

# Modular Covariance and the Algebraic PCT/Spin-Statistics Theorem

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## Abstract

In the theory of nets of observable algebras, the modular operators associated with wedge regions are expected to have a natural geometric action, a generalization of the Bisognano-Wichmann condition for nets associated with Poincaré-covariant fields. Here many possible such modular covariance conditions are discussed (in spacetime of at least three dimensions), including several conditions previously proposed and known to imply versions of the PCT and spin-statistics theorems. The logical relations between these conditions are explored: for example, it is shown that most of them are equivalent, and that all of them follow from appropriate commutation relations for the modular automorphisms alone. These results allow us to reduce the study of modular covariance to the case of systems describing non-interacting particles. Given finitely many Poincaré-covariant non-interacting particles of any given mass, it is shown that modular covariance and wedge duality must hold, and the modular operators for wedge regions must have the Bisognano-Wichmann form, so that the usual free fields are the only possibility. For models describing interacting particles, it is shown that if they have a complete scattering interpretation in terms of such non-interacting particles, then again modular covariance and wedge duality must hold, and the modular operators for wedge regions must have the Bisognano-Wichmann form, so that wedge duality and the PCT and spin-statistics theorems must hold.

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Running title: Modular Covariance

# I Introduction

Among the most important results of the axiomatic formulation of quantum field theory are the proofs of the existence of a PCT operator [15], and of the connection between spin and statistics [19, 8]. However, these results rely on complicated analytic continuation arguments that depend heavily on the detailed structure of Wightman fields. It would be strange if such highly physical properties did not have a simpler and more general proof. Here we present one such proof, actually a demonstration of the stronger property of modular covariance, for nets of local algebras satisfying asymptotic completeness with certain restrictions on their particle content.

It was the work of Bisognano and Wichmann [1, 2] that first introduced the notion that there should be a geometric interpretation attached to the modular conjugation and automorphisms with respect to the vacuum for the algebras of operators associated with certain highly symmetric spacetime regions. They worked with algebras associated with a complementary pair of wedge-shaped regions, within the context of a set of finite-component Lorentz-covariant Wightman fields. In this setting they showed that duality must hold for such a pair of algebras, that the corresponding modular automorphisms with respect to the vacuum must be the velocity transformations that leave the wedge invariant, and finally that the modular conjugations must be antiunitary reflections—essentially versions of the PCT operator, but with parity replaced by a reflection appropriate to the wedge. The property of duality for such regions, known as wedge duality, implies essential duality for the corresponding net of local algebras.

There has been a great deal of interest recently in abstracting these notions to nets of local algebras not necessarily associated with any Wightman field, for they seem to encode many of the desirable properties of fields in a more physically direct manner. In particular, they imply versions of the PCT and spin-statistics theorems (in the Bisognano-Wichmann theory, by contrast, the PCT operator is taken as a necessary input). This has been spurred by the proof due to Borchers that a weaker result, the covariance of the translations under modular conjugations and automorphisms, holds under very general assumptions [3]. This was then followed by a number of related results [22, 23, 24, 7] concerning the interrelations of the modular structure and the translations, for the most part summarized in [4]. In two spacetime dimensions, these are the only relationships required; also for conformally covariant nets they imply everything desired [11, 5, 25, 10]. We are concerned here, however, with the remaining

cases, for which it is still not clear what can be proved and what must be assumed.

The Bisognano-Wichmann conditions cannot hold generally as they stand, for it is easy to construct counterexamples using infinite-component fields [21, 18]. However, the essence of these examples is that one is free to specify a representation of the Poincaré group for which the Bisognano-Wichmann conditions do not hold. The modular operators retain their geometric interpretation, but they generate a different representation of the Poincaré group. For this reason one wishes to use a criterion that is independent of any specified representation of the Poincaré group: one that simply describes the geometric interpretation of the action of the modular structure on the local structure of a given net. Since both structures are somewhat complicated, there are a number of such criteria that might be and have been proposed, and an even wider variety of names for them. The first goal of this paper is to set out some of these criteria, which we will refer to generically as relations of modular covariance, and to clarify their interrelationships. In particular, we are concerned with the two papers [13] and [16], which derive related results from somewhat different modular covariance conditions. Here (in Theorem 7) we show that under natural assumptions these conditions (and many others) are all equivalent, and that all of them in fact follow from much weaker modular covariance premises, ones which can be expressed entirely in terms of the modular structure, without reference to the precise local structure of the net. This we regard as essential to further study of the possible modular structures of nets. As we will see, it allows us to reduce this to the study of nets without interaction.

The modular covariance conditions of Theorem 7 imply the existence of a PCT operator in even spacetime dimensions. As stated, however, they apply only to observable nets, for which they imply that the spin must be integral. For the full spin-statistics theorem it is necessary to extend the assumptions and modular covariance conditions slightly to cover field nets containing both bosonic and fermionic quantities, with normal commutation relations (Theorem 7'). These results are sufficiently general as to justify our calling a net 'modular covariant' if and only if it satisfies the conditions of Theorem 7'. This implies the existence of a natural representation of the Poincaré group (and, in even spacetime dimensions, a PCT operator), determined entirely by the modular structure, under which the field net is automatically covariant, and for which the Bisognano-Wichmann and spin-statistics relations hold. In addition, modular covariance implies wedge duality, the strongest duality condition that can be expected under these circumstances. It is also known that, under rather mild conditions, mod-

ular covariance for the observable net implies modular covariance for the field net, but we discuss this only briefly here.

We then turn to the study of certain nets without interaction, those described in terms of a one-particle space by means of Weyl operators. For these nets the entire structure is determined by the restriction of the modular operators to the one-particle space, which determines the localization properties of the one-particle states. If modular covariance holds on the one-particle space, then it holds for the entire net, which then arises from free fields of the usual sort; on the other hand, if modular covariance does not hold on the one-particle space, then the net cannot arise from a set of fields. We show that if the one-particle space carries a physically reasonable (positive energy, finite spin, finite multiplicity for each mass) representation of the Poincaré group, and if certain standard properties are obeyed, then modular covariance must hold. In these cases the usual one-particle localization is unique, and all such nets arise from free fields. The representation of the Poincaré group is unique, and the Bisognano-Wichmann condition holds. The PCT operator, however, is determined only up to unitary equivalence.

Finally, we consider Poincaré-covariant nets having a complete asymptotic particle interpretation in terms of non-interacting particles of the sort just described. Scattering theory in this case is well developed, at least provided all particles have discrete positive masses, and we now have the additional information that the asymptotic particles can be described only by free fields. We are able to adapt some previous results [18] to show that there is a close relationship between the modular operators for the in-fields, the out-fields, and the interacting net: they all satisfy modular covariance with respect to the unique representation of the Poincaré group, differing only in their choice of PCT operators, and this difference describes the scattering. Thus we see that the Bisognano-Wichmann condition, wedge duality, the PCT theorem, and the spin-statistics theorem hold not only for nets associated with fields, but in addition for all nets with reasonable (massive) scattering behavior.

## II Notions of Modular Covariance

We begin with Minkowski space  $\mathcal{M}$  of  $d$  spacetime dimensions, with coordinates  $x = (x_0, x_1, \dots, x_{d-1})$ , where  $x_0$  is the time coordinate. We will in general assume  $d > 2$ , since although some of our results hold also in the lower-dimensional cases, they

are no longer particularly relevant there. For the sake of concreteness, one might simply take  $d = 3$ , since the results are generally such that if they hold in three dimensions they hold also in higher dimensions, but we will avoid explicit references to  $d$ . The  $d$ -dimensional Poincaré group  $\mathcal{P}$  is the group of all inhomogeneous linear transformations of  $\mathcal{M}$  preserving the metric with diagonal elements  $(1, -1, \dots, -1)$ . We will be particularly interested in the subgroups  $\mathcal{P}_+$ , the proper Poincaré group, generated by the translations and all homogeneous transformations of determinant  $+1$ , and  $\mathcal{P}_+^\uparrow$ , the restricted Poincaré group, consisting of all proper orthochronous Poincaré transformations (those that preserve the sign of the time component). We will also make use of their universal covering groups  $\tilde{\mathcal{P}}_+$  and  $\tilde{\mathcal{P}}_+^\uparrow$ .

Within Minkowski space we distinguish the family of wedge regions: the particular complementary pair of wedges  $W_R = \{x \mid x_1 > |x_0|\}$  and  $W_L = \{x \mid x_1 < -|x_0|\}$  and their Poincaré transforms. (We will think of  $W_L$  as complementary to  $W_R$ , and write  $W'_R = W_L$ , even though strictly speaking  $W'_R = \overline{W_L}$ ). A complementary pair of wedges has a common vertex (for  $W_R$  and  $W_L$ , the hyperplane  $x_1 = x_0 = 0$ ) and opposite sets of directions (for  $W_R$ , a set of directions including  $\hat{x}_1$ ; for  $W_L$ , a set including  $-\hat{x}_1$ ). We will write  $W_1 \parallel W_2$  if the vertices of  $W_1$  and  $W_2$  are parallel, specialized as  $W_1 \parallel_s W_2$  if in addition they have the same directions, or  $W_1 \parallel_a W_2$  if they have opposite directions.

To each wedge  $W$  we associate certain Poincaré transformations (which we will actually use primarily as maps of the family of wedges): a reflection  $j(W)$  about the vertex of the wedge, and a one-parameter family of velocity transformations (in the appropriate reference frame, and with an appropriate scale)  $\lambda(W, t)$  in the direction of the wedge, both leaving the vertex fixed. For example,  $j(W_R)(x_0, x_1, x_2, \dots, x_d) = (-x_0, -x_1, x_2, \dots, x_d)$ , and  $\lambda(W_R, t)(x_0, x_1, x_2, \dots, x_d) = (x'_0, x'_1, x_2, \dots, x_d)$ , with  $x'_0 = x_0 \cosh 2\pi t + x_1 \sinh 2\pi t$  and  $x'_1 = x_1 \cosh 2\pi t + x_0 \sinh 2\pi t$ . In general,  $j(W)$  and  $\lambda(W, t)$  are conjugates of these within the Poincaré group. Thus  $j(W)$  is a proper but time-reversing involutory Poincaré transformation, which interchanges  $W$  with  $W'$ , while the  $\lambda(W, t)$  form a one-parameter group of proper orthochronous Poincaré transformations, leaving  $W$  and  $W'$  invariant, and such that  $\lambda(W', t) = j(W)\lambda(W, t)j(W) = \lambda(W, -t)$ . The  $\lambda(W, t)$  generate the entire restricted Poincaré group; since  $d > 2$ , the  $j(W)$  generate the entire proper Poincaré group.

For our purposes, the only data required from a net will be a map from the family of wedges  $W$  to a family  $\mathcal{A}(W)$  of von Neumann algebras of operators on a Hilbert space  $\mathcal{H}$ , with a distinguished vacuum vector  $\Omega$ . The conclusions drawn will also apply

directly only to the wedge algebras; if these results are to be applied to a local net, then to begin with it must satisfy essential duality, and the statements must be considered as referring to its dual net. In what follows we will make the following assumptions:

- (i)  $\mathcal{A}(W') \subset \mathcal{A}(W)'$  (locality);
- (ii)  $\mathcal{A}(W_1) \subset \mathcal{A}(W_2)$  whenever  $W_1 \subset W_2$  (isotony);
- (iii)  $\Omega$  is a cyclic vector for  $\mathcal{A}(W_1) \cap \mathcal{A}(W_2)$  whenever  $W_1 \cap W_2 \neq \emptyset$  (cyclicity);
- (iv) for all  $W_1$  and  $W_2$ ,  $\mathcal{A}(W_1) = \{A(W_1) \cap \mathcal{A}(W) \mid W \parallel_s W_2\}''$  (an additivity property).

If the wedge algebras are in fact derived from a local net, then (i)–(iv) follow from standard assumptions, but we prefer to state them here in the form required. We will show in many cases that a stronger version of (i) actually holds, namely  $\mathcal{A}(W') = \mathcal{A}(W)'$  (wedge duality). This implies essential duality, so that if the wedge algebras are derived from a local net, then there is some maximal local net consistent with them, which satisfies duality.

Assumptions (i)–(iv) imply, among other things, that for every wedge  $W$ , there are modular involution, conjugation, and automorphism operators  $S(W)$ ,  $J(W)$  and  $\Delta(W)$  for the pair of algebras  $\mathcal{A}(W)$ ,  $\mathcal{A}(W)'$  with respect to the vacuum  $\Omega$ . They may be defined by the unique polar decomposition  $S(W) = J(W)\Delta(W)^{1/2}$  of the closed antilinear operator  $S(W)$ , where  $S(W)$  is defined such that  $S(W)X\Omega = X^*\Omega$  for every  $X \in \mathcal{A}(W)$ . Then  $S(W)\psi = \psi$  if and only if  $\psi \in \overline{\mathcal{A}(W)^{\text{sa}}\Omega}$ .  $S(W)$  is an antilinear involution,  $J(W)$  is an antiunitary involution, and  $\Delta(W)$  is self-adjoint and positive. The modular conjugation is the adjoint action of  $J(W)$ , and the modular automorphism group is the one-parameter group given by the adjoint action of  $\Delta(W)^{it}$ . The properties of these operators and their actions are well known from the Tomita-Takesaki theory: for example,  $J(W)\Omega = \Delta(W)\Omega = 0$  and  $J(W)\Delta(W)J(W) = \Delta(W)^{-1}$ ; also  $J(W)\mathcal{A}(W)J(W) = \mathcal{A}(W)'$ , and  $\Delta(W)^{it}\mathcal{A}(W)\Delta(W)^{-it} = \mathcal{A}(W)$  for all real  $t$ .

Where necessary, we will also make the following additional assumption:

- (v)  $J(\Lambda W)$  is a weakly continuous function of  $\Lambda \in \mathcal{P}_+^\uparrow$  (a continuity property).

This property too typically holds for most nets that are usually considered; for example, it follows from the covariance of the net under any strongly continuous unitary representation of  $\mathcal{P}_+^\uparrow$ . For some of our results, however, it must be specifically assumed.

Then relations of modular covariance will connect the action of the modular conjugation operators  $J(W)$  with the transformations  $j(W)$ , and the action of the modular automorphism operators  $\Delta(W)^{it}$  with the transformations  $\lambda(W, t)$ . Let us list a number of possible conditions:

(a) covariance under modular conjugations of modular conjugations,

$$J(W_1)J(W_2)J(W_1) = J(j(W_1)W_2); \quad (1)$$

(b) covariance under modular conjugations of modular automorphisms,

$$J(W_1)\Delta(W_2)J(W_1) = \Delta(j(W_1)W_2); \quad (2)$$

(c) covariance under modular conjugations of modular involutions,

$$J(W_1)S(W_2)J(W_1) = S(j(W_1)W_2); \quad (3)$$

(d) covariance under modular conjugations of wedge algebras,

$$J(W_1)\mathcal{A}(W_2)J(W_1) = \mathcal{A}(j(W_1)W_2); \quad (4)$$

(e) covariance under modular automorphisms of modular conjugations,

$$\Delta(W_1)^{it}J(W_2)\Delta(W_1)^{-it} = J(\lambda(W_1, t)W_2); \quad (5)$$

(f) covariance under modular automorphisms of modular automorphisms,

$$\Delta(W_1)^{it}\Delta(W_2)\Delta(W_1)^{-it} = \Delta(\lambda(W_1, t)W_2); \quad (6)$$

(g) covariance under modular automorphisms of modular involutions,

$$\Delta(W_1)^{it}S(W_2)\Delta(W_1)^{-it} = S(\lambda(W_1, t)W_2); \quad (7)$$

(h) covariance under modular automorphisms of wedge algebras,

$$\Delta(W_1)^{it}\mathcal{A}(W_2)\Delta(W_1)^{-it} = \mathcal{A}(\lambda(W_1, t)W_2); \quad (8)$$

(i) the modular conjugations  $J(W)$  are the representatives of  $j(W)$  under some representation of the proper Poincaré group;

(j) the modular automorphisms  $\Delta(W)^{it}$  are the representatives of  $\lambda(W, t)$  under some representation of the restricted Poincaré group.

These ten statements, and their combinations, cover most of possibilities for modular covariance relations that apply to all combinations of wedges. (We exclude here those referring only to wedges related in a certain way—for example, the conditions of modular inclusion in [4], which apply only to parallel wedges.) In Theorem 7, we will demonstrate the equivalence of the majority of these modular covariance relations, including those of [13] (covariance under modular automorphisms of wedge algebras) and of [16] (covariance under modular conjugations of wedge algebras).

Note that when we speak of a representation of the Poincaré group, we must require the conditions appropriate to a group of symmetries: that is, we refer to a strongly continuous projective representation by unitary or antiunitary operators. This will therefore be a representation of the covering group, either  $\tilde{\mathcal{P}}_+$  or  $\tilde{\mathcal{P}}_+^\uparrow$ , in which the connected component of the identity  $\tilde{\mathcal{P}}_+^\uparrow$  must be represented by unitary operators. If the representation condition (i) above is to hold, then the time-reversing operators in  $\tilde{\mathcal{P}}_+$  must be represented by antiunitary operators (which is in fact what we would expect physically, due to the positivity of the energy). Notice also, for example, that since  $J(W)$  is antiunitary, (b) above implies that  $J(W_1)\Delta(W_2)^{it}J(W_1) = \Delta(j(W_1)W_2)^{-it}$ .

### III Equivalence of Strong and Weak Formulations

We begin by discussing the relationship between conditions dealing with the wedge algebras, and those dealing purely with the modular structure. The modular operators associated with a given algebra contain much less information than the algebra itself, but there is the following weak result: if  $\psi \in \mathcal{H}$  is such that  $S(W)\psi = \psi$ , then there is a closed symmetric operator  $\tilde{X}$  affiliated with  $\mathcal{A}(W)$  such that  $\tilde{X}\Omega = \psi$ . It is defined on the core  $\mathcal{A}(W)'\Omega$  by  $\tilde{X}Y\Omega = Y\psi$  for all  $Y \in \mathcal{A}(W)'$ . (The following results could of course be proved without this machinery, but only at the cost of a certain increase in notational complexity; furthermore, the method we use seems in accord with the modular spirit of our presentation.)

**Lemma 1:** *Suppose assumptions (i)–(iv) hold. Let  $U$  be a unitary (or antiunitary) operator such that  $U\Omega = \Omega$ , and  $\gamma$  be a Poincaré transformation such that  $US(W)U^* = S(\gamma W)$  for every wedge  $W$ . If there is some particular wedge  $W_0$  such that  $UA(W_0)U^* = \mathcal{A}(\gamma W_0)$ , then likewise  $UA(W)U^* = \mathcal{A}(\gamma W)$  for every wedge  $W$ .*



*Proof:* Let us first show that the statement holds when  $W \subset W_0$ . Then we have immediately  $U\mathcal{A}(W)U^* \subset U\mathcal{A}(W_0)U^* = \mathcal{A}(\gamma W_0)$ , and also  $\mathcal{A}(\gamma W) \subset \mathcal{A}(\gamma W_0)$ . If  $X \in \mathcal{A}(W)^{\text{sa}}$ , then  $UXU^* \in \mathcal{A}(\gamma W_0)^{\text{sa}}$ , and furthermore  $UX\Omega \in \overline{\mathcal{A}(\gamma W)^{\text{sa}}\Omega}$ . Thus there is a closed symmetric operator  $\tilde{X}$  affiliated with  $\mathcal{A}(\gamma W)$  (and thus also with  $\mathcal{A}(\gamma W_0)$ ) such that  $\tilde{X}\Omega = UX\Omega$ . But  $\tilde{X}$  and  $UXU^*$  agree on the dense set  $\mathcal{A}(\gamma W_0)'\Omega$ , from which it follows that  $\tilde{X}$  is in fact bounded and equal to  $UXU^*$ , which therefore lies in  $\mathcal{A}(\gamma W)$ . Thus  $U\mathcal{A}(W)U^* \subset \mathcal{A}(\gamma W)$ . On the other hand, we may apply the same argument to  $\gamma W_0$  and  $\gamma W$ , with  $U$  and  $U^*$  interchanged and  $\gamma^{-1}$  taking the role of  $\gamma$ , to show that  $U^*\mathcal{A}(\gamma W)U \subset \mathcal{A}(W)$ . Thus  $U\mathcal{A}(W)U^* = \mathcal{A}(\gamma W)$ .

Likewise we have  $U\mathcal{A}(W_0)'U^* = \mathcal{A}(\gamma W_0)'$ , so that, by the same reasoning, if  $\mathcal{A}(W)' \subset \mathcal{A}(W_0)'$ , we have  $U\mathcal{A}(W)'U^* = \mathcal{A}(\gamma W)'$  and again  $U\mathcal{A}(W)U^* = \mathcal{A}(\gamma W)$ . Thus the statement of the lemma also holds whenever  $W' \subset W_0'$ , i.e. whenever  $W \supset W_0$ . If we have merely that  $W \parallel_s W_0$ , then there is some  $W_1 \subset W_0 \cap W$ . The statement of the lemma holds for  $W_1$ , and hence we may repeat the argument with  $W_1$  in place of  $W_0$  to show that it holds for  $W$ . Thus  $U\mathcal{A}(W)U^* = \mathcal{A}(\gamma W)$  whenever  $W \parallel_s W_0$ .

Next let us consider the case in which neither  $W \cap W_1$  nor  $W' \cap W_1'$  is empty for any  $W_1 \parallel_s W_0$  (as is the case for most choices of  $W$  and  $W_0$ ). Then the vacuum is cyclic and separating for both  $\mathcal{A} = \mathcal{A}(W) \cap \mathcal{A}(W_1)$  and  $\mathcal{B} = \mathcal{A}(\gamma W') \cap \mathcal{A}(\gamma W_1')$ . By the results above,  $U\mathcal{A}U^* \subset \mathcal{A}(\gamma W_1)$ . If  $X \in \mathcal{A}^{\text{sa}}$ , then  $UXU^* \in \mathcal{A}(\gamma W_1)^{\text{sa}}$ , but also  $UX\Omega \in \overline{\mathcal{A}(\gamma W)^{\text{sa}}\Omega}$ . Thus there is a closed symmetric operator  $\tilde{X}$  affiliated with  $\mathcal{A}(\gamma W)$  such that  $\tilde{X}\Omega = UX\Omega$ . But  $\tilde{X}$  and  $UXU^*$  agree on the dense set  $\mathcal{B}\Omega$ , from which it follows that  $\tilde{X}$  is in fact bounded and equal to  $UXU^*$ , which therefore is in  $\mathcal{A}(\gamma W)$ . Thus  $U\mathcal{A}U^* \subset \mathcal{A}(\gamma W)$ . Letting  $W_1$  vary we generate all of  $\mathcal{A}(W)$ , so that  $U\mathcal{A}(W)U^* \subset \mathcal{A}(\gamma W)$ . But again as above we can use a similar argument to show that  $U^*\mathcal{A}(\gamma W)U \subset \mathcal{A}(W)$ , so that in fact  $U\mathcal{A}(W)U^* = \mathcal{A}(\gamma W)$ . For the remaining wedges, we may repeat these arguments to show that the result holds generally.

This becomes useful when combined with the Tomita-Takesaki theorem, as follows:

**Theorem 2:** *Under assumptions (i)–(iv), the following are equivalent:*

- (a) *covariance under modular conjugations of both modular conjugations and modular automorphisms;*
- (b) *covariance under modular conjugations of modular involutions;*
- (c) *covariance under modular conjugations of wedge algebras.*

*If these conditions hold, then so also does wedge duality.*

**Theorem 3:** *Under assumptions (i)–(iv), the following are equivalent:*

- (a) *covariance under modular automorphisms of both modular conjugations and modular automorphisms;*
- (b) *covariance under modular automorphisms of modular involutions;*
- (c) *covariance under modular automorphisms of wedge algebras.*

*If these conditions hold, then so also does wedge duality.*

*Proof:* In each case it is clear from the definition of  $S(W)$  that (c) implies (b), and from the uniqueness of the polar decomposition that (b) is equivalent to (a). Any of the assumptions of Theorem 2 implies that  $J(W') = J(j(W)W) = J(W)$  and  $\Delta(W') = \Delta(j(W)W) = \Delta(W)^{-1}$ . Since we already have that  $\mathcal{A}(W') \subset \mathcal{A}(W)'$ , this implies that  $\mathcal{A}(W') = \mathcal{A}(W)'$ . Likewise in Theorem 3,  $\mathcal{A}(W')$  is a subalgebra of  $\mathcal{A}(W)'$ , whose modular operators are invariant under the modular automorphism group for  $\mathcal{A}(W)'$ , and for which the vacuum is cyclic. It follows that  $\mathcal{A}(W')$  is invariant under the modular automorphism group for  $\mathcal{A}(W)$ , and by a standard result it is therefore equal to  $\mathcal{A}(W)'$ . It remains for us to show that under assumptions (i)–(iv), (b) implies (c). This follows from Lemma 1 using the Tomita-Takesaki results  $J(W)\mathcal{A}(W)J(W) = \mathcal{A}(W') = \mathcal{A}(j(W)W)$  and  $\Delta(W)^{it}\mathcal{A}(W)\Delta(W)^{-it} = \mathcal{A}(W) = \mathcal{A}(\lambda(W, t)W)$ .

Thus we see that the covariance of wedge algebras can always be reduced to appropriate statements referring purely to the relations of modular operators to one another, without reference to the algebras themselves.

## IV Representations of the Poincaré Group

We have now to discuss the relationships between the modular covariance conditions referring only to the modular structure and those calling for the existence of certain representations of the Poincaré group. In this we make use of the results of [6], which establish the existence of such representations under rather weak conditions.

**Theorem 4:** *Under assumptions (i)–(v), the following are equivalent:*

- (a) *covariance under modular conjugations of modular conjugations;*
- (b) *the modular conjugations  $J(W)$  are representatives of  $j(W)$  under a representation of the covering group  $\tilde{\mathcal{P}}_+$  of the proper Poincaré group;*
- (c) *the modular conjugations  $J(W)$  are the representatives of  $j(W)$  under a representation of the proper Poincaré group  $\mathcal{P}_+$  which represents orthochronous transformations*

by unitary operators and time-reversing transformations by antiunitary operators; the vacuum is invariant under it, and the modular conjugations are covariant, and wedge duality holds.

*Assumption (v) is not necessary if (b) or (c) holds.*

*Proof:* Clearly (c) implies (b), and (b) implies both (a) and assumption (v). We therefore assume (a) and assumptions (i)–(v) and seek to prove the rest. We begin with a single one-parameter subgroup  $\Lambda(t)$  of the Poincaré group (necessarily the restricted group), for which we suppose that there is some wedge  $W$  such that  $j(W)\Lambda(t)j(W) = \Lambda(-t)$ . From this it follows that  $j(\Lambda(t)W)\Lambda(t')W = \Lambda(2t - t')W'$ . (There are many such instances: for example,  $W_R$  will serve for the translations in the  $\hat{x}_1$  direction, the velocity transformations in the  $\hat{x}_2$  direction, or the rotations about  $\hat{x}_3$ . Thus there is such a  $W$  if  $\Lambda(t)$  is conjugate to any of these one-parameter subgroups, and in particular such subgroups generate all of  $\mathcal{P}_+^\uparrow$ .)

We can then apply the methods of [4], Proposition 3.1 and Lemma 3.2, the proof of which can be simplified as follows. If we write  $J_t = J(\Lambda(t)W)$ , then from our assumptions  $J_t J_{t'} J_t = J_{2t-t'}$ . First we wish to show by induction that  $J_{nt} J_{(n+1)t} = J_0 J_t$  for all integers  $n$ . This holds for  $n = 0$ , but also  $J_{nt} J_{(n+1)t} J_{nt} = J_{(n-1)t}$ , so that  $J_{nt} J_{(n+1)t} = J_{(n-1)t} J_{nt}$ , and the induction proceeds in either direction. Next we wish to show by induction that  $(J_0 J_t)^n = J_0 J_{nt}$  for all integers  $n$ . This is immediate for  $n = 0, \pm 1$ , and

$$(J_0 J_t)^{n+1} = (J_0 J_t)^n J_0 J_t = J_0 J_{nt} J_{nt} J_{(n+1)t} = J_0 J_{(n+1)t} \quad (9)$$

provides the induction for  $n > 0$ . But then by the same result  $(J_0 J_t)^{-n} = (J_0 J_{-t})^n = J_0 J_{-nt}$ . From this we have  $J_0 J_{nt} J_0 J_{mt} = (J_0 J_t)^{m+n} = J_0 J_{(n+m)t}$ , and in general if  $t$  and  $t'$  are rationally related then  $J_0 J_t J_0 J_{t'} = J_0 J_{t+t'}$ . Then by continuity it follows that  $J_0 J_t$  is a continuous one-parameter unitary group.

Thus also  $U(\Lambda(t)) = J(W)J(\Lambda(-t/2)W)$  is a continuous one-parameter unitary group, which can be seen to implement  $\Lambda(t)$  on the modular conjugations: for an arbitrary wedge  $W_1$ ,  $U(\Lambda(t))J(W_1)U(\Lambda(-t)) = J(\Lambda(t)W_1)$ . Likewise  $U(\Lambda(t))$  is covariant under the modular conjugations:  $J(W_1)U(\Lambda(t))J(W_1) = U(j(W_1)\Lambda(t)j(W_1))$ . Thus also the  $U(\Lambda(t))$  are covariant with respect to each other:  $U(\Lambda'(t'))U(\Lambda(t))U(\Lambda'(-t')) = U(\Lambda'(t')\Lambda(t)\Lambda'(-t'))$ . We may then apply the methods of [6], as also in [13], Proposition 2.4: the  $U(\Lambda(t))$  generate a central weak Lie extension of  $\mathcal{P}_+^\uparrow/H$ , where  $H$  is the normal subgroup  $H = \{\Lambda \in \mathcal{P}_+^\uparrow \mid J(\Lambda W) = J(W) \text{ for all } W\}$  of  $\mathcal{P}_+^\uparrow$ . If  $H$  is trivial

or the translation subgroup, then  $\mathcal{P}_+^\uparrow/H$  is the restricted Poincaré or Lorentz group, and we have a representation of the corresponding covering group, hence in either case a representation of  $\tilde{\mathcal{P}}_+^\uparrow$ . If  $H = \mathcal{P}_+^\uparrow$ , then  $J(W_1) = J(W_2)$  for all  $W_1, W_2$ , and the representation is trivial immediately. Thus in any case we have a representation of the covering group  $\tilde{\mathcal{P}}_+^\uparrow$ , under which the modular conjugations are covariant. It is then straightforward to extend this to a representation of the covering group  $\tilde{\mathcal{P}}_+$  of the proper Poincaré group, since as we have seen the representation is also covariant under the modular conjugations. Since the vacuum is invariant under each  $J(W)$ , it is also invariant under this representation. As we have remarked before, the subgroup  $\tilde{\mathcal{P}}_+^\uparrow$  must be represented by unitary operators, and the time-reversing transformations by antiunitary operators.

To show that this representation is in fact of the proper Poincaré group itself, we follow the procedure used in [13] and [16]: let  $R(\theta)$  be the representative of the rotation by the angle  $\theta$  about the  $\hat{x}_3$  axis, so that  $J(W_R)R(\theta)J(W_R) = R(-\theta)$ . We have  $J(W_R) = J(j(W_L)W_L) = J(W_L)$ , so  $1 = J(W_L)J(W_R) = R(\pi)J(W_R)R(-\pi)J(W_R) = R(2\pi)$ .

Note that an extension of a representation of the restricted Poincaré group to one of the proper Poincaré group, in which the time-reversing transformations are represented by antiunitary operators, is almost unique, but not quite. One has always a choice of phase—that is, the time-reversing transformations may always be multiplied by any common unitary operator  $V$  which commutes with the restricted Poincaré group and anticommutes ( $VU = UV^*$ ) with the time-reversing transformations. For example, if the representation is irreducible, then  $V$  can be any complex phase  $e^{i\theta}$ .

Next we introduce a lemma that allows us to connect the behavior of modular automorphisms with that of modular conjugations:

**Lemma 5:** *Suppose assumptions (i)–(iii) hold. If  $W_1, W_2$  are two wedges such that  $W_1 \cap W_2 \neq \emptyset$  and  $W_1' \cap W_2' \neq \emptyset$ , then*

$$\Delta(W_1)^{1/2}\Delta(W_2)^{-1/2} \subset J(W_1)J(W_2). \quad (10)$$

*That is, the operator on the left is densely defined and closable, and the bounded operator on the right extends it. This implies among other things that for every  $\psi \in D(\Delta(W_1)^{1/2})$ ,  $\phi \in D(\Delta(W_2)^{-1/2})$ , we have*

$$\langle \Delta(W_1)^{1/2}\psi \mid \Delta(W_2)^{-1/2}\phi \rangle = \langle J(W_1)\psi \mid J(W_2)\phi \rangle. \quad (11)$$

Let  $U$  be a unitary (or antiunitary) operator, and  $\gamma$  a Poincaré transformation such that  $U\Delta(W)U^* = \Delta(\gamma W)$  for every wedge  $W$ . Then also

$$UJ(W_1)J(W_2)U^* = J(\gamma W_1)J(\gamma W_2) \quad (12)$$

for any pair of wedges  $W_1, W_2$ , and there is a unitary operator  $V$  such that  $UJ(W)U^* = VJ(\gamma W)$  for every wedge  $W$ . If in addition there is some particular wedge  $W_0$  such that  $UJ(W_0)U^* = J(\gamma W_0)$ , then likewise  $UJ(W)U^* = J(\gamma W)$  for every wedge  $W$ .

*Proof:* By assumption, the vacuum is cyclic for  $\mathcal{A} = \mathcal{A}(W_1) \cap \mathcal{A}(W_2)$ . If  $X \in \mathcal{A}$ , then  $S(W_1)X\Omega = S(W_2)X\Omega = X^*\Omega$ . Thus  $S(W_1)S(W_2)$  agrees with the identity on the dense set  $\mathcal{A}\Omega$ . But  $S(W_1) = J(W_1)\Delta(W_1)^{1/2}$  and  $S(W_2) = \Delta(W_2)^{-1/2}J(W_2)$ . Thus the bounded operator  $J(W_1)J(W_2)$  agrees with the product  $\Delta(W_1)^{1/2}\Delta(W_2)^{-1/2}$  on the dense set  $J(W_2)\mathcal{A}\Omega$ . Applying the same reasoning to  $\mathcal{A}(W_2)' \supset \mathcal{A}(W_2')$  and  $\mathcal{A}(W_1)' \supset \mathcal{A}(W_1')$  shows that  $\Delta(W_2)^{-1/2}\Delta(W_1)^{1/2}$  agrees with  $J(W_2)J(W_1)$  on a dense set. In particular,  $\Delta(W_1)^{1/2}\Delta(W_2)^{-1/2}$  has a densely defined adjoint, and thus is closable. Then since  $\Delta(W_1)^{1/2}\Delta(W_2)^{-1/2}$  agrees with  $J(W_1)J(W_2)$  on a dense set, they must agree wherever defined. Furthermore since  $\langle \Delta(W_1)^{1/2}\psi | \Delta(W_2)^{-1/2}\phi \rangle = \langle J(W_1)\psi | J(W_2)\phi \rangle$  for a dense set of  $\psi$  and  $\phi$ , and since the right-hand side is a bounded function of  $\psi$  and  $\phi$ , equality must hold whenever the left-hand side is defined.

For the second part, let us first assume that  $W_1$  and  $W_2$  satisfy the condition of the first part. Then  $UJ(W_1)J(W_2)U^*$  extends  $U\Delta(W_1)^{1/2}\Delta(W_2)^{-1/2}U^*$ , but by assumption the latter is equal to  $\Delta(\gamma W_1)^{1/2}\Delta(\gamma W_2)^{-1/2}$ , which extends to the bounded operator  $J(\gamma W_1)J(\gamma W_2)$ . Thus  $UJ(W_1)J(W_2)U^*$  and  $J(\gamma W_1)J(\gamma W_2)$  are extensions of the same densely defined closable operator, and must in fact be equal. This is so provided that  $W_1 \cap W_2 \neq \emptyset$  and  $W_1' \cap W_2' \neq \emptyset$ , but in any case we can find some  $W_3$  such that none of  $W_1 \cap W_3$ ,  $W_2 \cap W_3$ ,  $W_1' \cap W_3'$ , or  $W_2' \cap W_3'$  is empty. Then again we have

$$\begin{aligned} UJ(W_1)J(W_2)U^* &= UJ(W_1)J(W_3)U^*UJ(W_3)J(W_2)U^* \\ &= J(\gamma W_1)J(\gamma W_3)J(\gamma W_3)J(\gamma W_2) = J(\gamma W_1)J(\gamma W_2), \end{aligned} \quad (13)$$

without restriction on  $W_1, W_2$ . We may rearrange this to obtain

$$UJ(W_1)U^*J(\gamma W_1) = UJ(W_2)U^*J(\gamma W_2) = V \quad (14)$$

where  $V$  is a single unitary operator independent of the choice of wedges. Thus  $UJ(W)U^* = VJ(\gamma W)$  for any wedge  $W$ . Then if  $UJ(W_0)U^* = J(\gamma W_0)$  we have  $VJ(\gamma W_0) = J(\gamma W_0)$  and  $V = I$ .

This result is a very strong condition on the modular operators, with several important consequences. First, the modular automorphisms must be such that the product of  $\Delta(W_1)^{1/2}$  and  $\Delta(W_2)^{-1/2}$  is densely defined and is the restriction of a unitary operator for every  $W_1, W_2$  with  $W_1 \cap W_2 \neq \emptyset$  and  $W'_1 \cap W'_2 \neq \emptyset$ . Second, the modular automorphisms almost determine the modular conjugations: they determine the products of pairs of modular conjugations, or the modular conjugations themselves up to a unitary phase operator  $V$ . This will be used several times in the next section, in the proof of our main theorem on modular covariance.

All that remains is to show that a corresponding condition holds for representations of the Poincaré group, as is necessary if the modular operators are to be derived from such a representation.

**Lemma 6:** *Let  $U(\lambda)$  be a representation of the restricted Poincaré group  $\mathcal{P}_+^\uparrow$  satisfying the spectrum condition. If  $W_1$  and  $W_2$  are two wedges such that  $W_1 \cap W_2 \neq \emptyset$  and  $W'_1 \cap W'_2 \neq \emptyset$ , and we write  $\Delta_0(W)^{it} = U(\lambda(W, t))$ , then*

$$\langle \Delta_0(W_1)^{1/2} \psi \mid \Delta_0(W_2)^{-1/2} \phi \rangle = \langle \psi \mid U(j(W_1)j(W_2))\phi \rangle \quad (15)$$

for every  $\psi \in D(\Delta_0(W_1)^{1/2})$ ,  $\phi \in D(\Delta_0(W_2)^{-1/2})$ .

*Proof:* From the proof of Theorem 1.1 of [13] we can see that

$$\begin{aligned} & \langle \Delta_0(W_1)^{1/2} \psi \mid \Delta_0(\lambda(W, t)W_1)^{-1/2} \phi \rangle \\ &= \langle \Delta_0(W_1)^{1/2} \psi \mid \Delta_0(W)^{it} \Delta_0(W_1)^{-1/2} \Delta_0(W)^{-it} \phi \rangle \\ &= \langle \psi \mid U(j(W_1)\lambda(W, t)j(W_1)\lambda(W, -t))\phi \rangle = \langle \psi \mid U(\lambda(W, -2t))\phi \rangle \end{aligned} \quad (16)$$

whenever  $W_1$  and  $W$  are orthogonal wedges, so that our condition holds for  $W_1$  and  $W_2 = \lambda(W, t)W_1$ . The same proof also can be adapted to give the same result if  $W_2 = \Lambda(t)W_1$  where  $\Lambda(t)$  is any one-parameter subgroup of the Lorentz group such that  $W_1$  and  $W_2$  satisfy the conditions of the present lemma. Thus the lemma holds whenever  $W_1$  and  $W_2$  are Lorentz transforms of each other.

If  $W_1$  and  $W_2$  are arbitrary wedges satisfying the conditions of our lemma, then there is some  $W_3 \parallel_s W_2$  such that  $W_1$  and  $W_3$  also satisfy the conditions of our lemma, and in addition are Lorentz transforms of each other. Thus there is some translation  $T(x)$  having no component parallel to the vertex of  $W_2$ , with  $W_3$  the translate by  $x$  of  $W_2$ , and

$$\langle \Delta_0(W_1)^{1/2} \psi \mid \Delta_0(W_2)^{-1/2} \phi \rangle = \langle \Delta_0(W_1)^{1/2} \psi \mid \Delta_0(W_3)^{-1/2} \Delta_0(W_3)^{1/2} \Delta_0(W_2)^{-1/2} \phi \rangle$$

$$\begin{aligned}
&= \left\langle \Delta_0(W_1)^{1/2} \psi \left| \Delta_0(W_3)^{-1/2} T(x) \Delta_0(W_2)^{1/2} T(-x) \Delta_0(W_2)^{-1/2} \phi \right\rangle \quad (17) \\
&= \left\langle \psi \left| U(j(W_1)j(W_3)) T(x) \Delta_0(W_2)^{1/2} T(-x) \Delta_0(W_2)^{-1/2} \phi \right\rangle
\end{aligned}$$

for all  $\psi \in D(\Delta_0(W_1)^{1/2})$  and all  $\phi$  for which the expression is defined.

At this point we employ a converse to Borchers' Theorem ([9], Theorem 3), a consequence of the analytic continuation made possible by the spectrum condition. The theorem shows, for example, that for the particular wedge  $W_R$ , the expression  $T(x)\Delta_0(W_R)^{1/2}T(-x)\Delta_0(W_R)^{-1/2}$  agrees with  $T(2x)$  wherever it is defined, provided  $x$  lies in the  $\hat{x}_0 + \hat{x}_1$  or  $\hat{x}_0 - \hat{x}_1$  directions. Thus this is true also for linear combinations of these directions, i.e., in general, whenever  $x$  has no component parallel to the vertex of the wedge. This means that

$$\begin{aligned}
&\left\langle \psi \left| U(j(W_1)j(W_3)) T(x) \Delta_0(W_2)^{1/2} T(-x) \Delta_0(W_2)^{-1/2} \phi \right\rangle \quad (18) \\
&= \left\langle \psi \left| U(j(W_1)j(W_3)) T(2x) \phi \right\rangle = \left\langle \psi \left| U(j(W_1)j(W_2)) \phi \right\rangle
\end{aligned}$$

for all  $\psi, \phi$  for which the expression is defined. Thus the lemma holds generally.

## V Modular Covariance

We are now ready for our main (and rather heavily overloaded) theorem on modular covariance.

**Theorem 7:** *Under assumptions (i)–(v), the following are equivalent:*

- (a) *covariance under modular conjugations of modular automorphisms;*
- (b) *covariance under modular conjugations of modular involutions;*
- (c) *covariance under modular conjugations of wedge algebras;*
- (d) *the modular conjugations  $J(W)$  are representatives of  $j(W)$  under a representation of the covering group  $\tilde{\mathcal{P}}_+$  of the proper Poincaré group, under which the modular automorphisms are covariant;*
- (e) *the modular conjugations  $J(W)$  are the representatives of  $j(W)$  under a representation of the proper Poincaré group  $\mathcal{P}_+$  satisfying the spectrum condition, under which the vacuum is invariant, the modular conjugations, modular automorphisms, modular involutions, and wedge algebras are all covariant, and wedge duality holds;*
- (f) *covariance under modular automorphisms of modular automorphisms;*
- (g) *covariance under modular automorphisms of modular involutions;*

- (h) covariance under modular automorphisms of wedge algebras;
- (i) the modular automorphisms  $\Delta(W)^{it}$  are the representatives of  $\lambda(W, t)$  under a representation of the covering group  $\tilde{\mathcal{P}}_+^\uparrow$  of the restricted Poincaré group;
- (j) the modular automorphisms  $\Delta(W)^{it}$  are representatives of  $\lambda(W, t)$  under a unitary representation of the restricted Poincaré group  $\mathcal{P}_+^\uparrow$  satisfying the spectrum condition, under which the vacuum is invariant, the modular conjugations, modular automorphisms, modular involutions, and wedge algebras are all covariant, and wedge duality holds;
- (k) the modular conjugations  $J(W)$  are the representations of  $j(W)$ , and the modular automorphisms  $\Delta(W)^{it}$  of  $\lambda(W, t)$ , under a representation of the proper Poincaré group  $\mathcal{P}_+$  which represents orthochronous transformations by unitary operators and time-reversing transformations by antiunitary operators; this representation satisfies the spectrum condition, under it the vacuum is invariant, the modular conjugations, modular automorphisms, modular involutions, and wedge algebras are all covariant, and wedge duality holds.

Assumption (v) is not necessary if any of (d)–(k) holds.

*Remark:* The paper [16] assumes the existence and uniqueness of a covariant representation of  $\tilde{\mathcal{P}}_+^\uparrow$ , and then shows essentially that (c) above implies (e). The paper [13] shows essentially that (h) above implies (k).

*Proof:* We have seen in the proof of Theorems 2 and 3 that wedge duality follows from most of these statements. Also, assumption (v) follows from (d), (e), (j), or (k). In fact, (k) implies all the other statements, (e) implies (a)–(d), and (j) implies (f)–(i). We will therefore begin by showing that (a)–(d) are all equivalent to (e), continue by showing that (f)–(i) are all equivalent, and finally show that (e), (i), (j), and (k) are all equivalent.

By Theorem 2, (c) is equivalent to (b), or to the conjunction of (a) above with (a) of Theorem 4. We show that (a) above implies (a) of Theorem 4. If (a) above holds, then by Lemma 5 we have  $J(W_1)J(W_2)J(W_1) = V(W_1)J(j(W_1)W_2)$  where  $V(W_1)$  depends only on  $W_1$  and not on  $W_2$ . But we may choose  $W_1 = W_2$ , from which we get  $J(W_1) = V(W_1)J(j(W_1)W_1) = V(W_1)J(W_1)$  and  $V(W_1) = I$ , so that modular conjugations are covariant under modular conjugations. Thus (a), (b), and (c) are all equivalent. Clearly (d) implies (a); on the other hand, by Theorem 4, (a) of Theorem 4 combined with assumption (v) implies that the modular conjugations generate a representation, which by (c) is covariant. Thus (d) is equivalent to (a)–(c). That the



spectrum condition holds follows from a converse to Borchers' Theorem [22, 6, 9]. The equivalence of (a)–(e) then follows using Theorems 2 and 4.

By Theorem 3, (h) is equivalent to (g), or to the conjunction of (f) with the covariance under modular automorphisms of modular conjugations. We will therefore first show that (f) implies the latter covariance. If (f) holds, then by Lemma 5 we have

$$\Delta(W_1)^{it} J(W_2) \Delta(W_1)^{-it} = V(W_1, t) J(\lambda(W_1, t) W_2) \quad (19)$$

where  $V(W_1, t)$  is independent of  $W_2$ . We may choose  $W_1 = W_2$ , obtaining

$$J(W_1) = \Delta(W_1)^{it} J(W_1) \Delta(W_1)^{-it} = V(W_1, t) J(\lambda(W_1, t) W_1) = V(W_1, t) J(W_1), \quad (20)$$

so that  $V(W_1, t) = I$  identically. From this follows the desired covariance and the equivalence of (f), (g), and (h). Clearly (i) implies (f), so we must show that (f) implies (i). This result is contained in [6], cf. also [13], and is analogous to that of Theorem 4. We have immediately one-parameter unitary groups  $\Delta(W)^{it}$ , which by the same procedure as in Theorem 4 give a central weak Lie extension, and thus a unitary representation of the covering group  $\tilde{\mathcal{P}}_+^\uparrow$  under which the modular automorphisms are covariant. Since the vacuum is invariant under each  $\Delta(W)^{it}$ , it is invariant under this representation.

Now we must show that (e), (i), (j), and (k) are all equivalent. Let us begin by showing that (e) implies (i), (j) and (k). In the representation of the Poincaré group generated by the  $J(W)$ , let  $\Delta_0(W)^{it}$  be the representative of  $\lambda(W, t)$ . Then the  $\Delta(W)$  are covariant under the  $\Delta_0(W)^{it}$ . Thus for any particular wedge  $W$ ,  $\Delta(W)$  commutes strongly with  $\Delta_0(W)$ , so the two positive operators have a common dense set  $D_\omega(W)$  of vectors  $\psi$  such that  $\Delta(W)^{iz}\psi$  and  $\Delta_0(W)^{iz}\psi$  are both entire analytic. Let us take  $W_1, W_2$  such that  $W_1 \cap W_2 \neq \emptyset$  and  $W_1' \cap W_2' \neq \emptyset$ . For any  $\psi \in D_\omega(W_1)$ ,  $\phi \in D_\omega(W_2)$  we may define a jointly entire analytic function

$$f(z, w) = \left\langle \Delta(W_1)^{i\bar{w}} \Delta_0(W_1)^{i\bar{z}} \psi \mid \Delta(W_2)^{iw} \Delta_0(W_2)^{iz} \phi \right\rangle \quad (21)$$

satisfying

$$|f(z, w)| \leq \left\| \Delta(W_1)^{i\bar{w}} \Delta_0(W_1)^{i\bar{z}} \psi \right\| \left\| \Delta(W_2)^{iw} \Delta_0(W_2)^{iz} \phi \right\| \quad (22)$$

$$= \left\| \Delta(W_1)^{\text{Im } w} \Delta_0(W_1)^{\text{Im } z} \psi \right\| \left\| \Delta(W_2)^{-\text{Im } w} \Delta_0(W_2)^{-\text{Im } z} \phi \right\|. \quad (23)$$

Then we may use Lemmas 5 and 6 to compute

$$\begin{aligned} f(z, w + i/2) &= \left\langle \Delta(W_1)^{1/2} \Delta(W_1)^{i\bar{w}} \Delta_0(W_1)^{i\bar{z}} \psi \mid \Delta(W_2)^{-1/2} \Delta(W_2)^{iw} \Delta_0(W_2)^{iz} \phi \right\rangle \\ &= \left\langle J(W_1) \Delta(W_1)^{i\bar{w}} \Delta_0(W_1)^{i\bar{z}} \psi \mid J(W_2) \Delta(W_2)^{iw} \Delta_0(W_2)^{iz} \phi \right\rangle \end{aligned} \quad (24)$$

and

$$\begin{aligned}
f(z + i/2, w) &= \left\langle \Delta_0(W_1)^{1/2} \Delta(W_1)^{i\bar{w}} \Delta_0(W_1)^{i\bar{z}} \psi \middle| \Delta_0(W_2)^{-1/2} \Delta(W_2)^{iw} \Delta_0(W_2)^{iz} \phi \right\rangle \\
&= \left\langle J(W_1) \Delta(W_1)^{i\bar{w}} \Delta_0(W_1)^{i\bar{z}} \psi \middle| J(W_2) \Delta(W_2)^{iw} \Delta_0(W_2)^{iz} \phi \right\rangle \quad (25)
\end{aligned}$$

from which we deduce that  $f(z, w) = f(z + i/2, w - i/2)$ . Let us consider  $f(z + \zeta, w - \zeta)$  as a function of  $\zeta$ : it is periodic in  $\zeta$  with period  $i/2$ , and satisfies a bound independent of  $\text{Re } \zeta$ , so it is bounded and, hence, constant. Thus  $f(z, w) = f(z + \zeta, w - \zeta)$  for all  $z, w, \zeta$ , and in particular  $f(t, 0) = f(0, t)$ , so that

$$\left\langle \Delta(W_1)^{it} \psi \middle| \Delta(W_2)^{it} \phi \right\rangle = \left\langle \Delta_0(W_1)^{it} \psi \middle| \Delta_0(W_2)^{it} \phi \right\rangle. \quad (26)$$

Since  $\psi$  and  $\phi$  may vary over dense sets, we conclude that for all real  $t$  we have  $\Delta(W_1)^{-it} \Delta(W_2)^{it} = \Delta_0(W_1)^{-it} \Delta_0(W_2)^{it}$ , and by suitably varying  $W_1$  and  $W_2$  we see that  $V(W, t) = \Delta_0(W)^{it} \Delta(W)^{-it}$  is in fact independent of  $W$ . But  $V(W, t)$  is a one-parameter unitary group, so from  $V(W, -t) = V(W', t) = V(W, t)$  we find that  $V(W, t) = 1$  identically and  $\Delta(W) = \Delta_0(W)$ . This implies (i), (j), and (k) directly.

It will then suffice to show that (i) implies (a). This result is contained in [6], and is closely connected with our Lemmas 5 and 6. This completes the proof.

## VI PCT and Spin-Statistics Theorems

If  $d$  is even, then the complete spacetime inversion is an element of  $\mathcal{P}_+$ . If the wedge algebras are covariant under a representation of  $\mathcal{P}_+$  (or of  $\tilde{\mathcal{P}}_+$ ) then the (antiunitary) representative of this inversion is just the PCT operator  $\Theta$ . Thus in even dimensions, Theorem 7 is a PCT theorem; in odd dimensions, on the other hand, it seems that it must suffice to have a representation of  $\mathcal{P}_+$  (or of  $\tilde{\mathcal{P}}_+$ ).

So far the argument has been stated entirely in terms of observable algebras, and thus necessarily in terms of bosonic quantities. In this case Theorem 7 guarantees representations of  $\mathcal{P}_+$  and  $\mathcal{P}_+^\uparrow$  rather than of their covering groups—that is, it guarantees that all spins are integral. Thus it is also a spin-statistics theorem for bosonic statistics. It may also be extended to fermionic statistics by the use of a standard notation [2]. We let  $\Gamma$  be a unitary involution such that  $\Gamma\Omega = \Omega$  and  $\Gamma\mathcal{A}(W)\Gamma = \mathcal{A}(W)$  for all  $W$ . Operators that commute with  $\Gamma$  are intended to be bosonic, while those that anticommute are to be fermionic. Let  $Z = (I + i\Gamma)/(1 + i)$ , and let us alter assumption

(i) as follows:

(i')  $Z\mathcal{A}(W')Z^* \subset \mathcal{A}(W)'$  (twisted locality).

Wedge duality is likewise altered to  $Z\mathcal{A}(W')Z^* = \mathcal{A}(W)'$  (twisted wedge duality).

The alterations correspond to normal commutation relations: commutation between spacelike separated operators, except that two spacelike separated fermionic operators anticommute. The argument proceeds much as before, save that in place of covariance under the modular conjugations we must substitute covariance under the twisted modular conjugation operators  $Z^*J(W)$ .  $\Gamma$  commutes with every  $J(W)$  and  $\Delta(W)$ , so  $Z^*J(W) = J(W)Z$  is again an antiunitary involution. The results are as before, except that (i') implies that  $Z^*J(W_R)Z = J(W_L)$ , so that the final argument in Theorem 4 now shows that  $R(2\pi) = Z^*J(W_R)ZJ(W_L) = Z^2 = \Gamma$ . Thus in each case we have representations of the covering groups  $\tilde{\mathcal{P}}_+$  or  $\tilde{\mathcal{P}}_+^\uparrow$ , but subject to the condition that  $R(2\pi) = \Gamma$ . The modified Theorem 7' as follows is therefore an algebraic PCT and spin-statistics theorem.

**Theorem 7':** *Under assumptions (i') and (ii)–(iv), any of the subparts corresponding to (a)–(j) of Theorem 7 is equivalent to the following:*

*(k') the twisted modular conjugations  $Z^*J(W)$  are the representations of  $j(W)$ , and the modular automorphisms  $\Delta(W)^{it}$  of  $\lambda(W, t)$ , under a representation of the covering group  $\tilde{\mathcal{P}}_+$  of the proper Poincaré group, subject to the condition that  $R(2\pi) = \Gamma$ , which represents orthochronous transformations by unitary operators and time-reversing transformations by antiunitary operators; this representation satisfies the spectrum condition, and under it the vacuum is invariant, and the modular conjugations, modular automorphisms, modular involutions, and wedge algebras are all covariant, and twisted wedge duality holds.*

However, there is more than this that can be said. Using Theorem 7' we can show only that if any one of the equivalent conditions (a')–(k') holds for the field net, then so also do all the rest. What is in fact true, with only a few additional assumptions, is that modular covariance for the observable net implies modular covariance for the field net. However, the present context does not appear to be the appropriate one for a discussion of these issues. Various aspects of the matter are treated in [12] and in [5, 13, 16, 14]. In some sense this is the true spin-statistics theorem in this context, but of course it depends on the modular covariance of the observable net, precisely the question studied

here so far. It is for this reason, as well as for clarity of exposition, that the presentation in the previous sections has been entirely in terms of observables: modular covariance for the observables implies modular covariance generally. The remainder of this paper will discuss modular covariance for field nets directly.

## VII Localized States for Elementary Systems

We must now explain our earlier statement that these theorems allow us to reduce the study of the possible modular structures of nets to the study of nets without interaction; this will give substance to the rather abstract results of Theorems 7 and 7'. We first note that the relevant portions of these theorems apply equally to families of 'modular operators' not necessarily associated with any von Neumann algebras. Let us consider a family of antiunitary involutions  $J(W)$  and unbounded positive operators  $\Delta(W)$ , or equivalently the corresponding unbounded antilinear involutions  $S(W) = J(W)\Delta(W)^{1/2}$ , acting on a Hilbert space  $\mathcal{H}_1$ . There is then a corresponding family of subsets  $R(W)$  of  $\mathcal{H}_1$  defined by  $R(W) = \{\psi \mid S(W)\psi = \psi\}$ . Conversely, the operators can be recovered from the  $R(W)$  by letting  $S(W)(\psi + i\phi) = \psi - i\phi$  for every  $\psi, \phi \in R(W)$ . Consider the following assumptions:

- (i)  $S(W)^*\psi = \psi$  for every  $\psi \in R(W')$ ;
- (ii)  $R(W_1) \subset R(W_2)$  whenever  $W_1 \subset W_2$ ;
- (iii)  $R_{12} = R(W_1) \cap R(W_2)$  is such that  $R_{12} + iR_{12}$  is dense in  $\mathcal{H}_1$  whenever  $W_1 \cap W_2 \neq \emptyset$ ;
- (iv) for all  $W_1$  and  $W_2$ ,  $R(W_1) = \{R(W_1) \cap R(W) \mid W \parallel_s W_2\}^-$ ;
- (v)  $J(\Lambda W)$  is a weakly continuous function of  $\Lambda \in \mathcal{P}_+^\uparrow$ ;

or, in the more general case, given a unitary involution  $\Gamma$ ,

- (i')  $ZS(W)^*\psi = \psi$  for every  $\psi \in R(W')$ .

These correspond directly to assumptions (i)–(v) and (i') for families of wedge algebras  $\mathcal{A}(W)$ . Wedge duality corresponds to  $S(W') = S(W)^*$ , and twisted wedge duality to  $S(W') = ZS(W)^*$ . Theorems 7 and 7' still hold, with the omission of (c), (h), and all other references to wedge algebras. Thus it is still reasonable to speak of modular covariance for such a family of modular operators. Notice that the  $\Delta(W)$  do not determine the  $J(W)$  uniquely, but only up to a phase operator  $V$ . This is entirely consistent with the results of Lemma 5.

Next, we point out that a construction analogous to that of the free fields can produce a family of wedge algebras describing non-interacting particles corresponding

to any such family of modular operators. Let us write  $\mathcal{H}_1 = \mathcal{H}_1^b \oplus \mathcal{H}_1^f$  where  $\mathcal{H}_1^b$  and  $\mathcal{H}_1^f$  are the eigenspaces of  $\Gamma$  with eigenvalues  $+1$  and  $-1$  respectively (the bosonic and fermionic one-particle spaces). Then over  $\mathcal{H}_1$  we construct a mixed bosonic/fermionic Fock space  $\mathcal{H} = \mathcal{H}^b \otimes \mathcal{H}^f$ , where  $\mathcal{H}^b$  is the bosonic (symmetric) Fock space over  $\mathcal{H}_1^b$ , and  $\mathcal{H}^f$  is the fermionic (antisymmetric) Fock space over  $\mathcal{H}_1^f$ . On  $\mathcal{H}$  it is possible to define bosonic field operators  $\phi^b(\psi)$  for  $\psi \in \mathcal{H}_1^b$ , fermionic field operators  $\phi^f(\psi)$  for  $\psi \in \mathcal{H}_1^f$ , and, by adding these, general self-adjoint field operators  $\phi(\psi)$  for  $\psi \in \mathcal{H}_1$ , defined such that  $\phi(\psi)\Omega = \psi$ . We then define algebras  $\mathcal{A}(W)$  generated by the field operators  $\phi(\psi)$  for every  $\psi \in R(W)$ . If the  $R(W)$  satisfy conditions (i)–(v) or (i') above, then the  $\mathcal{A}(W)$  satisfy the corresponding conditions (i)–(v) or (i') previously defined for them. Furthermore the modular conjugations and automorphisms for the  $\mathcal{A}(W)$  agree with the specified  $J(W)$  and  $\Delta(W)$  on the one-particle space in  $\mathcal{H}$ , which we may identify with  $\mathcal{H}_1$ . Such a family of wedge algebras describes a system without interaction, but it clearly gives examples of any phenomenon that occurs at the level of modular operators. If the  $J(W)$  and  $\Delta(W)$  satisfy the conditions of Theorem 7', then the  $\mathcal{A}(W)$  correspond to a (generalized) free field, and thus certainly arise from a net of local algebras. If they do not, however, then by the Bisognano-Wichmann theorem they cannot correspond to any set of Wightman fields, and they may or may not arise from a local net. We knew already that every family of wedge algebras produces a corresponding family of modular operators; what this shows is that a family of modular operators produces a family of wedge algebras. Thus for every result about families of wedge algebras, there is a corresponding result about families of modular operators, and vice versa. The study of modular structures for general families of wedge algebras is reduced to that of modular structures in the abstract, which correspond in this way to algebras without interaction.

Systems of algebras of this type are not the only ones to describe models without interaction, but they form an important class: given any one-particle space  $\mathcal{H}_1$ , these are those that describe all states of arbitrarily many such particles, present together without interaction. The Hilbert space  $\mathcal{H}$  is uniquely determined by  $\mathcal{H}_1$  and  $\Gamma$ , but the algebras  $\mathcal{A}(W)$  depend on the family of modular operators on  $\mathcal{H}_1$ —this family describes the localization properties of the one-particle states, and we will refer to it as the localization structure for such a model. For the free fields, the localization structure is determined entirely by the representation of  $\tilde{\mathcal{P}}_+$ , by the Bisognano-Wichmann condition, but it is not known whether there might be other possible localization structures

not corresponding to free fields. Here we study the question of uniqueness for families of modular operators on  $\mathcal{H}_1$ , assuming the existence of an appropriate representation of the Poincaré group. In this case there is a distinguished family  $J_0(W) = U(j(W))$ ,  $\Delta_0(W)^{it} = U(\lambda(W, t))$  of modular operators. For any other family  $J(W)$ ,  $\Delta(W)$  covariant under the representation, we may define  $J'(W) = J(W)J_0(W) = J_0(W)J(W)$  and  $\Delta'(W)^{it} = \Delta(W)^{it}\Delta_0(W)^{-it}$ , taking advantage of the commutation properties provided by the covariance. If our representation is only of the restricted Poincaré group, then we may at least define  $\Delta_0(W)$  and  $\Delta'(W)$  in the same way.

The most interesting case is that in which  $\mathcal{H}_1$  carries an irreducible representation of the Poincaré group, corresponding to Wigner's notion of an elementary system. Newton and Wigner [20] studied the possibilities for localization of states in the traditional quantum-mechanical sense on elementary systems; what we are studying here is a different sort of localization structure, one appropriate to the systems we describe here, and in particular to the free fields. We will treat not only elementary systems, but also reducible representations, provided they satisfy certain multiplicity restrictions.

**Lemma 8:** *Suppose assumptions (i'), (ii), and (iii) hold. Let  $U(\lambda)$  be a representation of the covering group  $\tilde{\mathcal{P}}_+^\uparrow$  of the restricted Poincaré group, satisfying the spectrum condition, under which the modular automorphisms  $\Delta(W)$  are covariant. For any two wedges  $W_1, W_2$  such that  $W_1 \cap W_2 \neq \emptyset$  and  $W'_1 \cap W'_2 \neq \emptyset$ , there is a unitary operator  $V(W_1, W_2)$  such that*

$$\langle \Delta'(W_1)^{1/2}\psi \mid \Delta'(W_2)^{-1/2}\phi \rangle = \langle \psi \mid V(W_1, W_2)\phi \rangle \quad (27)$$

for every  $\psi \in D(\Delta'(W_1)^{1/2})$ ,  $\phi \in D(\Delta'(W_2)^{-1/2})$  (and each of these sets is dense).

*Proof:* Since the  $\Delta(W)$  are covariant under the  $\Delta_0(W)^{it}$ , we see that for any particular wedge  $W$ ,  $\Delta_0(W)$  commutes strongly with  $\Delta(W)$ . Thus there is a dense domain  $D_\omega(W)$  on which  $\Delta'(W)^{iz} = \Delta(W)^{iz}\Delta_0(W)^{-iz}$  for all complex  $z$ . Thus for all  $\psi \in D_\omega(W_1)$ ,  $\phi \in D_\omega(W_2)$ , we may use Lemmas 5 and 6 (appropriately extended for the possibility of fermions) to compute

$$\begin{aligned} \langle \Delta'(W_1)^{1/2}\psi \mid \Delta'(W_2)^{-1/2}\phi \rangle &= \langle \Delta_0(W_1)^{-1/2}\Delta(W_1)^{1/2}\psi \mid \Delta_0(W_2)^{1/2}\Delta(W_2)^{-1/2}\phi \rangle \\ &= \langle \Delta(W_1)^{1/2}\psi \mid U(j(W_1)j(W_2))\Delta(W_2)^{-1/2}\phi \rangle \quad (28) \\ &= \langle \Delta(W_1)^{1/2}\psi \mid \Delta(j(W_1)W'_2)^{-1/2}U(j(W_1)j(W_2))\phi \rangle \\ &= \langle \psi \mid J(W_1)U(j(W_1)j(W_2))J(W_2)\phi \rangle, \end{aligned}$$

and the operator in the last expression is unitary. The result then follows by linearity for all  $\psi, \phi$  for which the expression is defined.

Next, we see that relations of this sort cannot be satisfied by bounded operators, by commuting operators, or even by matrices of commuting operators.

**Lemma 9:** *If the equality of Lemma 8 holds, and if each  $\Delta'(W)$  is a bounded operator, then  $\Delta'(W) = I$  for every wedge  $W$ .*

*Proof:* In this case,  $\Delta'(W_1)^{1/2}\Delta'(W_2)^{-1/2} = V(W_1, W_2)$  whenever  $W_1, W_2$  are as in Lemma 8. Thus  $V(W_1, W_2)^* = \Delta'(W_2)^{-1/2}\Delta'(W_1)^{1/2}$ , so that  $V(W_1, W_2)V(W_1, W_2)^* = \Delta'(W_1)^{1/2}\Delta'(W_2)^{-1}\Delta'(W_1)^{1/2} = I$ . Thus  $\Delta'(W_2)^{-1} = \Delta'(W_1)^{-1}$ , and using appropriate pairs of wedges we see that  $\Delta'(W)$  is independent of  $W$ . But then  $\Delta'(W)^{it} = \Delta'(W')^{-it} = \Delta'(W)^{-it}$  so that  $\Delta'(W) = I$ .

**Lemma 10:** *If the equality of Lemma 8 holds, and if there is an abelian von Neumann algebra  $\mathcal{N}$  and an embedding of the  $n \times n$  matrix algebra  $M_n(\mathcal{N})$  over  $\mathcal{N}$  in  $\mathcal{B}(\mathcal{H})$  such that every  $\Delta'(W)^{it}$  lies in  $M_n(\mathcal{N})$ , then  $\Delta'(W) = I$  for every wedge  $W$ .*

*Proof:* For simplicity we may consider  $\mathcal{N}$  as generated by a single self-adjoint operator  $X$ . For any unbounded measurable function  $f(X)$ , the sets  $E_a = f^{-1}([-a, a])$  form an increasing family of measurable sets on which  $f$  is bounded, and such that  $\cup_a E_a = \mathbf{R}$ . Likewise if we have a finite family  $f_i$  of such functions, then  $E_a = \cap_i f_i^{-1}([-a, a])$  has the same properties. For each wedge  $W$ ,  $\Delta'(W)$  is an  $n \times n$  matrix whose entries are unbounded measurable functions of  $X$ . Since there are only finitely many such entries, for any finite collection of wedges  $W$  there is such a family of sets  $E_a$  on which every entry in every  $\Delta'(W)$  is finite. If  $\Pi_a$  are the corresponding spectral projections for  $X$ , then  $\Delta'_a(W) = \Pi_a \Delta'(W) = \Delta'(W) \Pi_a$  is in  $M_n(\mathcal{N})$  for every  $a$ , and  $\Pi_a$  tends strongly to the identity as  $a \rightarrow \infty$ . Then if  $W_1$  and  $W_2$  are as in Lemma 8,  $\Delta'_a(W_1)$  and  $\Delta'_a(W_2)$  satisfy the relation of Lemma 8, but are both bounded, and hence the proof of Lemma 9 (which requires only a suitable finite collection of wedges) can be used to show that  $\Delta'_a(W)$  is the identity on  $\Pi_a \mathcal{H}$ . Thus in the limit we have  $\Delta'(W) = I$ .

These results imply uniqueness in case the representation of the Poincaré group satisfies certain multiplicity conditions. These will not hold on the full Fock space, but they can be satisfied on the one-particle space. In particular, they hold in the case of elementary systems. These conditions also suffice to guarantee the uniqueness of the representation of the Poincaré group under which the net is covariant.

**Theorem 11:** *If  $\mathcal{H}_1$  carries a representation  $U(\lambda)$  of the covering group  $\tilde{\mathcal{P}}_+^\uparrow$  of the restricted Poincaré group, satisfying the spectrum condition, and such that for any given mass there occur only finitely many finite-spin irreducible representations, each with finite multiplicity, then there is up to unitary equivalence at most one set of modular operators covariant under the  $U(\lambda)$  and satisfying assumptions (i'), (ii), and (iii). It must satisfy assumptions (iv) and (v), twisted wedge duality, and modular covariance. The modular automorphisms are uniquely determined, while the modular conjugations are determined only up to a unitary operator commuting with all  $U(\lambda)$ .*

*Proof:* There is a set of free fields with  $\mathcal{H}_1$  for one-particle space if and only if  $\mathcal{H}_1$  admits a PCT operator with respect to the given representation  $U(\lambda)$ , which thus extends to a representation of the proper group  $\tilde{\mathcal{P}}_+$ . Let us first assume that this is the case, so that there is a free-field net satisfying (i') and (ii)–(v), twisted wedge duality, and modular covariance, for which  $\mathcal{H}_1$  is the one-particle space. The PCT operator and the representation of  $\tilde{\mathcal{P}}_+$  are unique up to a Poincaré-invariant unitary operator. By the Bisognano-Wichmann results, the modular operators on  $\mathcal{H}_1$  for any free-field net must come from one of these representations.

Let us assume that there is some other set  $J(W), \Delta(W)$  of modular operators on  $\mathcal{H}_1$ , and let us use the notation of Lemma 8. We now require a version of Borchers' Theorem [3]; one directly applicable to the present situation may be found in Theorem 3 of [9]. This result, dependent on the spectrum condition, implies that  $\Delta(W)^{it}T(x)\Delta(W)^{-it} = \Delta_0(W)^{it}T(x)\Delta_0(W)^{-it}$  if  $T(x)$  is any translation. Thus  $\Delta'(W)$  commutes strongly with all translations, and with the von Neumann algebra  $\mathcal{T}$  generated by all translations. Since  $\Delta_0(W)$  and  $\Delta'(W)$  commute with the mass operator, so also does  $\Delta(W)$ , and without loss of generality we may restrict ourselves to the case of a single mass  $m$ . Then for this case we note that the multiplicity conditions imply that  $\mathcal{T}'$  is isomorphic to  $M_n(\mathcal{T})$ , where  $n$  is the total number of local degrees of freedom for all particles of mass  $m$ . Then Lemma 10 gives us our result immediately.

On the other hand, if  $\mathcal{H}_1$  does not admit a PCT operator, then we may substitute for it a direct sum  $\mathcal{H}_1 \oplus \mathcal{H}'_1$ , where  $\mathcal{H}'_1$  is the PCT conjugate of  $\mathcal{H}_1$ , without disturbing the multiplicity conditions, to obtain a representation which does admit a PCT operator. Any set of modular operators on  $\mathcal{H}_1$  gives a set of modular operators on  $\mathcal{H}_1 \oplus \mathcal{H}'_1$  by the same operation. But by the argument just given, there is precisely one set of modular operators for  $\mathcal{H}_1 \oplus \mathcal{H}'_1$ , and it satisfies modular covariance. Thus there can be no set of modular operators for  $\mathcal{H}_1$ .



Thus if the particle content of  $\mathcal{H}_1$  is sufficiently restricted, there is only one possible localization structure, that corresponding to the usual free fields. If we consider non-interacting systems of the sort described above, the only possibilities are those corresponding to free fields, which necessarily arise from local nets. Thus in these cases the Bisognano-Wichmann condition, wedge duality, the PCT theorem, and the spin-statistics theorem must hold. As the examples of [21] and [18] show, the net of modular operators and the representation of the Poincaré group need not be unique in the absence of multiplicity constraints. These examples still satisfy modular covariance, but it is not known whether there might be other Poincaré-covariant structures not satisfying modular covariance. If we omit the hypothesis of Poincaré covariance, there will be many possibilities not satisfying wedge duality [26]. One additional interesting possibility suggested by the result of [17] is that the modular automorphism group might have a geometric interpretation differing from that of Bisognano and Wichmann by a translation parallel to the vertex of the wedge. What we see here is that this possibility cannot occur in simple non-interacting models.

## VIII Asymptotic Locality

In this section we will assume that we are dealing with a Poincaré-covariant net of local algebras having a complete asymptotic interpretation via the Haag-Ruelle scattering theory, in terms of massive particles. Standard assumptions for nets of local algebras then imply that the wedge algebras must satisfy conditions (i'), (ii), and (iii). The restriction to massive particles is probably not necessary, but the scattering theory for massive particles is considerably simpler. The Haag-Ruelle theory assures us that the non-interacting behavior of the asymptotic particles is described by non-interacting systems of the sort constructed in the last section, but it does not specify any particular localization structure. The localization structure for the non-interacting systems must be determined from the interacting net after the manner described in [18] (asymptotic locality). However, if the theory is such that the asymptotic one-particle space  $\mathcal{H}_1$  satisfies the multiplicity conditions of Theorem 11, then we know that the only possible localization structures describing the free behavior of the asymptotic particles are those of free-field nets. There are in fact two sets of free fields relevant to scattering, the in-fields and the out-fields, and these differ by a Poincaré-invariant unitary operator, the S-matrix.

**Theorem 12:** *Let  $\mathcal{A}(W)$  be a family of wedge algebras derived from a local net covariant under a representation  $U(\lambda)$  of the covering group  $\tilde{\mathcal{P}}_+^\uparrow$  of the restricted Poincaré group, that satisfies the spectrum condition. Suppose the  $\mathcal{A}(W)$  satisfy conditions (i), (ii), and (iii), and also asymptotic completeness, with a one-particle subspace  $\mathcal{H}_1$  on which the mass spectrum is discrete and positive, and for which the multiplicity conditions of Theorem 11 hold. Then the  $\mathcal{A}(W)$  satisfy twisted wedge duality and modular covariance.*

*Proof:* We will use the same notation as in Theorem 11. As in Theorem 11, all the modular operators for wedge regions commute with the mass. The one-particle space  $\mathcal{H}_1$  can be distinguished as the discrete-mass subspace, so that each  $\Delta(W)^{it}$  must leave  $\mathcal{H}_1$  invariant, and must restrict to a set of modular operators on  $\mathcal{H}_1$  satisfying conditions (i'), (ii), and (iii). By Theorem 11, we see that  $\mathcal{H}_1$  must admit a PCT operator, and  $\Delta(W)$  must agree with  $\Delta_0(W)$  on  $\mathcal{H}_1$ . The modular conjugations may differ on  $\mathcal{H}_1$ , but only by a Poincaré-invariant unitary operator. This is consistent with the asymptotic locality result of [18]. What we now wish to show is that the same result holds on all of  $\mathcal{H}$ . We will do this by showing that the modular operators act multiplicatively on the asymptotic fields. Then the modular automorphisms must arise from the common representation of the Poincaré group, and both twisted wedge duality and modular covariance must hold.

We first notice that by the Tomita-Takesaki theorem, for any wedge  $W_0$ , we have

$$\Delta(W_0)^{it}\mathcal{A}(W_0)\Delta(W_0)^{-it} = \mathcal{A}(W_0) = \Delta_0(W_0)^{it}\mathcal{A}(W_0)\Delta_0(W_0)^{-it} \quad (29)$$

for all real  $t$ . Also, by Borchers' Theorem [3], for every translation  $T(x)$  we have

$$\Delta(W_0)^{it}T(x)\Delta(W_0)^{-it} = \Delta_0(W_0)T(x)\Delta_0(W_0)^{-it} \quad (30)$$

for all  $t$ . Thus by covariance we have

$$\Delta(W_0)^{it}\mathcal{A}(W_1)\Delta(W_0)^{-it} = \Delta_0(W_0)^{it}\mathcal{A}(W_1)\Delta_0(W_0)^{-it} \quad (31)$$

for all real  $t$  and any  $W_1 \parallel_s W_0$ , and by the same reasoning likewise

$$J(W_0)\mathcal{A}(W_1)J(W_0) = J_0(W_0)\mathcal{A}(W_1)J_0(W_0) \quad (32)$$

for every  $W_1 \parallel_s W_0$ . The collection of  $\mathcal{A}(W)$  and  $Z^*\mathcal{A}(W)'Z$  for each  $W \parallel_s W_0$  forms a two-dimensional net of wedge algebras, and the set of algebras  $\mathcal{A}(W_1) \cap Z^*\mathcal{A}(W_2)'Z$

for  $W_1, W_2 \parallel_s W_0$  forms a two-dimensional local net. It is highly degenerate, but it is still possible to construct its Haag-Ruelle scattering theory. The asymptotic fields will be generalized free fields, produced by the same sort of dimension-reducing procedure from the original asymptotic fields. Their one-particle space will again be  $\mathcal{H}_1$ , but now reinterpreted as carrying a highly reducible representation of the two-dimensional Poincaré group. With respect to this net,  $\Delta'(W_0)^{it}$  and  $J'(W_0)$  are local internal symmetries, and as in [18], they must act multiplicatively on the asymptotic fields. But we have already concluded in the previous paragraph that  $\Delta'(W_0)$  is trivial on  $\mathcal{H}_1$ , and  $J'(W_0)$  is a Poincaré-invariant unitary operator, so this must also be true on all of  $\mathcal{H}$ .

Thus we see that the Bisognano-Wichmann condition, modular covariance, wedge duality, the PCT theorem, and the spin-statistics theorem all hold for nets satisfying asymptotic completeness with appropriate restrictions on their asymptotic particle content: namely, that the particle spectrum be discrete and positive in mass, and finite in spin and total multiplicity for each mass.

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