Gauge Invariant Geometric Variables For Yang-Mills Theory^{*}

Peter E. Haagensen¹, Kenneth Johnson², and C.S. Lam¹

¹Physics Department, McGill University 3600 University St. Montréal H3A 2T8 CANADA.

²Center for Theoretical Physics, Laboratory for Nuclear Science Massachusetts Institute of Technology, Cambridge, MA 02139, USA.

Abstract

In a previous publication [1], local gauge invariant geometric variables were introduced to describe the physical Hilbert space of Yang-Mills theory. In these variables, the electric energy involves the inverse of an operator which can generically have zero modes, and thus its calculation is subtle. In the present work, we resolve these subtleties by considering a small deformation in the definition of these variables, which in the end is removed. The case of spherical configurations of the gauge invariant variables is treated in detail, as well as the inclusion of infinitely heavy point color sources, and the expression for the associated electric field is found explicitly. These spherical geometries are seen to correspond to the spatial components of instanton configurations. The related geometries corresponding to Wu-Yang monopoles and merons are also identified.

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1. Introduction

A new formulation of nonabelian gauge theory has recently been given [1] (hereafter referred to as I.), which is founded on a geometrical basis and seeks to define a setting in which gauge symmetry is implemented exactly and manifestly, even under approximations of the dynamics. Here, we will present further developments of this formulation as well as an example showing how it may allow one to obtain new insights into the reasons why the Yang-Mills field theory has a mass gap and produces a long range confining interaction between massive colored sources. We have introduced this geometric basis in the Hamiltonian formulation because in the Lagrangian path integral formalism for a nonabelian gauge theory, the local gauge invariance does not really manifest itself as a symmetry, but rather as a redundancy in the path integral measure. Any approximations in that formalism are likely to introduce gauge artifacts precisely because local gauge invariance is not acting as a symmetry. In the strong coupling region of the theory, an alternative and perhaps better procedure might be one that avoids such gauge artifacts. This is possible in the canonical Hamiltonian formulation, because in "temporal" gauge, $A_0^a = 0$, there is a remaining local gauge invariance restricted to space-dependent transformations at a fixed time, and one can achieve the goal of treating this gauge invariance like a true quantum mechanical symmetry. The local generators of such gauge transformations form an algebra and represent symmetries of the Hamiltonian which can be maintained exactly, even when the dynamics is done approximately. In view of this, one can understand better the statement that gauge invariance is not a symmetry away from a fixed-time formalism, and why approximations introduce gauge artifacts: the *full* classical gauge group includes time-dependent gauge transformations, and these are coupled to the dynamics. An approximate version of the dynamics is then likely to destroy any attempt to keep the full gauge invariance.

We implement the gauge symmetry by considering a change of variables in the Hilbert space such that any function of the transformed variables is a singlet under the gauge group, *i.e.*, is gauge invariant. In fact, this can be done in different ways [1]-[5]. In I., we have introduced such a transformation of variables, and here we will develop this formalism in more detail. Our basic procedure is straightforward and simple to state: rather than using the space components of the vector potential, A_i^a (a is a color index, i = 1, 2, 3 a space index), as fundamental coordinates in the Hilbert space of the theory, we use local quantities which transform covariantly under gauge transformations. Whereas the generator of gauge transformations in terms of A_i^a is complicated by the noncovariant transformation properties of the vector potential, when expressed in terms of gauge covariant variables it simply turns into a (color) rotation generator. Gauge covariant quantities can furthermore be contracted with themselves in color and lead to gauge invariant variables. In terms of such gauge invariant variables, Gauss' law, or gauge symmetry, becomes manifest.

Nevertheless, not any choice of gauge covariant variables is appropriate. An appropriate set of variables should describe the correct number of gauge invariant degrees of freedom at each point of space, and should also be free of ambiguities (such as, for instance, Wu-Yang ambiguities, where several gauge unrelated vector potentials may lead to the same color magnetic field [3],[6]). In I., the set of gauge covariant variables, u_i^a , we have chosen to define is given by the following differential equations:

$$\varepsilon^{ijk} D_j u_k^a \equiv \varepsilon^{ijk} (\partial_j u_k^a + f^{abc} A_j^b u_k^c) = 0 .$$
(1.1)

The linear operator $\varepsilon^{ijk}D_j \equiv (S^j)^{ik}D_j$, where S is the single gluon "spin" operator, plays a central role in our formulation. When A_i^a is a pure gauge, the eigenvalues of this operator are $\pm p$ and 0; in this case the zero modes are the "longitudinal" gluons. In general, the zero mode wavefunctions replace the vector potential as the dynamical coordinate. Further, the remaining spectrum and eigenfunctions of the operator enter in the process of obtaining an expression for the electric field conjugate to A_i^a in terms of the variables u_i^a . It is clear that the spectrum is gauge invariant and that the wavefunctions u_i^a transform as vectors under gauge changes. In I. and here we actually only consider in detail the SU(2) theory, $f^{abc} = \varepsilon^{abc}$ (although in I. the extension to SU(N) is also partially treated). Because most of the details of calculation needed for our purposes have been spelled out in I., here we will rather present a brief summary of previous results.

It turns out that for the SU(2) theory in canonical formalism, there is a natural symmetry under coordinate reparametrizations, which is respected by all commutators and basic formulas except for the Hamiltonian itself. We have purposefully maintained this symmetry in defining our new variables, so that a natural geometric picture arises as a guiding principle in the formalism at no extra cost. Under this reparametrization symmetry, the vector potential transforms as a covariant vector, while both the electric field $E^{ai} = -i\delta/\delta A_i^a$ and the magnetic field,

$$B^{ai} = \varepsilon^{ijk} (\partial_j A^a_k + \frac{1}{2} \varepsilon^{abc} A^b_j A^c_k) , \qquad (1.2)$$

transform as contravariant vector densities. The canonical commutators between coordinates A_i^a and momenta E^{bj} also transform covariantly, and the Gauss law generator,

$$\mathcal{G}^a = D_i E^{ai} \equiv \partial_i E^{ai} + \varepsilon^{abc} A^b_i E^{ci} , \qquad (1.3)$$

transforms as a scalar density. It is only because of this reparametrization covariance that we introduce the above seemingly peculiar placement of space indices. The only failure in reparametrization covariance comes in the Hamiltonian:

$$H = \frac{1}{2} \int d^3x \, \left(\frac{B^{ai} B^{ai}}{g_s^2} + g_s^2 E^{ai} E^{ai} \right) \,. \tag{1.4}$$

The integrand above is not a geometric scalar with the correct density weight, and the contraction in space indices is made with a Kronecker δ_{ij} rather than with a metric tensor, that is, the Hamiltonian is "committed" to a flat space.

It is straightforward to check that the definition of u_i^a in (1.1) is identical to the standard geometric equation defining the spin connection in terms of the Christoffel connection (or vice-versa):

$$\partial_j u_k^a + \varepsilon^{abc} A_j^b u_k^c - \Gamma_{jk}^s u_s^a = 0, \qquad (1.5)$$

where u_i^a is a 3-bein, $\varepsilon^{abc} A_i^b$ is a spin connection, and Γ_{jk}^i is the Christoffel connection of the metric $g_{ij} = u_i^a u_j^a$. Requiring that u_i^a transform as a vector under both gauge and reparametrization transformations then gives us a gauge and reparametrization covariant definition, and the above simple geometric picture. The "metric" tensor $g_{ij} = u_i^a u_j^a$ neatly organizes the six local gauge invariant degrees of freedom of the problem into a symmetric 3×3 matrix, and the next task would be to write the Hamiltonian in terms of these variables. One can then prove that any gauge invariant functional of A_i^a can be written as a function of g_{ij} only, and that any functional of g_{ij} is gauge invariant (cf. I.). This implements gauge invariance exactly. Further, it is also easy to include other color variables into the formalism when u_i^a is used as the independent variable.

Before we proceed to write down the gauge invariant, geometric expressions for the quantities of interest, we shall first observe that there are zero mode problems associated with the calculation of the electric field $E^{ai} = -i\delta/\delta A_i^a$. It is easy to see that, under the transformation of variables (1.1), the Jacobian matrix $\delta A_i^a/\delta u_j^b$ involves the operator $\varepsilon^{ijk}D_j$, which may have more than one remaining zero mode when the potential is not a pure gauge (where that is of course u_i^a itself). It then seems there will be an indefinition in

expressing the electric field through a chain rule in terms of derivatives $\delta/\delta u_i^a$. Providing a careful treatment of this problem is one of the main goals of this paper. To establish that such a redundancy can be handled, a detailed treatment will be presented of an example where there remain in the presence of a large set of vector potentials, an infinite set of such zero modes. We shall for that purpose propose an infinitesimal deformation of (1.1) which, as we shall verify explicitly, resolves the difficulties with zero modes. In the sections that follow, we shall first write down all the relevant geometrical formulas in the presence of the deformation. Then, as an example of the treatment where this care is required, we will specialize to those gauge field configurations for which the associated geometries are 3-spheres, where we shall be able to give the explicit expression for the electric field. In the limit in which the deformation is eliminated, we shall see that rather than an indefinition in the electric energy, there will be a restriction on the possible wavefunctionals describing states of the theory for such configurations. Finally, we shall give the expression for the electric field of a system of infinitely heavy point color sources immersed in these spherical configurations, using the formalism for introducing sources also presented in I..

In introducing a deformation to (1.1), again we are careful to preserve both reparametrization and gauge covariance, and we must verify that it indeed removes any zero mode ambiguities. We choose

$$\varepsilon^{ijk} D_j u_k^a = p \varepsilon^{ijk} \varepsilon^{abc} u_j^b u_k^c , \qquad (1.6)$$

where p is a small parameter with dimensions of mass. It is possible to find A_i^a explicitly as a functional of u_i^a . With manipulations similar to those found in I., one finds

$$A_i^a[u] = pu_i^a + \frac{(\varepsilon^{nmk}\partial_m u_k^b)(u_n^a u_i^b - \frac{1}{2}u_n^b u_i^a)}{\det u} .$$

$$(1.7)$$

One can already glean from the above why this eliminates zero mode problems: variations $\delta u, \delta A$ must satisfy

$$\varepsilon^{ijk} (\delta^{ac} \partial_j + \varepsilon^{abc} (A^b_j - 2pu^b_j)) \delta u^c_k = -\varepsilon^{ijk} \varepsilon^{abc} u^b_j \delta A^c_k .$$
(1.8)

To obtain δu_i^a in terms of δA_i^a , the operator acting on δu_i^a must be inverted. With p = 0 this operator would be $\varepsilon^{ijk}D_j$ itself, whose zero modes are the very solutions to (1.1), while for $p \neq 0$, on the other hand, possible zero modes of the operator on the l.h.s. of (1.8) are clearly not solutions to (1.6). In fact, the claim we shall make is that the operator in (1.8) has *no* zero modes for small enough nonzero p and can always be inverted, leading to an unambiguous definition of the electric field in the *u*-variables. We now proceed to present the relevant formulas in geometric variables.

2. Gauge Invariant Geometric Variables

With the definition (1.6), and using the gauge Ricci identity analogously to what was done in I., one can determine that the magnetic field, when expressed in terms of geometric variables, is

$$B^{ai} = \sqrt{g} \left(G^{ij} + p^2 g^{ij} \right) u^a_j .$$
 (2.1)

Here G_{ij} is the Einstein tensor of the metric g_{ij} , $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$, and throughout indices are raised and lowered with the metric g_{ij} and its inverse g^{ij} . By \sqrt{g} we mean det u, which can in principle take on both positive and negative values. The gauge Bianchi identity can be worked out to be

$$D_i B^{ai} = \sqrt{g} (\nabla_i G^{ij}) u_j^a = 0 ,$$
 (2.2)

so that it implies the geometric Bianchi identity and vice-versa. We note that this is a nontrivial consistency check of our geometric picture.

We now turn to Gauss' law and the electric field. If we define, following I., the gauge invariant tensor e^{ij} through

$$iE^{ai} = \frac{\delta}{\delta A_i^a} \equiv \sqrt{g} \ u_j^a e^{ij} , \qquad (2.3)$$

we can then verify that the Gauss law generator in geometric variables becomes

$$i\mathcal{G}^a = D_i\left(\frac{\delta}{\delta A_i^a}\right) = \sqrt{g}u_j^a(\tilde{\nabla}_i e^{ij}) = \sqrt{g}u_j^a(\nabla_i e^{ij}) + pu^{ia}\varepsilon_{ijk}e^{jk} , \qquad (2.4)$$

where $\tilde{\nabla}_i$ is the geometric covariant derivative with the torsion term introduced by $p \neq 0$, and u^{ia} is the matrix inverse of u_i^a (or, what amounts to the same thing, u_j^a with the space index raised by g^{ij}). Alternatively, it follows directly from (1.6) that if one makes a gauge variation of A_i^a ,

$$\delta A_i^a = -D_i \delta w^a , \qquad (2.5)$$

then

$$\varepsilon^{ijk} D_j^{(2p)} (\delta u_k^a - \varepsilon^{abc} \delta w^b u_k^c) = 0 , \qquad (2.6)$$

where the operator $D^{(2p)}$ contains in place of A the "potential" A - 2pu. Since $\varepsilon D^{(2p)}$ has no zero modes it follows that, as expected, a gauge variation of A is equivalent to a gauge variation of u transforming as a vector so \mathcal{G}^a is represented in terms of u by

$$i\mathcal{G}^a = \varepsilon^{abc} u^b_i \frac{\delta}{\delta u^c_i} \ . \tag{2.7}$$

To calculate the electric field in the *u*-variables, we begin from the definition (2.3) of e^{ij} :

$$e^{ij} = \frac{1}{\sqrt{g}} u^{aj} \frac{\delta}{\delta A^a_i} = \frac{1}{\sqrt{g}} u^{aj} \frac{\delta u^b_k}{\delta A^a_i} \cdot \frac{\delta}{\delta u^b_k} , \qquad (2.8)$$

where the dot stands for a generalized contraction including an integration over space. Variations in u_i^a can be further separated into variations of the six gauge invariant degrees of freedom g_{ij} and of three gauge degrees of freedom in the following simple way: first we split the summation over the index b by inserting the unit color matrix in the form $\delta^{bc} = u_m^b u^{cm}$. The resulting quantity $u^{cm} \frac{\delta}{\delta u_k^c}$ can then be written as the sum of its symmetric and antisymmetric pieces in m and k. Finally, it is easy to see that these correspond, respectively, to variations in the metric g_{ij} and gauge variations. The final expression we arrive at is:

$$e^{ij}(x) = \int d^3y \, \frac{1}{\sqrt{g}(x)} \left(u^{aj}(x) \frac{\delta u^b_k(y)}{\delta A^a_i(x)} u^b_m(y) \right) \left(2 \frac{\delta}{\delta g_{km}(y)} + \frac{i}{2} \frac{\varepsilon^{km\ell}}{\sqrt{g}(y)} u^a_\ell(y) \mathcal{G}^a(y) \right) \,, \quad (2.9)$$

where $i\mathcal{G}^a = \varepsilon^{abc} u_i^b \delta / \delta u_i^c$ is again the Gauss law generator, now in terms of the *u*-variables, and we make explicit the space integration.

We now need to write the Jacobian matrix $\delta u/\delta A$ in geometric form. To do so, we start by considering the following eigenvalue problem:

$$\varepsilon^{ijk} (\delta^{ac} \partial_j + \varepsilon^{abc} (A^b_j - pu^b_j)) w_{A^b_k} = \sqrt{g} \lambda_A w_A{}^{ia} , \qquad (2.10)$$

where again indices are raised with the inverse metric $g^{ij} = u^{ia}u^{ja}$. We note that, by definition, one solution to the above with $\lambda_A = 0$ is u_i^a itself. In the notation we are using, A is an index that labels all these eigenfunctions *except* the particular one given by u_i^a . The operator above is real and symmetric, and we assume $\{u_i^a, w_A_i^a\}$ forms a complete orthonormal spectrum of real eigenfunctions for it. By orthonormality we mean

$$\int d^3x \sqrt{g} g^{ij} u^a_i w_{Aj}^a = 0$$

$$\int d^3x \sqrt{g} g^{ij} w_{Ai}^a w_{Bj}^a = 3V \delta_{AB}$$

$$\int d^3x \sqrt{g} g^{ij} u^a_i u^a_j = 3 \int d^3x \sqrt{g} = 3V ,$$
(2.11)

where V is the volume of the space described by g_{ij} , and δ_{AB} is a Kronecker or Dirac delta depending on whether the spectrum is discrete or continuous. Because we will eventually concentrate on spherical geometries, we are only considering here configurations of finite volume (that is the "dynamical" volume V, and not the volume of space, which is infinite). The generalization to infinite V should not entail further conceptual difficulty.

If we now expand a generic variation δu_i^a in terms of this complete set,

$$\delta u_i^a = \eta u_i^a + \sum_A \eta_A w_A{}_i^a , \qquad (2.12)$$

substitute this in (1.8) and dot it on the left with the same complete set ("dot" meaning an inner product with the measure $\sqrt{g}g^{ij}$), we easily get the following relations:

$$3Vp\eta = \int d^3x \,\sqrt{g} u^{ia} \delta A^a_i$$

$$3V \sum_B I_{AB} \eta_B = -\int d^3x \,\varepsilon^{ijk} \varepsilon^{abc} u^a_i w_{Aj}^b \delta A^c_k \,, \qquad (2.13)$$

where

$$I_{AB} = \lambda_A \delta_{AB} - \frac{p}{3V} \int d^3x \ \varepsilon^{ijk} \varepsilon^{abc} u^a_i w_A^b_j w_B^c_k \ . \tag{2.14}$$

The origin of the zero mode problems alluded to above and their resolution through the $p \neq 0$ deformation of (1.1) now become manifest: for p = 0, the first of eqs. (2.13) actually represents a constraint on the variations δA_i^a for which a δu_i^a can be found. This constraint is a direct consequence of the fact that (1.7) is homogeneous in u_i^a for p = 0. For $p \neq 0$, this homogeneity is clearly broken, and there is no longer a constraint. Furthermore, essentially the same happens with the second set of equations in (2.13): for p = 0 further constraints on δA_i^a follow for each mode for which $\lambda_A = 0$. Again, these are eliminated by taking $p \neq 0$.

From here, it is straightforward to write the Jacobian matrix in explicit form:

$$\frac{\delta u_i^a(x)}{\delta A_j^b(y)} = \frac{1}{3Vp} \sqrt{g}(y) u_i^{jb}(y) u_i^a(x) - \frac{1}{3V} \sum_{AB} I_{AB}^{-1} \varepsilon^{jmn} \varepsilon^{bcd} u_m^c(y) w_{Bn}^{\ d}(y) w_{Ai}^{\ a}(x) \ . \tag{2.15}$$

This can be expressed in an entirely geometric form by using the important fact that if $w_{A_i^a}$ is a mode of (2.10), then the geometric modes $z_{A_i}^{j}$ defined through

$$w_{A_{i}^{a}} = z_{A_{i}}^{\ j} u_{j}^{a} \tag{2.16}$$

can be seen to be eigenmodes of the geometric curl operator $\varepsilon^{ijk}\nabla_j$ with the same eigenvalues λ_A :

$$\varepsilon^{ijk} \nabla_j z_{Ak}^{\ m} = \sqrt{g} \lambda_A z_A^{\ im} \ . \tag{2.17}$$

This leads to the fully geometric form we were seeking for the operator appearing in (2.9):

$$\left(u^{aj}(x)\frac{\delta u^{b}_{k}(y)}{\delta A^{a}_{i}(x)}u^{b}_{m}(y)\right) = \sqrt{g}(x)\left[\frac{1}{3Vp}g^{ij}(x)g_{km}(y) + \mathcal{H}^{ji}_{\ \ km}(x,y) - g^{ij}(x)\mathcal{H}^{\ \ s}_{s\ \ km}(x,y)\right],$$
(2.18)

where the Green's function $\mathcal{H}_{ijmn}(x, y)$ is defined to be:

$$\mathcal{H}_{ijmn}(x,y) \equiv \frac{1}{3V} \sum_{AB} z_{Aij}(x) I_{AB}^{-1} z_{Bmn}(y) . \qquad (2.19)$$

Assembling all these results leads to the following electric geometric tensor acting on functionals Ψ :

$$e^{ij}(x)\Psi = \int d^{3}y \left[\frac{1}{3Vp} g^{ij}(x)g_{mn}(y) + \left(\mathcal{H}^{ji}_{mn}(x,y) - g^{ij}(x)\mathcal{H}^{s}_{smn}(x,y)\right) \right] \cdot \left(2\frac{\delta\Psi}{\delta g_{mn}(y)} + \frac{i}{2}\frac{\varepsilon^{mn\ell}}{\sqrt{g}(y)}u^{a}_{\ell}(y)\mathcal{G}^{a}(y)\Psi\right) .$$

$$(2.20)$$

From this expression, one may already observe that, independent of the geometry, there is always at least one divergence in the electric energy as $p \to 0$. We eliminate it by requiring that gauge invariant functionals $\Psi[u]$ be invariant under global rescalings of the metric, *i.e.*,

$$\int d^3y \ u_i^a(y) \frac{\delta\Psi}{\delta u_i^a(y)} = 0 \ . \tag{2.21}$$

The Green's function $\mathcal{H}_{ijmn}(x, y)$ may also have divergences in the limit $p \to 0$, which again have to be eliminated. Generally speaking, the higher the degree of symmetry of a certain geometry (which is determined by its Killing vectors), the larger the number of zero modes of the curl operator and, due to (2.14), the larger the number of divergent terms in \mathcal{H}_{ijmn} as $p \to 0$. In the following section we will work out and analyze the electric field for those compact geometries with the maximum number of Killing vectors, namely, spheres. Because they are maximally symmetric spaces, for spheres it is possible to find the spectrum of $\varepsilon \nabla$ explicitly without too much difficulty, and therefore an explicit expression for \mathcal{H}_{ijmn} as well.

A final note on renormalization of divergences is in order here. It has been usual in the past literature on the subject to consider as the electric energy density expectation on a state Ψ the expression

$$<\Psi|(E^{ai}(x))^2|\Psi> = -\int [\mathcal{D}A] \ \Psi \frac{\delta^2}{\delta A_i^a(x)^2} \Psi , \qquad (2.22)$$

with $[\mathcal{D}A]$ an appropriately defined integration measure. However, another way to define the electric energy expectation is (cf. I.)

$$<\Psi|(E^{ai}(x))^2|\Psi> = \int [\mathcal{D}A] \frac{\delta\Psi}{\delta A^a_i(x)} \frac{\delta\Psi}{\delta A^a_i(x)}$$
 (2.23)

The former expression is inherently divergent due to the coincident points in the double functional derivative, and one must go to some lengths to properly define the operator. The latter expression, on the other hand, is easier to define and can be seen as an alternative prescription for the electric energy density. In our work we always use this second form. Of course, for systems with a finite number of degrees of freedom the two expressions are equivalent. Here, they differ formally by a total functional derivative.

3. Spherical Configurations

In order to get a clear picture of our restriction to spherical configurations, the first questions we will address are a) to what $A_i^a(x)$ configurations do spherical geometries correspond, and b) how much of the entire space of $A_i^a(x)$ do these geometries cover.

The direct way of answering the first question is of course to take a configuration $u_i^a(x)$ describing a 3-sphere and substitute it in (1.7) to find $A_i^a(x)$. We will do so below for a particular metric on S_3 . There is, however, a more indirect but extremely economical way, based on the following reasoning: the geometry of a sphere is that of an Einstein space, for which $G_{ij} \propto g_{ij}$; this implies, by (2.1), that the magnetic field for such geometries must be proportional to the matrix inverse of the 3-bein, $B^{ai} \propto u^{ai}$. If we now consider the standard expression (1.2) for B^{ai} as a function of A_i^a , and the fact that pure gauge configurations, say \bar{A}_i^a , have vanishing magnetic field, it follows immediately that for configurations which are global scalings of a pure gauge, say $A_i^a(x) = k\bar{A}_i^a(x), k \neq 1$, the magnetic field will turn out to be proportional to $B^{ai} \propto \varepsilon^{ijk} \varepsilon^{abc} A_j^b A_k^c$. But this is proportional to the matrix inverse of A_i^a and thus, for spherical geometries, $A_i^a(x)$ must be proportional to $u_i^a(x)$. Closer scrutiny of this argument shows that it indeed holds and furnishes all the proportionality constants missing above. We now list a series of results that derive from the above reasoning. In what follows, we take $\bar{A}_i^a(x)$ to be a pure gauge configuration, and α a real number $\neq 1$.

$$i) A_i^a(x) = \frac{1}{2(1-\alpha)} \bar{A}_i^a(x) \Longleftrightarrow u_i^a(x) = \frac{\alpha}{p} A_i^a(x) = \frac{\alpha}{2p(1-\alpha)} \bar{A}_i^a(x).$$

- *ii)* $A_i^a(x) = \frac{1}{2(1-\alpha)} \bar{A}_i^a(x) \implies u_i^a(x)$ is an Einstein space, with $G_{ij}(x) = -\frac{R}{6} g_{ij}(x) = -p^2 \frac{(1-\alpha)^2}{\alpha^2} g_{ij}(x)$.
- *iii)* $u_i^a(x)$ is an Einstein space, with $G_{ij}(x) = -\frac{R}{6}g_{ij}(x) = -p^2\frac{(1-\alpha)^2}{\alpha^2}g_{ij}(x) \Longrightarrow \mathcal{A}_i^a(x) \equiv A_i^a(x) \frac{p}{\alpha}u_i^a(x)$ is pure gauge.
- *iv)* $\mathcal{A}_{i}^{a}(x) \equiv A_{i}^{a}(x) \frac{p}{\alpha}u_{i}^{a}(x)$ is pure gauge $\implies (A^{g})_{i}^{a}(x) = \frac{p}{\alpha}(u^{g})_{i}^{a}(x)$ for some gauge transform (A^{g}, u^{g}) of (A, u).

Thus, up to gauge transformations, the circle of implications above flows freely in both directions. Moreover, we can now also answer the second question as well: the space of all spherical geometries corresponds to the space of all vector potentials that are rescalings of all possible pure gauges. More concretely, all possible pure gauges in SU(2) are spanned by three real functions $\xi^{a}(x)$, and rescalings are spanned by one real number.[†] This is indeed expected and consistent with the geometric picture, since all possible 3-sphere metrics are spanned by one real number (the inverse radius of the sphere, in units of p) and three real functions $y^{i}(x)$ (coordinate reparametrizations of a reference metric). This should be contrasted with six real, local functions, which parametrize the physical, gauge invariant, Hilbert space of the theory, so that, roughly speaking, 3-spheres span half the dimensions of this space. It is also possible to study the case of noncompact maximally symmetric spaces, *i.e.*, 3-hyperboloids, although we will not consider these geometries here. The vector potential can easily be found by substituting the appropriate bein $u_i^a(x)$ in (1.7). The procedure described above for spheres can be extended to hyperboloids by taking α complex, which would lead to complex potentials. Insofar as the symmetries are concerned, complex potentials do not spoil any of the reasoning above; however, in order to have real vector potentials in the end would require a complex gauge transformation in iv) above. Altogether we know this is possible since it is not difficult to obtain a real vector potential for this case.

To give a concrete example, we can consider the projective metric on S_3 used below (cf. (3.16) and below for the coordinate conventions). It is not difficult to find that in the limit of interest to us, $p \to 0$, the associated gauge field configuration is

$$A_i^a(x) = -2 \ \frac{\varepsilon_{iaj} x^j}{a^{-2} + |x|^2} \ . \tag{3.1}$$

[†] Incidentally, the special case of $\alpha = 1$ can be treated separately, and is seen to correspond to a flat geometry where, in an appropriate gauge, $u_i^a(x) = \delta_i^a$ and the vector potential is $A_i^a(x) = p\delta_i^a$.

Here, a^{-1} is the radius of the sphere. Two features of these configurations are worthwhile noting: first, if we take a^2 negative, we simply get the hyperbolic configurations mentioned above; unlike spherical configurations, they have singularities at finite x. Secondly, these configurations correspond precisely to the spatial components of the instanton of Belavin *et al.*, with a^{-1} the "size" of the instanton [7]. Further, a closely related type of configuration for which such a geometrical picture is also readily available correspond to Wu-Yang monopoles [6] (which are in turn related to merons [8]). They are gotten by taking $a^{-1} \rightarrow 0$ in the instanton configuration above, and multiplying it by a global factor $\frac{1}{2}$. The associated gauge invariant metric variable is $g_{ij} = \rho^2/|x|^2\delta_{ij}$, with ρ a constant parameter. This metric again describes a space of constant curvature, but closer inspection of its curvature invariants reveals that it actually corresponds to the space $S_2 \times \mathbb{R}$. It has been argued that (coherent states of) these magnetic monopoles are in fact one of the key ingredients underlying the color confinement mechanism in QCD [9]. We will not pursue such a geometry further in this paper, but we wanted nonetheless to illustrate the point that our geometric setting fits very nicely with specific gauge field configurations that are deemed to be important for the dynamics of Yang-Mills theory.

It is also curious to note that for spherical configurations, and again in the limit $p \to 0$, two of our fundamental equations are identical to the two equations defining the classical phase space of d = 3 gravity in the presence of a cosmological constant. For u_i^a seen as a dreibein and $\omega_i^{ab} \equiv \varepsilon^{abc} A_i^c$ seen as a spin connection, Eq. (1.1) states that the connection is torsion free, and (2.1) restricted to spheres states that the curvature built out of the spin connection is proportional to the inverse dreibein. These are just the equations of motion of d = 3 Einstein-Hilbert gravity with a cosmological constant (related to p^2 in (2.1)) or, equivalently, of an SO(4) Chern-Simons action with the gauge field being given by a combination of both u_i^a and ω_i^{ab} [10]. This analogy, of course, does not go any further since it makes no mention of the electric field or of the specific form of the Hamiltonian.

We now present our results on the spectrum of the curl operator [5]. The most straightforward way we found of calculating this spectrum was by writing ansätze for the modes z_{Aij} based on the scalar, vector and 2-tensor eigenmodes of the Laplacian on the 3-sphere and, after exhausting all possibilities for covariant ansätze, verifying that the resulting modes z_{Aij} satisfy a completeness relation. The eigenmodes we find are:

1. Exact zero modes.

$$z_{Aij} = (Z_N)_{ij} = \frac{1}{a^2 \sqrt{(\omega_N^2 - 3)(\omega_N^2 - 1)}} (\nabla_i \nabla_j + a^2 g_{ij}) y_N , N = 0, 2, 3, \dots$$
(3.2)

Here and in what follows, $\omega_N^2 = N(N+2)$ is the spectrum of the Laplacian acting on scalars on the 3-sphere, y_N are the associated eigenmodes (the hyperspherical harmonics) normalized to 3V, and a is the inverse radius of the sphere (with scalar curvature $R/6 = a^2$), so that

$$(\nabla^2 + a^2 \omega_N^2) y_N = 0 . (3.3)$$

The index A represents the three quantum numbers (N, ℓ, m) labelling these modes. Because ℓ and m do not affect the spectrum, but only its degeneracy, they are of secondary importance here, and will be omitted and understood whenever possible. The normalization is such that geometrical modes satisfy the normalization conditions following from (2.11). For all these modes, $\lambda_A = 0$. We also note that N = 1 is missing: it vanishes identically due to the structure of the operator $(\nabla \nabla + a^2 g)$. Moreover, the N = 0 term will also be eliminated from the calculation of the Green's function in (2.19) because it corresponds to the metric zero mode g_{ij} , which is to be treated separately (cf. discussion above (2.11)).

2. Scalar-based modes. These are nonzero modes based on the spectrum of the Laplacian acting on scalars:

$$z_{Aij} = (S_{N\alpha})_{ij} = \frac{1}{2a^2\omega_N\sqrt{\omega_N^2 - 1}} \left(\nabla_i\nabla_j + a^2\omega_N^2 g_{ij} - \alpha a\sqrt{\omega_N^2 - 1}\frac{\varepsilon_{ijk}}{\sqrt{g}}\nabla^k\right) y_N \ . \ (3.4)$$

Here, $\alpha = \pm$, and the associated eigenvalues are $\lambda_A = \lambda_{N\alpha} = \alpha a \sqrt{\omega_N^2 - 1}$, N = 1, 2, 3, ...3. Vector-based modes. These modes are based on the spectrum of the Laplacian acting on vectors,

$$(\nabla^2 + a^2(\omega_N^2 - 1))(v_N^\alpha)_i = 0 , \qquad (3.5)$$

with $\nabla^i (v_N^{\alpha})_i = 0$. Such vectors are also eigenvectors of the curl operator,

$$\frac{\varepsilon_{ijk}}{\sqrt{g}}\nabla^j (v_N^{\alpha})^k = \alpha a \sqrt{\omega_N^2 - 1} (v_N^{\alpha})_i , \qquad (3.6)$$

with $\alpha = \pm$. The vector-based modes are then given by

$$z_{Aij} = (V_N^{\alpha\beta})_{ij} = \frac{1}{2} \left[A_N^{\alpha\beta} \left(\nabla_i (v_N^{\alpha})_j + \nabla_j (v_N^{\alpha})_i \right) - \frac{\varepsilon_{ijk}}{\sqrt{g}} (v_N^{\alpha})^k \right] , \qquad (3.7)$$

with eigenvalues

$$\lambda_N^{\alpha\beta} = \frac{\alpha a}{2} \left(\sqrt{\omega_N^2 + 1} + \beta \sqrt{\omega_N^2 - 3} \right) = \frac{\alpha a}{2} \left(N + 1 + \beta \sqrt{(N+3)(N-1)} \right) , \qquad (3.8)$$

where $\beta = \pm$, $A_N^{\alpha\beta} = \alpha\beta/(a\sqrt{\omega_N^2 - 3})$, and $N = 2, 3, 4, \ldots$ (the N = 1 modes built in this way vanish identically, similarly to what happens in the exact zero mode case).

4. Tensor-based modes. These modes are based on the spectrum of the Laplacian acting on tensors,

$$(\nabla^2 + a^2(\omega_N^2 - 2))(T_{N\alpha})_{ij} = 0 , \qquad (3.9)$$

with $\nabla^i (T_{N\alpha})_{ij} = 0$, $(T_{N\alpha})_i^i = 0$ and $(T_{N\alpha})_{ij}$ symmetric. They are given by the $(T_{N\alpha})_{ij}$ themselves, with $\alpha = \pm, N = 2, 3, 4...$ and eigenvalues $\lambda_{N\alpha} = \alpha a \sqrt{\omega_N^2 + 1} = \alpha a (N+1)$. Again, the putative N = 1 mode vanishes identically, and the spectrum starts from N = 2.

It is now simple (but lengthy!) to calculate the matrix I_{AB} , through (2.14) (with the gauge modes w referred to geometric modes z through (2.16)). If we organize the matrix into five sectors, T (for tensor-based modes), V (for vector-based modes), S+ (for scalar-based modes with $\alpha = +$), S- (for scalar-based modes with $\alpha = -$), and Z (for zero modes), a simple structure emerges: the matrix is diagonal in the T and V sectors, and the only non-diagonal couplings appear between the Z, S+ and S- sectors. Furthermore, its subblocks are diagonal in each and all of its nonvanishing sectors (e.g., TT, VV, ZS+, ZS-, ZZ, etc.). This allows for an explicit inversion, even though the matrix is infinite. The nonvanishing entries of I_{AB} in the different sectors are:

$$I_{N\alpha,M\beta}^{(TT)} = (\alpha a \sqrt{\omega_N^2 + 1} + p) \delta_{MN} \delta_{\alpha\beta}$$
(3.10)

$$I_{N\alpha\beta,M\alpha'\beta'}^{(VV)} = \frac{\alpha a}{2} \left(\sqrt{\omega_N^2 + 1} + \beta \sqrt{\omega_N^2 - 3} \right) \delta_{MN} \delta_{\alpha\alpha'} \delta_{\beta\beta'} , \qquad (3.11)$$

(where δ_{MN} includes orthogonality in ℓ, m as well) and

$$I = \frac{p}{(\omega_N^2 - 1)} \begin{pmatrix} 2 & \omega_N \sqrt{\omega_N^2 - 3} & \omega_N \sqrt{\omega_N^2 - 3} \\ \omega_N \sqrt{\omega_N^2 - 3} & \frac{a}{p} (\omega_N^2 - 1)^{3/2} - \omega_N^2 & -1 \\ \omega_N \sqrt{\omega_N^2 - 3} & -1 & -\frac{a}{p} (\omega_N^2 - 1)^{3/2} - \omega_N^2 \end{pmatrix} , \quad (3.12)$$

in the Z, S+, S- sectors, where the first, second and third row (or column) refers to, respectively, Z, S+ and S- sectors. Each entry represents an infinite diagonal matrix and because of this, inversion can be accomplished by simply inverting the 3×3 matrix. This inverse is

$$I^{-1} = \frac{a^2}{2(p^2 - a^2)\xi_N^2} \cdot \begin{pmatrix} p(\omega_N^2 + 1) - \frac{\xi_N^4}{a^2 p} & (p + \xi_N)\omega_N\sqrt{\omega_N^2 - 3} & (p - \xi_N)\omega_N\sqrt{\omega_N^2 - 3} \\ (p + \xi_N)\omega_N\sqrt{\omega_N^2 - 3} & -2\xi_N - p\omega_N^2 & p(\omega_N^2 - 2) \\ (p - \xi_N)\omega_N\sqrt{\omega_N^2 - 3} & p(\omega_N^2 - 2) & 2\xi_N - p\omega_N^2 \end{pmatrix},$$
(3.13)

with $\xi_N = a\sqrt{\omega_N^2 - 1}$.

These results can finally be substituted in (2.19) in order to calculate $\mathcal{H}_{ijmn}(x, y)$. The TT and VV contributions can easily be gleaned from (3.10) and (3.11), while the contribution from the Z, S+, S- sectors is quite lengthy. In fact, a series of simplifications take place, and the final result is:

$$\mathcal{H}_{ijmn}(x,y) = \frac{1}{2a^2p} (\nabla_i \nabla_j + a^2 g_{ij})^x (\nabla_m \nabla_n + a^2 g_{mn})^y G_3(x,y) + \\ -\frac{p}{2a^2(p^2 - a^2)} \left[(\nabla_i \nabla_j)^x (\nabla_m \nabla_n)^y + \left(\frac{a\varepsilon_{ijk} \nabla^k}{\sqrt{g}}\right)^x \left(\frac{a\varepsilon_{mn\ell} \nabla^\ell}{\sqrt{g}}\right)^y \right] G_0(x,y) \\ -\frac{a}{2a^2(p^2 - a^2)} \left[(\nabla_i \nabla_j)^x \left(\frac{a\varepsilon_{mn\ell} \nabla^\ell}{\sqrt{g}}\right)^y + \left(\frac{a\varepsilon_{ijk} \nabla^k}{\sqrt{g}}\right)^x (\nabla_m \nabla_n)^y \right] G_0(x,y) + \dots$$

$$(3.14)$$

The dots represent the TT and VV contributions, and G_0 and G_3 are the Green's functions for the operators ∇^2 and $(\nabla^2 + 3a^2)$, respectively, acting on scalars on the sphere:

$$G_{0}(x,y) \equiv \frac{1}{3V} \sum_{N=1}^{\infty} \frac{y_{N}(x)y_{N}(y)}{a^{2}\omega_{N}^{2}}$$

$$G_{3}(x,y) \equiv \frac{1}{3V} \sum_{N=2}^{\infty} \frac{y_{N}(x)y_{N}(y)}{a^{2}(\omega_{N}^{2}-3)} .$$
(3.15)

Since $(\nabla^2 + 3a^2)$ has zero modes given by $y_{1\ell m}(x)$, it does not strictly speaking have a Green's function; what we mean by the above is of course the Green's function on the subspace of functions on S_3 that is orthogonal to this zero mode. In fact, both Green's functions above are also lacking the trivial $y_0 = \text{const.}$ mode in their spectral sum. This will be reflected in the differential equations they satisfy.

It is in fact possible, with some effort, to find closed expressions for these propagators. We present them in what follows and briefly describe how they are gotten since these are useful in the calculations envisaged in Sec. 4. We use the standard projective metric on S_3 :

$$g_{ij}(x_1, x_2, x_3) = \frac{4}{(1+a^2|x|^2)^2} \delta_{ij} , \qquad (3.16)$$

where x_1, x_2, x_3 are projective coordinates, with range $-\infty$ to ∞ , $|x|^2 = x_1^2 + x_2^2 + x_3^2$, and a is the inverse radius of the sphere. From this, the Laplacian acting on scalars on S_3 can be built, leading to the following differential equations for G_0 and G_3 :

$$\nabla^2 G_0(x,y) = -\frac{\delta(x-y)}{\sqrt{g}(x)} + \frac{1}{3V} y_0(x) y_0(y) = -\frac{\delta(x-y)}{\sqrt{g}(x)} + \frac{1}{V}$$

$$(\nabla^2 + 3a^2) G_3(x,y) = -\frac{\delta(x-y)}{\sqrt{g}(x)} + \frac{1}{3V} \sum_{N=0}^1 \sum_{\ell=0}^N \sum_{m=-\ell}^\ell y_{N\ell m}(x) y_{N\ell m}(y) ,$$
(3.17)

where $V = 2\pi^2/a^3$ is the volume of the sphere. The extra terms on the r.h.s. represent the lack of completeness of these Green's functions, as alluded to above. We will not go into the lengthy details of calculation, but rather just present the final result for G_0 and G_3 :

$$G_0(x,y) = \frac{a}{8\pi} \left(\frac{\sqrt{1-\eta^2}}{\eta} - \frac{\eta}{\sqrt{1-\eta^2}} \right) \left(1 - \frac{2}{\pi} \sin^{-1} \eta \right) - \frac{a}{8\pi^2}$$
(3.18)

$$G_3(x,y) = \frac{a}{16\pi} \left(\frac{1}{\eta\sqrt{1-\eta^2}} - 8\eta\sqrt{1-\eta^2} \right) \left(1 - \frac{2}{\pi} (\sin^{-1}\eta - \cos^{-1}\eta)\right) + \frac{a}{24\pi^3} (6\eta^2 + 1)$$
(3.19)

where

$$\eta \equiv \frac{ad}{2} = \frac{a|x-y|}{\sqrt{1+a^2|x-y|^2}}$$
(3.20)

is one half the chordal distance d between the points x and y in units of a^{-1} .

We are in fact interested in the $p \to 0$ limit of (3.14), and there are a few important features to note regarding this limit. Firstly, in the TT and VV sectors, the $p \to 0$ limit is perfectly smooth; in particular, we would have obtained the same result had we taken p = 0 from the beginning. Secondly, we find in the ZZ sector another 1/p divergence exactly like the one associated with global scalings (cf. (2.20) and (2.21)). This divergence appears in the first term in (3.14), and it will likewise entail a constraint on finite energy physical wavefunctionals. What this constraint is can be seen from the term $g^{ij}\mathcal{H}_s {}^s{}_{mn}$ in (2.20): the trace in the first two indices leads to the operator ($\nabla^2 + 3a^2$) acting on G_3 , and this leads to three types of terms, as can be seen from (3.17). The first term is a $\delta(x - y)$, and vanishing of this term leads to the constraint

$$\left(\nabla_m \nabla_n + a^2 g_{mn}\right)^y \frac{\delta \Psi}{\delta g_{mn}(y)} \bigg|_{g_{ij} = \text{sphere}} = 0 \tag{3.21}$$

in order not to have a divergence in the limit $p \to 0$. The second term contains $y_0(x)y_0(y) =$ const., and the second operator $(\nabla \nabla + a^2 g)$ acting on it kills it because of the constraint (2.21). The third type of term contains $y_{1\ell m}(x)y_{1\ell m}(y)$, and these vanish automatically under the action of the second operator $(\nabla \nabla + a^2 g)$. Thus, only the first term leads to a constraint, (3.21), on physical wavefunctions. This constraint must be satisfied by physical wavefunctionals in order for their energy to be finite in the limit $p \to 0$. This is again a result we would obtain directly in the p = 0 case from a similar requirement of finiteness of the electric energy. Finally, we note that the last term in (3.14) above is finite and nonvanishing in the $p \to 0$ limit. This result is different from what one would obtain by treating the p = 0 case directly. The correct result is the one presented here, since it properly takes into account the mixing of the zero and scalar modes.

In the $p \to 0$ limit and with the above finiteness constraints in place, the Green's function \mathcal{H}_{ijmn} is

$$\begin{aligned} \mathcal{H}_{ijmn}(x,y) &= \frac{1}{3V} \sum_{N} \left\{ \frac{1}{a\sqrt{\omega_{N}^{2}-1}} \left[(T_{N+})_{ij}(x)(T_{N+})_{mn}(y) - (T_{N-})_{ij}(x)(T_{N-})_{mn}(y) \right] \right. \\ &+ \frac{2}{a(\sqrt{\omega_{N}^{2}+1}+\sqrt{\omega_{N}^{2}-3})} \left[(V_{N++})_{ij}(x)(V_{N++})_{mn}(y) - (V_{N-+})_{ij}(x)(V_{N-+})_{mn}(y) \right] \\ &+ \frac{2}{a(\sqrt{\omega_{N}^{2}+1}-\sqrt{\omega_{N}^{2}-3})} \left[(V_{N+-})_{ij}(x)(V_{N+-})_{mn}(y) - (V_{N--})_{ij}(x)(V_{N--})_{mn}(y) \right] \right\} \\ &+ \frac{1}{2a^{2}} \left[\left(\nabla_{i}\nabla_{j} \right)^{x} \left(\frac{\varepsilon_{mn\ell}\nabla^{\ell}}{\sqrt{g}} \right)^{y} + \left(\frac{\varepsilon_{ijk}\nabla^{k}}{\sqrt{g}} \right)^{x} \left(\nabla_{m}\nabla_{n} \right)^{y} \right] G_{0}(x,y) , \end{aligned}$$

$$(3.22)$$

while its trace reduces to a single term

$$\mathcal{H}_{s}^{s}{}^{s}{}_{mn}(x,y) = -\frac{1}{2a^2} \left(\frac{\varepsilon_{mn\ell}\nabla^{\ell}}{\sqrt{g}}\right)^y \frac{\delta(x-y)}{\sqrt{g}(x)} . \tag{3.23}$$

From now on we assume this limit unless stated otherwise. We now have an explicit expression for both electric and magnetic energy densities. Although the electric energy is still in a rather unwieldy form, it is possible already to make an important observation regarding the vacuum state of the theory: spherical configurations introduce a scale into the problem, as any other explicit configuration would. Such a scale must be dynamically determined and, although we will not perform such a calculation here, we can already observe that this scale *a* enters the magnetic energy with a positive power and the electric energy in negative powers. This will cause the ground state wave functional to fall rapidly for large amplitude magnetic densities which vary slowly in space. At the same time, it will also become small for low amplitude magnetic energy densities with slow spatial variation. The correspondingly reduced fluctuations in the magnetic energy density will presumably fill the role of what is meant by a magnetic "condensate". If this is correct then one should be able to get at least a semi-quantitative estimate of the long range color electric fields produced by static sources in such surroundings. To do this we suggest looking at the electric field by dropping terms which should mainly be associated with short scales.

It would also be possible to perform manipulations in the vector and tensor sectors similarly to what has been done for the scalar and zero sectors, in order to simplify their respective contributions to \mathcal{H}_{ijmn} . However, as we shall argue, it is these latter sectors (*i.e.*, the $\nabla \nabla \nabla G_0$ term in (3.22)) which will give rise to the main contribution to the potential between color sources, and therefore one may in fact drop the TT and VV terms in a first approximation. To identify particular terms in \mathcal{H}_{ijmn} that lead to large electric energy densities, we must look both for eigenvalues λ_A such that $1/\lambda_A$ becomes large, and for modes which do not oscillate much, since highly oscillating modes cannot contribute to long distance effects. The third term in (3.22) (the last VV term) does have asymptotically large inverse eigenvalues as $N \to \infty$; however, these are also associated to highly oscillatory modes. On the other hand, all these volumes T and V modes do not have inverse eigenvalues that become asymptotically large. The only term satisfying both conditions we are seeking is the last one, associated to the scalar and zero sectors, and therefore it is reasonable to keep only this term as a leading approximation. Naturally, our formalism automatically guarantees, as announced in Sec. 1, that this represents a gauge invariant approximation to the dynamics.

4. Static Point Color Sources

We now consider the energy density of infinitely heavy point color sources immersed in the Yang-Mills configurations associated to spherical geometries. The formalism for introducing point color sources has been developed in I..

Introducing color sources at isolated points in space entails a local modification of Gauss' law only at these points. Then, rather than introducing an additional set of variables at every point in space, one can accomodate these isolated inhomogeneities in Gauss' law by simply considering wavefunctionals that carry the appropriate representation for each source, but that are still functionals of u_i^a only. To be specific, let us consider, for instance, the insertion of two sources at points x_1 and x_2 . The generalization to more sources is entirely trivial, but we consider here this specific case for clarity of presentation. Then, wavefunctionals describing states of this system should take the form

$$\Psi_{\alpha\beta}[u_i^a] , \qquad (4.1)$$

where α is an index in some SU(2) representation transforming at point x_1 and β likewise, but transforming at x_2 . The modification in Gauss' law is

$$\mathcal{G}^{a}(x) \to \bar{\mathcal{G}}^{a}(x)_{\alpha\alpha'\beta\beta'} = \mathcal{G}^{a}(x)\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \Lambda^{a}_{\alpha\alpha'}\delta_{\beta\beta'}\delta(x-x_{1}) + \delta_{\alpha\alpha'}\Lambda^{a}_{\beta\beta'}\delta(x-x_{2}) , \quad (4.2)$$

where Λ^a are the appropriate SU(2) generators. Again, to be specific, we consider the two sources to be in the fundamental representation, in which case the Λ^a are proportional to the Pauli matrices σ^a . The statement of gauge invariance becomes

$$\bar{\mathcal{G}}^a(x)_{\alpha\alpha'\beta\beta'}\Psi_{\alpha'\beta'}[u_i^a] = 0.$$
(4.3)

At this point it may not be clear whether or how one can build a color singlet wavefunctional, satisfying the local constraint (4.2), exactly at the locations x_1 and x_2 of the sources, since at each of these points the total color has contributions coming only from the combination of "integer spin" variables in the adjoint representation (the u_i^a), and a "half-integer spin" variable (α or β) coming from the source, which is in the fundamental representation. This turns out to be possible because there are sufficient variables u_i^a in order to build a half-integer spin representation of SU(2) at x_1 and at x_2 with these variables alone, even though they are in the adjoint representation. The way this is done is by realizing that u_1^a, u_2^a and u_3^a form three vectors, and thus comprise nine degrees of freedom at the point x_1 or x_2 . While six of these degrees of freedom, $u_i^a u_i^a$ (*i.e.*, the gauge invariant ones), give three lengths and three angles with which to uniquely define a tetrahedron, the three remaining degrees of freedom can be used to define the Euler angles uniquely fixing the orientation of the tetrahedron in color space. We then make use of the fact that it is possible to build half-integer spin representations of SU(2) with three Euler angles. With the appropriate transformation of variables from u_i^a to Euler angles, the angular momentum operator becomes precisely \mathcal{G}^a , and the appropriate eigenfunctions are the Wigner $\mathcal{D}^{(j)}$ -functions, with $j = \frac{1}{2}$ for angular momentum one-half. Equations (4.2),(4.3) then express the fact that under the usual addition of angular momentum in quantum mechanics, the wavefunctional at x_1 and at x_2 is a singlet, of total angular momentum zero, built out of two spin one-half representations. Because this procedure is to be done at the isolated points x_1 and x_2 , that is, because of the delta functions $\delta(x - x_1)$ and $\delta(x-x_2)$, the wavefunctional, besides being a functional of $u_i^a(x)$ everywhere, must now also be a regular function of the variables $u_i^a(x_1)$ and $u_i^a(x_2)$:

$$\Psi = \Psi_{\alpha\beta}[u_i^a; u_i^a(x_1), u_i^a(x_2)] , \qquad (4.4)$$

so that the functional differentiation in \mathcal{G}^a at those points becomes a regular derivative, and automatically incorporates the two delta functions.

With the introduction of sources, the change in magnetic energy is due not to any modification in (2.1) but rather to the constraints on Ψ engendered by (4.3). For the electric energy, on the other hand, there are modifications coming from the Gauss law term in (2.20), which would otherwise be absent for gauge invariant functionals. Neglecting the *TT* and *VV* contributions to \mathcal{H}_{ijmn} in (3.22), the contribution of the sources to the geometric electric tensor is

$$a^{2}e_{ij}^{\text{source}}(x)\Psi = \int d^{3}y \left\{ \left[(\nabla_{i}\nabla_{j})^{x} \left(\frac{\varepsilon_{mn\ell}\nabla^{\ell}}{\sqrt{g}} \right)^{y} - \left(\frac{\varepsilon_{ijk}\nabla^{k}}{\sqrt{g}} \right)^{x} (\nabla_{m}\nabla_{n})^{y} \right] G_{0}(x,y) - g_{ij}(x) \left[\nabla_{x}^{2} \left(\frac{\varepsilon_{mn\ell}\nabla^{\ell}}{\sqrt{g}} \right)^{y} \right] G_{0}(x,y) \right\} \left(\frac{\varepsilon^{mnp}}{\sqrt{g}(y)} \mathcal{G}_{p}(y)\Psi \right) ,$$

$$(4.5)$$

which simplifies to

$$2a^{2}e_{ij}^{\text{source}}(x)\Psi[g] = [(\nabla_{i}\nabla_{j})^{x}\nabla_{k}^{x_{1}}G_{0}(x,x_{1})]\Lambda^{k}(x_{1})\Psi + g_{ij}(x)\frac{1}{\sqrt{g}(x)}(\nabla_{k}^{x_{1}}\delta(x-x_{1}))\Lambda^{k}(x_{1})\Psi$$
(4.6)

plus an identical contribution at x_2 . Here, $\Lambda_k(x) \equiv -iu_k^a(x)\Lambda^a$, and we have ommitted the SU(2) indices. The action of \mathcal{G}^a on Ψ has been such as to satisfy (4.3).

One could now calculate an expression for the "potential" associated with static sources just as one evaluates the static Coulomb energy in the abelian gauge theory. It is clear that as well as being more complicated, the static potential is a function of the gauge field configuration and hence even in an approximation of the Born-Oppenheimer type it must be averaged with the ground state wavefunctional of the gauge field. Here we have obtained an explicit expression only for those gauge field configurations which are related to each other by GL(3) transformations of equal curvature geometries. We shall postpone a more complete discussion, and an explicit evaluation of the "potential" associated with the electric field in (4.6) for a later publication.

5. Conclusions

In this paper we have pursued further the formalism developed in I., where a set of local gauge invariant variables were introduced to describe the physical Hilbert space of Yang-Mills theory. We have chosen to do this in a Hamiltonian, fixed-time formalism because there one can identify the subset of the full gauge group that truly acts as a quantum mechanical symmetry of the theory, and one can implement it in a manifest and exact way. We have furthermore showed that the present treatment allows for approximations to the dynamics that do not spoil this exact gauge symmetry.

As a first step towards concrete calculations, we have worked out in detail the expression for the electric and magnetic energies for those gauge field configurations corresponding to spherical geometries. This has furnished indications of the mechanism through which a dynamically determined scale enters the theory and leads to a nonvanishing magnetic energy density of the vacuum and a mass gap. We have also indicated explicitly how such a geometry is related to instanton configurations, and how magnetic monopole configurations also correspond to a fairly simple, constant curvature space $-S_2 \times I\!\!R$ – in our geometric formulation of the theory. Moreover, for spherical configurations, we have studied particular terms in the electric field energy in the presence of heavy point sources that lead to the main contribution to the potential for these sources. We have also identified the manner in which exact gauge symmetry is maintained locally in the presence of half-integral spin sources, whereby one must construct half-integer spin representations from the gauge field variables in order to construct total angular momentum zero from the addition to the color sources.

Our calculations are by no means complete, and a number of important issues must still be considered: for instance, we have not studied the Jacobian determinant appearing in the measure after the change of variables, det $|\delta A/\delta u|$, and we have not considered the effects of renormalization. We also expect infrared effects to appear once noncompact geometrical configurations are considered, and these must be properly treated. Such issues would form an integral part of a more detailed computation of, for instance, the potential energy between two static color sources. A more detailed study of the $S_2 \times I\!\!R$ geometry would also be of interest. All such computations are part of our plans for future work.

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