

Modular Invariance and the Odderon

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September 24, 2018

Abstract

We identify a new symmetry for the equations governing odderon amplitudes, corresponding in the Regge limit of QCD to the exchange of 3 reggeized gluons. The symmetry is a modular invariance with respect to the unique normal subgroup of $SL(2, \mathbb{Z})$ of index 2. This leads to a natural description of the Hamiltonian and conservation-law operators as acting on the moduli space of elliptic curves with a fixed “sign”: elliptic curves are identified if they can be transformed into each other by an *even* number of Dehn twists.

PACS: 12.38.Bx;02.10.Rn

Keywords: Odderon, BFKL Pomeron, Modular Invariance, Elliptic Curves

1 Introduction

Recent experimental data in small Bjorken x region have gained much attention in the study of the Regge limit of QCD. Already in the late 70’s the amplitude corresponding to the exchange of two reggeized gluons was calculated, and the formula for the intercept of the renowned BFKL pomeron was derived (see [1] [2]). Lipatov’s solution depends in a crucial way on the global conformal symmetry of the problem. The BFKL equations were generalized to the case of the exchange of 3 reggeized gluons (so-called odderon) by Kwiciński, Praszalowicz [3] and Bartels [4]. Later, this approach was extended to the case of arbitrary number of reggeons in the form of the Generalised Leading Logarithm Approximation (GLLA) [4].

In the large N_c limit, the intriguing connection between the GLLA equations and exactly solvable lattice models was established. The GLLA equations are reduced in this limit to a Schroedinger equation with a two-body interaction Hamiltonian. Due to the holomorphic separability of this Hamiltonian the

problem reduces further to the exactly solvable Heisenberg XXX spin $s = 0$ chain ([5], [6], [7]). However, despite the richness of mathematical structures involved (global $SL(2, \mathbb{C})$ -invariance and abundance of integral of motions) as well as diversity of approaches to the problem (Yang-Baxter equations, Bethe ansatz([5], [6], [7], [8]) quasiclassical approximation[9]), the explicit expression for the intercept of the odderon has not been found yet.

In this letter we show that the odderon possesses (albeit in a somewhat hidden form) the modular symmetry, ubiquitous in conformal field theories (CFT) and string theory. In the next section we recall the theory of the odderon, then, following Lipatov, the consequences of global $SL(2, \mathbb{C})$ invariance. As new results, we analyse the role of cyclic symmetry in this framework and we explicitly demonstrate the link to modular invariance through two alternative descriptions of elliptic curves. This link may lead to the effective, two-dimensional string theory for QCD in the moduli space of elliptic curves with fixed “parity”, corresponding to the transformations of the torus through the even number of Dehn twists.

2 The Odderon

The Regge limit of QCD is defined as the kinematical region

$$s \gg -t \approx M^2 \tag{1}$$

where M is the hadron mass scale, or, in the case of Deep Inelastic Scattering, as the small $x = Q^2/s$ limit. Here we sketch, following [7], the equivalence between the Regge intercept of amplitudes and energy levels of a two-body Hamiltonian.

The aim is to find the Regge behaviour of the amplitude $A(s, t) \sim s^{\omega_0+1}$. Mellin transformation leads to

$$A(s, t) = is \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{M^2} \right)^\omega A(\omega, t) \tag{2}$$

so the Regge behavior corresponds to finding the poles of the amplitude $A(\omega, t)$. Rewriting this amplitude as the convolution of “hadron” wave functions $\Phi_{A,B}$ and a kernel $T(\{k_i\}, \{k'_j\}, \omega)$ we get:

$$A(\omega, t) = \int d^2 k_i \int d^2 k'_j \Phi_A(\{k_i\}) T(\{k_i\}, \{k'_j\}, \omega) \Phi_B(\{k'_j\}) \tag{3}$$

where $\{k_i\}$ and $\{k'_j\}$ are the transverse momenta of the N exchanged reggeons (in the case of odderon $N = 3$). The next step amounts to writing the Bethe-Salpeter equations for the kernel T :

$$\omega T(\omega) = T_0 + \mathcal{H}T(\omega) \tag{4}$$

where T_0 is the free propagator and \mathcal{H} is the operator corresponding to the insertion of single gluonic interactions between all pairs of reggeons. This equation can be formally solved:

$$T(\omega) = \frac{T_0}{\omega - \mathcal{H}} \quad (5)$$

Therefore the poles of the amplitude correspond to the eigenvalues of the hamiltonian operator \mathcal{H} . After performing Fourier transformation ($k_i \rightarrow b_i$) and using the complex notation $z_j := x_j + iy_j$, the Hamiltonian splits into a sum of a holomorphic part and an antiholomorphic part. In the large N_c limit the two commute. It is therefore sufficient to consider only the holomorphic part, which in the case of odderon reads

$$(H(z_1, z_2) + H(z_2, z_3) + H(z_3, z_1))\Psi(z_1, z_2, z_3) = E\Psi(z_1, z_2, z_3) \quad (6)$$

where

$$H(z_1, z_2) = \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) - L_{12}^2} - \frac{2}{l+1} \quad (7)$$

with

$$L_{12}^2 := -z_{12}^2 \frac{d}{dz_1} \frac{d}{dz_2} \quad (8)$$

being the holomorphic Casimir operator of the group $SL(2, \mathbb{C})$. The eigenvalue E of the holomorphic hamiltonian and the corresponding eigenvalue \bar{E} of the antiholomorphic one are related to the Regge intercept by the formula:

$$\omega_0 = \frac{\alpha_s N_c}{4\pi} (E + \bar{E}) \quad (9)$$

The celebrated BFKL solution ($N = 2$ case) corresponds in this language to finding the maximal eigenvalue of the equation

$$H(z_1, z_2)\Psi(z_1, z_2) = E\Psi(z_1, z_2) \quad (10)$$

and has the known solution

$$E = -4[\psi(m) - \psi(1)] \quad (11)$$

where ψ is the derivative of the logarithm of the Euler Γ function and m is a conformal weight. The maximum of (11) is achieved at $m = 1/2$ and reproduces the BFKL slope

$$\omega_0^{BFKL} = \frac{\alpha_s N_c}{\pi} 4 \ln 2 \quad (12)$$

2.1 Conservation laws

The Hamiltonian H is invariant with respect to the action of $SL(2, \mathbb{C})$ on holomorphic functions given by:

$$(g \cdot \Psi)(z_1, z_2, z_3) = \Psi \left(\frac{az_1 + b}{cz_1 + d}, \frac{az_2 + b}{cz_2 + d}, \frac{az_3 + b}{cz_3 + d} \right) \quad \text{for } g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, \mathbb{C}) \quad (13)$$

Therefore it commutes with the holomorphic Casimir operator for this representation:

$$\hat{q}_2 := -z_{12}^2 \frac{d}{dz_1} \frac{d}{dz_2} - z_{23}^2 \frac{d}{dz_2} \frac{d}{dz_3} - z_{31}^2 \frac{d}{dz_3} \frac{d}{dz_1} \quad (14)$$

This enables us to consider functions transforming under the unitary representations of $SL(2, \mathbb{C})$ labelled by $n \in \mathbb{N}$ and $\nu \in \mathbb{R}$. In this case the eigenvalue q_2 is $((1+n)/2 + i\nu)((-1+n)/2 + i\nu)$.

It has been shown [10] that the system possesses another integral of motion — an operator \hat{q}_3 :

$$\hat{q}_3 = z_{12}z_{23}z_{31}\partial_1\partial_2\partial_3 \quad (15)$$

which commutes with the hamiltonian H .

One of the strategies for solving the odderon problem, proposed by Lipatov [10], was to diagonalize the conservation laws \hat{q}_2 and \hat{q}_3 and to substitute the solution into the Schroedinger equation in order to find the energy eigenvalue.

2.2 Conformal ansatz

Lipatov [11] has chosen an ansatz, which automatically diagonalizes \hat{q}_2 :

$$\Psi_{z_0}(z_1, z_2, z_3) = \left(\frac{z_{12}z_{23}z_{31}}{z_{10}^2 z_{20}^2 z_{30}^2} \right)^{m/3} \varphi(\lambda) \quad (16)$$

where $m = 1/2 + i\nu + n/2$, n is an integer and ν is a real number. Here, $z_0 \in \mathbb{C}$ is just a parameter and λ is the anharmonic ratio:

$$\lambda = \frac{z_{12}z_{30}}{z_{13}z_{20}} \quad (17)$$

Lipatov further derived the form of the operator \hat{q}_3 within this ansatz. Inserting $\Psi_{z_0}(z_1, z_2, z_3)$ into the equation

$$\hat{q}_3 \Psi(z_1, z_2, z_3) = q_3 \cdot \Psi(z_1, z_2, z_3) \quad (18)$$

and canceling the factor $(\dots)^{m/3}$ he obtained:

$$\nabla_1 \frac{1}{\lambda(1-\lambda)} \nabla_2 \nabla_3 \varphi(\lambda) = q_3 \varphi(\lambda) \quad (19)$$

where

$$\nabla_1 = \frac{m}{3}(1 - 2\lambda) + \lambda(1 - \lambda)\partial, \quad (20)$$

$$\nabla_2 = \frac{m}{3}(1 + \lambda) + \lambda(1 - \lambda)\partial, \quad (21)$$

$$\nabla_3 = -\frac{m}{3}(2 - \lambda) + \lambda(1 - \lambda)\partial, \quad (22)$$

$$(23)$$

The Hamiltonian (6) has also been rewritten in terms of λ .

3 Cyclic invariance

It is easy to see that both the Hamiltonian H and \hat{q}_3 are invariant under cyclic permutations of the gluonic coordinates z_1, z_2, z_3 . We show now how this symmetry manifests itself in the formalism of the preceding section. Under the permutation $z_1 \rightarrow z_2 \rightarrow z_3$ the anharmonic ratio transforms as follows:

$$\lambda \rightarrow 1 - \frac{1}{\lambda} \rightarrow \frac{1}{1 - \lambda} \quad (24)$$

We postulate, that the ground state is symmetric under this transformation and so

$$\varphi(\lambda) = f(s_1, s_2, s_3, \tilde{j}) \quad (25)$$

where s_i are the symmetric polynomials in $x_1 = \lambda, x_2 = 1 - 1/\lambda$ and $x_3 = 1/(1 - \lambda)$, and \tilde{j} is the Vandermonde determinant. Namely

$$s_1 = x_1 + x_2 + x_3 = \frac{\lambda^3 - 3\lambda + 1}{\lambda(\lambda - 1)} \quad (26)$$

$$s_2 = x_1x_2 + x_2x_3 + x_3x_1 = \frac{\lambda^3 - 3\lambda^2 + 1}{\lambda(\lambda - 1)} \quad (27)$$

$$s_3 = x_1x_2x_3 = -1 \quad (28)$$

$$\tilde{j} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \quad (29)$$

It turns out that the only independent quantity is $A = s_1 + s_2 = \frac{(\lambda+1)(2\lambda-1)(\lambda-2)}{\lambda(\lambda-1)}$ related to \tilde{j} by the equation $4\tilde{j} = A^2 + 27$. It is convenient to introduce the notation:

$$B := 8A = \sqrt{j - 1728} \quad (30)$$

$$j := 256\tilde{j} = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \quad (31)$$

At this moment we make a refinement of Lipatov's ansatz, namely

$$\begin{aligned}\Psi_{z_0}(z_1, z_2, z_3) &= \left(\frac{z_{12}z_{23}z_{31}}{z_{10}^2z_{20}^2z_{30}^2}\right)^{m/3} f(B) \\ &= \left(\frac{z_{12}z_{23}z_{31}}{z_{10}^2z_{20}^2z_{30}^2}\right)^{m/3} f\left(8\frac{(\lambda+1)(2\lambda-1)(\lambda-2)}{\lambda(\lambda-1)}\right)\end{aligned}\quad (32)$$

where $m = 1/2 + i\nu + n/2$, n is an integer and ν is a real number. λ is the anharmonic ratio:

$$\lambda = \frac{z_{12}z_{30}}{z_{13}z_{20}}\quad (33)$$

Now we insert the function $\varphi(\lambda) = f(B)$ into the conservation law (19). After reexpressing the result in terms of j and $B = \sqrt{j - 1728}$ we get:

$$\begin{aligned}\left\{\frac{j^2}{2}\frac{d^3}{dB^3} + 2\sqrt{j-1728}j\frac{d^2}{dB^2} + \left(j\left(1 + \frac{m(1-m)}{6}\right) - 3 \cdot 2^8\right)\frac{d}{dB} + \right. \\ \left.\frac{(m-3)m^2}{27}\sqrt{j-1728} - 8q_3\right\}f(\sqrt{j-1728}) = 0\end{aligned}\quad (34)$$

The original Hamiltonian (6) expressed by Lipatov in terms of λ can also be recast using the functions $B = \sqrt{j - 1728}$ (although obtaining an explicit expression seems to be highly non-trivial).

In the next section we will show that the variable j can indeed be considered as a modular invariant and we give a geometric interpretation of the symmetry considered here.

4 Elliptic curves

According to one of the many possible definitions (see e.g. [12]), an elliptic curve is a complex curve of genus one. There are two alternative descriptions of these objects. The first one is the Weierstrass parametrization which labels each elliptic curve by a complex number $\lambda \in \mathbb{C}$. The curve given by λ is given by the equation

$$y^2 = x(x-1)(x-\lambda)\quad (35)$$

where x and y are complex coordinates. Two such curves are conformally isomorphic if and only if their j -invariants coincide. The j -invariant is given by the formula:

$$j = 2^8 \cdot \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}\quad (36)$$

Note that this expression is identical to the Vandermonde determinant considered before (31). The only problem that we encounter here in providing a geometric interpretation to our variables (30) and (31) is the fact that $\sqrt{j-1728}$ may differ in sign for isomorphic elliptic curves. Therefore one must consider elliptic curves with some additional structure, which we will define after presenting the other description of genus one curves.

We see here that, since the Hamiltonian can be expressed through the invariants $\sqrt{j-1728}$, it can be identified in a natural way with an operator acting on the moduli space of elliptic curves with that additional structure.

Alternatively one can view elliptic curves as complex tori $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ parametrized by $\tau \in \mathbb{C}$ in the upper half-plane. Another way of looking at this quotient space is to consider it as the torus obtained by identifying opposite edges in the parallelogram bounded by $0,1,\tau$ and $1+\tau$. The j -invariant is now a transcendental function of τ . This description is linked to the preceding one by the correspondence [13]:

$$\lambda(\tau) = \left(\frac{\Theta_2(0; \tau)}{\Theta_3(0; \tau)} \right)^4 \quad (37)$$

where $\Theta_2(0; \tau)$ and $\Theta_3(0; \tau)$ are the Jacobi theta functions. Moreover, the symmetry which leaves j invariant corresponds in this description to modular invariance in the τ -plane i.e.

$$j(\tau) = j(\tau') \iff \tau' = \left(\frac{a\tau + b}{c\tau + d} \right) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (38)$$

In our case, the symmetry $\lambda \rightarrow 1 - \frac{1}{\lambda} \rightarrow \frac{1}{1-\lambda}$ corresponds to modular transformations belonging to Γ^2 — the unique normal subgroup of $SL(2, \mathbb{Z})$ of index 2 [13]. This is an infinite group generated by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. We still have to define the additional geometric structure on the torus which is left invariant by the subgroup Γ^2 .

The two elementary Dehn twists, which generate the full modular group, are associated to the two noncontractible loops of winding number one. The “Dehn twist” operation consists of cutting the torus along the chosen loop, twisting one boundary by 2π , and gluing it back (see e.g. [14]). Although the number of Dehn twists (n_D) is not well defined, given an isomorphism corresponding to a modular transformation between equivalent tori, the parity of number of Dehn twists ($(-1)^{n_D}$) is well defined. The subgroup Γ^2 corresponds precisely to the transformations of the torus through an even number of Dehn twists. We may therefore attach a kind of “sign” to each torus.

Using the correspondence between λ and τ one can reexpress the Hamiltonian and the integral of motion in terms of τ .

5 Conclusions

In this letter we have shown that the odderon problem possesses a new symmetry - *i.e.* modular symmetry with respect to Γ^2 — an index 2 normal subgroup of $SL(2, \mathbb{Z})$. Expressing the conservation law (19) through modular invariants (34) leads in a natural way to the new methods of solving 3^{rd} order Fuchsian differential equations proposed by B.H. Lian and S-T Yau's in the framework of mirror symmetry [15]. This analogy may be an aid in carrying out Lipatov's strategy mentioned in section 2.1 and may lead to the analytical solution of the odderon problem.[16]

Apart from the practical applications of this symmetry, we hope that it may lead to deeper understanding of the Regge limit of QCD. The modular invariance of the odderon leads to a natural interpretation of all the operators as acting on the moduli space of genus one curves with fixed 'sign', *i.e.* the even-parity of the number of Dehn twists. In particular this may be a further step in establishing the relation between QCD and effective string theory.

Acknowledgments

I would like to thank Dr. Maciej A. Nowak for suggesting this investigation and for fruitful discussions. I would like to express my gratitude to Prof. Don Zagier for e-mail correspondence which enhanced my understanding of the $SL(2, \mathbb{Z})$ structures.

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