqrt{x^2+b^2}}e^{-i\mathcal{H}_b t}\frac{e^{ipx}}{\sqrt{x^2+b^2}}\,dx \ ,$$

with analogous expressions holding for higher-order multi-time correlations. The procedure illustrated here is analogous to that to be followed in our discussion of φ^4 models in Sections 2 and 3.

On the other hand, for the sake of completeness, we should also illustrate an alternative dynamical extension analogous to the first proposal to have a Poisson distribution for fields over all space and time. In this alternative dynamical extension we modify the second characteristic so that

$$\langle 0|e^{ip'Q}e^{-i\mathcal{H}t}e^{ipQ}|0\rangle \equiv \exp[\frac{b}{\pi}\int \frac{1}{|x|}(e^{ip'x}e^{-iht}e^{ipx} - 1)\frac{1}{|x|}\,dx] \ ,$$

where in the present case

$$
\begin{aligned}
\mathsf{h} &\equiv -\tfrac{1}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}|x|\frac{\partial^2}{\partial x^2}|x|^{-1} \\
&= -\tfrac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \ .
\end{aligned}
$$

At $t = 0$, of course, the two dynamical extensions coincide. The Hamiltonian h has a positive, purely absolutely continuous spectrum, and as a consequence, the asymptotic behavior of both dynamical expressions exhibits

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clustering as $t \to \pm\infty$. Thus on general grounds both dynamical expressions may seem viable. However, as argued earlier, one of these extensions leads to manifestly trivial scattering in the field theory case. Consequently, our further discussion will be aimed at converting the sharp-time field distribution into a generalized Poisson distribution and proceeding in analogy with the former of the two dynamical toy models.

1.2   Summary of Proposed Lattice Model for φ_n^4

Here we summarize the results of the next two sections and present in as concrete a form as possible our proposal for a lattice model designed to lead to a nontrivial continuum limit and to have the possibility of nontrivial scattering as a relativisitic theory. In the language with which the conventional lattice models are formulated, our proposal is very much the same save for two aspects. The first difference is the presence of the auxiliary, nonclassical potential that enforces a generalized Poisson distribution on the sharp-time field distribution, and the second difference is the wholly new form taken by the bare parameters of the theory in terms of the renormalized (or finite, nonzero) parameters that eventually determine the theory properly.

The model is characterized by the Euclidean lattice form of the correlation generating function as given by

$$S(h) \;\; = \;\; N_0 \int \exp[\Sigma\, h_k \phi_k a^n - \tfrac{1}{2} Y(a) \Sigma (\phi_{k^*} - \phi_k)^2 a^{(n-2)} - \tfrac{1}{2} m_0^2(a) \Sigma \phi_k^2 a^n$$
$$-g_0(a)\Sigma \phi_k^4 a^n - \Sigma\, P(\phi_k, a) a^n]\, \Pi d\phi_k \; .$$

In this expression [9]

$$P(\phi_k, a) = \frac{\mathsf{A}(a)\phi_k^2 - \mathsf{B}(a)}{[\phi_k^2 + \mathsf{C}(a)]^2} \; ,$$

where

$$\mathsf{A}(a) = Y(a)^{-1} a^{-2s}\, 8^{-1}(1+\xi)(3+\xi) \; ,$$
$$\mathsf{B}(a) = Y(a)^{-1} a^{-2s}\, 4^{-1}(1+\xi)\mathsf{C}(a) \; ,$$
$$\mathsf{C}(a) = M^{n-2}(Ma)^{2s(1-\xi)/\xi} \; .$$

In these relations $\xi > 0$ is a free parameter labeling inequivalent quantizations; as we shall show, the cases $0 < \xi < 2$ and $2 \le \xi$ are qualitatively

different. For the present, as well as for the most part, we shall confine attention to $0 < \xi < 1$. Our analysis indicates that the bare parameters are related to renormalized parameters according to the relation

$$
\begin{aligned}
m_0^2(a) &= (Ma)^{n-2}\, m^2 \;, \\
g_0(a) &= (Ma)^{3n-4} g \;, \\
Y(a) &= (Ma)^n \;,
\end{aligned}
$$

where M denotes a mass parameter chosen to ensure that engineering dimensions are preserved. Observe that these three bare parameter renormalizations are *multiplicative* as opposed to *subtractive;* this consequence is a direct reflection of the profound difference that the auxiliary potential introduces into the integrand. Moreover, all three bare parameters are *small* when $Ma \ll 1$, i.e., as the continuum limit is approached. This means that in the integration variables indicated the essential support of ϕ extends roughly to $O((Ma)^{-s})$, namely to large values. On the other hand, the contribution of the auxiliary potential is significant for *much smaller* values of the integration variables, namely when $\phi^2 \simeq \mathsf{C}(a) = M^{n-2}(Ma)^{2s(1-\xi)/\xi}$. In brief, each integration variable sees two qualitatively different regions, one for small values dominated by the auxiliary potential, and one for large values controlled by the terms of the usual model. Of course, "small" and "large" are relative terms and may be influenced by a rescaling of the variables of integration; for example, if we introduce $\rho_k = (Ma)^s \phi_k$, then it follows that ρ_k has an essential support of order one or less. But in this case the influence of the auxiliary potential occurs for extremely small values of ρ_k. As the continuum limit is approached the disparate scales for the two effects become even more pronounced. Incidentally, the importance of the very small and very large integration domains for each variable would make a naive application of either renormalization group arguments or Monte Carlo calculations appear rather difficult.

In a formal continuum limit the density of the auxiliary potential term is invariably of the form

$$
\frac{c\hbar^2}{\phi^2(x)}
$$

for any value of ξ and any space-time dimension n, where c denotes a suitable constant. We will present arguments that show that this term should not be interpreted as a renormalization counterterm for the nonlinear interaction

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term, but rather as a necessary renormalization for the *kinetic energy term* with which it shares an identical scaling behavior. (The analog of the centrifugal potential arising out of the kinetic energy in nonrelativistic quantum mechanics is an appropriate one here.)

Such renormalization counterterms have been encountered in previous models studied by the author, and one may expect that some of the general discussion that held true for previous models may well hold true in the present case [10]. In particular, we have in mind the implications of such a counterterm for: (i) the presence of a hard-core interaction; (ii) the limiting behavior as the nonlinear coupling constant is turned off being not the usual free theory but rather a pseudo-free theory which retains the essential effects of the hard-core potential; and (iii) the existence of a meaningful perturbation theory in the nonlinear coupling constant, not about the free theory, but instead about the pseudo-free theory.

2   Lattice Regularized Models: First Formulation

In this section we shall present the first of two formulations of a class of models each of which we believe has the right behavior so that in the continuum limit a non-Gaussian Euclidean quantum field theory emerges. Based on the discussion in the preceding section it is possible that in Minkowski space these models exhibit nontrivial scattering as well. Among such models are natural candidates for nontrivial theories of φ_n^4 in space-time dimensions $n \geq 5$ and perhaps $n = 4$ as well. We begin this section with some remarks on the difference between locality on the lattice and locality in the continuum.

Lattice and continuum locality

Let us illustrate the point of this subsection with a very simple example, namely one that deals with Gaussian white noise. Suppose we want to find a lattice prescription for the Gaussian functional integral given by

$$e^{-\frac{1}{2}\int h^2(x)\,d^nx} = \int e^{i\int h(x)\phi(x)\,d^nx}\,d\mu(\phi)\;.$$

The natural lattice formulation of this problem is

$$e^{-\frac{1}{2}\Sigma_k h_k^2 a^n} = N \int e^{i\Sigma_k h_k \phi_k a^n} e^{-\frac{1}{2}\Sigma_k \phi_k^2 a^n} \, \Pi d\phi_k \ ,$$

in which the manifest locality—even ultralocality—of the continuum result is already apparent in the lattice form. However, it is by no means necessary that the lattice formulation needs to be local in order for the continuum limit to be local. As an example we study the following alternative lattice formulation for the same problem.

Let us consider

$$N_\Xi \int e^{i\Sigma_k h_k \phi_k a^n} e^{-\frac{1}{2}\Sigma_{k,l}\Xi_{k;l}\phi_k \phi_l a^n} \, \Pi d\phi_k \ ,$$

where Ξ is a positive-definite matrix, is independent of the lattice spacing, and has the property that $\Sigma_k \Xi_{k;l} = 1$ for each fixed l. For general Ξ, the lattice exponent is certainly not local on the lattice; however, the continuum limit leads to the same result as before. Specifically,

$$\lim_{a \to 0} N_\Xi \int e^{i\Sigma_k h_k \phi_k a^n} e^{-\frac{1}{2}\Sigma_{k,l}\Xi_{k;l}\phi_k \phi_l a^n} \, \Pi d\phi_k = e^{-\frac{1}{2}\int h^2(x) \, d^n x} \ .$$

An acceptable example for Ξ is given by

$$\Xi_{k;l} = \tilde{M} \, \Pi_{j=1}^n \sin^2(k_j - l_j)/(k_j - l_j)^2 \ ,$$

where $\tilde{M}$ is a normalizing constant. This expression has the apparent feature of being nonlocal. However, in the continuum any two points a finite distance apart are actually an infinite number of (continuum) lattice steps apart, and so there is no real nonlocality after all. Insofar as the continuum limit is concerned, the matrix $\Xi_{k;l}$ is just as good as the choice $\delta_{k,l}$. One matrix may be more natural than the other, but it would be incorrect to say that one is "right" and the other is "wrong". Since the continuum limit is the ultimate goal they are both equally "right".

The purpose of this comment is to assuage any concern about continuum nonlocality that may be raised in the lattice formulation to follow. What appears to be nonlocal is in fact perfectly local in the continuum limit.

2.1 Determining the Second Characteristic

As a preliminary we choose a "time" direction in the Euclidean lattice; this will become the time direction when one changes from a Euclidean to a Minkowski theory. For this purpose we single out the first component k_1 of the lattice label as the future time direction, but for convenience we shall simply refer to k_1 as the time direction. Evidently there are $s = n - 1$ space components, $k_2, \ldots, k_n$, remaining. We assume the original lattice is a hypercube with an odd number, $2L+1$, of lattice sites in each direction. Thus the total number of lattice points is $N \equiv (2L + 1)^n$, while the total number of lattice points in a spatial slice (at constant k_1) is given by $N' \equiv (2L+1)^s$. We adopt periodic boundary conditions in the spatial directions and Dirichlet boundary conditions in the time direction. Let us next introduce a family of real constants $\{\beta_k\}$ (with k a lattice site label) each of which is positive, $\beta_k > 0$, and subject to the normalization condition

$$\Sigma'_k \beta_k = 1 \ ,$$

where the sum is over the N' sites in any spatial slice in the lattice, i.e., only $n - 1$ dimensions of the lattice. In doing the sum we hold k_1 fixed, and the normalization holds for any k_1. We shall use a "prime" to signify when a sum, or a factor, refers to a lattice of codimension one. A suitable choice of constants β_k is given in the form

$$\beta_k = \overline{\beta}_k \equiv M' \, \Pi^n_{j=2} K^{-|k_j|}$$

where $K > 1$ and M' is a normalization factor given by

$$M' = [(K - 1)/(K + 1 - 2K^{-L})]^s \ .$$

Armed with these definitions we introduce an ansatz for the weight function appropriate to the second characteristic. In particular we let

$$\psi_a(\phi) = J \frac{e^{-w(\phi)}}{\Pi'_k [\Sigma'_l \beta_{k-l} \phi_l^2]^{\gamma/2}} \ ,$$

where J is a normalization factor (to be chosen below), $w(\phi)$, with $w(0) = 0$, is a function designed to control the large field behavior, while the factor in

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the denominator controls the small field behavior. We choose the parameter $\gamma = 1/2 + \xi/(2N')$, $0 < \xi < 1$, in order, as we shall see, that

$$\int |\psi_a(\phi)|^2 \, \Pi' d\phi_k = \infty \; ,$$

while on the other hand, along with a suitable w,

$$\int (\Sigma' g_k \phi_k a^s)^p |\psi_a(\phi)|^2 \, \Pi' d\phi_k < \infty$$

for all $p \geq 1$. In terms of these expressions, the (negative of the) second characteristic reads

$$C_S(g) \equiv J^2 \int [1 - \cos(\Sigma' \phi_k g_k a^s)] \frac{e^{-2w(\phi)}}{\Pi_k'[\Sigma_l' \beta_{k-l} \phi_l^2]^\gamma} \, \Pi_k' d\phi_k \; .$$

Normalization is readily dispensed with. We choose a special (sp) test sequence, for example we may choose

$$g_k = g_k^{sp} \equiv M^{n/2} e^{-Ma \Sigma_{j=2}^n |k_j|} \; ,$$

for a suitable constant M. This special sequence is chosen so that

$$\Sigma'(g_k^{sp})^2 a^s = M + O(a, L^{-1})$$

is finite and nonzero in the continuum limit. We declare that $C_S(g^{sp}) = 1$, a condition that fixes the value of J, in particular, so that

$$J^{-2} \equiv \int [1 - \cos(\Sigma' \phi_k g_k^{sp} a^s)] \frac{e^{-2w(\phi)}}{\Pi_k'[\Sigma_l' \beta_{k-l} \phi_l^2]^\gamma} \, \Pi_k' d\phi_k \; .$$

In order to better ascertain the continuum limit for the second characteristic, it is convenient to introduce a new set of integration variables which we shall refer to as *hyper-extreme spherical coordinates*. Based on the definition $\phi_k \equiv \kappa \eta_k$, $\kappa \geq 0$, with $\Sigma' \phi_k^2 \equiv \kappa^2$, let us consider the measure

$$\begin{aligned}
\Pi_k'[\Sigma_l' \beta_{k-l} \phi_l^2]^{-\gamma} \, \Pi_k' d\phi_k &= \Pi_k'[\Sigma_l' \beta_{k-l} \phi_l^2]^{-\gamma} \, \Pi_k' d\phi_k \, \delta(\kappa^2 - \Sigma_k' \phi_k^2) \, d\kappa^2 \\
&= \kappa^{N'-2} \, \kappa^{-N'-\xi} \, d\kappa^2 \, \delta(1 - \Sigma' \eta_k^2) \Pi_k'[\Sigma_l' \beta_{k-l} \eta_l^2]^{-\gamma} \, \Pi_k' d\eta_k \\
&= 2\kappa^{-1-\xi} \, d\kappa \, d\tau(\eta) \; ,
\end{aligned}$$

where
$$d\tau(\eta) \equiv \delta(1 - \Sigma'_k \eta_k^2)\, [\Pi'_k [\Sigma'_l \beta_{k-l} \eta_l^2]^{-\gamma}\, \Pi'_k d\eta_k\ .$$

In terms of these variables it follows that

$$C_S(g) = 2J^2 \int [1 - \cos(\kappa \Sigma' \eta_k g_k a^s)]\, e^{-2w(\kappa \eta)} \kappa^{-1-\xi}\, d\kappa\, d\tau(\eta)\ ,$$

and more particularly for the moments that

$$
\begin{aligned}
< (\Sigma'\, g_k \phi_k a^s)^p > \ &= \ 2J^2 \int (\Sigma'\, g_k \eta_k a^s)^p\, \Pi'_k [\Sigma'_l \beta_{k-l} \eta_l^2]^{-\gamma}\, e^{-2w(\kappa \eta)} \\
&\qquad \times\, \kappa^{p-1-\xi}\, d\kappa\, \delta(1 - \Sigma' \eta_k^2)\, \Pi'_k d\eta_k\ .
\end{aligned}
$$

Here we restrict attention to $0 < \xi < 1$ which is readily seen to lead to the desired conditions that $\int (\Sigma' \phi_k g_k a^s)^p |\psi_a(\phi)|^2 \Pi' d\phi_k$ diverges for $p = 0$ and is finite for $p \geq 1$. The case where ξ is larger is treated below.

First look at acceptable expressions for $w(\phi)$

When moments exist, an acceptable expression for $w(\phi)$ is one for which moments of the second characteristic are all of the same order of magnitude. We can assure this by studying the quotient of any two moments and ensuring that it neither vanishes nor diverges in the continuum limit. We focus our initial attention on the quotient

$$\frac{< (\Sigma' g_k \phi_k a^s)^4 >}{< (\Sigma' g_k \phi_k a^s)^2 >} = \frac{\int (\Sigma'\, g_k \eta_k a^s)^4 \kappa^{3-\xi}\, e^{-2w(\kappa \eta)}\, d\kappa\, d\tau(\eta)}{\int (\Sigma'\, g_k \eta_k a^s)^2 \kappa^{1-\xi}\, e^{-2w(\kappa \eta)}\, d\kappa\, d\tau(\eta)}\ .$$

Suppose that $w = \kappa^2/T$, then it follows that

$$
\begin{aligned}
\frac{< (\Sigma' g_k \phi_k a^s)^4 >}{< (\Sigma' g_k \phi_k a^s)^2 >} \ &= \ cT\, \frac{\int (\Sigma'\, g_k \eta_k a^s)^4\, d\tau(\eta)}{\int (\Sigma'\, g_k \eta_k a^s)^2\, d\tau(\eta)} \\
&= \ 3c\, T\, a^s \frac{(\Sigma'_k g_k^2 a^s)(\Sigma'_l g_l^2 a^s) \int \eta_k^2 \eta_l^2\, d\tau(\eta)}{(\Sigma'_k g_k^2 a^s) \int \eta_k^2\, d\tau(\eta)}\ ,
\end{aligned}
$$

where c denotes a fixed, nonzero, dimensionless constant. Now, as discussed below, it is reasonable to assume something like

$$\frac{\int \eta_k^2 \eta_l^2\, d\tau(\eta)}{\int \eta_m^2\, d\tau(\eta)} \approx \beta_{k-l}\ .$$

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As a consequence it is necessary to choose $T \propto (Ma)^{-2s}$—for some fixed parameter M with the dimension of a mass—in order to have a ratio that stays finite and away from zero as $a \to \infty$ and $N' \to \infty$. Note that T has the mass dimensions of ϕ^2, namely $n-2$. It is readily seen that if $w = \kappa^4/T^2$, with $T \propto (Ma)^{-2s}$, a suitable behavior is also found. In addition all other moment ratios are maintained correctly with these same choices. Consequently, we conclude that the proper choice for w is given by

$$w(\phi) = \Upsilon(\kappa\eta(Ma)^s)$$

for a large class of functions Υ that may involve dimensional constants, but are (effectively) *independent* of either the lattice spacing a or the number of lattice sites in a spatial slice N'. This remark has strong consequences for the form of the regularized coefficients in the lattice Hamiltonian and therefore in the lattice action.

Elementary exercises in many-variable integrations

There are a number of significant differences in many-dimensional integrations between Gaussian-like integrands and Poisson-like integrands of the general kind that appear in the second characteristic. For this remark we have in mind integrals of the form

$$I_G(2p) = \int (\Sigma\phi_k^2)^p \, e^{-A\Sigma\phi_k^2} \, \Pi \, d\phi_k$$

as well as those of the form

$$I_P(2p) = \int (\Sigma\phi_k^2)^p \, e^{-A\Sigma\phi_k^2} [\Sigma\phi_k^2]^{-N/2} \, \Pi \, d\phi_k \;,$$

where $p \geq 1$. Here, in this and the following exercise section, we do not distinguish between N and N', and for simplicity in notation we drop the prime. Note that the integrals in question are simple "caricatures" of the more complicated integrals of interest; nevertheless these integrals will illustrate an important point. More relevant integrals are studied below.

If we use the variables κ and $\{\eta_k\}$ introduced previously, it follows that

$$I_G(2p) = 2 \int \kappa^{2p} \, e^{-A\kappa^2} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k \;.$$

Evaluation of this expression for large N is dominated by the factor κ^{N-1}, and a steepest descent method can be used to evaluate to a suitable accuracy the integral over κ. It follows that the stationary point for each integral of this form is given to leading order by $\kappa = \sqrt{N/(2A)}$. As a consequence, for each different value of A the integrand lives on a different set—a disjoint set of field values—as $N \to \infty$ [11]. This fact is responsible for divergences in perturbation calculations. For example, if we shift the value of A and attempt to calculate

$$I'_G(2) = 2 \int \kappa^2 \, e^{-A'\kappa^2} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma\eta_k^2)\Pi \, d\eta_k$$

in terms of $I_G(2p)$ by means of a perturbation series in $\Delta A \equiv A' - A$, then we are led to the series

$$I'_G(2) = I_G(2) - \Delta A I_G(4) + \tfrac{1}{2}(\Delta A)^2 I_G(6) - \tfrac{1}{6}(\Delta A)^3 I_G(8) + \cdots$$

which exhibits *divergences* as $N \to \infty$ since $I_G(2p)/I_G(2) \propto N^{(p-1)}$. In the same spirit if one attempted to evaluate

$$I''_G = 2 \int \kappa^2 \, e^{-A\kappa^2 - B\kappa^4 \Sigma\eta_k^4} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k$$

by means of a perturbation in B, then one would be led to the series

$$
\begin{aligned}
I''_G \;=\; & I_G(2) - B I_G(6) \int (\Sigma\eta_k^4) \, d\sigma(\eta) \\
& + \tfrac{1}{2} B^2 I_G(10) \int (\Sigma\eta_k^4)^2 \, d\sigma(\eta) + \cdots ,
\end{aligned}
$$

where we have introduced

$$d\sigma(\eta) \equiv \frac{\delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k}{\int \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k} \; .$$

Here again we encounter divergences, e.g., because $I_G(6)/I_G(2) \propto N^2$ while

$$\int (\Sigma\eta_k^4) \, d\sigma(\eta) \propto N^{-1}$$

and so the net value for the first-order correction relative to the initial term is of order N. In point of fact, the kind of divergences under discussion here are

extremely simplified examples—analogous to the "caricatures" spoken about above—of the kind of divergences that arise in perturbative calculations in relativistic φ^4 theory!

We now turn our attention to a study of I_P. The expressions that we shall study are once again caricatures of the more complicated ones studied in previous sections; nevertheless they contain the essential ingredients for present purposes. In particular, for $p \geq 1$, let us examine

$$
\begin{aligned}
I_P(2p) &= \int (\Sigma\phi_k^2)^p \, e^{-A\Sigma\phi_k^2} [\Sigma\phi_k^2]^{-N/2} \, \Pi \, d\phi_k \\
&= 2 \int \kappa^{2p} \, e^{-A\kappa^2} [\kappa^2]^{-N/2} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k \\
&= 2 \int \kappa^{2p-1} e^{-A\kappa^2} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k \; .
\end{aligned}
$$

Now we see no large power of κ in the integrand to lead to N-dependent factors. Indeed,

$$
\begin{aligned}
\frac{I_P(4)}{I_P(2)} &= \frac{\int \kappa^3 e^{-A\kappa^2} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k}{\int \kappa e^{-A\kappa^2} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k} \\
&= \frac{\int \kappa^3 e^{-A\kappa^2} \, d\kappa}{\int \kappa e^{-A\kappa^2} \, d\kappa} = \frac{1}{A} \; ,
\end{aligned}
$$

which is N-independent and finite. With

$$
I_P'(2) = 2 \int \kappa e^{-A'\kappa^2} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k
$$

a perturbation expansion in ΔA as before is given by

$$
I_P'(2) = I_P(2) - \Delta A I_P(4) + \tfrac{1}{2}(\Delta A)^2 I_P(6) - \tfrac{1}{6}(\Delta A)^3 I_P(8) + \cdots
$$

which exhibits no divergences whatsoever. In a similar way

$$
I_P'' = 2 \int \kappa e^{-A\kappa^2 - B\kappa^4 \Sigma\eta_k^4} \, d\kappa \, \delta(1 - \Sigma\eta_k^2) \, \Pi \, d\eta_k
$$

admits an expansion in B of the form

$$
\begin{aligned}
I_P'' &= I_P(2) - B I_P(6) \int (\Sigma\eta_k^4) \, d\sigma(\eta) \\
&\quad + \tfrac{1}{2} B^2 I_P(10) \int (\Sigma\eta_k^4)^2 \, d\sigma(\eta) + \cdots \; .
\end{aligned}
$$

The I_P coefficients are all finite in this case, but we still have

$$\int (\Sigma \eta_k^4) \, d\sigma(\eta) \propto N^{-1} \, ,$$

$$\int (\Sigma \eta_k^4)^2 \, d\sigma(\eta) \propto N^{-2} \, .$$

Consequently, if we reinterpret B as being proportional to N, say $B = \overline{B} N$, then each term of the perturbation series makes perfectly good sense as a power series in $\overline{B}$.

These simple examples serve to illustrate one aspect of the profound difference that exists between many-variable integrations involving Gaussian-like integrands compared with those that arise with Poisson-like integrands. However, there is one glaring deficiency in our discussion, namely we have not included any examples in which field derivatives play a role. Let us remedy that situation.

Advanced exercises in many-variable integrations

Let us again return to Gaussian-like integrals, this time some that resemble scalar field models, namely

$$
\begin{aligned}
\mathcal{I}_G &= \int e^{-A\Sigma(\phi_{k^*} - \phi_k)^2 a^{(n-2)} - B\Sigma \phi_k^2 a^n} \, \Pi d\phi_k \\
&= 2 \int e^{-A\kappa^2 \Sigma(\eta_{k^*} - \eta_k)^2 a^{(n-2)} - B\kappa^2 a^n} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma \eta_k^2) \, \Pi \, d\eta_k \, .
\end{aligned}
$$

The subsequent calculation depends on the number of space-time dimensions involved. Let us start simply and take $n = 1$, namely

$$\mathcal{I}_G = 2 \int e^{-A\kappa^2 \Sigma(\eta_{k+1} - \eta_k)^2 a^{-1} - B\kappa^2 a} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma \eta_k^2) \, \Pi \, d\eta_k \, .$$

With $A \approx B \approx 1$, it follows that $\kappa \approx \sqrt{N}$, $\eta_k^2 \approx 1/N$, and $(\eta_{k+1} - \eta_k)^2 \approx a/N$. This is the case of Brownian motion and it is interesting to observe that in the continuum limit the time derivative of η is almost everywhere finite. This behavior is unlike that for ϕ for which

$$< (\phi_{k+1} - \phi_k)^2 > = < \kappa^2(\eta_{k+1} - \eta_k)^2 > \approx N(a/N) = a \, .$$

It is only when the amplitude factor (κ) is included that the paths pass from almost everywhere differentiable to nowhere differentiable! Now let

us turn our attention to higher dimensions, in particular $n \geq 3$ (ignoring logarithmic behavior characteristic of $n = 2$). In that case—and assuming for simplicity that $Na^n \approx 1$—it follows that $\phi_k^2 \approx a^{-(n-2)}$, $\kappa \approx N^{(1-1/n)}$, $\eta_k^2 \approx N^{-1} \approx a^n$, and $(\eta_{k*} - \eta_k)^2 \approx a^n$ as well. It follows, for example that $<(\phi_{k*} - \phi_k)^2> \approx a^{-(n-2)}$, which shows the divergence of the field difference comes about solely from the radius factor (κ) and is not at all due to the variables (η_k).

If we add a quartic interaction term we are led to consider

$$\mathcal{I}_G'' = 2 \int e^{-A\kappa^2 \Sigma (\eta_{k*} - \eta_k)^2 a^{n-2} - B\kappa^2 a^n - C\kappa^4 \Sigma \eta_k^4 a^n} \kappa^{N-1} \, d\kappa \, \delta(1 - \Sigma \eta_k^2) \, \Pi d\eta_k \ .$$

In this case the relative order of magnitude of the terms in the principal part of the integrand is given by

$$e^{-A\kappa^2 \Sigma (\eta_{k*} - \eta_k)^2 a^{n-2} - B\kappa^2 a^n - C\kappa^4 \Sigma \eta_k^4 a^n} \approx e^{-AN - BN^{(1-2/n)} - CN^{(2-4/n)}} \ ,$$

which has the interesting corollary that the quartic interaction term becomes comparable with the kinetic term when $n = 4$ and actually dominates the kinetic term when $n \geq 5$. This is how (strictly) renormalizable and non-renormalizable models appear in this abbreviated language.

Now let us make a comparable study for Poisson-like integrals. Specifically, and with $p \geq 1$, we consider

$$\mathcal{I}_P(2p) = 2 \int (\Sigma h_k \eta_k a^n)^{2p} \, e^{-A\kappa^2 \Sigma (\eta_{k*} - \eta_k)^2 a^{n-2} - B\kappa^2 a^n} \kappa^{2p-1} \, d\kappa \, \delta(1 - \Sigma \eta_k^2) \, \Pi d\eta_k \ .$$

Our criterion of quality in the present case will relate to the comparable magnitude of the even moments. To study this issue we examine

$$\frac{\mathcal{I}_P(4)}{\mathcal{I}_P(2)} = \frac{\int (\Sigma h_k \eta_k a^n)^4 \, e^{-A\kappa^2 \Sigma (\eta_{k*} - \eta_k)^2 a^{n-2} - B\kappa^2 a^n} \kappa^3 \, d\kappa \, \delta(1 - \Sigma \eta_k^2) \, \Pi d\eta_k}{\int (\Sigma h_k \eta_k a^n)^2 \, e^{-A\kappa^2 \Sigma (\eta_{k*} - \eta_k)^2 a^{n-2} - B\kappa^2 a^n} \kappa \, d\kappa \, \delta(1 - \Sigma \eta_k^2) \, \Pi d\eta_k} \ .$$

To approximate this quotient, we use the estimates that $\eta_k^2 \approx (\eta_{k*} - \eta_k)^2 \approx 1/N$, which for $n \geq 3$ hold just as before. Thus, in a self-evident notation, the quotient is approximated by

$$\begin{aligned}
\frac{\mathcal{I}_P(4)}{\mathcal{I}_P(2)} &\approx 3a^n \frac{\int (\Sigma_{k,l} h_k^2 a^n h_l^2 a^n) < \eta_k^2 \eta_l^2 > \int e^{-A\kappa^2 a^{n-2} - B\kappa^2 a^n} \kappa^3 \, d\kappa}{\int (\Sigma_k h_k^2 a^n) < \eta_k^2 > \int e^{-A\kappa^2 a^{n-2} - B\kappa^2 a^n} \kappa \, d\kappa} \\
&\propto \frac{a^n}{N} \frac{\int e^{-A\kappa^2 a^{n-2} - B\kappa^2 a^n} \kappa^3 \, d\kappa}{\int e^{-A\kappa^2 a^{n-2} - B\kappa^2 a^n} \kappa \, d\kappa} \ .
\end{aligned}$$

For small a and large N, the prefactor a^n/N is tiny, and the only way to counter this tiny factor is to allow significant weight to large κ values by making A and B small. To obtain the desired result for the quotient, first imagine that $A = 0$. In that case $B \propto 1/N$ leads to a quotient that is independent of a and N. If instead $B = 0$ then we may choose $A \propto a^2/N$ which again leads to a quotient that is a- and N-independent. When both terms are present the same values prove satisfactory. Observe that no reweighting of field support to disjoint sets occurs if either parameter is rescaled by a finite amount. This fact is reflected in the statement that an expansion in powers of B or of A would not lead to divergences order by order. (If this property were to carry over to the type of integrals that arise for the models that are the main subject of this paper, then it may well happen that for such models a perturbation expansion in an interaction term does not lead to divergences order by order.) For completeness, for the very special examples of this section, we observe that if we had added another interaction term in the exponent, such as $C\kappa^4 \Sigma \eta_k^4 a^n$, then the correct choice in that case would be $C \propto a^n/N^2$. If instead the added term were $D\kappa^{44} \Sigma \eta_k^{44} a^n$, then the correct choice would be $D \propto a^{21n}/N^{22}$!

In summary, except for the term involving lattice derivatives, each of the factors is correctly given by the observation that κ enters in the form $\kappa\sqrt{(Ma)^n/N}$. For the kinetic term, besides $\kappa^2 a^{(n-2)}/N$, an additional factor of $(Ma)^2$ is needed as well. Insofar as an acceptable caricature of the lattice action suitable for "large" field values in the Poisson case goes, we conclude that we may choose the integrand to be

$$e^{-\frac{1}{2}((Ma)^2/N)\Sigma(\phi_{k*}-\phi_k)^2 a^{(n-2)} - \frac{1}{2}(m^2/N)\Sigma\phi_k^2 a^n - g((Ma)^n/N^2)\Sigma\phi_k^4 a^n} \quad ,$$

where m^2 and g are cutoff-independent parameters.

Further analysis of acceptable expressions for $w(\phi)$

We note that the examples just studied are not directly relevant for the main problems of interest because we have, for clarity, omitted from consideration in this simplified model the basic factor $\Pi'_k [\Sigma'_l \beta_{k-l} \eta_l^2]^{-\gamma}$. It is straightforward to see that this factor reweights the $\{\eta_k\}$ variables already constrained to lie on the unit sphere in N' dimensions in such a way that points where $\eta_k^2 = \delta_{kl}$, for each lattice site label l, are significantly enhanced. That is, the measure

$d\tau$ is heavily weighted on the unit sphere along each coordinate axis, and significantly reduced away from any coordinate axis. As a consequence, and in the absence of field gradients, we have [with $\langle(\cdot)\rangle \equiv \int(\cdot)d\tau(\eta)/\int d\tau(\eta)$]

$$
\begin{aligned}
\langle\eta_k^2\rangle &\equiv \frac{1}{N'} , \\
\langle\eta_k^2\eta_l^2\rangle &\approx \frac{\beta_{k-l}}{N'} + \text{l.o.t.} , \\
\langle\eta_k^2\eta_l^2\eta_m^2\rangle &\approx \frac{\beta_{k-l}\beta_{l-m}\beta_{k-m}}{N'} + \text{l.o.t.} ,
\end{aligned}
$$

etc., which crudely represent the fact that only very short range correlations exist due to the strong peaking along coordinate axes, and the only factors available to characterize that short range dependence are the β_k terms. The first equality is an identity, and the second one is scaled to satisfy the first as a sum rule. The third equality fails to satisfy a sum rule but is of the right order of magnitude. There are other terms not explicitly written and denoted by l.o.t. (lower order terms) and these involve potentially long range contributions but which are of a very small magnitude, so small in fact that they will make *no* contribution in the continuum limit. In the exercises treated above, for which the β-reweighting factor was omitted, the weight of the η variables was distributed uniformly over the unit sphere, and that gave rise to qualitatively different correlation functions than we are here discussing. The present correlation functions have the feature that as the points separate, i.e., $|k-l| \to L$, the correlations effectively vanish, while in the uniformly distributed case that was not the case. In the present case it is actually *essential* that the correlations vanish so that our ultimate field theory of interest can satisfy the cluster property (here in spatial directions only).

Now let us consider the effect of introducing field derivatives. The effect of field derivatives is to introduce additional, and longer-range correlations into the direction field. Thus, as a consequence, we may expect that the correlations discussed above are transformed into

$$
\begin{aligned}
\langle\eta_k^2\rangle &\equiv \frac{1}{N'} , \\
\langle\eta_k^2\eta_l^2\rangle &\approx \frac{e^{-\overline{M}a|k-l|}(\overline{M}a)^s}{N'} + \text{l.o.t.}
\end{aligned}
$$

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$$\langle \eta_k^2 \eta_l^2 \eta_m^2 \rangle \approx \frac{e^{-\overline{M}a(|k-l|+|l-m|+|k-m|)}(\overline{M}a)^{2s}}{N'} + \text{l.o.t.} \, ,$$

for some mass parameter $\overline{M}$.

We are now in a position to discuss the addition of field derivatives to the factor $w(\phi)$ in the second characteristic weight function. Based on the way that w enters the lattice potential, it is clear that our original identification of $w(\phi) = \Upsilon(\phi(Ma)^s)$ remains a valid conclusion when nonderivative terms are involved. The behavior of lattice derivatives for space-time dimensions $n \geq 3$ is such that the derivative of the direction field η diverges, even without the need for large κ values; this conclusion stems from the fact that $(\eta_{k^*} - \eta_k)^2 \approx \eta_k^2$ itself. Thus the classically motivated factor $a^{(n-2)}$ is incorrect on the lattice (and more particularly in the continuum limit) with respect to the factor a^{-2}. For dimensions $n \geq 3$, this factor must be renormalized away by an additional factor of a^2, just as was the case in the "advanced exercises". For the nonderivative terms we are instructed to replace ϕ by $\phi(Ma)^s$; to augment this recipe in the presence of derivatives we need only *drop* the classically motivated a^{-2} factor in forming the lattice derivative. Consequently, even in the presence of derivatives, the renormalization of ϕ to become $\phi(Ma)^s$ is the only modification required to obtain a model with cutoff-independent parameters! With this prescription accepted then the inclusion of terms with derivatives does not qualitatively change the discussion given earlier about how to choose w.

The route to these conclusions has not been straightforward because the road traveled is not a familiar one. Indeed, it is quite possible that we have been too cavalier in our analysis of the needed renormalization for terms involving the field gradient; the appropriate factors can no doubt be determined by a more careful study. However, for the sake of further discussion we shall adopt our present arguments.

Relaxation of sharp time requirement

Up to this point we have limited the parameter ξ so that $0 < \xi < 1$. By our consistent use of "cosine" in the second characteristics, it is not difficult to see that we could also allow $0 < \xi < 2$ with no real change. Beyond this value we run into serious difficulty because the lowest order even moment is no longer finite. Suppose, however, that we dealt not with the characteristic

function for the field but for a "renormalized" *cube* of the field. By this expression we mean to consider

$$\exp\{-J^2 \int [1 - \cos(\Sigma' g_k \phi_k^3 a^s)] \frac{e^{-2w(\phi)} \Pi' d\phi_k}{\Pi'[\Sigma' \beta_{k-l} \phi_l^2]^\gamma}\} \ .$$

In this relation it is clear that the parameter ξ that enters into γ in the usual way can be extended now so that $0 < \xi < 6$ and the indicated expression will still be well defined. The larger values of ξ do not support sharp-time fields, although they do support sharp-time fields cubed in the manner shown. Even larger values of ξ may be entertained by considering still higher-order renormalized field powers. In brief, therefore, the class of models can be extended beyond those that support sharp-time fields merely by extending the range of the parameter ξ.

However, for convenience, we shall continue to restrict our analysis to the case $0 < \xi < 1$ for which the smeared sharp-time field is an operator.

Continuum limit

We now have accumulated enough information in order to take a continuum limit for certain of the models in question. First we consider a "primordial" model in which $w(\phi) \equiv 0$, and we consider the second characteristic given in this case by the expression

$$C_S^o(g) \equiv 2J^{o\,2} \int [1 - \cos(\kappa \Sigma' \eta_k g_k a^s)] \kappa^{-1-\xi} \, d\kappa \, d\tau(\eta) \ .$$

To make further progress we need to investigate the nature of the support of the measure τ. Recall that earlier we have asserted that the weighting of the measure τ is concentrated along each coordinate axis on the unit sphere. Therefore, it is not difficult to argue that in the continuum limit, the measure $d\tau$ would be effectively the same as

$$d\tau(\eta) \simeq d\tau_0(\eta) \equiv F \, \Sigma'_l \, \delta(1 - \eta_l^2) \, d\eta_l \, \Pi''_{k \neq l} \, \delta(\eta_k) \, d\eta_k \ ,$$

where F denotes a suitable normalization. Use of this expression for the second characteristic leads to

$$C_S^o(g) \ = F J^{o\,2} \Sigma'_l \int [1 - \cos(\kappa g_l a^s)] \kappa^{-1-\xi} \, d\kappa$$

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$$= F J^{o\,2} \Sigma'_l (|g_l| a^s)^\xi \int [1 - \cos(v)] v^{-1-\xi}\, dv$$

$$= \frac{\Sigma'_l |g_l|^\xi a^s}{\Sigma'_l |g_l^{sp}|^\xi a^s} \; ,$$

which in the continuum limit becomes

$$\lim_{a \to 0} C_S^o(g) = K^o \int |g(\mathbf{x})|^\xi\, d^s x \; ,$$

where $(K^o)^{-1} \equiv \int |g^{sp}(\mathbf{x})|^\xi\, d^s x$.

Next, as a further example, we include one particular term in w and again take the continuum limit. In particular, we examine the case where

$$C_S(g) = 2 J^2 \int [1 - \cos(\kappa \Sigma' \eta_k g_k a^s)] e^{-(Ma)^{2s} \kappa^2} \, \kappa^{-1-\xi}\, d\kappa\, d\tau(\eta) \; .$$

The same discussion regarding the general behavior of the measure τ holds again, and we conclude that

$$C_S(g) = F J^2 \Sigma'_l \int [1 - \cos(\kappa g_l a^s)] e^{-(Ma)^{2s} \kappa^2} \, \kappa^{-1-\xi}\, d\kappa \; ,$$

which implies for the continuum limit that

$$\lim_{a \to 0} C_S(g) = K \int d^s x \int \{1 - \cos[\varrho g(\mathbf{x})]\} e^{-M^{2s} \varrho^2} \, \varrho^{-1-\xi}\, d\varrho \; ,$$

where

$$K^{-1} \equiv \int d^s x \int \{1 - \cos[\varrho g^{sp}(\mathbf{x})]\} e^{-M^{2s} \varrho^2} \, \varrho^{-1-\xi}\, d\varrho \; .$$

3   Lattice Regularized Models: Second Formulation

In the preceding section we have made an ansatz for the weight function of the second characteristic by choosing the form of $\psi_a(\phi)$. We also made considerable progress in characterizing acceptable functions $\psi_a(\phi)$, as preliminaries to finding the lattice ground state $\Psi_a(\phi)$ which are related by formulas presented in Section 1. Unfortunately, it is all but impossible to find analytic expressions for Ψ_a from the given form of ψ_a. Thus to proceed further we must find an alternative way to analytically express the lattice ground state

Ψ_a so that we can define the lattice Hamiltonian and the lattice action. We will do so by choosing a different starting point, but one which has the *same continuum limit* for the second characteristic as just discussed.

In our second formulation we do not start with the weight function for the *second characteristic* of the sharp-time characteristic function but rather with the weight function for the *characteristic function* itself. In other words, rather than initially assume a form for ψ_a we shall make an ansatz regarding Ψ_a directly. We are unable (and not necessarily interested) to determine whether or not the distribution of the sharp-time field is a generalized Poisson distribution on the lattice (as was the case previously), but since the continuum limit will agree for the two different starting points it follows that the continum limit in the second case is indeed described by a generalized Poisson distribution.

In the present case we postulate the form of the lattice ground state to be

$$\Psi_a(\phi) = \frac{e^{-W(\phi)}}{\Pi'_k[\phi_k^2 + \mathsf{C}(a)]^{(1+\xi)/4}} \, , \qquad \mathsf{C}(a) > 0 \, ,$$

where W includes the necessary constant to ensure that $\int |\Psi_a(\phi)|^2 \Pi'_k \, d\phi_k = 1$. This starting point may look fairly standard accept for the presence of the product factor in the denominator. Observe in the present case there is no appearance of the β parameters. Our goal now is to show that this choice leads to the same continuum limit in the two special cases discussed previously.

First we assume that $W(\phi) \equiv c$, c a constant, a case that corresponds to our "primordial" situation. In that case we consider the characteristic function given by

$$C^o(g) = \mathsf{N}^o \int e^{i\Sigma' g_k \phi_k a^s} \, \Pi'_k \frac{d\phi_k}{[\phi_k^2 + \mathsf{C}(a)]^{(1+\xi)/2}} \, ,$$

where N^o, which arises from c, ensures normalization. By a change of integration variables it follows that

$$
\begin{aligned}
C^o(g) \;\; &= \prod'_k K_C^o \int e^{i\varrho g_k} \frac{d\varrho}{[\varrho^2 + a^{2s}\mathsf{C}(a)]^{(1+\xi)/2}} \\
&= \prod'_k \{1 - K_C^o \int [1 - \cos(\varrho g_k)] \frac{d\varrho}{[\varrho^2 + a^{2s}\mathsf{C}(a)]^{(1+\xi)/2}} \} \, .
\end{aligned}
$$

Observe that
$$(K_C^o)^{-1} = \int [\varrho^2 + a^{2s}\mathsf{C}(a)]^{-(1+\xi)/2} \, d\varrho = a^{-s\xi}\mathsf{C}(a)^{-\xi/2} F \,,$$

where $F \equiv \int (u^2 + 1)^{-(1+\xi)/2} \, du$ is a fixed number depending only on $\xi > 0$. We next choose the a dependence of $\mathsf{C}(a)$ so that $K_C^o \equiv 2J^2 a^s$ for some constant J. This leads to
$$\mathsf{C}(a) = M^{n-2}(Ma)^{2s(1-\xi)/\xi} \,,$$

for some mass M. With this choice for C it follows that
$$C^o(g) = \prod_k{}' \{1 - 2J^{o\,2}a^s \int [1 - \cos(\varrho g_k)] \, \frac{d\varrho}{[\varrho^2 + a^{2s}\mathsf{C}(a)]^{(1+\xi)/2}}\} \,.$$

Although $\mathsf{C}(a)$ may, depending on the value of ξ, converge to zero, remain constant, or diverge to infinity as $a \to 0$, the combination $a^{2s}\mathsf{C}(a) \propto a^{2s/\xi}$ always goes to zero as $a \to 0$. Therefore, we finally conclude that
$$\lim_{a \to 0} C^o(g) \;= \exp\bigl(-2J^{o\,2} \int \{1 - \cos[\varrho g(\mathbf{x})]\}|\varrho|^{-1-\xi} \, d\varrho\bigr)$$
$$= \exp[-K^o \int |g(\mathbf{x})|^{\xi} \, d^s x] \,.$$

If we again assert that $C^o(g^{sp}) = \exp(-1)$, then it follows that *the continuum limit is exactly the same as was previously obtained.* Hence the ansatz for the second form of regularization has been verified for the choice $W = $ constant.

We next study the case $W = (Ma)^{2s}\Sigma'\phi_k^2 + c$, where c, which is different than before, is chosen to ensure normalization. The expression to be studied reads
$$C(g) \;= \mathsf{N} \int e^{i\Sigma'\phi_k g_k a^s} e^{-(Ma)^{2s}\Sigma'\phi_k^2} \, \Pi' \frac{d\phi_k}{[\phi_k^2 + \mathsf{C}(a)]^{(1+\xi)/2}}$$
$$= \prod_k{}' K_C \int e^{i\varrho g_k} e^{-M^{2s}\varrho^2} \, \frac{d\varrho}{[\varrho^2 + a^{2s}\mathsf{C}(a)]^{(1+\xi)/2}}$$
$$= \prod_k{}' \{1 - K_C \int [1 - \cos(\varrho g_k)]e^{-M^{2s}\varrho^2} \, \frac{d\varrho}{[\varrho^2 + a^{2s}\mathsf{C}(a)]^{(1+\xi)/2}}\} \,.$$

In the present case
$$(K_C)^{-1} \;= \int e^{-M^{2s}\varrho^2} \, \frac{d\varrho}{[\varrho^2 + a^{2s}\mathsf{C}(a)]^{(1+\xi)/2}}$$
$$= a^{-s\xi}\mathsf{C}(a)^{-\xi/2} \int e^{-M^{2s}[a^{2s}\mathsf{C}(a)]u^2} \, \frac{du}{(u^2 + 1)^{(1+\xi)/2}} \,,$$

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and in the continuum limit the exponent in the integrand of the last integral makes no contribution and thus asymptotically K_C is exactly what it was in the case when $W = $ constant! Hence we may determine that the continuum limit in the present case is given by

$$\lim_{a \to 0} C(g) = \exp\left(-2J^2 \int \{1 - \cos[\varrho g(\mathbf{x})]\} \, e^{-M^s \varrho^2} \, \frac{d\varrho}{|\varrho|^{1+\xi}}\right),$$

and once again the value of J is fixed by appealing to the special test sequence g^{sp}. As a consequence, the continuum limit in the present case cooincides with the second example studied at the end of Section 2. Observe additionally that this equality of the continuum limit has been acheived with *exactly the same renormalization of the additional term*—namely $\exp[-(Ma)^{2s}\Sigma'\phi_k^2]$—as was used in the previous section.

Thus, based on the identical form of the continuum limits and the appearance of an identical renormalization, we are persuaded in favor of adopting this second and alternative regularization based on the choice of the lattice ground state $\Psi_a(\phi)$ rather than choosing the weight function $\psi_a(\phi)$ that entered the second characteristic in the first approach. Although we have not studied any further choices of weight functions w and W in the two cases, we strongly believe that the second regularization captures the essence of what is contained in the first regularization. Hereafter, we choose the second form of regularization exclusively.

3.1 Determination of the Auxiliary Potential

Having agreed that the ansatz

$$\Psi_a(\phi) = \frac{e^{-W(\phi)}}{\Pi'_k[\phi_k^2 + \mathsf{C}(a)]^{(1+\xi)/4}}$$

provides an acceptable choice for a lattice ground state, we are in position to determine the form of the auxiliary potential P. To focus wholly on the auxiliary potential let us return to the "primordial" model for which $W(\phi) = c$, namely for a ground state given by

$$\Psi_a^o(\phi) = \frac{\mathsf{N}^{o\,1/2}}{\Pi'_k[\phi_k^2 + \mathsf{C}(a)]^{(1+\xi)/4}}.$$

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With this form of ground state the auxiliary potential P may be identified by way of the reduced Hamiltonian

$$
\begin{aligned}
\mathcal{H}_a^o & \equiv -\tfrac{1}{2}\hbar^2 Y(a)^{-1} a^{-s} [\Sigma' \frac{\partial^2}{\partial \phi_k^2} - \frac{1}{\Psi_a^o(\phi)} \Sigma' \frac{\partial^2 \Psi_a^o(\phi)}{\partial \phi_k^2}] \\
& \equiv -\tfrac{1}{2}\hbar^2 Y(a)^{-1} a^{-s} \Sigma' \frac{\partial^2}{\partial \phi_k^2} + \Sigma' P(\phi_k, a) a^s .
\end{aligned}
$$

The result is that

$$
\begin{aligned}
P(\phi_k, a) & \equiv \tfrac{1}{2}\hbar^2 Y(a)^{-1} a^{-2s} \frac{1}{\Psi_a^o(\phi)} \frac{\partial^2 \Psi_a^o(\phi)}{\partial \phi_k^2} \\
& = \tfrac{1}{2}\hbar^2 Y(a)^{-1} a^{-2s} \frac{4^{-1}(1+\xi)(3+\xi)\phi_k^2 - 2^{-1}(1+\xi)\mathsf{C}(a)}{[\phi_k^2 + \mathsf{C}(a)]^2} \\
& \equiv \frac{\mathsf{A}(a)\phi_k^2 - \mathsf{B}(a)}{[\phi_k^2 + \mathsf{C}(a)]^2} ,
\end{aligned}
$$

which is just the expression previously quoted in Section 1.2. Although we have normally assumed units such that $\hbar = 1$, in the preceding formula we have incorporated the correct $\hbar^2$ factor explicitly in order to establish—for this one time only—its proper role.

We note in passing that the ground state expectation of P diverges in the continuum limit when it formally becomes proportional to $\phi^{-2}(\mathbf{x})$. Therefore the ground state expectation of the kinetic energy term must also diverge so that the sum is zero in the present case. This compensation of divergences demonstrates that the auxiliary potential serves as a renormalization counterterm for the kinetic energy. This same interpretation remains true even with the addition of other potential terms.

3.2 Lattice Model Proposal

It is now a small step to propose a lattice model for the φ_n^4 theory based on the auxiliary potential just identified. Let us first recall the discussion related to the toy model in Section 1.1. There we learned that the model without moments (based on the Cauchy distribution) led to a one-dimensional potential analogous to the auxiliary potential in the lattice model as identified with the help of the "primordial" model ($W = $ constant). We also recall that

for the toy model involving the modified Bessel function the weight function for the characteristic function has, for very small ρ values, a functional form for small x identical to the Cauchy distribution. This fact suggests that we define the model by a lattice action which for small field values is equivalent to that of the auxiliary potential and for large field values has the functional form of the conventional action up to suitable bare parameters. In particular, we choose the generating functional for the lattice Euclidean-space φ_n^4 given by

$$
\begin{aligned}
S(h) \;\;=\;\; & N_0 \int \exp\big(\Sigma\, h_k \phi_k a^n - \tfrac{1}{2} Y(a) \Sigma (\phi_{k^*} - \phi_k)^2 a^{(n-2)} - \tfrac{1}{2} m_0^2(a) \Sigma \phi_k^2 a^n \\
& -g_0(a) \Sigma \phi_k^4 a^n - \Sigma\, \{[\mathsf{A}(a)\phi_k^2 - \mathsf{B}(a)]/[\phi_k^2 + \mathsf{C}(a)]^2\} a^n \big)\, \Pi d\phi_k \,,
\end{aligned}
$$

where A, B, and C are given above. Observe that we have not chosen to define the model by picking W for the simple reason that we do not have any analytic proposal for that quantity; rather we have returned to the lattice action which preserves its simple analytic form even in the presense of the auxiliary potential. Observe also that for small Ma the coefficients of the conventional action terms (Y, m_0^2, and g_0) are all small and this fact makes for a greater isolation of the large and small field behavior. Analogously to the case of small ρ in the second toy model, this isolation of the two regimes gives even greater credence to our choice of the auxiliary potential determined from the "primordial" model as the form adopted when the conventional terms are present.

It may be natural for the reader to ask why we have not started with the second formulation of Section 3 and dispensed with the first formulation of the model in Section 2 all together. The reason we have included the first formulation is because experience has shown that it is much easier to motivate unusual forms of the weight function in the second characteristic than in the characteristic function itself. Only after gaining familiarity with the strange denominators in the weight function of the second characteristic—*factors which are mandated in order to satisfy the physically dictated moment requirements for the weight function of the second characteristic*—and seen the consequences of these denominator terms in the continuum limit, only then do the unusual denominators of the weight function for the characteristic function of the sharp-time field itself become reasonable and therefore acceptable.

A remark on O($\overline{N}$) symmetric models

It is interesting to observe that if one deals with an $\overline{N}$-component scalar field with an O($\overline{N}$) symmetry, then the principal change involved is the replacement of all factors ϕ_k^2 by the O($\overline{N}$)-symmetric term $\Sigma_\alpha(\phi_k^\alpha)^2$, $1 \le \alpha \le \overline{N}$, etc., along with a corresponding change of γ or equivalently ξ. How this additional symmetry fits into the present scheme may be partially traced from previous work [12].

Acknowledgments

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