Solving The N-Body Problem in (2+1)-Gravity *

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Abstract

We construct a non-perturbative, single-valued solution for the metric and the motion of N interacting particles in 2 + 1-Gravity. The solution is explicit for two particles with any speed and for any number of particles with small speed. It is based on a mapping from multivalued Minkowskian coordinates to single-valued ones, which solves the non-abelian monodromies due to particles' momenta. The two and three-body cases are treated in detail.

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1 Introduction

We address, in this paper, a classic issue in gravitation theory [1], namely the one of finding the self-consistent metric and the corresponding motion of N interacting particles. This problem turns out to be solvable in 2 + 1 dimensions [2]-[11], and the solution that we find [12] shows several nontrivial features.

Firstly, our solution is regular, i.e., metric and coordinates are single-valued, or in other words, the metric is singular only at the particle sites. This is to be contrasted with the spurious singularities, found in previous studies [3],[7] by using the existence of locally Minkowskian coordinates and / or the relation of (2 + 1) gravity to Chern-Simons theory [8],[9]. In fact in (2 + 1) dimensions the space is flat outside the (pointlike) sources, but the corresponding Minkowskian coordinates are not single-valued, due to the localized curvature at the particle sites. Therefore, in such solutions [10],[11], the metric has spurious singularity tails departing from each particle.

Secondly, we are able to treat particles moving with arbitrary speed and with arbitrary masses, in some "physical" range consistent with an open universe. Thus we generalize in a nontrivial way the well known [2], [3] conformal metric of the static limit, which in 2 + 1 dimensions is a rather simple one, due to the lack of a Newtonian force.

Finally, the classical phase space emerging in the explicit form of our metric shows several links with conformal and Liouville theories, which suggest a way towards an N-body quantum mechanics [5], [6], [9] and perhaps a full quantum theory.

A basic reason why the N-body problem is solvable in the extended sense just explained, is that there is no graviton radiation in 2 + 1 dimensions. In fact, physical tensor waves (unlike photons) are not possible with only one transverse space dimension. Therefore, the gravitational degrees of freedom are longitudinal, and can propagate instantaneously in the gauge of Coulomb type [13]-[16] that we have proposed in I.

Our method of solution exploits both the existence of multivalued Minkowskian coordinates and the instantaneous propagation to construct a mapping to single-valued coordinates with nontrivial metric. In a generalized conformal gauge, such mapping is based on holomorphic and antiholomorphic representations of the Minkowskian monodromies, in which the analyticity properties of the mapping function are a consequence of the instantaneous propagation.

The Minkowskian monodromies are provided by the Deser-Jackiw and 't-Hooft [3] (DJH) "matching conditions" and define the particle sources. Together with proper boundary conditions at the particle singularities and at space infinity they determine both the metric and the motion, up to some residual gauge freedom, which allows one arbitrary trajectory and one scale parameter.

Since the DJH matching conditions involve Lorentz (or Poincarè) transformations which leave the particle's Minkowskian momentum invariant, they form in general a non-abelian group, depending on the N particles' momenta, with a quite complicated algebra. Nevertheless, it is possible to construct a spin $\frac{1}{2}$ (or projective) representation of such monodromies in terms of independent solutions of a second order differential equation with Fuchsian singularities [17] (the "Riemann-Hilbert" problem [18]).

The solution for the mapping function is here explicitly given in the two-body case, in terms of proper hypergeometric functions, and is given for $N \geq 3$ also, but only in the quasi-static limit, i.e., to first nontrivial order in the velocities.

Given the mapping function, all components of the metric and all the motion parameters are provided in terms of quadratures.

We have already given in I an account of our method and of the main features of our solution. Similar ideas have been, later on, discussed by Welling [19]. The purpose of the present paper is to describe the method and the interesting features of the two-body problem in full detail, and to investigate the novel features arising for $N \geq 3$, by describing the explicit solutions in the quasi-static limit (not to be confused with the static one) and some other simple N-body example.

An interesting feature arising for $N \geq 3$ is that the mapping function is determined not only by the particle singularities, but also by some "apparent singularities" [18] which have an invariant meaning (because they occur in the Schwarzian derivative [20] of the mapping function) but have, nevertheless, trivial monodromies. Such apparent singularities carry accessory parameters which are needed to match the non-abelian particle monodromies, including the one at space infinity. Here they are studied in detail in the quasi-static case.

The contents of the paper are as follows. In the introductory Section 2 we review the known singular solutions and, following I, we define the mapping problem from Minkowskian to single valued coordinates in our conformal Coulomb gauge. In Sec. 3 we treat, following I, the two-body problem, and in particular, the solution for the mapping function, the metric, and the motion. We also set up the conditions for the mapping to be non-singular, and to avoid closed timelike curves [20]. In Sec. 4 we treat the many-body case, the apparent singularities, and the explicit solution in the quasi-static limit. In Sec. 5 we describe in more detail the three-body case, its decoupling properties, and an interesting N-body case characterized by a symmetric particle configuration. We discuss our results in the conclusive Section 6, where we also give a few suggestions for the left-over problems. Some details of

the quadratures leading to the motion parameters are given in the Appendix.

2 Minkowskian vs single-valued coordinates

2.1 General features [2-11]

A basic property of classical (2 + 1)-Gravity is that the Riemann tensor is proportional to the Einstein tensor, and thus to the energy-momentum tensor. More precisely, due to the existence of the invariant ϵ -symbol, we have

$$R^{\alpha\beta}_{\mu\nu} = -\epsilon_{\mu\nu\lambda} \epsilon^{\alpha\beta\gamma} T^{\lambda}_{\gamma}, \qquad (2.1)$$

where, for pointlike particles, the energy-momentum density is a sum of delta-functions at the particle sites $x^{\mu} = \xi_i^{\mu}(\tau)$:

$$\sqrt{g} T^{\mu\nu} = \sum_{i=1}^{N} m_i \int d\tau_i \frac{d\xi^{\mu}}{d\tau_i} \frac{d\xi^{\nu}}{d\tau_i} \delta^{(3)}(x - \xi_i(\tau)).$$
(2.2)

Therefore, the space is flat everywhere, except at the particles sites, in which a singular curvature exists, related to the particles' momenta. This means that local Minkowskian coordinates can be extended all around the particles, but are in general multivalued, i.e., carry non-trivial monodromy transformations for parallel transport in a closed loop around each particle site.

The simplest example is for one (spinless) particle at rest in the origin. The space is flat everywhere else, but the loop integral of the connection is nontrivial, i.e., at a given time,

$$\oint_{C_0} dx^{\mu} (\Gamma_{\mu})_{\alpha\beta} = \epsilon_{\alpha\beta0} m, \quad (8\pi G = 1)$$
(2.3)

where m is the particle mass and C_0 encircles the origin. This situation admits several descriptions, according to the coordinate choice. In Minkowskian coordinates $X^a \equiv (T/Z/\bar{Z})$ the line element is trivial

$$ds^{2} = \eta_{ab} dX^{a} dX^{b} = dT^{2} - |dZ|^{2}, \quad Z = X + iY, \qquad (2.4)$$

but there is a cut-out sector, or a branch cut, corresponding to a deficit angle m [2],[3] :

$$|argZ| < \pi \alpha, \quad \alpha \equiv 1 - \frac{m}{2\pi}$$
 (2.5)

so that values of the Z coordinate above and below the cut are related by

$$Z_{II} = e^{-im} Z_I. (2.6)$$

The connection is localized on this cut, so that (2.3) is satisfied.

On the other hand, the branch point can be eliminated by a coordinate transformation to single-valued variables $x^{\mu} \equiv (t/z/\bar{z})$, defined by

$$Z = z^{\alpha} , T = t \tag{2.7}$$

in such a way that, for $z \to e^{2\pi i} z$, $Z \to e^{-im} Z$, as required by (2.6). The corresponding line element is now nontrivial

$$ds^{2} = dt^{2} - \alpha^{2} |z|^{-\frac{m}{\pi}} |dz|^{2}, \qquad (2.8)$$

yielding the conformal gauge metric, for which the connection is isotropically distributed around the particle.

The scale change $\rho = |z|^{\alpha}$ brings (2.8) to the conical gauge form

$$ds^{2} = dt^{2} - d\rho^{2} - \alpha^{2} \rho^{2} d\theta^{2}, \quad (0 \le \theta < 2\pi),$$
(2.9)

where the well-known conical geometry is transparent.

The discontinuity relation (2.6) is called the DJH matching condition [3]. It is just a rotation for a particle at rest in the origin. If the particle moves and is located at $X = X_1$, the monodromy (2.6) is boosted to a Poincare' transformation (Fig. 1):

$$X_{II} - X_1 = L(P_1)(X_I - X_1), (2.10)$$

where the O(2, 1) matrix

$$L(P_1) = exp(-iJ_a \cdot P_1^a) = Pexp(-\oint_{C(X_1)} \omega_\mu dx^\mu),$$

($i J_a)_{bc} = \epsilon_{abc},$ (2.11)

is the holonomy of the (spin) connection, related to the particle's Minkowskian momentum P_1^a , constant of motion. Correspondingly, the particles carry string singularities, or tails, which are needed to yield a precise determination of the X^a 's.

While eliminating the branch cut (2.6) was trivial, eliminating all tails is in general difficult, for the following reasons:

(i) If there are at least two particles, with a relative speed, the problem is non abelian, i.e. the monodromies do not commute

$$[L(P_1), L(P_2)] \neq 0 \tag{2.12}$$

and therefore cannot be brought together to the form of a phase transformation.

(ii) The rest frame complex planes #1, #2, etc... are inequivalent, i.e., they are related in general by Lorentz transformations which mix space and time, and have a complicated time dependence in a coordinate frame in which, say, tail #1 is eliminated.

Fortunately, there is a second simplifying feature in 2 + 1 dimensions, i.e., there are no transverse gravitons. For a given wave vector, there is only one transverse dimension, which cannot accomodate tensor waves. As a consequence, there are only longitudinal degrees of freedom, which can propagate instantaneously, in a properly chosen gauge.

Indeed, here we shall use the conformal gauge of Coulomb type that we have proposed in I, which provides an instantaneous propagation. This gauge allows to deal with all monodromies at a given time, in the same complex plane, and allows to treat the tails as true branch cuts of analytic functions.

2.2 Singular solutions

In order to make the above reasoning more precise, let us recall the class of singular solutions that were found [10], [11] in the first order formalism, which exhibits the known relation to a Chern-Simons Poincare' gauge theory. By defining dreiben and spin connection in the usual way

$$g_{\mu\nu} \equiv E^a_{\mu} E^b_{\nu} \eta_{ab},$$

$$\Gamma_{\lambda,\mu\nu} = E^a_{\lambda} \left(\partial_{\mu} + \omega_{\mu} \right)_{ab} E^b_{\nu},$$
(2.13)

such solutions turn out to be additive with the particles, provided the tails do not overlap. The spin connection is given by

$$(\omega_{\mu})_{ab} = \sum_{i=1}^{N} \epsilon_{abc} \Theta_{i}^{X} P_{i}^{c} \partial_{\mu} \Theta_{i}^{Y} \equiv \sum_{i=1}^{N} \omega_{\mu}^{(i)}$$
(2.14)

where the P^{a} 's are the (conserved) Minkowskian momenta and the Θ -functions define the particles' trajectories and tails. For instance, for momenta P_i in the x direction, one can take

$$\Theta_i^X = \Theta(V_i T(x^{\mu}) - X(x^{\mu})),$$

$$\Theta_i^Y = \Theta(Y(x^{\mu}) - B_i)$$
(2.15)

where $V_i = P_i^x/P_i^0$ ($P_i^y = 0$) and the $X^a(x)$ are arbitrary functions of x^{μ} , which parametrize the trajectories in the form

$$X(\xi_i) = V_i T(\xi_i), \quad Y(\xi_i) = B_i.$$
 (2.16)

The $\partial_{\mu}\Theta^{Y}$ derivative in Eq. (2.14) yields a δ -function singularity on the tails, so that the spin connection is localized, as anticipated. Since $\omega_{i} \simeq J \cdot P_{i}$, it is easy to verify by loop integration of (2.14) that the monodromies are given precisely by Eq. (2.11). In particular, if the X^{a} themselves are chosen as coordinates, one has the Minkowskian picture of Sec.(2.1), with the *DJH* matching conditions (2.10).

Corresponding to the spin connection (2.14), the dreibein solution takes the form

$$E^{a}_{\mu} = (\partial_{\mu} + \omega_{\mu})^{a}_{b} X^{b}(x) - \sum_{i} (\omega^{(i)}_{\mu} B_{i})^{a}$$
(2.17)

where the B_i 's are the translational parameters occuring in Eq. (2.16). Outside the tails, Eq. (2.17) reduces to

$$E^a_\mu = \partial_\mu X^a(x), \quad \text{(outside tails)}, \qquad (2.18)$$

and the dreibein defines the coordinate transformation from the x's to the Minkowskian coordinates X^a . Our purpose here is to look for single-valued x_{μ} 's by restricting the arbitrary functions (2.16) by a gauge choice.

2.3 The conformal Coulomb gauge

According to Eq. (2.18), the Minkowskian coordinates X^a and the single-valued ones $x^{\mu} \equiv (t/z/\overline{z}), \quad z \equiv x + iy$, are related, outside particle tails, by

$$dX^{a} = E^{a}_{\mu}dx^{\mu} = A^{a}dt + B^{a}dz + \tilde{B}^{a}d\bar{z}, \qquad (2.19)$$

with the consistency conditions

$$\partial_{[\nu} E^a_{\mu]} = 0. \tag{2.20}$$

We shall fix the gauge by a Coulomb condition

$$\partial_z E^a_{\bar{z}} + \partial_{\bar{z}} E^a_z = 0. \tag{2.21}$$

and by a conformal one for the space part of the metric:

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{2.22}$$

Because of Eqs. (2.20) and (2.21) the dreibein components satisfy the equations

$$\partial_{\bar{z}}B^a = \partial_z \tilde{B}^a = 0 \tag{2.23}$$

and

$$\partial_z A^a = \partial_0 B^a(z,t), \quad \partial_{\bar{z}} A^a = \partial_0 \bar{B}^a(\bar{z},t). \tag{2.24}$$

Therefore, $B^a(z,t)(\tilde{B}^a(\bar{z},t))$ are analytic (anti-analytic) functions and $A^a(z,\bar{z},t)$ are harmonic functions i.e., sums of analytic and antianalytic ones. These analyticity properties arising from Eq. (2.21) in two space dimensions, are the counterpart of instantaneous propagation in a second order formalism [13],[14], and are in fact fundamental to solve the monodromy problem.

Furthermore, because of Eq. (2.22), B^a and \tilde{B}^a are null vectors. By using straightforward conjugation properties, we can parametrize

$$B^{a} = N(z,t) W^{a}(z,t), \qquad \widetilde{B}^{a} = \overline{N}(\overline{z},t) \widetilde{W}^{a}(\overline{z},t)$$

$$(2.25)$$

where we have defined the null vectors

$$W^a = (f')^{-1} (f/1/f^2), \quad \widetilde{W}^a = (\bar{f}')^{-1} (\bar{f}/\bar{f}^2/1)$$
 (2.26)

in terms of the analytic function f(z,t) that we shall call the mapping function, of its complex conjugate, and of its derivative $f'(z,t) \equiv df/dz$.

We can also write

$$A^{a} = (a/A/\bar{A}), \quad a = \bar{a},$$
 (2.27)

where a(A) are real (complex) harmonic functions satisfying, by Eq. (2.24), the conditions

$$\partial_z a = \partial_t \left(\frac{N}{f'} f \right),$$

$$\partial_z A = \partial_t \left(\frac{N}{f'}\right), \quad \partial_{\bar{z}} A = \partial_t \left(\frac{Nf^2}{f'}\right).$$
 (2.28)

On the whole, we have now seven real variables (N, f, a, A) in terms of which we can express, in a straightforward way, the metric tensor in Eq. (2.13) as follows:

$$-2g_{z\bar{z}} \equiv e^{2\phi} = \left|\frac{N}{f'}\right|^{2} (1 - |f|^{2})^{2} = |N|^{2} (-2W_{a} \cdot \widetilde{W}^{a})$$

$$g_{0z} \equiv \frac{1}{2}\bar{\beta}e^{2\phi} = NW_{a} \cdot A^{a}, \qquad g_{0\bar{z}} = \frac{1}{2}\beta e^{2\phi} = \bar{N}\widetilde{W}_{a} \cdot A^{a}$$

$$g_{00} = \alpha^{2} - |\beta|^{2}e^{2\phi}, \quad \alpha = V_{a} \cdot A^{a} = \bar{\alpha} \qquad (2.29)$$

where the *a* indices are lowered by the minkowskian metric η_{ab} with non-vanishing components $\eta_{00} = -2\eta_{z\bar{z}} = 1$, and we have defined the Lorentz vector

$$V^{a} = (1 - |f|^{2})^{-1} (1 + |f|^{2} / 2\bar{f} / 2f) = \epsilon^{a}_{bc} W^{b} \widetilde{W}^{c} (W \cdot \widetilde{W})^{-1}.$$
(2.30)

Eq. (2.29) expresses the four real variables of the metric $(\phi, \alpha, \beta, \overline{\beta})$ in terms of the seven variables of the dreibein. This is because the metric determines the dreibein only up to local Lorentz transformations, in this case the three-parameter O(2, 1) group. Shortly we shall take advantage of this fact in order to define a single valued metric.

2.4 The Mapping Function

Note now that a non-trivial, i.e, non-Minkowskian metric is obtained because, due to the matching conditions, the X^a coordinates are multivalued and, in particular, for $(z - \xi_i) \rightarrow e^{2\pi i} (z - \xi_i), \ dX^a)_I \rightarrow dX^a)_{II}$, with

$$dX^{a})_{II} = (L_{i})^{a}_{b}(dX^{b})_{I}, (2.31)$$

according to the matching conditions (2.10). We should thus require that the dreibein components to be multivalued also, and to transform as Lorentz vectors for loops around the particles' singularities $z = \xi_i(t)$.

The corresponding metric (2.29) will be at this point single-valued, because it is not affected by a pure Lorentz transformation on the a indices.

This remark suggests a method to solve the monodromy problem. Il f(z,t) has branch points at $z = \xi_i(t)$ such that, when z turns around ξ_i , f transforms as a projective representation of the monodromies (2.11), then W^a , \widetilde{W}^a , V^a , will transform as Lorentz vectors by construction, and so will the dreibein, thus leading to a single-valued metric. More precisely, by defining a spin- $\frac{1}{2}$ representation of the holonomies (2.11), i.e.

$$L_i^{-1} \to \ell_i^{-1} \equiv \begin{pmatrix} a_i & b_i \\ \bar{b}_i & \bar{a}_i \end{pmatrix}$$
(2.32)

with

$$a_{i} = \cos \frac{m_{i}}{2} + i\gamma_{i} \sin \frac{m_{i}}{2} , \qquad b_{i} = -i\gamma_{i} \bar{V}_{i} \sin \frac{m_{i}}{2},$$

$$\gamma_{i} \equiv (1 - |V_{i}|^{2})^{-1/2}, \qquad V_{i} = (P_{i}^{x} + iP_{i}^{y})/E_{i} , \qquad (2.33)$$

we require that the mapping function f(z, t) transforms as

$$f(z,t) \to \frac{a_i f(z,t) + b_i}{\bar{b}_i f(z,t) + \bar{a}_i}, \quad (i = 1, ..., N)$$
 (2.34)

for $(z - \xi_i) \rightarrow e^{2\pi i} (z - \xi_i)$.

Since the generators of the transformation (2.34) on analytic (antianalytic) functions are

$$L^{a} = \left(f \frac{\partial}{\partial f} / \frac{\partial}{\partial f} / f^{2} \frac{\partial}{\partial f} \right), \quad \left[\bar{L}^{a} = \left(\bar{f} \frac{\partial}{\partial \bar{f}} / \bar{f}^{2} \frac{\partial}{\partial \bar{f}} / \frac{\partial}{\partial \bar{f}} \right) \right], \quad (2.35)$$

respectively, it follows that the W^a , \widetilde{W}^a in Eq. (2.26) transform according to the adjoint (vector) representation, and so does V^a in Eq. (2.30). It follows that N(z,t) should be single-valued (i.e., at most meromorphic, with poles at $z = \xi_i$) and that A^a will also transform as a vector, because of Eqs. (2.24) and (2.28).

Our program to solve for the single-valued metric and the corresponding motion in the conformal Coulomb gauge will thus involve the following steps:

(i) Find the mapping function f(z,t) by solving the monodromy problem in Eqs.(2.32)-(2.34).

(ii) Find the meromorphic function N(z,t) and the harmonic functions $A^a(z, \bar{z}, t)$ by integrating the consistency conditions (2.28) under proper asymptotic conditions, to be defined below.

(iii) Find the metric from Eq. (2.29).

(iv) Find the motion by mapping the Minkowskian trajectories (2.16), or more precisely, from the equations

$$Z(\xi_i, \bar{\xi}_i, t) = B_i + V_i T(\xi_i, \bar{\xi}_i, t), \qquad (2.36)$$

where the Minkowskian complex velocities V_i are defined in Eq. (2.33) and $B_i = B_i^x + iB_i^y$.

3 The Two-Body problem

This is the simplest non-trivial monodromy problem because there are two non-commuting monodromies, L_1 and L_2 , one for each particle, with

$$L_i = e^{-iJ \cdot P_i}, \quad (iJ_a)_{bc} = \epsilon_{abc}. \tag{3.1}$$

Here the P_a 's are the conserved minkowskian momenta which will be assumed to have a parallel space part, i.e.,

$$P_i^a = (E_i/P_i/\bar{P}_i) = m_i \gamma_i (1/V_i/\bar{V}_i) , \qquad (3.2)$$

with $V_2 \sim V_1$ (e.g. $P_2 = -P_1$ in the naive c.m. Lorentz frame).

In order to determine the analyticity properties of the mapping function we will also assume that initially the tails run outwards and are parallel. However, it will be clear in a while that the final solutions in single-valued coordinates will not depend on the fact that the tails may cross, and the Minkowskian momenta may jump during the motion [10]. Rather, our proviso on the tails is to be regarded as an asymptotic initial condition on the motion, which specifies P_1^a , P_2^a and the determination of the mapping function.

As stated in Sec. 2, we shall first determine the mapping to single-valued coordinates at a given time t, and given particle coordinates $z = \xi_i(t)$, and we shall find their motion later on. By defining the conformally rescaled variable

$$\zeta(z,t) = \frac{z - \xi_1(t)}{\xi_2(t) - \xi_1(t)}$$
(3.3)

the particles' positions are mapped to $\zeta = 0$ and $\zeta = 1$, around which the monodromies are given by (3.1).

However, since in an open universe the composite loop operator around both particles, for instance $L_{21} = L_2 L_1$ is non-trivial, then, $\zeta = \infty$ is also a singularity point of the problem. By explicit computation [9], e.g. on the spin $\frac{1}{2}$ representation (2.32) we find

$$L_2L_1 = L_{21} = e^{-iP_{21} \cdot J}, \quad P_{21}^a = (\sqrt{m^2 + |p_{21}|^2}/p_{21}/\bar{p}_{21}), \quad (3.4)$$

where

$$\cos\frac{\mathcal{M}}{2} = \cos\frac{m_1}{2} \cos\frac{m_2}{2} - \frac{P_1 \cdot P_2}{m_1 m_2} \sin\frac{m_1}{2} \sin\frac{m_2}{2}, \qquad (3.5)$$

and

$$\frac{p_{21}}{\mathcal{M}}\sin\frac{\mathcal{M}}{2} = \frac{p_1}{m_1}\sin\frac{m_1}{2}\cos\frac{m_2}{2} + \frac{p_2}{m_2}\sin\frac{m_2}{2}\cos\frac{m_1}{2} + i(V_2 - V_1)\gamma_1\gamma_2\sin\frac{m_1}{2}\sin\frac{m_2}{2}.$$
(3.6)

Thus, note that the monodromy is now dependent on whether we start the $\zeta = \infty$ loop anticlockwise on the upper or lower z-plane because $p_{21} \neq p_{12}$, as it appears from the third term in Eq. (3.6). This is not surprising, due to the non-commutativity. However L_{12} is related to L_{21} by a similarity transformation which then leaves the total invariant mass \mathcal{M} unchanged, i.e., independent of the order of the particles.

3.1 Solution for the mapping function

The problem of finding a (projective) representation, as in Eq. (2.34) of given monodromies, as in Eq. (2.32), is well known in the mathematical literature and is referred to as the Riemann-Hilbert problem [18]. It is related to the theory of (second-order) ordinary differential equations with Fuchsian singularities.

In fact, if $y_+(\zeta)$ and $y_-(\zeta)$ are independent solutions of the differential equation

$$y'' + q(\zeta)y = 0 (3.7)$$

with Fuchsian singularities at $\zeta = \zeta_i$, then it is known [17] that the ζ_i 's are branch points of $y_{\alpha}(\zeta)$ ($\alpha = +, -$) around which they transform linearly according to a subgroup of SL(2, C). If we are then able to choose $q(\zeta), y_+, y_-$, such that, for $(\zeta - \zeta_i) \to e^{2\pi i}(\zeta - \zeta_i)$,

$$\begin{pmatrix} y_+\\ y_- \end{pmatrix} \rightarrow \begin{pmatrix} a_i & b_i\\ \bar{b}_i & \bar{a}_i \end{pmatrix} \begin{pmatrix} y_+\\ y_- \end{pmatrix}, \quad (i = 1, ..., N)$$
(3.8)

(where the *a*'s and *b*'s parametrize a spin $\frac{1}{2}$ representation of our O(2, 1) monodromies in Eq. (2.33)), then the mapping function is given by

$$f(z,t) \rightarrow f(\zeta) = \frac{y_+(\zeta)}{y_-(\zeta)}$$

$$(3.9)$$

where we have incorporated the t-dependence in the rescaled variable $\zeta(z,t)$ in Eq. (3.3).

It is useful to note, for further reference, that in the canonical form (3.7), the "potential" $q(\zeta)$ can be expressed as

$$2q(\zeta) = \{f, \zeta\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2,$$
(3.10)

in terms of the Schwarzian derivative $\{f,\zeta\}$, which is invariant under projective transformations

$$\left\{\frac{af+b}{cf+d},\zeta\right\} = \{f,\zeta\}.$$
(3.11)

Furthermore, since the Wronskian of two solutions of (3.7) is constant, it is easy to realize, by using (3.10), that a particular basis of solutions can be expressed in terms of $f(\zeta)$ itself, as follows

$$Y_{+} = f(\zeta)(f'(\zeta))^{-1/2}, \quad Y_{-} = f'(\zeta)^{-1/2}, \quad (Y'_{+}Y_{-} - Y_{+}Y'_{-} = 1).$$
 (3.12)

For N = 2, we have three Fuchsian singularities, at $\zeta = 0$, $\zeta = 1$, $\zeta = \infty$, as remarked before, and the y's are expected to be expressible in terms of hypergeometric functions. The most general form of $q(\zeta)$ consistent with the Fuchsian requirements is (Cfr. Sec. 4)

$$q(\zeta) = \frac{1}{4} \left(\frac{1 - \mu_1^2}{\zeta^2} + \frac{1 - \mu_2^2}{(1 - \zeta)^2} + \frac{1 - \mu_1^2 - \mu_2^2 + \mu_\infty^2}{\zeta(1 - \zeta)} \right),$$
(3.13)

where

$$\lambda_1^{\pm} = \frac{1}{2} (1 \pm \mu_1), \quad \lambda_2^{\pm} = \frac{1}{2} (1 \pm \mu_2), \quad \lambda_{\infty}^{\pm} = \frac{1}{2} (1 \pm \mu_{\infty})$$
(3.14)

represent the pairs of "exponents" at $\zeta = 0, 1, \infty$ respectively, which parametrize the behaviour of the solutions around $\zeta = \zeta_i$ as follows

$$y_{\alpha} \simeq A_{\alpha}^{+}(i)(\zeta - \zeta_{i})^{\lambda_{i}^{+}} + A_{\alpha}^{-}(i)(\zeta - \zeta_{i})^{\lambda_{i}^{-}}, \quad (i = 1, 2, \infty, \quad \alpha = \pm).$$
 (3.15)

It is then straightforward to realize that an explicit basis of solutions is provided by

$$y_{+} = k_{+}\zeta^{\frac{1}{2}(1+\mu_{1})}(1-\zeta)^{\frac{1}{2}(1-\mu_{2})}\widetilde{F}(\frac{1}{2}(1+\mu_{\infty}+\mu_{1}-\mu_{2}), \frac{1}{2}(1-\mu_{\infty}+\mu_{1}-\mu_{2}); 1+\mu_{1}; \zeta),$$

$$y_{-} = k_{-}\zeta^{\frac{1}{2}(1-\mu_{1})}(1-\zeta)^{\frac{1}{2}(1-\mu_{2})}\widetilde{F}(\frac{1}{2}(1+\mu_{\infty}-\mu_{1}-\mu_{2}), \frac{1}{2}(1-\mu_{\infty}-\mu_{1}-\mu_{2}); 1-\mu_{1}; \zeta), \quad (3.16)$$

where y_- differs from y_+ for $\mu_1 \to -\mu_1$, and we have defined for convenience a modified hypergeometric function

$$\widetilde{F}(a,b,c;z) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a,b,c;z).$$
(3.17)

We also understand, with the usual determination of F, that the branch cuts of y_{\pm} run on the intervals $(-\infty, 0)$ and $(1, \infty)$, which thus correspond in the ζ -plane, to the outwards directed initial tails.

We have now to see whether the expressions (3.16), replaced in Eq. (3.9), match the monodromy properties of the mapping function. This means that we have to identify the exponent differences μ_1 , μ_2 , μ_{∞} , and the coefficient ratio $k \equiv k_+/k_-$ so as to satisfy the monodromy conditions (3.8), or (2.34).

Since the F's in Eq. (3.16) are regular at $\zeta = 0$, it is clear that y_{\pm} are relevant solutions in the rest frame of particle #1, in which the Lorentz transformation (2.33) is diagonal, and reduces to a rotation of the mass m_1 . But $y_+/y_- \simeq \zeta^{\mu_1}$, so that we can identify

$$\mu_1 = \pm \frac{m_1}{2\pi}, \pmod{n_1}.$$
(3.18)

The monodromy matrix of y_+/y_- will then be non-diagonal at both $\zeta = 1$ and $\zeta = \infty$, and determined by the coefficients in Eq. (3.15), which are calculable by analytic continuation [17], and given in Table I. It is clear beforehand that, similarly to Eq. (3.18), one should have

$$\mu_2 = \pm \frac{m_2}{2\pi} \pmod{n_2}, \quad \mu_\infty = \pm \left(-1 + \frac{\mathcal{M}}{2\pi}\right), \pmod{2n - n_1 - n_2},$$
 (3.19)

where \mathcal{M} is the invariant mass in Eq. (3.5), characterizing $\zeta = \infty$, because the exponent differences μ_i have an invariant meaning which can be established, e.g., by transforming linearly y_+ and y_- to the particle #2 rest frame, or to an overall c.m. frame, in which the corresponding monodromy is diagonal.

The determination of $k_+/k_- \equiv k$ is more subtle, and in a sense fundamental, because it characterizes the O(2, 1) monodromies, compared to other subgroup of SL(2, C). From the coefficients $A^{\beta}_{\alpha}(2)$ in Table I we find the monodromy matrix

$$\ell_2^{-1} = \begin{pmatrix} \cos \pi \mu_2 + i \sin \pi \mu_2 \frac{\rho^+ + \rho^-}{\rho^+ - \rho^-} & -i \sin \pi \mu_2 \frac{2\rho^+}{\rho^+/\rho^- - 1} \\ i \sin \pi \mu_2 \frac{2/\rho^-}{\rho^+/\rho^- - 1} & \cos \pi \mu_2 - i \sin \pi \mu_2 \frac{\rho^+ + \rho^-}{\rho^+ - \rho^-} \end{pmatrix}, \quad \rho^\beta = \frac{A_+^\beta(2)}{A_-^\beta(2)},$$
(3.20)

which in general is an $SL(2, \mathbb{C})$ matrix. It matches the O(2, 1) form in eq. (3.8) provided

$$\rho^+ = \frac{\gamma_{12}\bar{V}_{21}}{\gamma_{12}-1}, \quad 1/\rho^- = \frac{\gamma_{12}V_{21}}{\gamma_{12}-1}$$
(3.21)

where the velocities are assumed to be collinear and $\gamma_{12} \equiv (P_1P_2)/m_1m_2$ is the γ -factor of particle #2 in the particle #1 rest frame.

Table I	behaviour at $\zeta = 0, 1$	
	λ_1^+ λ_1^-	λ_2^+ λ_2^-
y_+	$k_{+} \frac{\Gamma(a')\Gamma(b')}{\Gamma(1+\mu_{1})} \qquad 0$	$k_+ \Gamma(-\mu_2) k_+ \frac{\Gamma(a')\Gamma(b')\Gamma(\mu_2)}{\Gamma(c'-a')\Gamma(c'-b')}$
y_{-}	$0 k_{-} \frac{\Gamma(a)\Gamma(b)}{\Gamma(1-\mu_{1})}$	$k_{-} \Gamma(-\mu_2) = k_{-} \frac{\Gamma(a)\Gamma(b)\Gamma(\mu_2)}{\Gamma(c-a)\Gamma(c-b)}$
	behaviour at $\zeta = \infty$	
	λ^+_∞	λ^∞
y_+	$k_+ e^{\pm i\pi\mu_1/2} \left(\frac{\Gamma(b')\Gamma(\mu_{\infty})}{\Gamma(1-b)}, \frac{\Gamma(a')\Gamma(-\mu_{\infty})}{\Gamma(1-a)}\right)$	
y_{-}	$k_{-} e^{\mp i\pi\mu_{1}/2} \left(\frac{\Gamma(i)}{\Gamma}\right)$	$(1-b') \Gamma(\mu_{\infty}) \over \Gamma(1-a')$, $(1-a') \Gamma(1-a')$
$a(\mu_1) =$	$\frac{1}{2}(1 + \mu_{\infty} - \mu_1 - \mu_2), b(\mu_1) = a' = a(-\mu_1), b' = b$	$\begin{vmatrix} \frac{1}{2}(1-\mu_{\infty}-\mu_{1}-\mu_{2}), \ c(\mu_{1}) = 1-\mu_{1} \\ p(-\mu_{1}), \ c' = c(-\mu_{1}) \end{vmatrix}$

By the explicit expressions in Table I we find the conditions

$$\frac{\rho^{-}}{\rho^{+}} = \frac{\sin \pi a}{\sin \pi a'} \frac{\sin \pi b}{\sin \pi b'} = \frac{\gamma_{12} - 1}{\gamma_{12} + 1} = th^{2} \frac{1}{2} (\eta_{1} - \eta_{2}),$$

$$k = \rho^{+} = \frac{\gamma_{12} \bar{V}_{21}}{\gamma_{12} - 1}$$
(3.22)

where a, b (a', b') are the indices of the *F*'s in Eq. (3.16) defined in table *I*, and $\eta_i = th^{-1}|V_i|$, $\eta_{12} = \eta_1 - \eta_2$ are the velocity boosts in a general collinear frame. From the expression (3.22) of k, we conclude, as in I, that

$$f_{(1)}(\zeta) = \frac{\gamma_{12}\bar{V}_{12}}{\gamma_{12} - 1} \frac{\zeta^{\mu_1} \tilde{F}(a', b', c'; \zeta)}{\tilde{F}(a, b, c; \zeta)}$$
(3.23)

where the subscript means that it refers to the particle #1 rest frame.

It is straightforward to check that the condition (3.22), by Eq. (3.19), is actually equivalent to the definition of the invariant mass in Eq. (3.5), thus confirming the expression (3.19) of μ_{∞} . We shall furthermore specify Eqs. (3.18) and (3.19) by taking the positive signs, or masses, and by setting $n_i = n = 0$, in order to meet the boundary conditions to be discussed below.

3.2 Solution for N, A and boundary conditions

The function N(z,t) is single-valued (meromorphic) and, according to Eq. (2.25), appears as a coefficient of $W^a(f)$ in the expression of $B^a = E_z^a$. Its meaning is better seen in a second-order formalism [13] in which its polar behaviour appears determined by the type of distribution occurring in the energy-momentum tensor, a single pole corresponding to a δ -function, a double pole to a δ' , and so on.

In the present approach, in which only global monodromy conditions are set, the complete determination of N and A requires boundary conditions at the singularity points, which also help clarifying the determination of the μ_i indices in (3.18) and (3.19), to be used in the following.

The single particle metric in Eq. (2.8) shows the following features:

(i) The minkowskian coordinates $Z(\xi_i) = Z_i$, $T(\xi_i) = T_i$ are well defined in the spinless case, and

(ii) The monodromy behaviour is of type

$$Z \simeq (z - \xi_i)^{1 - \frac{m_i}{2\pi}} + Z_i \quad (|z - \xi_i| << 1).$$
(3.24)

Both conditions will turn out to be verified around the singularities if we set $n_i = n = 0$ in Eqs. (3.18) and (3.19) and we also take the ansatz of I, i.e.,

$$N(z,t) = \frac{R(\xi(t))}{(z-\xi_1)(z-\xi_2)} = \frac{R(\xi)}{\xi^2} \frac{1}{\zeta(1-\zeta)},$$
(3.25)

where $\xi = \xi_2 - \xi_1$. In Eq. (3.25) we have single poles (corresponding to δ -function singularities) and the residues are related so as to avoid a pole at infinity (which is unphysical) and also a zero of the determinant

$$2\sqrt{|g|} = \alpha \ e^{2\phi} = \left|\frac{N}{f'}\right| (1 - |f|^2)(V \cdot A).$$
(3.26)

We can now discuss the form of the mapping and check the boundary conditions. By integrating Eq. (2.19) out of particle #1, say, we obtain

$$X^{a} = X_{1}^{a}(t) + \int_{\xi_{1}}^{z} dz \ N \ W^{a}(z,t) + \int_{\bar{\xi}_{1}}^{\bar{z}} d\bar{z} \ \bar{N} \ \widetilde{W}^{a}(\bar{z},t).$$
(3.27)

The behaviour of W^a close to the singularity points is better seen by using the basis (3.12) and Eq. (3.15) in the form

$$W^{a}(\zeta) = \begin{pmatrix} y_{+}y_{-} \\ y_{-}^{2} \\ y_{+}^{2} \end{pmatrix} \simeq \Delta^{(1-\mu)} \begin{pmatrix} (A_{+}^{-} + A_{+}^{+}\Delta^{\mu})(A_{-}^{-} + A_{-}^{+}\Delta^{\mu}) \\ (A_{-}^{-} + A_{-}^{+}\Delta^{\mu})^{2} \\ (A_{+}^{-} + A_{+}^{+}\Delta^{\mu})^{2} \end{pmatrix}$$
(3.28)

where $\Delta \equiv \zeta - \zeta_i$, we have dropped the *i* index for simplicity, and the coefficients $A^{\beta}_{\alpha}(i)$ are given in Table I for $f_{(1)}$. From eqs. (3.28) and (3.25) we see that

$$NW^{a} \simeq (\zeta - \zeta_{i})^{-\frac{m_{i}}{2\pi}} \quad (\zeta \to \zeta_{i}, \quad i = 1, 2),$$
$$\simeq \zeta^{-\frac{M}{2\pi}}, \quad (\zeta \to \infty) \tag{3.29}$$

It follows that the endpoint integrals at $z = \xi_1, \xi_2$ are indeed well defined, provided

$$0 \le m_{1,2} < 2\pi, \qquad (8\pi G = 1)$$
 (3.30)

and that Eq. (3.24) is verified also.

The point $\zeta = \infty$ requires further care. Let us first rewrite Eq. (3.27) in more detail as follows

$$X^{a} = B_{1}^{a} + V_{1}^{a}T_{1}(t) + R(\xi(t)) I^{a}(0,\zeta(z,t)) + \bar{R}(\bar{\xi}) \tilde{I}^{a}(0,\bar{\zeta}), \qquad (3.31)$$

where

$$I^{a}(0,\zeta) = \int_{0}^{\zeta} \frac{d\zeta}{\zeta(1-\zeta)} W^{a}(\zeta) \stackrel{\zeta \to \infty}{\simeq} (A_{-}^{-})^{2} \zeta^{1-\frac{M}{2\pi}} \begin{pmatrix} f(\infty) \\ 1 \\ f^{2}(\infty) \end{pmatrix} + A_{-}^{-} A_{-}^{+} \log \zeta \begin{pmatrix} \rho^{+} + \rho^{-} \\ 2 \\ 2\rho^{+}\rho_{-} \end{pmatrix}$$
(3.32)

and

$$\rho^{\alpha} \equiv \frac{A^{\alpha}_{+}(\infty)}{A^{\alpha}_{-}(\infty)} , \quad \rho^{-} = f(\infty).$$
(3.33)

We thus see that the behaviour $\sim \zeta^{1-\frac{M}{2\pi}}$ is in general translated by (3.32) in the time component $E_0^a = A^a$ of the dreibein also, where it would indicate a rotating frame at space infinity.

More precisely we obtain, by a time derivative of Eq. (3.31)

$$A^{a} = V_{1}^{a} \dot{T}_{1} + \partial_{t} [R(\xi(t)) I^{a}(0, \zeta(z, t)) + \bar{R}(\bar{\xi}(t)) \tilde{I}^{a}(0, \bar{\zeta}(z, t))].$$
(3.34)

Since $\zeta \simeq z/\xi(t)$, the coefficient of the diverging behaviour will vanish provided

$$R(\xi) = C \, \xi(t)^{1-} \frac{\mathcal{M}}{2\pi},$$
 (3.35)

thus cancelling the time dependence of the leading term.

We thus see that, as a consequence of the asymptotic condition that A^a be at most logarithmic for $\zeta \to \infty$, we are able to determine $R(\xi)$, and thus A^a in Eq. (3.34), N in Eq. (3.25) and the metric in Eq. (2.29).

The logarithmic behaviour of A^a has a second important consequence. From Eq. (3.34) and the asymptotic form (3.32) we find, after some algebra, the asymptotic metric

$$ds^{2} \stackrel{|z|>>1}{\simeq} \left[dRe\left(\frac{C}{|\mu_{\infty}|}\xi_{21}(t)^{1-\frac{M}{2\pi}} \log z\right) \right]^{2} - \text{const.} |z|^{-\frac{M}{\pi}} |dz|^{2}.$$
(3.36)

Since $\log z = \log |z| + i\theta$, it appears from (3.36) that the time $T(t, z, \bar{z})$, in a Lorentz c.m. frame, is asymptotically multivalued, with monodromy related to the particle motion as follows

$$T(t, z \ e^{2\pi i}, \bar{z} \ e^{-2\pi i}) - T(t, z, \bar{z}) = \frac{2\pi}{\mu_{\infty}} Im(C\xi_{21}^{1-\frac{\mathcal{M}}{2\pi}}), \qquad (3.37)$$

where the quantity in the r.h.s. will be related in the next subsection to the total angular momentum of the system.

3.3 Solution for the motion

We have already used the equation of motion for particle #1 in order to normalize the inhomogeneous part of A^a in (3.34) to the time function $T_1(t)$. The second one will determine the relative motion trajectory $\xi(t)$, up to some residual gauge freedom.

In fact, from the trajectory equations (2.36), and the coordinate mapping (3.31), we obtain

$$B_2^a - B_1^a + T_2 V_2^a - T_1 V_1^a = C \xi^{1-} \frac{\mathcal{M}}{2\pi} I^a(0,1) + \bar{C} \xi^{1-} \frac{\mathcal{M}}{2\pi} \tilde{I}^a(0,1).$$
(3.38)

Since I^a and \tilde{I}^a are calculable constants (see Appendix), eq. (3.38) determines $\xi(t)$ and the relative time variable $T_1(t) - T_2(t)$, up to an overall time reparametrization and a scale freedom provided by C.

More in detail, by using a time parametrization in which

$$T_1(t) = t - \Delta(t), \quad T_2(t) = t + \Delta(t),$$
 (3.39)

it is straightforward to solve for ξ Eq. (3.38), in terms of the relative impact parameter B(in Minkowskian coordinates) and of the integrals $I^0, I^z, I^{\bar{z}}$ and their complex conjugates. After some algebra we find

$$C\xi(t)^{1-\frac{\mathcal{M}}{2\pi}} = t \frac{V_2 - V_1}{I^z + I^{\bar{z}} - (V_1 + V_2)I^0} + i \frac{B}{I^z - I^{\bar{z}}},$$

$$\Delta(t) = t \frac{I^0(V_2 - V_1)}{I^z + I^{\bar{z}} - (V_1 + V_2)I^0}$$
(3.40)

where we have assumed that the Minkowskian velocities run along the x-axis (so that V_1 , V_2 , and I^a are real) and the relative impact parameter is along the Y axis (so that iB is imaginary).

Eq. (3.40) determines completely the form of the mapping in Eq. (3.31) in the time parametrization (3.39), but does not determine the form of $\xi_1(t)$, which thus remains a residual gauge freedom, in the class of single-valued solutions. The reason for this freedom is that our gauge choice in Eqs. (2.20) and (2.21) is preserved by holomorphic time dependent conformal transformations. However, if we add the boundary conditions (3.24) at $\zeta = 0, 1$ and $\zeta = \infty$, this freedom reduces to a linear time dependent transformation of the form

$$z \to a(z - b(t)) \tag{3.41}$$

which precisely allows one arbitrary trajectory.

From Eq. (3.40) one can read off the scattering properties. Since $\xi(t)^{1-\frac{M}{2\pi}}$ has the same phase as roughly $(iB + (V_2 - V_1)t)$, the scattering angle is clearly

$$\theta = \pm \frac{\mathcal{M}/2}{1 - \mathcal{M}/2\pi}, \quad (\text{sign } B = \pm)$$
 (3.42)

that is, the same as for a test particle [3] moving in the field of the total invariant mass, as suggested by 't-Hooft [5].

Note that the result (3.42) is valid for any value of the speed, provided the coefficient of time keeps its sign, i.e.,

$$I^{z} + I^{\bar{z}} > 2I^{0} > I^{0}(V_{1} + V_{2}), \qquad (3.43)$$

which is true because $1 + f^2 > 2f$, f being real for $0 < \zeta < 1$. Furthermore it is also independent of the details of the time parametrization (3.39), provided monotonicity of time is preserved.

Note also that this result for the scattering angle is much different from the one found in a covariant gauge of Aichelburg-Sexl type for massless particles [10],[21]. The instantaneous gauge in the present case forces the particles to interact at any time, even in the massless limit (Cfr. Sec. 3.5) and shows no sign of a shock-wave picture.

Finally, let us remark that the result (3.40) allows the evaluation of the asymptotic time shift (3.37). By using the explicit expressions for the I^{a} 's (see Appendix) we find

$$I^{z} - I^{\bar{z}} = \frac{\pi \sin \pi \mu_{\infty}}{\mu_{\infty} \sin \pi \mu_{1} \sin \pi \mu_{2} \gamma_{12} |V_{12}|}$$
(3.44)

and therefore, by Eq. (3.37)

$$\Delta T \stackrel{|z|>>1}{\simeq} \frac{2\pi}{\mu_{\infty}} \frac{B}{I^z - I^{\bar{z}}} = 2B \frac{\sin\frac{m_1}{2}\sin\frac{m_2}{2}}{\sin\frac{M}{2}} \gamma_{12} |V_{12}| \equiv J$$
(3.45)

For small masses, the r.h.s. reduces to $Bp_{rel} = J$, where $p_{rel} \simeq \frac{m_1 m_2}{\mathcal{M}} \gamma_{12} |V_{12}|$ is the relative momentum. It is thus natural to define the r.h.s. of (3.45) as the total angular

momentum of the system for any mass, with a peculiar identification of the "reduced mass" parameter.

3.4 Avoiding closed time-like curves

Having in mind the general two-body dynamics, let us now discuss in more detail the form of the conformal mapping induced by $f(\zeta)$, and also its possible breakdown, for particular mass values.

Since the expression (3.23) of $f_{(1)}$ has no poles in the (upper) ζ plane, and has branch cuts at $\zeta = 0, 1, \infty$ with known behaviour (Eq. 3.15), it is straightforward to see that $f_{(1)}(\zeta)$ maps the upper half ζ -plane into a Schwarz triangle [20] of type in Fig. 2(*a*).

In drawing Fig. 2(a) we have used the boundary values

$$f_{(1)}(0) = 0, \ f_{(1)}(1) = \rho^{-}(1) = \frac{\gamma_{12} - 1}{\gamma_{12} + 1},$$
 (3.46)

and

$$e^{\mp i\pi\mu_1} f_{(1)}^{\pm}(\infty) = \rho^{-}(\infty) = \left[\frac{\sin \frac{(\mathcal{M} + m_1 - m_2)}{4} \sin \frac{(\mathcal{M} - m_1 - m_2)}{4}}{\sin \frac{(\mathcal{M} - m_1 + m_2)}{4} \sin \frac{(\mathcal{M} + m_1 + m_2)}{4}} \right]^{1/2}, \text{ (sign Im}\zeta = \pm),$$
(3.47)

which are valid in the mass range

$$0 \le m_1, m_2, \qquad \mathcal{M} < 2\pi. \tag{3.48}$$

We see that the upper half plane is mapped on a triangle whose edges are circular arcs, and whose internal angles are $m_1/2$, $m_2/2$ and $\pi - \mathcal{M}/2$, for $m_1 + m_2 < \mathcal{M} < 2\pi$. The lower half plane is obtained by Schwarz's reflection of this triangle, thus obtaining the region of Fig. 2(*a*).

It is interesting to note that, in the mass range (3.48), the whole region satisfies |f(z)| < 1, and the same inequality is satisfied by f(z) on any other Riemann sheet, because of the elliptic monodromy (2.34). Therefore, the determinant of the metric in Eq. (3.26) is nonvanishing in the whole ζ -plane.

On the other hand, if $P_1 \cdot P_2$ in Eq. (3.5) exceeds a critical value, $\cos(\mathcal{M}/2)$ becomes smaller than -1, the mass takes the form $\mathcal{M} = 2\pi(1 + i\sigma)$, and closed timelike curves appear [22]. In this situation, the behaviour of $f(\zeta)$ for $\zeta \to \infty$, provided by Eq. (3.15), becomes oscillatory because $\mu = i\sigma$ is pure imaginary and does in fact cross the |f(z)| = 1 value an infinite number of times, as is apparent from its explicit form (Fig. 2(b))

$$e^{\mp i\pi\mu} f_{\pm}(\zeta) \stackrel{\zeta \to \infty}{\simeq} \rho^{-}(\infty) \left(\frac{1 + (-\zeta)^{i\sigma} e^{i\phi_{+}}}{1 + (-\zeta)^{i\sigma} e^{i\phi_{-}}} \right) , \quad \mathcal{M} = 2\pi \left(1 + i\sigma \right)$$
(3.49)

where $|\rho^{-}(\infty)| = 1$, and ϕ_{\pm} are proper phases which can be derived from Table I.

We conclude that the restriction $\cos M/2 > -1$ is needed to avoid both CTC's , and a pathological situation for the gauge choice.

One can look at the behaviour at $\zeta = \infty$ also from another point of view. If a finite limit $f(\infty)$ exists, in some analyticity sector (upper and lower ζ -plane in the present case), then it must be the same in all directions of the sector by the Phragmèn-Lindelöf theorems [23]. In our case we will have two values, $f_{\pm}(\infty)$. By applying the monodromies counterlockwise, we can relate values above and below the cuts as follow

$$f_{-}(\infty) = \ell_1 f_{+}(\infty), \quad f_{+}(\infty) = \ell_2 f_{-}(\infty).$$
 (3.50)

We thus find that

$$f_{+}(\infty) = \ell_{2}\ell_{1}f_{+}(\infty), \quad f_{-}(\infty) = \ell_{1}\ell_{2}f_{-}(\infty),$$
 (3.51)

that is, $f_+(f_-)$ are fixed points of the composite loop operators $\ell_{21}(\ell_{12})$. By parametrizing it as in (3.4), we find

$$V_{21}(\infty) f_{+}^{2}(\infty) - 2f_{+}(\infty) + \bar{V}_{21}(\infty) = 0$$
(3.52)

where $V_{21}(\infty) = \frac{p_{21}}{M}$ denotes the velocities of the (upper) c.m. frame in Eq. (3.6). Therefore,

$$f_{+}(\infty) = \frac{1}{V_{21}(\infty)} (1 - \sqrt{1 - |V_{21}(\infty)|^2})$$
(3.53)

where we have chosen the root such that $|f(\infty)| < 1$. For $V_1 = 0$, Eq. (3.53) reduces to Eq. (3.47).

However, the square-root in (3.53) exists properly only if $|V_{12}|$, $|V_{21}| < 1$. For too large $P_1 \cdot P_2$ this is no longer the case, and we end up with $\cos \mathcal{M}/2 < -1$, $|f_{\pm}(\infty)| = 1$, a pathological $\zeta \to \infty$ limit, and closed timelike curves. The nonexistence of a sensible speed for the Lorentz c.m. frames is yet another hint that such large values of $P_1 \cdot P_2$ have no physical meaning.

3.5 Massless and Single Particle Limits

In our two-body solution we may take the limit of one particle being massless ($m_2 = 0$, say) with fixed value of $\gamma_{12} = (P_1 \cdot P_2)/m_1m_2$. In this case, the total mass \mathcal{M} becomes just m_1 , and we are describing the single particle limit.

Since f is normalized by Eq. (3.23) and N by Eq. (3.35) and (3.40) we can check that both f and N vanish, in this limit, so that

$$\frac{N(z,t)}{f'(z,t)} \to \text{ const. } (z - \xi_1(t))^{-\frac{m_1}{2\pi}}$$
(3.54)

is a finite quantity. Thus, apart from the already metioned arbitrariness of $\xi_1(t)$, we end up with a single particle metric of type

$$ds^{2} = dt^{2} - \text{const.} |z - \xi_{1}(t)|^{-\frac{m_{1}}{\pi}} |d(z - \xi_{1}(t))|^{2}, \qquad (3.55)$$

which is just a reparametrization of the static one in Eq. (2.8).

This result, surprising at first sight for a particle that can "move", is due to the asymptotic conditions that we have set, which are appropriate for the configuration space "center of mass" (not to be confused with the Lorentz ones) and are in fact inspired to the single particle metric (2.8). In other words, if we only have one particle, it cannot move in an absolute sense, and we end up with a metric equivalent to the static one.

The massless limit has instead a nontrivial two-body meaning if we let $m_1, m_2 \to 0$, but also $\gamma_{12} \to \infty$ so that the Minkowskian energies E_1 and E_2 are finite. In such case

$$\cos\frac{\mathcal{M}}{2} = 1 - \frac{(P_1 \cdot P_2)}{4} = 1 - \frac{(E_1 E_2)}{2}$$
(3.56)

provides the invariant mass. In a general frame, with particle boosts η_1, η_2 one has

$$f(\zeta) = \frac{f_{(1)}(\zeta) - th(\eta_1/2)}{1 - th(\eta_1/2)f_{(1)}(\zeta)}.$$
(3.57)

We then let $m_i \to 0$ with $m_i \gamma_i = E_i$ fixed, so that $\mu_i = E_i/2\pi\gamma_i \to 0$ and $th \eta_1/2 \to 1 - m_1/E_1$ in Eqs. (3.23) and (3.57). After some algebra we find, in a collinear frame defined by the energies E_1 and E_2 ,

$$f(\zeta) = \frac{1 - f_0(\zeta)}{1 + f_0(\zeta)}, \quad f_0(\zeta) = \sqrt{\frac{E_1}{E_2}} \frac{\widetilde{F}(\frac{1}{2}(1 + \mu_\infty), \frac{1}{2}(1 - \mu_\infty), 1, 1 - \zeta)}{\widetilde{F}(\frac{1}{2}(1 + \mu_\infty), \frac{1}{2}(1 - \mu_\infty), 1, \zeta)}, \quad (3.58)$$

where $\mu_{\infty} = \mathcal{M}/2\pi - 1$ as usual.

Note that both hypergeometric functions have c - a - b = 0, so that they provide logarithmic singularities at $\zeta = 0$ and $\zeta = 1$, as expected. The behaviour at $\zeta = \infty$ is normal, and we have to require $E_1E_2 < 4$ in order to avoid CTC's and a pathological metric.

We also obtain, from Eq. (3.45), the limiting form of the asymptotic time shift in the massless case

$$\Delta T = J = 2Btg\frac{\mathcal{M}}{4} = B\mathcal{E}$$
(3.59)

where we have defined the "effective energy" $\mathcal{E} = 2tg(\frac{M}{4})$. It is amusing to note that in the Aichelburg-Sexl gauge of Ref. [10] the massless scattering angle and energies are

$$\theta_{A.S.} = \frac{\mathcal{M}}{2}, \quad E_{A.S.} = \mathcal{E} = 2tg\frac{\mathcal{M}}{4}$$
(3.60)

and that, therefore, the bound $\mathcal{M} < 2\pi$ is built in, in the physical energy range.

In the limiting case $\mathcal{M} = 2\pi$ (which requires care in most formulas), the mapping function $f_0(\zeta)$ reduces to the well known function

$$f_0 = \sqrt{\frac{E_1}{E_2}} \frac{\tilde{F}(1/2, 1/2, 1; 1-\zeta)}{\tilde{F}(1/2, 1/2; 1; \zeta)} , \qquad (3.61)$$

which is the inverse of the automorphic function $\zeta = \kappa^2(\tau)$, occuring in the theory of elliptic integrals [20].

4 Main Features of the N-Body Problem

Our method of solution for the metric works for $N \ge 3$ as well, once the mapping function is found. However, unlike the N = 2 case, we have found no explicit general form of the mapping for $N \ge 3$, except to first order in the relative speed, i.e. in the "quasi static case", that will be discussed in a moment. The main difficulty that we find is the fact that the Fuchsian problem for the mapping shows additional "apparent singularities" besides the N+1 expected ones. For instance, the three-body problem requires the solution of a Fuchsian equation with five singularities, which is not explicitly known.

In this section, we set up the general problem and we solve the quasi-static case. In the next, we discuss some explicit examples that we are able to treat in full detail.

4.1 The Fuchsian Problem

The method for finding the mapping function works exactly as in the two-body case, and is based on the fuchsian differential equation (3.7), i.e.

$$y'' + q(\zeta)y = 0. (4.1)$$

However, the potential $q(\zeta)$ is now dependent on some set of singularities $\zeta = \zeta_i$ (i = 1,...,n), yet to be found. Because of the Fuchsian constraints, we can parametrize $q(\zeta)$ in terms of double and single poles at the singularities, i.e.

$$2q(\zeta) = \sum_{i=1}^{n} \left(\frac{1 - \mu_i^2}{2(\zeta - \zeta_i)^2} + \frac{\beta_i}{\zeta - \zeta_i} \right) , \qquad (4.2)$$

where

$$\lambda_i^{\pm} = \frac{1}{2} (1 \pm \mu_i) \tag{4.3}$$

are the exponents at the singularities, and the residues β_i are accessory parameters which could accomodate our nonabelian monodromies. The condition that $\zeta = \infty$ be a Fuchsian singularity yields two constraints on the β 's, i.e.,

$$\sum_{i=1}^{n} \beta_i = 0 , \quad 1 - \mu_{\infty}^2 = \sum_{i=1}^{n} (1 - \mu_i^2 + 2\beta_i \zeta_i) , \quad (4.4)$$

where μ_{∞} is the corresponding difference of exponents.

In the two body case, the particles can be set at $\zeta = 0$ and $\zeta = 1$ by the conformal transformation (3.3) and β_1 , β_2 are completely determined by (4.4) to be

$$\beta_1 = -\beta_2 = \frac{1}{2}(1 + \mu_{\infty}^2 - \mu_1^2 - \mu_2^2), \qquad (4.5)$$

so that the potential (3.13) emerges. No additional singularity is needed, because the number of invariants required by the momenta is precisely three [†].

In the general case, we have N three-momenta, and 3N - 3 invariants. Assuming N particle singularities and one at $\zeta = \infty$ we only have 2N - 1 free parameters, N - 2 coming from the β 's. Thus, we need additional singularities which yield trivial monodromy properties for the mapping function. These are the "apparent singularities" [18].

Since an apparent singularity ζ_j $(j \ge N+1)$ has trivial monodromy, the difference of exponents μ_j should be an integer, the simplest case being $\mu_j = 2$ ($\mu_j = 1$ means no

[†]There is a hidden parameter, the ratio of coefficients of two independent solutions, which however is needed to match the O(2, 1) nature of the monodromies (Cfr. Sec. 3.1).

singularity). A $\mu_j = 2$ singularity, corresponding to exponents -1/2 and 3/2 (Eq. (4.3)), yields a simple zero of f', because one of the solutions is $\simeq (f')^{-1/2}$ (Eq. (3.12)).

Setting $\mu_j = 2$, say, is however not enough to insure the absence of $\log(\zeta - \zeta_j)$ terms in the solution, which yield nontrivial monodromy. Around $\zeta = \zeta_j$ we can write

$$2q(\zeta) = \frac{-3/2}{(\zeta - \zeta_i)^2} + \frac{\beta_j}{\zeta - \zeta_j} + 2\bar{q}_j(\zeta) , \quad (j = N+1, ..., n)$$
(4.6)

where \bar{q}_j is regular at ζ_j . Then a simple analysis shows that the "non-logarithmic condition" is

$$-\frac{1}{2}\beta_j^2 = 2\bar{q}_j(\zeta_j) = \sum_{i\neq j}^n \left[\frac{1-\mu_i^2}{2(\zeta_j-\zeta_i)^2} + \frac{\beta_i}{\zeta_j-\zeta_i}\right], \quad (\mu_j = 2).$$
(4.7)

By assuming now that all apparent singularities have $\mu_j = 2$, i.e., are simple zeros of f' (multiple zeros being a limiting case of this one) and satisfy the nonlogarithmic condition (4.7), it is easy to realize that we need N-2 of them to solve our problem.

In fact, having fixed the N + 1 exponents

$$\mu_i = \frac{m_i}{2\pi}, \quad i = 1, ..., N ; \ \mu_\infty = \frac{\mathcal{M}}{2\pi} - 1 ,$$
(4.8)

we need 2(N-2) parameters to accomodate N-2 complex relative velocities. On the other hand, we have (N-2) normal residues β_i (i = 3, ..., N), and (n - N - 1) residues β_j and positions ζ_j (j = N + 1, ..., n) of the apparent singularities, of which only one per singularity is to be counted, because of the nonlogarithmic conditions. This determines

$$n = 2N - 1 \tag{4.9}$$

and the number of apparent singularities to be (N-2).

Note that we have not counted as parameters the N-2 particle positions $\zeta_i(t)$, (i = 3, ...N), because the latter should be determined dynamically from the equations of the motion of the problem. In order to distinguish them from the apparent singularities, we shall sometimes use for the latter the notation $\eta_k = \zeta_{k+N}$, $\gamma_k = \beta_{k+N}$, with k = 1, ..., N-2. Then, the above counting of conditions and parameters means that, given the set of particle singularities

$$0, 1, \zeta_3, ..., \zeta_N, \infty, \tag{4.10}$$

and corresponding exponents, the residues β_{2+i} and γ_k (i, k = 1, ..., N - 2) should be determined from the relative velocities in the monodromy matrices, while the apparent singularities $\eta_k(\beta_i, \gamma_i; \zeta_i)$ would then be dependent variables because of the non-logarithmic conditions.

The really awkward part of this program is the determination of the monodromy matrices by an analytic continuation of the solutions, which however is not yet explicitly available. General results are instead available in the mathematical literature on the functional dependence $\eta_k(\zeta_i)$, at constant monodromy matrices $L_i(P_i)$.

In fact, if we decide to solve the (N - 2) non-logarithmic conditions (4.7) for the $\beta_{2+i}(\gamma_k, \eta_k)$ (because they are linear in such quantities), we obtain [24],[18] the so called "Garnier systems", hamiltonian systems in which $\{\eta_k, \gamma_k\}$ is a set of conjugate variables, and $\{\zeta_{i+2}, \beta_{i+2}\}$ is a corresponding set of time-hamiltonian pairs, with equations:

$$\frac{\partial \eta_k}{\partial \zeta_{i+2}} = \frac{\partial \beta_{i+2}(\gamma, \eta)}{\partial \gamma_k}, \quad \frac{\partial \gamma_k}{\partial \zeta_{i+2}} = -\frac{\partial \beta_{i+2}(\gamma, \eta)}{\partial \eta_k}, \quad (i, k = 1, ..., N-2).$$
(4.11)

This (quite nonlinear) set of equations with N - 2 "times" $\zeta_3, ..., \zeta_N$ constrains the dependence $\eta_k(\zeta_{i+2})$ so as to insure that the Minkowskian momenta are constants of motion. For N = 3, the system (4.11) reduces to the VI-th Painlevè equation [25] for $\eta(\zeta_3)$.

Thus, we conclude that the monodromy problem for $N \geq 3$ is in principle solvable, but the general solution is not yet explicit.

4.2 The Quasi Static Case

The N-body problem can be explicitly solved to first order in the velocities of the particles. In this case, the singularity exponents turn out to be the static ones, but the particles can move, and the solution shows rather nontrivial features.

The basic observation is that the mapping function is itself of first order in the velocities, as it can be seen from Eq. (3.23), by taking care of the fact that $a \rightarrow 0$ in this limit. Therefore, the projective transformation (2.34) linearizes in the form

$$f(\zeta) \to e^{2i\pi\mu_i}f + \frac{\bar{V}_i}{2}(e^{2i\pi\mu_i} - 1).$$
 (4.12)

for $(\zeta - \zeta_i) \to e^{2i\pi}(\zeta - \zeta_i)$.

Although (4.12) is still non commutative, it becomes trivially commutative for the derivative, i.e.,

$$f'(\zeta) \to e^{2i\pi\mu_i} f'(\zeta). \tag{4.13}$$

We shall thus solve (4.13) first, in the form

$$f'(\zeta) = K \prod_{i=1}^{2N-1} (\zeta - \zeta_i)^{\mu_i - 1} = K \prod_{i=1}^{N} (\zeta - \zeta_i)^{\mu_i - 1} \cdot \prod_{k=1}^{N-2} (\zeta - \eta_k),$$
(4.14)

where $\zeta_1 = 0, \ \zeta_2 = 1,$

$$\mu_i = \frac{m_i}{2\pi}, \quad (i = 1, ..., N), \quad \mu_i = 2, \quad (i = N+1, ..., 2N-1),$$
(4.15)

and we have explicitly shown the zeros at $\zeta = \eta_k \equiv \zeta_{k+N}$.

Then, we obtain the mapping function $f_{(1)}(\zeta)$ in the particle #1 rest frame in the form

$$f_{(1)}(\zeta) = K \int_0^{\zeta} d\zeta \prod_{i=1}^{2N-1} (\zeta - \zeta_i)^{\mu_i - 1}, \qquad (4.16)$$

where the cuts at the branch points are assumed to run outwards and to not overlap, for a given cyclic initial ordering of the particles (Fig. 3).

The function (4.6) changes by just the phase $exp(im_1)$ around particle #1, but has nontrivial monodromies around the remaining ones. In order to find them, we use additivity of the integrals in their analyticity domain, to write

$$f_{(1)}(\zeta) = f_{(1)}(\zeta_i) + f_{(i)}(\zeta).$$
(4.17)

Since $f_{(i)}(\zeta)$ changes by $exp(i \ m_i)$ around ζ_i , we find the monodromy at ζ_i to be

$$f_{(1)}(\zeta) \to e^{im_i} f_{(1)}(\zeta) + f_{(1)}(\zeta_i)(1 - e^{im_i}).$$
 (4.18)

By matching (4.18) to (4.12) we find the conditions

$$f_{(1)}(\zeta_i) = K \int_0^{\zeta_i} d\zeta \prod_{i=1}^N (z - \xi_i)^{\mu_i - 1} \prod_{k=1}^{N-2} (z - \eta_k) = \frac{\bar{V}_1 - \bar{V}_i}{2}, \qquad (4.19)$$

where we have restored V_1 , in a general frame. The (N-1) equations (4.19) determine $K(\zeta_i)$ and $\eta_k(\zeta_i)$ in terms of the V_i 's and of the particle positions $\zeta_3, ..., \zeta_N$. Note that we could not have matched the monodromies without the zeros η_k , which seemed perhaps useless in Eq. (4.14). Of course, since we have the solutions (4.14) and (4.16), all parameters of the potential $q(\zeta)$ are also determined. By using the relation (3.10) with the Schwarzian derivative, we find

$$2q(\zeta) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \sum_{i=1}^{2N-1} \left(\frac{(1-\mu_i^2)}{2(\zeta-\zeta_i)^2} + \frac{\beta_i}{\zeta-\zeta_i}\right),$$
(4.20)

where

$$\beta_i = -(1-\mu_i) \sum_{j\neq i}^{2N-1} \frac{1-\mu_j}{\zeta_{ij}} , \quad (\zeta_{ij} \equiv \zeta_i - \zeta_j).$$
(4.21)

The accessory parameters just found satisfy $\sum_i \beta_i = 0$ identically, and define μ_{∞} by the equation (4.4), yielding after a simple algebra

$$\mu_{\infty} = -\sum_{i=1}^{2N-1} (1-\mu_i) + 1 = \sum_{i=1}^{N} \mu_i - 1, \qquad (4.22)$$

thus confirming the expectation that the invariant mass is just static, to first order in V_i 's.

Finally, one can also check with some algebra that the expressions (4.21) satisfy the (N-2) nonlogarithmic conditions (4.7). Eqs. (4.16), (4.19) and (4.21) represent the complete solution of the mapping problem in the quasi-static case. Their explicit expressions for N = 3 are given in Sec. (5.1).

The qualitative form of the mapping induced by $f(\zeta)$ is pictured in Fig. 4 for N = 3, where the values $f_{ii+1}(\infty)$ are discussed in the next subsection.

4.3 Metric and motion

Given the mapping function, we have still to find N and A. Since the zeros of f' would imply poles in NW^a , we require the function N to cancel them. By assuming N to have zeros at $\zeta = \eta_k$, simple poles at $\zeta = \zeta_i$, (as for N = 2) and no pole at $\zeta = \infty$, we obtain

$$N(z,t) = \frac{R(\xi_{21},\zeta_i)}{\xi_{21}^2} \frac{\prod_{k=1}^{N-2}(\zeta-\eta_k)}{\prod_{i=1}^{N}(\zeta-\zeta_i)}, \quad (\xi_{21} = \xi_2 - \xi_1), \quad (4.23)$$

where $\zeta = (z - \xi_1)/\xi_{21}$ is again the rescaled variable (3.3), and $\zeta_i = \xi_{i1}/\xi_{21}$. The form (4.23) satisfies the boundary conditions (i) and (ii) of Sec. (3.2).

We can then write the detailed form of the mapping as in Eq. (3.31), namely

$$X^{a} = B_{1}^{a} + V_{1}^{a}T_{1}(t) + R(\xi_{12},\zeta_{i})I^{a}(0,\zeta(z,t)) + \bar{R}\tilde{I}^{a}(0,\bar{\zeta}), \qquad (4.24)$$

where now

$$I^{a}(0,\zeta) = \int_{0}^{\zeta} d\zeta \frac{\prod(\zeta - \eta_{k})}{\prod(\zeta - \zeta_{i})} W^{a}(\zeta).$$

$$(4.25)$$

By inserting in (4.25) the asymptotic behaviour of $f'(\zeta)$ that we write in the form

$$f'(\zeta) \xrightarrow{\zeta \to \infty} K(\zeta_i) \ \zeta^{\frac{\mathcal{M}}{2\pi} - 2} (1 - \frac{\mathcal{M}}{2\pi}), \tag{4.26}$$

we find

$$I^{a}(0,\zeta) \stackrel{\zeta \to \infty}{\sim} \left(\frac{z}{\xi_{21}}\right)^{1-\frac{\mathcal{M}}{2\pi}} \frac{1}{K(\zeta_{i})} \begin{pmatrix} f(\infty) \\ 1 \\ f^{2}(\infty) \end{pmatrix}, \qquad (4.27)$$

where $f(\infty)$ is the asymptotic value of $f(\zeta)$ in one of its analyticity sectors (Fig. 3), to be discussed shortly.

From Eqs. (4.24) and (4.27) it follows that A^a is badly behaved at infinity, unless we set, similarly to the two-body case

$$R(\xi_{12},\zeta_i) = C K(\zeta_i)(\xi_{21})^{1-\frac{M}{2\pi}}.$$
(4.28)

We see that the asymptotic condition $A^a \sim \log z$ determines this time the dependence of R on both ξ_{12} and ζ_i , in terms of $K(\zeta_i)$ in Eq. (4.26).

As a consequence, the metric is fully determined and so are the equations of motion

$$B_{i}^{a} - B_{1}^{a} + V_{i}^{a} T_{i} - V_{1}^{a} T_{1} = C \xi_{21}^{1 - \frac{\mathcal{M}}{2\pi}} K\left(\frac{\xi_{i1}}{\xi_{21}}\right) I^{a}\left(0, \frac{\xi_{i1}}{\xi_{21}}\right) + \bar{C}\bar{\xi}_{21}^{1 - \frac{\mathcal{M}}{2\pi}} \bar{K}\tilde{I}^{a} \quad (i = 2, ..., N).$$
(4.29)

These (N-1) equations determine the relative times $T_i - T_1$ and the relative motion trajectories $\xi_{i1}(t)$. Of particular interest is the motion in the dimensionless parameters ξ_{i1}/ξ_{21} discussed in Sec. (5.1) for N = 3.

So far, our discussion on N and A has been general, and applies to the N-body case for any speed. In the quasi-static case, the invariant mass \mathcal{M} takes the static value of Eq.(4.22) and $K(\zeta_i)$ is determined by the eqs. (4.19).

Furthermore, the value of $f(\infty)$ in Eq. (4.27) can be determined as follows. As in the two-body case (Sec. 2.4), we expect N different values of $f(\infty)$ in the various analyticity sectors (N1), (12), ..., (N-1, N) of Fig. (3). They correspond to the fixed points of the various monodromies at infinity that we can have, for instance, to

$$\ell_N \dots \ell_1 f^{(N_1)}(\infty) = f^{(N_1)}(\infty), \qquad (4.30)$$

and , for the others, to

$$f^{(12)}(\infty) = \ell_1 f^{(N1)}(\infty), \dots$$
(4.31)

and so on.

Referring to Eq. (4.30) for definiteness, and combining the monodromies (4.12), we easily obtain

$$f^{(N1)}(\infty) = -\frac{\bar{V}(N1)}{2},$$
 (4.32)

where

$$V(N1) = \frac{\prod_{2}^{N} e^{im_{i}}(e^{im_{1}}-1)V_{1} + \prod_{3}^{N} e^{im_{i}}(e^{im_{2}}-1)V_{2} + \dots (e^{im_{N}}-1)V_{N}}{\prod_{1}^{N} e^{im_{i}}-1}$$
(4.33)

is the velocity of the corresponding "center-of-mass" Lorentz frame. We have thus N c.m. systems and N values at $\zeta = \infty$, for a given initial cyclic ordering. This fact was used in drawing the qualitative mapping of Fig. (4), and is a consequence of the noncommutativity of the monodromies, even in this simplified limit.

Since the velocities are assumed to be small, we have in principle no problems with the requirement $|f(\infty)| < 1$, needed to have a nonsingular mapping. However, Eq. (4.33) shows that something wrong happens if $\sum_i m_i = 2\pi$, because the total velocity shows a pole. We thus expect that a condition similar to $\mathcal{M} < 2\pi$ should be imposed in general.

Note, however, that for $N \geq 3$ and finite velocities even the invariant mass will be dependent on the inequivalent cyclic orderings of the particles which insure initially nonoverlapping tails. This is yet another feature of the general N-body problem which requires further study.

5 Some explicit solutions

5.1 The Quasi-static three-body case

The detailed study of the N = 3 motion shows, even to first order in the velocities, some features of general interest, for instance related to the decoupling limit of the two-body subsystems from the three body one, which are worth looking at in detail. In the N = 3 case there is only one nontrivial position $\zeta_3 = \xi_{31}/\xi_{21}$, for particle # 3, and one apparent singularity $\eta(\zeta_3)$, that we can determine explicitly. The total number of singularities is thus five. As a consequence, the system of two equations in (4.19) is linear in η and has the following solution

$$\eta(\zeta_3) - \zeta_3 = \frac{\bar{V}_{31}I_{12}(\mu_3 + 1, \zeta_3) - \bar{V}_{21}I_{13}(\mu_3 + 1, \zeta_3)}{\bar{V}_{31}I_{12}(\mu_3, \zeta_3) - \bar{V}_{21}I_{13}(\mu_3, \zeta_3)} = \frac{\sum \bar{V}_i I_{jk}(\mu_3 + 1, \zeta_3)}{\sum \bar{V}_i I_{jk}(\mu_3, \zeta_3)}$$
(5.1)

where we have used the notation

$$I_{ij}(\mu_3,\zeta_3) = \int_{\zeta_i}^{\zeta_j} dz \ z^{\mu_1-1} \ (z-1)^{\mu_2-1} (z-\zeta_3)^{\mu_3-1} \ , \quad V_{ij} \equiv V_i - V_j.$$
(5.2)

The corresponding solution for K is

$$K(\zeta_3) = \frac{\mu_3}{2} \frac{\sum_{\text{cyclic}} V_i I_{jk}(\mu_3, \zeta_3)}{W[I_{12}(\mu_3 + 1, \zeta_3), I_{13}(\mu_3 + 1, \zeta_3)]}, \qquad (5.3)$$

where $W(y_1, y_2) \equiv y'_1 y_2 - y_1 y'_2$ is a Wronskian.

The integrals $I_{12}(\mu_3 + 1, \zeta)$ and $I_{13}(\mu_3 + 1, \zeta)$ are expressible in terms of hypergeometric functions with singularities at $\zeta = 0, 1, \infty$ (the remaining particle singularities) and difference of exponents $\mu_1 + \mu_2$, $\mu_2 + \mu_3$, $\mu_1 + \mu_2 - 1$ respectively. Their explicit form, for the branch-cut structure of Fig. (3) is

$$I_{12}(\mu_3 + 1, \zeta) = e^{i\pi(\mu_2 - 1)}B(\mu_2, \mu_1 + \mu_3)F(\mu_3, 1 - \mu_1 - \mu_2 - \mu_3; \zeta) -e^{i\pi(\mu_2 - \mu_1 - \mu_3)}B(\mu_1, -\mu_1 - \mu_3)\zeta^{\mu_1 + \mu_3}F(\mu_1, 1 - \mu_2, 1 + \mu_1 + \mu_3; \zeta),$$

$$I_{13}(\mu_3 + 1, \zeta) = -e^{i\pi(\mu_2 - \mu_3)}B(\mu_1, 1 + \mu_3)\zeta^{\mu_1 + \mu_3}F(\mu_1, 1 - \mu_2, 1 + \mu_1 + \mu_3; \zeta),$$
(5.4)

and we also have the Wronskian

$$W = e^{i\pi(2\mu_2 - \mu_3)} \frac{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(1 + \mu_3)}{\Gamma(\mu_1 + \mu_2 + \mu_3)} \zeta^{\mu_1 + \mu_3 - 1} (1 - \zeta)^{\mu_2 + \mu_3 - 1}, \qquad (5.5)$$

in terms of which $\eta(\zeta_3)$ and $K(\zeta_3)$ are explicitly found.

An interesting point to notice in Eq. (5.1) is the limiting behaviour of $\eta(\zeta_3)$ for ζ_3 close to the singularity points $0, 1, \infty$. It is easy to check that they approach the same limit. For instance if

$$\zeta_3 = \frac{\xi_{31}}{\xi_{21}} \to 0 \quad \eta = \frac{\xi_{41}}{\xi_{21}} \simeq \left(\frac{\xi_{31}}{\xi_{21}}\right)^{1-\mu_1-\mu_3} \to 0.$$
(5.6)

This means that if ξ_3 becomes degenerate with ξ_1 , the location of the zero does too. This is needed in order to have a correct two-body limit of the mapping function. In fact, in the same limit we have

$$f'(\zeta) \to K(\zeta_3) \ \zeta^{\mu_1 + \mu_3 - 1} (\zeta - 1)^{\mu_2 - 1} ,$$
 (5.7)

as expected from a system of masses $m_1 + m_3$ and m_2 . Correspondingly, f and N have the behaviour typical of three singularities, instead of five: one pole and one zero have disappeared. This behaviour is expected to hold for general velocity configurations.

One can check that the expression (5.1) of $\eta(\zeta_3)$, that we have found by solving explicitly for the monodromies (Eq. 4.19), is also a solution of the VI-th Painlevè equation for the value $\mu_{\infty} = \sum_{i=1}^{3} \mu_i - 1$ of the exponent at infinity. This nonlinear second-order equation comes from the fact that the monodromies do not change when ζ_3 varies ("isomonodromic problem") [25] and is therefore a consistency check of the present approach.

Let us now look in more detail to the three-body relative motion. According to Eq.(4.29) we have now two equations in which, to first nontrivial order in v, we have, by (4.25),

$$R I^{a}(0,\zeta_{i}) = C(\xi_{21})^{1-\frac{\mathcal{M}}{2\pi}} \int_{0}^{\zeta_{i}} d\zeta \zeta^{-\mu_{1}}(\zeta-1)^{-\mu_{2}}(\zeta-\zeta_{3})^{-\mu_{3}} \begin{pmatrix} O(v) \\ 1 \\ O(v^{2}) \end{pmatrix}.$$
 (5.8)

From Eq. (4.29) we then obtain $T = t + O(v^2)$, and

$$Z_{21}^{0} + V_{21}t = C\xi_{21}(t)^{1-\sum \mu_{i}} \int_{0}^{1} d\zeta \zeta^{-\mu_{1}}(\zeta - 1)^{-\mu_{2}}(\zeta - \zeta_{3}(t))^{-\mu_{3}}$$
$$Z_{31}^{0} + V_{31}t = C\xi_{21}(t)^{1-\sum \mu_{i}} \int_{0}^{\zeta_{3}} d\zeta \zeta^{-\mu_{1}}(\zeta - 1)^{-\mu_{2}}(\zeta - \zeta_{3}(t))^{-\mu_{3}}$$
(5.9)

It follows, remarkably, that the ζ_3 motion decouples from that of ξ_{21} , in the form

$$\frac{Z_{21}^0 + V_{21}t}{Z_{31}^0 + V_{31}t} = \alpha + \frac{\beta}{g(\zeta_3)}, \quad (\zeta_3 = \frac{\xi_{31}}{\xi_{21}}), \tag{5.10}$$

where

$$g(\zeta) = \frac{\zeta^{1-\mu_1-\mu_3} \widetilde{F}(1-\mu_1,\mu_2,2-\mu_1-\mu_3;\zeta)}{\widetilde{F}(\mu_3,\sum\mu_i-1,\mu_1+\mu_3;\zeta)}$$

$$\alpha = e^{i\pi\mu_1} \frac{\sin\pi\mu_3}{\sin\pi(\mu_1 + \mu_3)}, \quad \beta = -e^{-i\pi\mu_3} \frac{\sin\pi\mu_3 \sin\pi(\mu_1 + \mu_2 + \mu_3)}{\sin\pi\mu_2 \sin\pi(\mu_1 + \mu_3)}$$
(5.11)

We thus see that g represents the "mapping function" for a problem with difference of exponents $1 - \mu_1 - \mu_3$, $1 - \mu_2 - \mu_3$, $\mu_1 + \mu_2 - 1$ at $\zeta_3 = 0, 1, \infty$ respectively.

This fact has an interesting interpretation, which is better seen in one of the degenerate limits, for instance $\zeta_3 = \frac{\xi_{13}}{\xi_{12}} \ll 1$. This situation corresponds to close crossing of particles #1 and #3 in minkowskian coordinates (Fig. (5)), so that the subsystem 13 performs an "internal" scattering. Since $\zeta_3 \ll 1$, and $g(\zeta) \sim \zeta^{1-\mu_1-\mu_3}$, Eq. (5.10) reduces in this limit to the expression

$$\left(\frac{\xi_{13}}{\xi_{12}}\right)^{1-\mu_1-\mu_3} \simeq \text{ const. } \frac{V_{13}t}{Z_{12}} ,$$
 (5.12)

in which $\xi_{12}(Z_{12})$ is slowly varying with respect to $\xi_{13}(Z_{13})$. Therefore particles #1 and #3 scatter much in same way as in the two-body case, with the static mass $m_1 + m_3$ playing the role of total mass of the subsystem. One can also verify, by using hypergeometric identities, that Eq. (5.10) can be rewritten as

$$\frac{Z_{13}}{Z_{12}} = \frac{g(1)}{g(1) - g(\infty)} \left(1 - \frac{g(\infty)}{g(\zeta_3)} \right).$$
(5.13)

Therefore, a behaviour of type (5.12) holds for $\zeta_3 \to 1$ ($\xi_{23} << \xi_{12}$) and $\zeta_3 \to \infty$ ($\xi_{12} << \xi_{13}$) also, the relevant mass being $m_2 + m_3$ and $m_1 + m_2$ respectively. In other words, the equations for the relative shape motion, Eq. (5.10), is clever enough to be consistent with the decoupling properties of the two-body subsystems in the relevant limits.

A second point to be noticed is the possibility of "fixed points" of the mapping (5.10). For arbitrary initial conditions, the quantity Z_{21}/Z_{31} in the l.h.s., varying with time, describes a circle starting and ending at V_{21}/V_{31} . For proper initial conditions, however, it will be just a constant, thus yielding the equations

$$\frac{V_{21}}{V_{31}} = \alpha + \frac{\beta}{g(\zeta_0)} , \ \frac{\xi_{31}(t)}{\xi_{21}(t)} = \zeta_0 = const.$$
(5.14)

In this situation, the ζ_3 variable does not move, neither does $\eta(\zeta_3)$. Therefore, the three particles will sit at the vertices of a triangle, whose angular shape is fixed by ζ_0 , given implicitly by Eq. (5.14), the only freedom being the overall scale $\xi_{21}(t)$, which by (5.9) follows a two-body motion

$$\widetilde{C}(\xi_{21})^{1-\sum \mu_i} = Z_{21}^0 + V_{21}t ,$$

$$\widetilde{C}/C = \int_0^1 d\zeta \, \zeta^{-\mu_1} (\zeta - 1)^{-\mu_2} (\zeta - \zeta_0)^{-\mu_3}.$$
(5.15)

Therefore, in this case there is only one "scattering angle", corresponding to the total mass, as in Eq. (3.40). This feature also is expected to have a generalization to arbitrary speed, an example of which will be shown in the next subsection.

Finally, let us note that so far the explicit form of $\eta(\zeta_3)$ has played no role for the motion. However, this feature disappears to next order in the V_i 's, because the entries O(V) and $O(V^2)$ in Eq. (5.8) contain the explicit form (4.16) of $f(\zeta)$, with its $\eta(\zeta_3)$ dependence. Some further insight on the role of the η 's will appear in the following example.

5.2 A symmetric N-body case

We have noticed that at the fixed points of Eq. (5.14), the variable $\zeta_3 = \xi_{31}(t)/\xi_{21}(t) = \zeta_0$ stays fixed, and the motion is effectively of two-body type, with fixed triangular configuration of the ζ 's in the three-body case. This fact admits a generalization to the *N*-body case by just splitting the singularities of a two-body mapping function by a change of variables.

Consider in fact the two-body mapping function in Eq. (3.23) with $V_1 = 0$ and $V_2 = th^{-1}\eta_{21}$, and perform on it the change of variables $z = \zeta^N$, as follows

$$F(\zeta) = cth \frac{1}{2} \eta_{21} \frac{\zeta^{N\mu_1} \widetilde{F}(\frac{1}{2}(1+\mu_{\infty}+\mu_1-\mu_2), \frac{1}{2}(1-\mu_{\infty}+\mu_1-\mu_2), 1+\mu_1; \zeta^N)}{\widetilde{F}(\frac{1}{2}(1+\mu_{\infty}-\mu_1-\mu_2); \frac{1}{2}(1-\mu_{\infty}-\mu_1-\mu_2), 1-\mu_1; \zeta^N)}.$$
 (5.16)

The singularity at z = 1 is now split into N ones, at $\zeta = \omega_k = exp(2\pi i k/N)$, the N-th roots of unity (Fig. 6).

Correspondingly the potential, which transforms as the Schwarzian derivative, becomes

$$2Q(\zeta) = \{f, z\} \left(\frac{dz}{d\zeta}\right)^2 + \{z, \zeta\} = 2q(z)(N\zeta^{N-1})^2 + (1 - N^2)/2\zeta^2 = \frac{1}{2} \left[\frac{1 - (N\mu_1)^2}{\zeta^2} + \frac{(1 - \mu_2^2)N^2\zeta^{2N-2}}{(1 - \zeta^N)^2} + \frac{N^2(1 - \mu_1^2 - \mu_2^2 + \mu_\infty^2)\zeta^{N-2}}{1 - \zeta^N}\right].$$
 (5.17)

As expected, this expression shows singularities at $\zeta = 0$ ($\zeta = \omega_k$) with difference of exponents $N\mu_1(\mu_2)$ respectively. In fact, by using the identities

$$\sum_{k=1}^{N} (\zeta - \omega_k)^{-1} = \frac{N\zeta^{N-1}}{\zeta^N - 1}, \sum_{k=1}^{N} (\zeta - \omega_k)^{-2} = \frac{N\zeta^{N-2}(\zeta^N + N - 1)}{(\zeta^N - 1)^2},$$

Eq. (5.17) can be rewritten as

$$2Q(\zeta) = \frac{1}{2} \frac{(1 - (N\mu_1)^2)}{\zeta^2} + \sum_{n=1}^N \frac{1}{2} \frac{(1 - \mu_2^2)}{(\zeta - \omega_k)^2} + \sum_{n=1}^N \frac{\beta_k}{(\zeta - \omega_k)},$$
(5.18)

where

$$\beta_k \equiv \frac{1}{2\omega_k} [N(\mu_1^2 - \mu_\infty^2) - (1 - \mu_2^2)]$$
(5.19)

are the accessory parameters.

Furthermore, by either the constraint in Eq. (4.4) or by direct inspection of the asymptotic behaviour in Eq. (5.16) we find

$$\frac{\mathcal{M}^{(N)}}{2\pi} - 1 = \mu_{\infty}^{(N)} = N\mu_{\infty} = N\left(\frac{\mathcal{M}^{(2)}}{2\pi} - 1\right) , \qquad (5.20)$$

where we have defined the total mass according to Eq. (3.19) (Eq. (4.26)) for the two - (many -) body case. It follows that

$$\mathcal{M}^{(N)} = N \mathcal{M}^{(2)} - 2\pi (N-1)$$
(5.21)

is the total mass of the system.

Finally one can check that the monodromies at $\zeta = \omega_k$ are those expected for a velocity $exp(-2\pi i\mu_1 k)V_2$, and that the total monodromy is

$$L^{(N)} = R_1^N R_1^{-(N-1)} L_2 R_1^{(N-1)} \dots (R_1^{-1} L_2 R_1) \cdot L_2 = (R_1 L_2)^N, \qquad (5.22)$$

corresponding to a total momentum

$$P^{(N)} = NP^{(2)}, \quad (\text{mod}2\pi \ n \ \frac{P^{(2)}}{\mathcal{M}^{(2)}}), \qquad (5.23)$$

n = -(N-1) being the choice (5.21).

Then it would seem that we have found for free an N-body system with masses $N\mu_1$ at $\zeta = 0$, μ_2 at $\zeta = \omega_k (k = 1, ..., N)$ and a nontrivial mass $\mathcal{M}^{(\mathcal{N})}$ at $\zeta = \infty$, the basic simplification being the fixed symmetric positions of the singularities, corresponding to a two-body motion of the overall scale.

A closer look shows, however, that the interpretation of the singularity at $\zeta = 0$ should be corrected. First, notice that the total mass (5.22) is correctly smaller than 2π if $\mathcal{M}^{(2)}$ is. However, its threshold value is

$$\mathcal{M}^{(N)} \ge Nm_2 + N(m_1 - 2\pi(1 - \frac{1}{N})).$$
 (5.24)

This suggests that the problem makes sense only if $m_1 \ge 2\pi(1-\frac{1}{N})$, $m_2 \le 2\pi/N$, and that the actual mass at the origin is

$$\mathcal{M}^{(1)} = Nm_1 - 2\pi(N-1) \le 2\pi \tag{5.25}$$

much as in Eq. (5.22). Furthermore the function $N(\zeta, t)$ for the N-body system is easily calculated from its transformation properties

$$N^{(N)}(\zeta) = N^{(2)}(\zeta^{N}) \left(\frac{dz}{d\zeta}\right)^{2} = C \left(\xi_{21}\right)^{\frac{\mathcal{M}^{(2)}}{2\pi} - 1} \frac{N^{2} \zeta^{N-2}}{\prod_{k} (\zeta - \omega_{k})}$$
(5.26)

and shows no pole at $\zeta = 0$, but rather a zero of order (N-2). As a consequence

$$\frac{N}{f'} \stackrel{\zeta \to 0}{\simeq} \zeta^{-\frac{Nm_1}{2\pi} + N - 1} , \qquad (5.27)$$

and this is, according to the boundary condition (3.24), the behaviour appropriate for the mass in Eq. (5.24).

Therefore the behaviour at the origin $F' \simeq \zeta^{N\mu_1-1}$ arises because of the mass (5.24), degenerate with a zero of order (N-2). In particular, if m_1 takes the value

$$m_1 = 2\pi (1 - \frac{1}{N}), \quad F' \simeq \zeta^{N-2}$$
 (5.28)

there is no physical mass at the origin, but just (N-2) degenerate apparent singularities !

The above interpretation is confirmed by the explicit calculation of $\eta(\zeta)$ in the quasistatic three-body case in Eq. (5.1). If we take equal masses, and we set $\zeta_3 = e^{i\pi/3}$, so that the singularities form an equilater triangle, then $\eta(e^{i\pi/3})$ becomes the center of such triangle, which corresponds to the origin in the present example.

6 Discussion

We have given, in this paper, a clear picture of the two-body system in our coordinates with instantaneous propagation, and we have also illustrated some interesting features of the classical N-body problem which may have an impact on the quantized theory. The two-body motion (Sec. 3.3) is actually equivalent to a single body one, for the relative trajectory $\xi(t) \equiv \xi_2(t) - \xi_1(t)$ which follows the test body geodesics in the field of the total invariant mass \mathcal{M} . Thus, the latter plays the role of hamiltonian of the system, and this presumably justifies previous derivations of quantum mechanical amplitudes [5],[6], provided the classical scattering angle is correctly identified, as in Eq. (3.42).

Some novel features appear in the many-body problem, which are related to the "internal" motion of the dimensionless shape parameters $\zeta_i = \frac{\xi_{2i}}{\xi_{21}}$. Remarkably, the internal motion decouples, at least in the quasi-static case, from the one of the overall scale $\xi(t)$, which follows a two-body dynamics in the field of the total mass. The internal dynamics is markedly different, however. In fact, it is consistent with decoupling limits, when a two-body subsystem is singled out, but it also admits a motion with constant shape parameters, for proper initial condition. It is thus possible that quantization of the internal motion may yield several surprises, including the existence of resonances in the quantum amplitudes. This issue requires further study.

Looking now at the metric in Eq. (2.29), we see that it is expressed in terms of Liouville fields which have various links to conformal type theories. For instance, the Schwarzian derivative of the mapping function, which defines particle masses and momenta in an invariant way, is related to the classical energy-momentum tensor of such field. In fact, it is easy to show that

$$\{f, z\} = -2\partial_z^2 \tilde{\phi} - 2(\partial_z \tilde{\phi})^2 , \quad \tilde{\phi} \equiv \phi - \log|N|.$$
(6.1)

Furthermore, the accessory parameters of the quasi-static limit in Sec. 4.2 have the same form as for vertex functions of properly defined [26] conformal fields, provided that the masses take over some limiting values. A similar relationship has been advocated in Ref. [19], on the basis of previous work [27] on conformal parametrization of the general Riemann-Hilbert problem, in terms of Green's functions of nonlocal vertex operators. Although such remarks do not help much making the solution explicit, they show that more attention should be devoted to direct quantization of the longitudinal degrees of freedom of the problem [28], coupled to matter fields.

We should also point out some left over problems, even at the classical level.

We have not treated at all spinning particles, because in this case the Minkowskian coordinates are not well defined at the particle sites, due to the known time shift [7] in their rest frame. This suggests that the mapping to single-valued coordinates becomes singular close to the particles. In fact the δ' singularity in the energy momentum tensor representing the localized spin, implies a double pole in the meromorphic N-function, and then a singularity of the mapping and a Gribov horizon (|f| = 1). Thus, a more refined analysis is needed.

The results of this paper in the instantaneous gauge do not help understanding the issue of matter asymptotic states in (2+1)-gravity, if any. In fact, there is no way of decoupling particles for large times in the two-body case, and only in a limited way for the limit of N+1bodies to N. This is to be contrasted to the yet partial results obtained [10] in covariant gauges, where such decoupling is possible, but a redefinition of energy [21] is needed, and thus of scattering parameters (Sec. 3.5). The trouble is that the definition of the localized energy-momentum and of the scattering matrix is dependent on the coordinate frame and thus on the gauge choice.

On the whole, we feel however that the present treatment of the N-body problem has helped clarifying a lot the degrees of freedom of 2+1 Gravity with matter and thus may be the basis of further progress in the quantum theory much as already occured in conformal models.

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A Appendix - Integrals for the two-body motion

We show in the following the derivation of the relevant integrals I^0 , I^z , $I^{\bar{z}}$ necessary to make explicit the geodesic equation (2.35) for the two-body problem.

The three integrals have the following structure

$$I^{a}(0,\zeta) = \int_{0}^{\xi} \frac{dz}{z(1-z)} y_{\alpha} y_{\beta} c^{a}_{\alpha\beta}, \qquad (A.1)$$

where the only non-vanishing entries for the coefficients $c^a_{\alpha\beta}$ are

$$c_{+-}^0 = 1, \quad c_{--}^z = 1, \quad c_{++}^{\bar{z}} = 1.$$
 (A.2)

The integrals I^a can be computed exactly by deriving Eq (3.7) with respect to μ_{∞}^2 to obtain the special measure $\frac{1}{z(1-z)}$ and therefore the following identity:

$$\frac{y_{\alpha}y_{\beta}}{z(1-z)} = \frac{d}{dz}W\left(y_{\alpha}, \frac{dy_{\beta}}{d\mu_{\infty}^2}\right)$$
(A.3)

Specializing Eq. (A.1) to the case of $f_{(1)}$ in Eq. (3.23), we obtain

$$I^{0}(0,\xi) = \int_{0}^{\xi} \frac{dz}{z(1-z)} y_{+}y_{-} = +\frac{2}{\mu_{\infty}} (\log F(a,b,c,\xi))_{/\mu_{\infty}} - \frac{k_{+}k_{-}}{\mu_{\infty}} \xi \tilde{F}(a,b,c,\xi) \tilde{F}(a',b',c',\xi) \left(\frac{ab}{c} \frac{F(c-a,c-b,c+1;\xi)}{F(a,b,c;\xi)}\right)_{/\mu_{\infty}},$$

$$I^{z}(0,\xi) = \int_{0}^{\xi} \frac{dz}{z(1-z)} y_{-}^{2} = -2 \frac{k_{-}^{2}}{\mu_{\infty}} \xi^{2\lambda_{1}^{-}} \tilde{F}^{2}(a,b,c,\xi) \left(\frac{ab}{c} \frac{F(c-a,c-b,c+1;\xi)}{F(a,b,c;\xi)}\right)_{/\mu_{\infty}},$$

$$I^{\bar{z}}(0,\xi) = \int_{0}^{\xi} \frac{dz}{z(1-z)} y_{+}^{2} = -2 \frac{k_{+}^{2}}{\mu_{\infty}} \xi^{2\lambda_{1}^{+}} \tilde{F}^{2}(a',b',c',\xi) \left(\frac{ab}{c} \frac{F(c'-a',c'-b',c'+1;\xi)}{F(a,b,c;\xi)}\right)_{/\mu_{\infty}}$$
(A.4)

where

$$k_{-}^{2} = \frac{\gamma_{12}V_{21}}{\gamma_{12} + 1} \frac{\pi}{\sin \pi \mu_{1}} \frac{1}{\Gamma(a)\Gamma(b)\Gamma(a')\Gamma(b')} ,$$

$$k_{+}^{2} = \frac{\gamma_{12}\bar{V}_{21}}{\gamma_{12} - 1} \frac{\pi}{\sin \pi \mu_{1}} \frac{1}{\Gamma(a)\Gamma(b)\Gamma(a')\Gamma(b')} .$$
(A.5)

By then setting $\xi = 1$ we obtain the final result

$$I_{(1)}^{0} = \frac{1}{\mu_{\infty}} \frac{\sin \pi a \sin \pi b}{\sin \pi \mu_{1} \sin \pi \mu_{2}} [\psi(a) - \psi(b)] + \frac{1}{\mu_{\infty}} \frac{\sin \pi a' \sin \pi b'}{\sin \pi \mu_{1} \sin \pi \mu_{2}} [\psi(1 - a') - \psi(1 - b')] ,$$

$$I_{(1)}^{z} = \frac{1}{\mu_{\infty}} \frac{\gamma_{12} V_{21}}{\gamma_{12} + 1} \frac{\sin \pi a' \sin \pi b'}{\sin \pi \mu_{1} \sin \pi \mu_{2}} [\psi(a) - \psi(b) + \psi(1 - a') - \psi(1 - b')] ,$$

$$I_{(1)}^{\bar{z}} = \frac{1}{\mu_{\infty}} \frac{\gamma_{12} \bar{V}_{21}}{\gamma_{12} - 1} \frac{\sin \pi a \sin \pi b}{\sin \pi \mu_{1} \sin \pi \mu_{2}} [\psi(a') - \psi(b') + \psi(1 - a) - \psi(1 - b)] , \qquad (A.6)$$

where the subscript (1) refers to the rest frame of particle #1, and $\psi(z) = \frac{d \log \Gamma(z)}{dz}$. Since $\psi(z) - \psi(1-z) = \pi \cot \pi z$, we derive from (A.6) the simpler formula (3.44) used in the text.

Let us comment about the vector property of $I^a(0,\xi)$. First, note that $I^a_{(1)}(0,\zeta)$ changes by just a phase when z turns around ξ_1 , this being the relevant monodromy. So does $I^a_{(2)}$ when z turns around ξ_2 . But $f_{(2)}$ and $f_{(1)}$ are related by a boost to the rest frame of particle # 2. Therefore $I^a_{(1)}$ has the correct monodromy around each particle, and the mapping in Eq. (3.31) automatically satisfies the complete DJH matching conditions in Eq. (2.10), including the translational part.

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Figure Captions

Fig.1: Particles with tails, and related monodromies.

Fig.2: a) Schwarz triangle, for the mapping function $f_{(1)}(z)$ in the case $\mathcal{M} < 2\pi$, where $f_{-}(\infty)$ is the Schwarz reflection of $f_{+}(\infty)$ through the segment [f(o), f(1)]. b) The same for $\mathcal{M} = 2\pi (1 + i\sigma)$; the intermediate closed curve is image of a straight line with constant $\theta = \arg(\zeta)$. The critical value $f\bar{f} = 1$ is crossed an infinity of times.

Fig.3: Tails and values of $f(\infty)$ for a given cyclic ordering of N particles.

Fig.4: f - Mapping for N = 3. The deficit angles at particle sites are shown together with the ones at space infinity, whose sum is $2\pi - \mathcal{M}$.

Fig.5: Scattering of a two-body subsystem in the three-body motion.

Fig.6: Configuration of tails for the symmetric N-body problem.

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