

Black Hole Condensation and the Web of Calabi-Yau Manifolds [†]

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Abstract

We review recent work concerning topology changing phase transitions through black hole condensation in Type II string theory. We then also briefly describe a present study aimed at extending the known web of interconnections between Calabi-Yau manifolds. We show, for instance, that all 7555 Calabi-Yau hypersurfaces in weighted projective four space are mathematically connected by extremal transitions.

I. Introduction

The distinction between classical and quantum geometry becomes most apparent when some subspace of a classical background ‘shrinks down to small size’. The precise meaning of the latter depends on details of the situation being studied, but for example in Calabi-Yau string compactifications the work of [1] [2] showed markedly new quantum geometric behaviour when certain rational curves (two-spheres) on a Calabi-Yau shrink to zero area. Whereas classical geometry changes discontinuously through such flop transitions, conformal field theory changes in a perfectly smooth manner. In this way, the above mentioned papers established that certain topology changing transitions are physically sensible in string theory. We recall that in these transitions, the Hodge numbers of the Calabi-Yau space do not change; rather, more subtle topological invariants such as the intersection form change.

One way of summarizing the above work is to say that some points in the moduli space of a Calabi-Yau manifold which correspond to singular geometric configurations actually correspond to non-singular conformal field theories. Now, these points — corresponding to Calabi-Yau manifolds in which some rational curves have degenerated — are not the only singular geometric configurations into which a Calabi-Yau manifold can degenerate. A natural question, then, is whether some (or all) of the other kinds of degenerations are, in fact, physically sensible. We certainly hope that all such degenerations which are at finite distance in the moduli

[†] Based in part on talk delivered by B.R.G at the Trieste Duality conference, (June 1995).

are physically sensible as there is no apparent mechanism for avoiding these theories in, for example, a cosmological setting.

For a degeneration of a Calabi-Yau manifold to a so-called ‘conifold’ — something we describe in greater detail below but for now can simply be thought of as a degeneration in which some S^3 ’s shrink to zero size — it was shown in [3] that type II string theory is physically sensible and it was shown in [4] that certain special kinds of conifold degenerations lead to remarkable physical consequences. Namely, as in the case of the flop transitions, these conifold degenerations provide the means for topology changing transitions. However, in this case, even the Hodge numbers of the Calabi-Yau jump — in a physically sensible manner. A key difference between these transitions and the previous flop transitions is that the latter can be understood perturbatively in string theory while the former can not.

These conifold transitions thereby take us one step closer to ameliorating the long standing vacuum degeneracy problem in string theory. Rather than each Calabi-Yau giving us an isolated distinct vacuum, it appears that many and possibly all of them are linked together in a single ‘universal’ moduli space.

In section II we will briefly recall the structure of Calabi-Yau moduli spaces found in [1] and [2] with an emphasis on the geometrically singular points. We then describe, in section III, Strominger’s proposal for making sense of the simplest kind of conifold degeneration, along the lines of Seiberg and Witten [5]. In section IV we discuss the extension of these results to the case in which the degenerating S^3 ’s are subject to nontrivial homology relations and show that this yields the topology changing phenomenon of *conifold transitions* [4]. In section V we briefly discuss a study presently underway to extend the known Calabi-Yau manifolds which are connected to the web through *extremal* transitions.

II. $N = 2$ Moduli Space

Here we briefly recount the phase description of $N = 2$ moduli space found in [1] [2]. The reader familiar with these ideas might want to go directly to section III.

For an $N = 2$ superconformal theory realized as a nonlinear sigma model on a Calabi-Yau target space, the data necessary to specify the model is a choice of complex structure and complexified Kähler class on the manifold. The moduli space of such theories, therefore, is given in terms of the moduli space of complex structures and complexified Kähler classes. We discuss each of these in turn. Of these two, the simplest to describe is the former, especially in the familiar context of a Calabi-Yau manifold realized as a complete intersection in a toric variety (such as a product of (weighted) projective spaces). For ease of discussion, we consider a hypersurface in some weighted projective four space, although the notions we

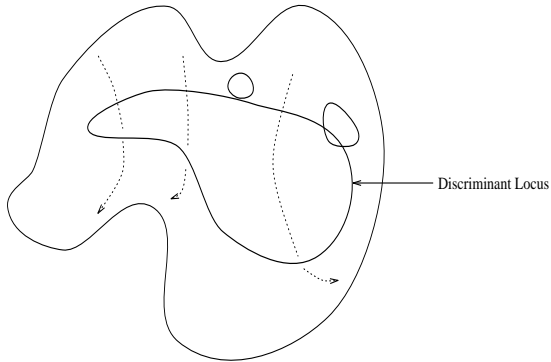


Figure 1. Complex structure moduli space showing discriminant locus.

present are well known to be far more general.

a) Complex Structure Moduli Space

The typical form of the Calabi-Yau spaces we shall consider is given by the vanishing locus of homogeneous polynomial equations in (products of) projective spaces. For ease of discussion, consider the case of a single equation $P = 0$ with

$$P = \sum a_{i_1 \dots i_n} z_1^{i_1} \dots z_n^{i_n}. \quad (1)$$

It is known that by varying the coefficients in such equations (modulo coordinate redefinitions) we vary the choice of complex structure on the the Calabi-Yau space. There is one constraint on the choice of coefficients a which we must satisfy in order to have a classically smooth Calabi-Yau manifold: they must be chosen so that P and its partial derivatives do not have a common zero in the defining projective space. If they did have such a common zero, the choice of complex structure would be singular (non-transverse). More concretely, for such choices of complex structure certain topological S^3 's on the Calabi-Yau are degenerated down to zero size — that is, their periods with respect to the holomorphic three form vanish. It is straightforward to see that having a common zero places one complex constraint on the choice of coefficients and hence we can think of the space of smooth complex structures as the space of all a 's (modulo those which are equivalent via coordinate transformations on the z 's) less this one complex constraint. Schematically, this space can be illustrated as in figure 1 where the “bad” choice of a 's is correctly denoted as the *discriminant locus*. Although geometrically singular, a natural question to ask is whether Calabi-Yau's corresponding to points on the discriminant locus are

physically singular. We will come back to this question shortly.

b) The Classical Kähler Moduli Space

Classically, the Kähler form on a Calabi-Yau space is a closed two form J related to the metric g via

$$J = ig_{i\bar{j}}dX^i \wedge dX^{\bar{j}}. \tag{2}$$

As such, J may be thought of as an element of the vector space of all closed two forms (modulo exact forms) $H^2(M, \mathbb{R})$. In fact, J lies in a special subspace of this vector space known as the Kähler cone by virtue of its relation to the metric. In particular, since the metric measures non-negative lengths, areas and volumes, J satisfies

$$\int_M J \wedge J \wedge J > 0 \tag{3}$$

$$\int_S J \wedge J > 0 \tag{4}$$

$$\int_C J > 0. \tag{5}$$

where S and C are nontrivial 4 and 2-cycles on the manifold respectively.

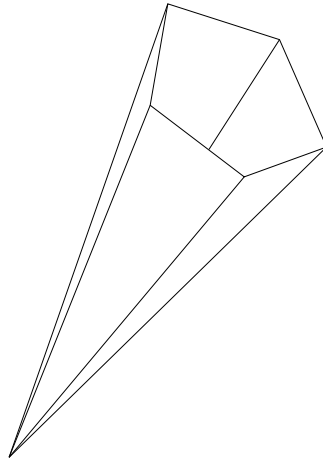


Figure 2a. Schematic diagram of Kähler cone.

The space of all J in $H^2(M, \mathbb{R})$ that satisfy these requirements has a cone structure because if J satisfies these conditions, so does the positive ray generated by J – hence the name Kähler cone. In figure 2a we schematically show a Kähler cone. A well known aspect of string theory is that it instructs us to combine the

Kähler form J with the antisymmetric tensor field B into the complexified Kähler class $K = B + iJ$. The physical model is invariant under integral shifts of B (more precisely, shifts of B by elements of $H^2(M, \mathbb{Z})$) which motivates changing variables to

$$w_l = e^{2\pi i(B_l + iJ_l)} \tag{6}$$

where (B_l, J_l) are coefficients in the expansion of B and J with respect to an integral basis of $H^2(M, \mathbb{Z})$. These new variables have the invariance under integral shifts built in.

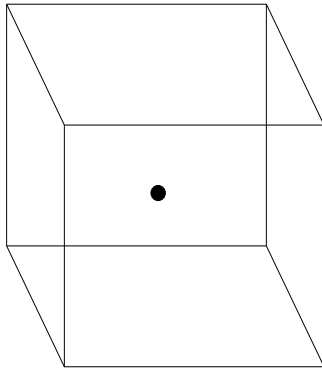


Figure 2b. Complexified Kahler cone.

The imaginary part of K satisfies the conditions on J just discussed and hence the Kähler cone of figure 2a becomes the bounded domain of $H^2(M, \mathbb{C})$ in the w variables as depicted in figure 2b. We note that the boundary of this region denotes those places in the parameter space where the Kähler form J degenerates in the sense that some of the positivity requirements are violated.

c) The Stringy Kähler Moduli Space

The above description of the respective parameter spaces led to a puzzling issue for mirror symmetry: mirror symmetry tells us that figure 1 and figure 2b are isomorphic if the former is for M and the latter for its mirror W . However, manifestly they are not. This is not a product of our schematic drawings as there are genuine qualitative distinctions. Most prominently, note that the locus of geometrically singular Calabi-Yau spaces is real codimension one in the Kähler parameter space, occurring on the walls of the domain where the classical Kähler form degenerates. On the contrary, the locus of geometrically singular Calabi-Yau's in the complex structure moduli space, as just discussed, is real codimension two (complex codi-

mension one). What is going on? The answer to this question was found in [1] and [2] and implies that:

- 1: Figure 2b for W is only a *subset* of figure 1 for M . To be isomorphic to figure 1 of M , it must be augmented by numerous other regions, of a similar structure, all adjoined along common walls. This yields the *enlarged Kähler moduli space* of W .

- 2: Some of these additional regions are interpretable as the complexified Kähler moduli space of *flops* of W along rational curves. In essence, to flop a rational curve (topologically an S^2) we shrink it down to a point (by varying the Kähler structure) and then subsequently give it positive volume but with respect to a different topology. This new topology is such that it agrees with the old in terms of, for instance, the Hodge numbers, but differs with respect to the topological intersection form.

- 3: Other regions may not have a direct sigma model interpretation, but rather are the parameter spaces for Landau-Ginzburg theory, Calabi-Yau orbifolds, and various relatively unfamiliar hybrid combination conformal theories.

- 4: Whereas classical reasoning suggests that theories whose complexified Kähler class lies on the wall of a domain such as that in figure 2b are ill defined, quantum conformal field theory reasoning shows that the generic point on such a wall corresponds to a perfectly well behaved conformal theory. Thus, the conformal field theory changes *smoothly* if the parameters defining it change in a generic manner from one region to another by crossing through such a wall. As some such regions correspond to sigma models on topologically distinct target spaces, this last point established the first concrete example of physically allowed spacetime topology change.

A particularly useful way of summarizing this is as follows: classical reasoning suggests that our physical models will be badly behaved if the complex structure is chosen to lie on the discriminant locus or if the Kähler class is chosen to lie on a wall of the classical Kähler moduli space. The fully quantum corrected conformal field theory corresponding to such points (yielding genus zero string theory), though, proves to be generically *non-singular* on walls in the Kähler moduli space. The pronounced distinction between the classical and stringy conclusions arises because such points are strongly coupled theories (as the coupling parameter α'/R^2 gets big as we shrink down R — the radius of an S^2). Analyzing such strongly coupled theories directly is hard; however, by mirror symmetry we know they are equivalent to weakly coupled field theories on the mirror Calabi-Yau space where we can directly show them to be well behaved.

So much for the generic point on a wall in the Kähler parameter space: classically they look singular but in fact they are well defined. What about choosing the complex structure to lie on the discriminant locus (which by mirror symmetry corresponds to a non-generic point on a wall in the Kähler parameter space of the mirror)? Might it be that these theories are well behaved too? At first sight the answer seems to be “no”. By taking the Kähler class to be deep inside a smooth phase (i.e. a smooth large radius Calabi-Yau background) we trust perturbation theory and can directly compute conformal field theory correlation functions. Some of them diverge as we approach the discriminant locus. This establishes that the conformal field theory is badly behaved. It is, however, important to distinguish between conformal field theory and string theory. Conformal field theory is best thought of as the effective description of string degrees of freedom which are light in the $\lambda \rightarrow 0$ limit, with λ being the string coupling constant. This includes all of the familiar perturbative string states, but effectively integrates out nonperturbative states whose most direct description is in terms of solitons in the low energy effective string action¹.

We are thus faced with the moduli space for an effective string description that contains points where physics appears to be singular. A close analog of this situation plays a central role in the celebrated work of Seiberg and Witten [5] where it is argued that the apparent singularity is due to the appearance of new massless nonperturbative degrees of freedom at those singular moduli space points. A natural guess in the present setting, then, is that the apparent singularity encountered on the discriminant locus is due to previously massive nonperturbative string states becoming massless. This solution was proposed by Strominger and we review its success in the next section.

III. Strominger’s Resolution of the Conifold Singularity

To quantitatively understand the proposed resolution of conifold singularities, we must introduce coordinates on the complex structure moduli space. As is familiar, we introduce a symplectic homology basis of $H^3(M, \mathbb{Z})$ denoted (A_I, B^J) where $I, J = 0, \dots, h^{2,1}(M)$ and by definition $A_I \cap B^J = \delta_I^J$, $A_I \cap A_J = B^I \cap B^J = 0$. We let $z^J = \int_{B^J} \Omega$ and $G_I = \int_{A_I} \Omega$, where Ω is the holomorphically varying three-form on the family of Calabi-Yau’s being studied. It is well known that the z^J provide a good set of local projective coordinates on the moduli space of complex structures and that the G_I can be expressed as functions of the z^J .

In terms of these coordinates, a conifold point in the moduli space can roughly

¹Recently, a description of these states in terms of Dirichlet-branes has been proposed [6].

be thought of as a point where some z^J vanishes (we will be more precise on this in the next section). The corresponding B^J is called a vanishing cycle as the period of Ω over it goes to zero. For our purposes, there is one main implication of the vanishing of, say, z^J , that we should discuss: the metric on the moduli space is singular at such a point. The easiest way to see this is recall that special geometry governs these moduli spaces and therefore the Kähler potential on the moduli space can be written

$$K = -\ln(i\bar{z}^I G_I - iz^I \bar{G}_I) \quad (7)$$

If we knew the explicit form for $G_J(z)$ we would thus be able to calculate the local form of the metric near the conifold point. Considerations of monodromy are sufficient to do this: as we will discuss in greater generality below, if we follow a path in the moduli space that encircles $z^J = 0$, the period G_J is not single valued but rather undergoes a nontrivial monodromy transformation

$$G_J \rightarrow G_J + z^J. \quad (8)$$

Near $z^J = 0$ we can therefore write

$$G_J(z^J) = \frac{1}{2\pi i} z^J \ln(z^J) + \text{single valued}. \quad (9)$$

Using this form one can directly compute that the metric $g_{J\bar{J}}$ has a curvature singularity at $z^J = 0$.

The reason that the singularity of the metric on the moduli space is an important fact is due to its appearing in the Lagrangian for the four-dimensional effective description of the moduli for a string model built on such a Calabi-Yau. Namely, the nonlinear sigma model Lagrangian for the complex structure moduli ϕ^K is of the form $\int d^4x g_{I\bar{J}} \partial\phi^I \partial\phi^{\bar{J}}$. Hence, when the metric on the moduli space degenerates, so apparently does our physical description.

This circumstance — a moduli space of theories containing points at which physical singularities appear to develop — is one that has been discussed extensively in recent work of Seiberg and Witten [5]. The natural explanation advanced for the physical origin of the singularities encountered is that states which are massive at generic points in the moduli space become massless at the singular points. As the Lagrangian description is that of an effective field theory in which massive degrees of freedom have been integrated out, if a previously massive degree of freedom becomes massless then we will be incorrectly integrating out a massless mode and hence expect a singularity to develop. In the case studied in [5], the states that became massless were BPS saturated magnetic monopoles or dyons. Strominger proposed that in compactified type IIB string theory there are analogous electrically or magnetically charged black hole states that become massless at conifold points.

The easiest way to understand these states is to recall that in ten-dimensional type IIB string theory there are $3 + 1$ dimensional extremally charged extended soliton solutions with a horizon: so called black three-branes [7]. These solitons carry Ramond-Ramond charge that can be detected by integrating the five-form field strength over a surrounding Gaussian five cycle Σ_5 : $Q_{\Sigma_5} = \int_{\Sigma_5} F^{(5)}$. Now, our real interest is in how this soliton appears after compactification to four dimensions via a Calabi-Yau three fold. Upon such compactification, the three spatial dimensions of the black soliton can wrap around nontrivial three-cycles on the Calabi-Yau and hence appear to a four dimensional observer as black holes states. More precisely, they yield an $N = 2$ hypermultiplet of states. The effective electric and magnetic charges of the black hole state are then obtained by integrating $F^{(5)}$ over $A_I \times S^2$ and $B^J \times S^2$. Explicitly, making the natural assumption of charge quantization, we can write

$$\int_{A_I \times S^2} F^{(5)} = g_5 n_I \qquad \int_{B^J \times S^2} F^{(5)} = g_5 m_J \qquad (10)$$

where g_5 is five form coupling and n_I and m_J are integers. Of prime importance is the fact that these are BPS saturated states and hence are subject to the mass relation [8]

$$M = g_5 e^{K/2} |m^I G_I - n_I z^I|. \qquad (11)$$

Lets consider the case in which $n_I = \delta_{IJ}$ and $m^I = 0$ for all I with J fixed. In the conifold limit for which z^J goes to zero we see that the mass of the corresponding electrically charged black hole vanishes. Hence, it is no longer consistent to exclude such states from direct representation in the Wilsonian effective field theory action describing the low energy string dynamics.

The claim is that the singularity encountered above is due precisely to such exclusion. Curing the singularity should therefore be achieved by a simple procedure: include the black hole hypermultiplet in the low energy effective action. There is a simple check to test the validity of this claim. Namely, if we incorrectly integrate out the black hole hypermultiplet from the Wilsonian action, we should recover the singularity discussed above. This is not hard to do. By the structure of $N = 2$ supersymmetry, the effective Lagrangian is governed by a geometrical framework which is identical to that governing Calabi-Yau moduli space. Namely, we can introduce holomorphic projective coordinates on the moduli space of the physical model z^J , and the model is determined by knowledge of holomorphic functions $G_I(z)$. In particular, the coupling constant for the J^{th} $U(1)$ is given by $\tau_{JJ} = \partial_J G_J$. We can turn the latter statement around by noting that knowledge of the coupling constant effectively allows us to determine G_J . We can determine the behaviour of the coupling by a simple one-loop Feynman diagram, which again by $N = 2$ is all we need consider. Integrating out a black hole hypermultiplet in a neighborhood of the

$z^J = 0$ conifold point yields the standard logarithmic contribution to the running coupling τ_{JJ} and hence we can write

$$\tau_{JJ} = \frac{1}{2\pi i} \ln z^J + \text{single valued.} \quad (12)$$

From this we determine, by integration, that

$$G_J = \frac{1}{2\pi i} z^J \ln z^J + \text{single valued.} \quad (13)$$

We note that this is precisely the same form as we found for G_J earlier via monodromy considerations. This, in fact, justifies our having referred to them by the same symbol. Now, by special geometry, everything about the mathematics and physics of the system follows from knowledge of the G_J . For our purposes, therefore, the singularity encountered previously (by determining the metric on the moduli space from the G_J) has been precisely reproduced by incorrectly integrating out the massless soliton states. This justifies the claim, therefore, that we have identified the physical origin of the singularity and also that by including the black hole field in the Wilsonian action (and therefore not making the mistake of integrating them out when they are light) we cure the singularity.

IV. Conifold Transitions and Topology Change

In the previous section we have seen how the singularity that arises when an S^3 shrinks to a point is associated with the appearance of new massless states in the physical spectrum. By including these new massless states in the physical model, the previous singularity is cured. In this section we consider a simple generalization of this discussion which leads to dramatic new physical consequences [4]. Concretely, we consider a less generic degeneration in which:

- More than one, say P , three-cycles degenerate.
- These P three-cycles are not homologically independent but rather satisfy R homology relations.

As we will now discuss, this generalization implies that:

- The bosonic potential for the scalar fields in the hypermultiplets that become massless at the degeneration has R flat directions.
- Moving along such flat directions takes us to another branch of type II string moduli space corresponding to string propagation on a topologically distinct Calabi-Yau manifold. If the original Calabi-Yau has Hodge numbers $h^{1,1}$ and $h^{2,1}$ then the new Calabi-Yau has Hodge numbers $h^{1,1} + R$ and $h^{2,1} - P + R$.

In order to understand this result, there are a couple of useful pieces of background information we should review. First, let's discuss a bit more precisely the mathematical singularities we are considering [9]. As we have discussed, the discriminant locus denotes those points in the complex structure moduli space of a Calabi-Yau where the space fails to be a complex manifold. We focus on cases in which the degenerations occur at some number of isolated points on the Calabi-Yau. In particular, we consider singularities that are known as “ordinary double points”. These are singular points which can locally be expressed in the form

$$\sum_{i=1}^4 w_i^2 = 0 \tag{14}$$

in \mathbb{C}^4 . This local representation is a cone with singular point at the apex, namely the origin. To identify the base of the cone we intersect it with a seven sphere in \mathbb{R}^8 , $\sum_{i=1}^4 |w_i|^2 = r^2$. Introducing the complex vector $\vec{w} = \vec{x} + i\vec{y} = (w_1, w_2, w_3, w_4)$ the equation of the intersection can be expressed as $\vec{x} \cdot \vec{x} = r^2/2$, $\vec{y} \cdot \vec{y} = r^2/2$ and $\vec{x} \cdot \vec{y} = 0$. The first of these is an S^3 , the latter two equations give an S^2 fibered over the S^3 . As there are no nontrivial such fibrations, the base of the cone is $S^2 \times S^3$. Calabi-Yau's which have such isolated ordinary double point singularities are known as conifolds and the corresponding point in the moduli space of the Calabi-Yau is known as a conifold point. The ordinary double point singularity is also referred to as a node.

Having described the singularity in this way we immediately discern two distinct ways of resolving it: either we can replace the apex of the cone with an S^3 , known as a deformation of the singularity, or we can replace the apex with an S^2 , known as a small resolution of the singularity. The deformation simply undoes the degeneration by re-inflating the shrunk S^3 to positive size. The small resolution, on the other hand, has a more pronounced effect: it repairs the singularity in a manner that changes the topology of the original Calabi-Yau. In essence, we shall find the *physical* interpretation of these two ways of resolving conifold singularities.

A second piece of background information is a mathematical fact concerning monodromy. Namely, if γ^a for $a = 1, \dots, k$ are k vanishing three-cycles at a conifold point in the moduli space, then another three-cycle δ undergoes monodromy

$$\delta \rightarrow \delta + \sum_{a=1}^k (\delta \cap \gamma^a) \gamma^a \tag{15}$$

upon transport around this point in the moduli space.

With this background, we can now proceed to discuss the result quoted at the beginning of this section. We will do so in the context of a particularly instructive

example, although it will be clear that the results are general. We begin with the quintic hypersurface in \mathbb{P}^4 , which is well known to have Hodge numbers $h^{2,1} = 101$ and $h^{1,1} = 1$. We then move to a conifold point by deforming the complex structure to the equation

$$x_1 g(x) + x_2 h(x) = 0 \quad (16)$$

where x denotes the five homogeneous \mathbb{P}^4 coordinates (x_1, \dots, x_5) and g and h are both generic quartics. We note that (16) and its derivative vanish at the sixteen points

$$x_1 = x_2 = g(x) = h(x) = 0. \quad (17)$$

It is straightforward to check, by examining the second derivative matrix, that these are sixteen ordinary double points. And, of primary importance to our present discussion, the sixteen singular points lie on the \mathbb{P}^2 contained in \mathbb{P}^4 given by $x_1 = x_2 = 0$. This implies that the sixteen vanishing cycles $\gamma^a, a = 1, \dots, 16$ that degenerate to the double points satisfy the nontrivial homology relation

$$\sum \gamma^a = 0 \quad (18)$$

[10]. We are thus in the desired situation. We proceed with the analysis in two steps. First, we check that inclusion of the appropriate massless hypermultiplets cures the singularity, as it did in the simpler case studied in [3]. Second, we then analyse the physical implication of the existence of a nontrivial homology relation.

i) *Singularity resolution:*

We introduce a symplectic homology basis (A_I, B^J) with $I, J = 1, \dots, 204$. By suitable change of basis we can take our sixteen vanishing cycles γ^a ($a = 1, \dots, 16$) to be B^1, \dots, B^{15} and $-\sum_{a=1}^{15} B^a$. As usual, we define $z^I = \int_{B^I} \Omega$ and $G_J = \int_{A_J} \Omega$. Now, for any cycle δ we have, as discussed before, the monodromy $\delta \rightarrow \delta + \sum_{a=1}^{16} (\delta \cap \gamma^a) \gamma^a$. From this we learn that the local form of the period over δ is given by

$$\int_{\delta} \Omega = \frac{1}{2\pi i} \sum_{a=1}^{16} (\delta \cap \gamma^a) \left(\int_{\gamma^a} \Omega \right) (\ln \int_{\gamma^a} \Omega) + \text{single valued}. \quad (19)$$

Specializing this general expression, we therefore see

$$G_J = \frac{1}{2\pi i} (z^J \ln(z^J) + \left(\sum_{I=1}^{15} z^I \right) (\ln(\sum_{I=1}^{15} z^I))) + \text{single valued}. \quad (20)$$

By special geometry, this latter expression determines the properties of the singularity associated with the conifold degeneration being studied. Thus, the question

we now seek to answer is: if we incorrectly integrate out the black hole states which become massless at this conifold point, do we reproduce the form (20)?

To address this issue we must identify the precise number and charges of the states that are becoming massless at the degeneration point. As discussed in [3], the counting of black hole states is a delicate issue for which there is as yet no rigorous algorithm. In [3], one homology class in H^3 degenerated at the conifold singularity and it was hypothesized that this implies one fundamental black hole state — the one of minimal charge — needs to be included in the Wilsonian action. In the present example, though, we have sixteen three cycles in fifteen homology classes in H^3 degenerating. In [4] it was argued that this should imply sixteen fundamental black hole fields need to be included in the Wilsonian action. Physically speaking, the black three-brane can wrap around any of the sixteen degenerating three-cycles, which at large overall radius of the Calabi-Yau would be widely separated. It thus seems sensible that even though there are only fifteen homology classes degenerating, we actually get sixteen massless black hole states. The charges of these states are easy to derive. If we let H^a be the black hole hypermultiplet associated with the vanishing cycle γ^a then the charge of H^a under the I^{th} $U(1)$ is given by $Q_I^a = A_I \cap \gamma^a$ where we write $F^{(5)}$ as the self dual part of $\sum_I \alpha^I F_I^{(2)}$ with α^I dual to A_I . We immediately learn from this that the black holes states have charges

$$Q_I^a = \delta_I^a, 1 \leq a \leq 15 \quad \text{and} \quad Q_I^{16} = -1, 1 \leq I \leq 15 \quad (21)$$

with all other charges zero. This is enough data to determine the running of the gauge couplings:

$$\tau_{IJ} = \frac{1}{2\pi i} \sum_{a=1}^{16} Q_I^a Q_J^a \ln(m^a) \quad (22)$$

where the mass m^a of H^a is proportional to $\sum_I Q_I^a z^I$. Using the above charges we therefore have

$$\tau_{IJ} = \frac{1}{2\pi i} \delta_{IJ} \ln(z^J) + \frac{1}{2\pi i} \ln\left(\sum_{k=1}^{15} z^k\right) + \text{single valued}. \quad (23)$$

Integrating we find therefore

$$G_J = \frac{1}{2\pi i} z^J \ln z^J + \frac{1}{2\pi i} \left(\sum_{k=1}^{15} z^k\right) \left(\ln\left(\sum_{k=1}^{15} z^k\right)\right) + \text{single valued}. \quad (24)$$

We note that this matches (20) and hence we have shown that inclusion of the sixteen black hole soliton states which become massless cures the singularity.

Having shown that a slight variant on Strominger’s original proposal is able to cure the singularity found in this more complicated situation, we now come to the main point of the discussion:

ii) *What is the physical significance of nontrivial homology relations between vanishing cycles?*

To address this question we consider the scalar potential governing the black hole hypermultiplets. It can be written as

$$V = \sum E_{\alpha\beta}^I E_I^{\alpha\beta} \quad (25)$$

where

$$E_{\alpha\beta}^I = \sum_{a=1}^{16} Q_a^I \epsilon_{\alpha\gamma} h_\gamma^{*(a)} h_\beta^{(a)} - (\alpha \rightarrow \beta) \quad (26)$$

in which the indices satisfy $I = 1, \dots, 15$; $\alpha, \beta, \gamma = 1, 2$. The fields $h_1^{(a)}$ and $h_2^{(a)}$ are the two complex scalar fields in the hypermultiplet H^a .

We consider the possible flat directions which this potential admits. The most obvious flat directions are those for which $\langle h_\beta^{(a)} \rangle = 0$ with nonzero values for the scalar fields in the vector multiplets. Physically, moving along such flat directions takes us back to the Coulomb phase in which the black hole states are massive. Mathematically, moving along such flat directions gives positive volume back to the degenerated S^3 ’s and hence resolves the singularity by deformation.

The nontrivial homology relation implies that there is another flat direction. Since $Q_a^I = A_I \cap \gamma^a$, we see that the homology relation $\sum_{a=1}^{16} \gamma^a = 0$ implies $\sum_a Q_a^I = 0$ for all I . This then implies that we have another flat direction of the form $\langle h_\beta^{(a)} \rangle = v^\beta$ for all a with v constant. In fact, simply counting degrees of freedom shows that this solution is unique up to gauge equivalence. What happens if we move along this flat direction? It is straightforward to see that this takes us to a Higgs branch in which fifteen vector multiplets pair up with fifteen hypermultiplets to become massive. This leaves over one massless hypermultiplet from the original sixteen that become massless at the conifold point. We see therefore that the spectrum of the theory goes from 101 vector multiplets and 1 hypermultiplet (ignoring the dilaton and graviphoton) to $101 - 15 = 86$ vector multiplets and $1 + 1 = 2$ hypermultiplets. Now precisely these Hodge numbers arise from performing the other means of resolving the conifold singularity (besides the deformation) — the small resolution described earlier! Hence, we appear to have found the physical mechanism for affecting a small resolution and in this manner changing the topology of the Calabi-Yau background.

Although we have focused on a specific example, it is straightforward to work out what happens in the more general setting of P isolated vanishing cycles satisfying R homology relations. Following our discussion above, we get P black hole hypermultiplets becoming massless with R flat directions in their scalar potential. Performing a generic deformation along these flat directions causes $P - R$ vectors to pair up with the same number of hypermultiplets. Hence the Hodge numbers change according to

$$(h^{21}, h^{11}) \rightarrow (h^{21} - (P - R), h^{11} + R). \quad (27)$$

The Euler characteristic of the variety thus jumps by $2P$.

So, in answer to the question posed above: homology relations amongst the vanishing cycles give rise to new flat directions in the scalar black hole potential. Moving along such flat directions takes us smoothly to new branches of the type II string theory moduli space. These other branches correspond to string propagation on topologically distinct Calabi-Yau manifolds. We have therefore apparently physically realized the Calabi-Yau conifold transitions discussed some years ago — without a physical mechanism — in insightful papers of Candelas, Green and Hubsch [11]. In the type II string moduli space with thus see that we can smoothly go from one Calabi-Yau manifold to another by varying the expectation values of appropriate scalar fields.

There is another aspect of these topology changing transitions which is worthy of emphasis. In the Coulomb phase, the black hole soliton states are massive. At the conifold point they become massless. As we move into the Higgs phase some number of them are eaten by the Higgs mechanism with the remainder staying massless. Now, with respect to the topology of the new Calabi-Yau in the Higgs phase, these massless degrees of freedom are associated with elements of $H^{1,1}$. Such states, as is well known, are perturbative string excitations — commonly referred to as elementary “particles”. Thus, a massive black hole sheds its mass, becomes massless and then re-emerges as an elementary particle-like excitation. There is thus no invariant distinction between black hole states and elementary perturbative string states: they smoothly transform into one another through the conifold transitions.

V. The Web of Connected Calabi-Yau Manifolds

In the previous section we have seen that type II string theory provides us with a mechanism for physically realizing topology changing transitions through conifold degenerations. This naturally raises two related questions:

- Are all Calabi-Yau manifolds interconnected through a web of such transitions?

- Are there other kinds of singularities, besides the ordinary double points discussed above, which might have qualitatively different physics and which might also have an important role in extending the Calabi-Yau web?

We now briefly report on work presently being carried out which is relevant to these two questions. A more detailed discussion will appear elsewhere.

In an important series of papers [11] it was argued some time ago that all Calabi-Yau manifolds realized as complete intersections in products of (ordinary) projective space are mathematically connected through conifold degenerations. As we mentioned above, although an intriguing prospect, it previously seemed that string theory did not avail itself of these topology changing transitions — as discussed, perturbative string theory is inconsistent at conifold points. The recent work described above shows that inclusion of nonperturbative effects cures the physical inconsistencies, at least in type II string theory, and hence the physical theory does allow such topology changing transitions to occur.

Since the time of [11], the class of well studied Calabi-Yau manifolds has grown. Initially inspired by work of Gepner [12], the class of hypersurfaces in weighted projective four spaces has received a significant amount of attention [13]. It was shown in [14] that there are 7555 Calabi-Yau's of this sort. Inspired by mirror symmetry, another class of Calabi-Yau's (containing these 7555 hypersurfaces) that have been under detailed study are complete intersections in toric varieties [15] [2] [1]. Understanding the structure of the moduli space of type II vacua requires that we determine if all of these Calabi-Yau's are interconnected through a web of topology changing transitions.

In the following we will briefly describe a procedure for finding transitions between Calabi-Yau manifolds realized as complete intersections in toric varieties. The method is elementary although at the present time we do not have any general results on its range of applicability. Rather, we have shown its usefulness by directly applying it to a subclass of the Calabi-Yau's realized in this manner. We have shown, for instance, that all 7555 Calabi-Yau hypersurfaces in weighted projective four space are *mathematically* connected to the web². We say mathematically because the transitions our procedure yields are not all of the conifold sort. Rather, there are Calabi-Yau's connected through more complicated singularities than the ordinary double points used in [4]. For example, some of these singularities are such that electrically *and* magnetically charged black hole states become simultaneously massless giving us an analog of the phenomenon discussed in [17]. Arguing for physical transitions through these theories requires more care than those involving

²We are aware that a similar conclusion has been reached by P. Candelas and collaborators using different methods [16].

conifold points. Whereas the term conifold transition refers to Calabi-Yau's linked through conifold degenerations, the term *extremal* transitions [18] refers to analogous links through any of the singularities encountered on the discriminant locus. At present we only have a satisfying physical understanding of the conifold subclass of extremal transitions.

The procedure we describe is relevant for Calabi-Yau's embedded in toric varieties. The reader requiring background in toric geometry should consult [2] [19]. To keep our discussion here concise, we shall focus on the case of hypersurfaces in weighted projective four spaces, although we shall briefly mention some generalization at the end of this section. As discussed in [20], the data describing such Calabi-Yau manifolds is:

- A lattice $N \simeq \mathbb{Z}^4$ and its real extension $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.
- A lattice $M = \text{Hom}(N, \mathbb{Z})$ and its real extension $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.
- A reflexive polyhedron $P \subset M_{\mathbb{R}}$.
- The dual or polar polyhedron $P^\circ \subset N_{\mathbb{R}}$.

Now, given the above sort of toric data for two different families of Calabi-Yau's in two different weighted projective four spaces, how might we perform a transition from one to the other? Well, given the polyhedra (P, P°) for one Calabi-Yau and (Q, Q°) for the other, one has the natural manipulations of set theory to relate them: namely, the operations of taking intersections and unions³. Consider, then, for instance, forming new toric data by taking the intersection $R = \text{convex hull}((P \cap M) \cap (Q \cap M))$. Further assume that R (and its dual R°) are reflexive polyhedra so that the singularities encountered are at finite distance in the moduli space [21]. How are the three Calabi-Yau's X, Y, Z associated to (P, P°) , (Q, Q°) and (R, R°) , respectively, related? The toric data contained in the polyhedron in $M_{\mathbb{R}}$ is well known to describe the complex structure deformations of the associated Calabi-Yau realized via monomial deformations of its defining equation⁴. Concretely, the lattice points in $P \cap M$ are in one-to-one correspondence with monomials in the defining equation of X , and similarly for Y and Z ⁵. Thus, in going from X to Z we have specialized the complex structure by restricting ourselves to a subset of

³The idea of trying to manipulate the toric data of one Calabi-Yau to produce another was first suggested by D. Morrison. Here we present one systematic procedure, that proves to be surprisingly robust, for doing so.

⁴By mirror symmetry, of course, it can also be used to describe the Kähler structure on the mirror Calabi-Yau.

⁵More precisely, some subset of these points correspond to the toric complex struc-

the monomial deformations. This is reminiscent of the example studied in [4], and described earlier, in which we specialized the complex structure of the quintic from its original 101 dimensional moduli space to an 86 dimensional subspace. This is not the end of the story. Clearly the dual R° contains P° . It is also well known that the toric data contained in the polar polyhedron describes the Kähler structure deformations of the associated Calabi-Yau. Concretely, lattice points in $P^\circ \cap N$ correspond to toric divisors which are dual to elements in $H^2(X, \mathbb{Z})^6$. Thus, in passing from X to Z we have also added toric divisors, i.e. we have performed a blow-up. This again is reminiscent of the example studied earlier: after specializing the complex structure we performed a small resolution. All of the discussion we have just had relating X to Z can be similarly applied to relate Y to Z . Hence, by using the toric data associated to X and to Y to construct the toric data of Z , we have found that Z provides a new Calabi-Yau that both X and Y are linked to in the web.

Of course, the key assumption in the above discussion is that (R, R°) provides us with toric data for a Calabi-Yau, i.e. they are reflexive polyhedra. At present, we have not developed a general method for picking (P, P°) and (Q, Q°) such that this is necessarily the case. In fact, the toric data for a given Calabi-Yau is not unique but, for instance, depends on certain coordinate choices. Thus the reflexivity of (R, R°) or lack thereof depends sensitively on the coordinate choices used in representing (P, P°) and (Q, Q°) . Hence, a more appropriate question is whether there exists suitable representations of (P, P°) and (Q, Q°) such that (R, R°) is reflexive. In our work we have arbitrarily chosen (P, P°) and (Q, Q°) , from the set of 7555 hypersurfaces, considered a variety of coordinate representations for each (related by $SL(5, \mathbb{Z})$ transformations and coordinate permutations) and directly checked to see if (R, R°) is reflexive. If it is, then X and Y are (mathematically) connected through the Calabi-Yau Z . We note that, in general, Z is not associated to a Calabi-Yau hypersurface in a weighted projective space — but rather a Calabi-Yau embedded in a more general toric variety.

In this manner, by direct computer search, we have checked that all 7555 hypersurfaces in weighted projective four space are linked (and through the process described we have actually linked them up to numerous other Calabi-Yau's — the Z -type Calabi-Yau's above). The main physical question, then, is what is the nature of the singularities encountered when we specialize the complex structure in the manner dictated by the intersection of P and Q . Analysis of the simplest examples shows that one often encounters singularities which are qualitatively different from

ture deformations. For details see [2] [22].

⁶More precisely some subset of the lattice points correspond to nontrivial elements in $H^2(X, \mathbb{Z})$. For details see [2] [22].

the well-understood case of several ordinary double points studied in [4].

To illustrate this point, and the discussion of this section more generally, lets consider two explicit examples.

Example 1:

Let's take X to be the family of quintic Calabi-Yau hypersurfaces in \mathbb{P}^4 and Y to be the family of Calabi-Yau hypersurfaces of degree 6 in $\mathbb{P}_{1,1,1,1,2}^4$. The Hodge numbers of X are $(h_X^{2,1}, h_X^{1,1}) = (101, 1)$ and those of Y are $(h_Y^{2,1}, h_Y^{1,1}) = (103, 1)$. Following the procedure described above, and recalling that $P^\circ \cap N$ is given by

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \quad \begin{pmatrix} 0, & 1, & 0, & 0 \\ -1, & -1, & -1, & -1 \end{pmatrix} \quad \begin{pmatrix} 0, & 0, & 1, & 0 \end{pmatrix} \quad (28)$$

and $Q^\circ \cap N$ by

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \quad \begin{pmatrix} 0, & 1, & 0, & 0 \\ -1, & -1, & -1, & -2 \end{pmatrix} \quad \begin{pmatrix} 0, & 0, & 1, & 0 \end{pmatrix} \quad (29)$$

we find that the toric data for family Z , R° , is the convex hull of

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \quad \begin{pmatrix} 0, & 1, & 0, & 0 \\ -1, & -1, & -1, & -1 \end{pmatrix} \quad \begin{pmatrix} 0, & 0, & 1, & 0 \\ -1, & -1, & -1, & -2 \end{pmatrix} \quad (30)$$

Note that for ease of presentation we are taking unions of data in N space which is dual to taking intersections in M space⁷, discussed above. Consider first the transition from Y to Z . One can show that the singular subfamily obtained by specializing the complex structure of Y , in the manner discussed above, consists of Calabi-Yau's which generically have 20 ordinary double points all lying on a single \mathbb{P}^2 and hence obeying one nontrivial homology relation. This, therefore, is another example of the conifold transitions described in [4], reviewed in the previous sections. Thus, we can pass from Y to Z in the manner discussed and the Hodge numbers change to $(h_Z^{2,1}, h_Z^{1,1}) = (103 - 20 + 1, 1 + 1) = (84, 2)$. The relation between X and Z , though, is more subtle. In specializing the complex structure of X dictated by the toric manipulation, we find a singular family of Calabi-Yau's, each generically having one singular point. The local description of this singularity, however, is *not* an ordinary double point, but rather takes the form

$$x^2 + y^4 + z^4 + w^4 = 0. \quad (31)$$

⁷The duality is only generally valid when considering intersections and unions in \mathbb{R}^4 instead of \mathbb{Z}^4 .

This singularity is characterized by Milnor number 27 which corresponds to 27 homologically independent S^3 's simultaneously vanishing at the singular point. Using standard methods of singularity theory [23] one can show that the intersection matrix of these S^3 's is *non-trivial* and has rank 20. Mathematically, it is straightforward to show that the transition from X to Z through such a degeneration causes the Hodge numbers to make the appropriate change.

Physically, in contrast to the previous cases, not only are A -type cycles shrinking down, but some dual B -type cycles are shrinking down as well. From this we see a phenomenon akin to that studied in [17]: we appear to have electrically and magnetically charged states simultaneously becoming massless⁸. It is such degenerations that require more care in establishing the existence of physical transitions. This also raises the interesting question of whether the web of Calabi-Yau's requires such transitions for its connectivity, or if by following suitable paths conifold transitions would suffice.

Example 2:

We take X to be the family of quintic Calabi-Yau hypersurfaces in \mathbb{P}^4 and we take Y to be the family of Calabi-Yau hypersurfaces of degree 8 in $\mathbb{P}_{1,1,1,1,4}^4$. As in the previous example, the transition from Y to Z just involves ordinary double points, so the discussion of [4] suffices. However, in passing from X to Z we encounter another type of singularity, known as a triple point. Namely, the generic Calabi-Yau in the subfamily of X obtained by specialization of the complex structure contains a single singular point whose local description is

$$x^3 + y^3 + z^3 + w^3 = 0. \tag{32}$$

The Milnor number for this singularity is equal to 16, and thus in this example we have 16 vanishing 3-cycles (homological to S^3 's) simultaneously shrinking to one point. The intersection matrix in this case has rank 10 and we thus again are dealing with a physical situation with massless electrically and magnetically charged particles.

For ease, in our discussion above, we have focused on hypersurfaces in weighted projective four space (which naturally led to hypersurfaces in more general toric varieties). We can carry out the same program on codimension d Calabi-Yau's. As we will discuss elsewhere, for these it is best to use the full reflexive Gorenstein cone

⁸In [24] it was independently noted that the phenomenon of [17] could be embedded in string theory in such a manner.

associated with the Calabi-Yau, but basically the idea is the same. For instance, the union of the Gorenstein toric fan (in the N lattice) for $\mathbb{P}_{3,3,2,2,1,1}^6(5,7)$ and $\mathbb{P}_{3,3,2,2,2,1}^6(5,8)$ is Gorenstein with index 2. Hence, these codimension two Calabi-Yau's are linked through such transitions.

In this manner we have established numerous links between Calabi-Yau's of codimension two and between Calabi-Yau's of codimension three. Furthermore, as in each of these classes its not hard to construct Calabi-Yau's with simple toric representations of various codimension (toric representations, of course, are not unique), we can link together the webs of different codimension as well. For instance, the quintic hypersurface, which is a member of the 7555 hypersurface web, is also linked to the web of complete intersections in products of ordinary projective spaces. Hence, all such Calabi-Yau's are so linked.

We therefore do not know the full answer to the two questions that motivated the discussion of this section, but we have gained some insight into each and hope to report on further progress shortly.

Acknowledgements

B.R.G thanks D. Morrison and A. Strominger with whom the results we have discussed on conifold transitions were found. The authors thank P. Aspinwall, D. Morrison and R. Plesser for helpful discussions related to the results presented in section V. This work was supported by the Alfred P. Sloan Foundation, a National Young Investigator Award, and by the National Science Foundation.

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