EXACT RENORMALIZATION GROUP WITH FERMIONS

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The development of the Exact Renormalization Group for fermionic theories is presented, together with its application to the chiral Gross-Neveu model. We focus on the reliability of various approximations, specifically the derivative expansion and further truncations in the number of fields. The main differences with bosonic theories are discussed.

1 Introduction

The set of ideas enclosed in the Renormalization Group (RG, hereafter) has led to a variety of developments in many fields, as this Conference has made apparent. From a particle theorist point of view they englobe a bunch of ideas form which we may understand what a quantum field theory is. Moreover, it provides a framework for nonperturbative calculations.

In recent years, there has been an intensive development of the field mainly for scalar theories.[1] It can probably be said that we thoroughly understand all the subtleties the RG reserves for us in this case.

Nevertheless what would ultimately justify the whole approach, as applied for Particle Physics, is the construction of Lorentz and gauge invariant nonperturbative equations, manageable for reliable approximations. The hope of RG practitioners is that we are really not that far from there.[a]

Particle physicists may imagine themselves, thus, applying in the nearby future the powerfulness of the approach to atack long-standing nonperturbative problems for, say, quantum gluodynamics. And the next step would probably be the introduction of matter to have the full physical theory.

At this point it will for sure be helpful to have already understood the peculiarities associated to fermions on their own, both conceptual and technical. This is the reason we believe that we should try, as it has been done for bosons, to master as deeply as we can fermionic theories, even with no additional fields.

We do not want to suggest that the features of fermionic equations have to be significantly different from the ones for bosons. On the contrary, we believe that they have to be ultimately a direct translation of ideas from one subject

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[a]We have to mention that although the search for nonperturbative gauge invariant equations is still open, there are a variety of results for perturbative definitions of a gauge invariant theory based on these grounds.[2]

to the other. What we do mean is that these features may not be noticeable in any obvious way. On the other hand, new technicalities may also appear. And before embarking ourselves in a more ambitious project these peculiarities must be previously worked out and clearly understood.

We thus want to present some recent work in that direction.[3] Namely, the study of a two dimensional sample model (the so-called Gross-Neveu model[4]) which we will define below. This model is sufficiently simple in order to be able to carry out the algebra as far as we need but that it still captures the essentials of the approach for the Grassman case.

Let us briefly review the main ideas involved in bosonic theories as they are studied by Polchinski.[5] We first have to choose a regulator adequated for our purposes. It is done by simply modifying the propagator $P(p)$

$$P(p) = \frac{1}{p^2} \tag{1}$$

to

$$P_\Lambda(p) = \frac{K(p^2/\Lambda^2)}{p^2} \tag{2}$$

where Λ is a momentum-space cutoff and $K(x)$ a regulating function which decays sufficiently rapid to zero when $x \to \infty$.

With this kind of regulator, a quick (and somehow sloppy) argument that leads to an appropriate RG equation is to identify all the Λ-dependences in a partly integrated action by signaling all possible occurrences of the propagator and multiplying by the Λ-derivative of it. In this manner we immediately obtain

$$-\Lambda\frac{d}{d\Lambda}S_{int} \equiv \dot{S}_{int} = \frac{1}{2}\frac{\delta S_{int}}{\delta\phi} \cdot \dot{P}_\Lambda \cdot \frac{\delta S_{int}}{\delta\phi} - \frac{1}{2}\mathrm{tr}\left(\dot{P}_\Lambda \cdot \frac{\delta^2 S_{int}}{\delta\phi\delta\phi}\right) \tag{3}$$

with

$$S_{int} = S - \frac{1}{2}\phi \cdot P_\Lambda^{-1} \cdot \phi \tag{4}$$

and S the full action. The first term takes into account tree-type propagators and the second one loop-type propagators. We are using a compact notation regarding the propagator as a matrix with a dot standing for matrix multiplication.

The equation for a pure fermionic theory can be written in a similar form. From the propagator

$$P_\Lambda = i \not{p} \frac{K(p^2/\Lambda^2)}{p^2} \tag{5}$$

we obtain the RG equation

$$\dot{S}_{int} = \frac{\delta S_{int}}{\delta \psi} \cdot \dot{P}_\Lambda \cdot \frac{\delta S_{int}}{\delta \bar{\psi}} - \text{tr} \left(\frac{\delta}{\delta \psi} \cdot \dot{P}_\Lambda \cdot \frac{\delta S_{int}}{\delta \bar{\psi}} \right) \qquad (6)$$

Returning to S and expressing the resultant equation in dimensionless variables it is obtained

$$
\begin{aligned}
\dot{S} &= 2K'(p^2)\frac{\delta S}{\delta \psi} \cdot i\, \not{p} \cdot \frac{\delta S}{\delta \bar{\psi}} - \text{tr}\left(2K'(p^2)\frac{\delta}{\delta \psi} \cdot i\, \not{p}\, \frac{\delta S}{\delta \bar{\psi}} \right) \\
&\quad - 2p^2 \frac{K'(p^2)}{K(p^2)} \left(\bar{\psi} \cdot \frac{\delta S}{\delta \bar{\psi}} + \psi \frac{\delta S}{\delta \psi} \right) \\
&\quad + dS \\
&\quad + \frac{1-d+\eta(t)}{2} \left(\bar{\psi}\frac{\delta S}{\delta \bar{\psi}} + \psi \frac{\delta S}{\delta \psi} \right) - \left(\bar{\psi} \cdot p^\mu \frac{\partial'}{\partial p^\mu} \frac{\delta S}{\delta \bar{\psi}} + \psi \cdot p^\mu \frac{\partial'}{\partial p^\mu} \frac{\delta S}{\delta \psi} \right)
\end{aligned}
\qquad (7)
$$

where we work on a d-dimensional Euclidean space; η is the anomalous dimension (needed to obtain a physically interesting fixed point); $t \equiv -\ln \Lambda$; and the prime in $\frac{\partial'}{\partial p^\mu}$ means that the derivative does not act on the momentum conservation delta functions and thus only serves to count powers of momenta.

Note the first difference between bosons and fermions. Due to the different structure of the propagators, the fermionic equation presents an explicit factor of p in the first two terms of the right hand side of the RG equation 7 while this is not the case for bosons (Eq. 3). This may just look like a technical remark without any relevance. However it turns out that the seemingly most powerful approximations to these equations are based on the so-called derivative expansion[6] whose first order term is obtained by restricting the action to be a kinetic term plus a general potential term with no derivatives. In a fermionic theory this approximation will not be feasible, because we will be left only with a fairly simple linear equation. The derivative expansion should nevertheless be applicable, but it would lead to more complicated structures even at first non-trivial order.

2   The model

Let us now apply the RG equation 7 to the so-called chiral Gross-Neveu model.[4]

As any other field theory, it is best defined through its symmetries. We will consider thus an Euclidean invariant N-flavoured model, with an $U(N) \times U(N)$ internal symmetry group. It is also chosen to obey the discrete symmetries of parity, charge conjugation and reflection hermiticity.[7]

When one imposes these restictions and further use Fierz reorderings, it is easily sown that there appear only three basic structures,

$$
\begin{aligned}
V_{12}^j &\equiv \bar{\psi}^a(p_1)\gamma^j\psi^a(p_2) \\
S_{12}S_{34} - P_{12}P_{34} &\equiv \bar{\psi}^a(p_1)\psi^a(p_2)\bar{\psi}^b(p_3)\psi^b(p_4) \\
&- \bar{\psi}^a(p_1)\gamma_s\psi^a(p_2)\bar{\psi}^b(p_3)\gamma_s\psi^b(p_4) \\
S_{12}P_{34} - P_{12}S_{34} &\equiv \bar{\psi}^a(p_1)\psi^a(p_2)\bar{\psi}^b(p_3)\gamma_s\psi^b(p_4) \\
&- \bar{\psi}^a(p_1)\gamma_s\psi^a(p_2)\bar{\psi}^b(p_3)\psi^b(p_4)
\end{aligned}
\tag{8}
$$

which have to be combined in an arbitrary way with combinations of momenta.

The next step is to define a reasonable approximation to handle the above functional-derivative equation. We would like to choose one that closely resembles the bosonic derivative approximation. Nevertheless, due to the number of different structures it is not that easy to parametrize the general action up to, say, two derivatives while maintaining arbitrary the number of fields. Moreover, we should keep in mind that the allowed action is, as long as we are working with a finite number of different species, composed by a finite number of operators: the Grassman character of our variables constraints the number of fields allowed at one point of space.

We have already commented that derivative terms should also be included. In fact, this is an important point because in $d = 2$ the anomalous dimension η usually plays an important role: we would probably be too naive if we try to obtain numbers without letting it to be nonzero. In fact two derivatives may seem to do the job. However, once one goes through the calculations, it turns out to be quite clear that $\eta = 0$ is the only consistent value. This implies that we need at least three derivatives.

The maximum number of fields was chosen to be six. This seems a number both sufficiently low in order to keep the action relatively simple and sufficiently high to let non-trivial results appear.

The action thus obtained has the usual kinetic term; one term with three derivatives and only two fields; two derivative-free four-fields operators

$$
g_1(S_{12}S_{34} - P_{12}P_{34}) , \qquad g_2 V_{12}^j V_{34}^j
\tag{9}
$$

with coupling constants g_1 and g_2; eleven operators with also four fields but two derivatives; and ninety-two six-fermions operators, five of them with only one derivative and the rest with three derivatives.

After some algebra we can now obtain the set of beta functions. The fixed points are the solutions for these functions to vanish. They are a set of 106 non-linear algebraic equations.

Up to this point, the function $K(p^2)$ can be mainted arbitrary, thus keeping some freedom of chosing a scheme. The fixed point solutions in our approximation will in general depend on two parameters which serve as a scheme parametrization. In principle, this should not worry us, because it is well known that the actual expression of the fixed point action has no intrinsic physical meaning.

For Particle Physics it is specially interesting the value of the relevant directions from the fixed points. That is, we linearize the RG transformations,

$$\dot{g}_i = \mathcal{R}_i(g_j) \tag{10}$$

to

$$\dot{g}_i = \mathcal{R}_{ij} \cdot \delta g_j \ , \qquad \mathcal{R}_{ij} \equiv \left. \frac{\partial \mathcal{R}_i}{\partial g_j} \right|_{g^0} \tag{11}$$

where g^0 is the fixed-point solution and δg_j are the deviations from it. The number of positive eigenvalues of the matrix $\mathcal{R}_{ij}$ coincide with the number of possible parameters we can fine-tune in the corresponding cutoff-free theory[8] and the actual value of these eigenvalues gives the speed of departure from the fixed point.

These eigenvalues are directly related to the so-called critical exponents in the terminology of second-order phase transitions. They are universal and, therefore, they should be free from schemes dependences. In our approximation, however, this is not so, as often happens with truncations. The scheme ambiguities are solved by a translation of the principle of minimal sensitivity used in perturbative calculations[9].

3   Results

We now sketch the main results.

The equations simplify enormously when $N \to \infty$. Two fixed points can be clearly identified. One of them with vanishing anomalous dimension (it is of order N^{-1}) and with the most relevant eigenvalue $\sqrt{17} - 3$ in this approximation. Moreover the coupling constant g_2 which corresponds to $U(1)$ Thirring-like excitations becomes free (we have, in fact, a line of fixed points) and g_1 is also of order N^{-1}. All these features but the anomalous dimension remind the fixed point solution found by Dashen and Frishman[10].

The other solution, which corresponds to a different definition of the large N limit (different assumed N dependences of the coupling constants) has a non-vanishing anomalous dimension. It is scheme dependent with a range of variation of 1.11–1.14 for most of the schemes. The most relevant eigenvalue is

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also scheme-dependent with a range of 2.1–2.3 and, unlike the previous case, there are no free parameters.

Before going on we must comment on a quite unpleasant feature of this kind of approximations. By now it is generally believe that any approximation based on truncations leads to a system of fixed-point equations with many spurious solutions.[11] It seems that a pure derivative expansion (that is, one with a truncation in the number of derivatives but without any further truncation in the number of fields) cures all this kind of problems. We work with a truncated action and thus we expect on general grounds that this unwanted peculiarity appears and, actually, it does. The solution of the puzzle is not always simple. One usually tries to discriminate among solutions by checking the stability of the obtained ones either going one step beyond in the approximation or else tuning some parameters. In our case we are lucky to have nitid results in the large N limit. Therefore, we take as reliable solutions only those whose limit when $N \to \infty$ concides with one of the solutions found above.

This procedure will probably not be available in all cases and one should wonder if there is any systematic procedure to deal with the problem without relying on technical details of the studied model. Of course, one can always try to perform a true derivative expansion instead of mutilating each term as we have done. It should eliminate at once the spurious results. In fact we have a special case, which we will refer later on that suggests that this is true. Nevertheless the expansion proposed is not that simple, specially for N moderately large. Moreover, it seems to be difficult to deal with different values of N simultaneously and still preserving each term in the expansion without truncating it at some arbitrary point.

One solution for finite N matches the first one discussed above, with the most relevant eigenvalue smoothly decreasing to $\sqrt{17} - 3$ and with $N \cdot \eta$ increasing with N to 4.87... Unlike the strict large N limit, we do not find, nevertheless, a line of fixed points but an isolated one. We blame this feature to the crudeness of the approximation.

The solution that matches the second one above has a much more conspicuous behaviour. In fact it is valid only for $N > 142.8$. At this value it matches another branch of solutions, which exists even when $N \to \infty$ although the couplings do not scale with N as integer powers but as noninteger ones. Both the anomalous dimension and the most relevant eigenvalue present strong scheme dependences, quite difficult to disentangle.

Finally, we can consider separately the $N = 1$ case. It is worth going through it because it is a simple case where we can treat the equations in a purely derivative approximation: due to Fermi statistics, we cannot have more than six fields if we consider terms up to three derivatives. Our action is thus

exact in this sense. Also, we have to work out further relations imposed by Fierz reorderings, not present for $N \neq 1$. Our action is, therefore, even shorter than that. The results are nevertheless very messy and, probably, not reliable. When considering terms up to two derivatives, we find a line of fixed points (as it is expected in the Thirring model) with $\eta = 0$ as stated previously. But once we go to three derivatives, $\eta \neq 0$ but, unexpectedly, the line of fixed points disappears and we obtain only an isolated one. Nevertheless one piece of nice news comes out: the spurious solutions disappear, as expected.

A final comment is in order. We have found η by imposing that the normalization of the kinetic term is fixed at some standard value. This is surely not the most general way to proceed. If an exact computation is performed we know that this normalization does not matter and we will be able to fix it safely to whatever value we want: we will find a whole line of physically equivalent fixed points. It is generally known that this kind of symmetry is broken for most of the approximations[12] (in particular it is broken by the derivative expansion). Moreover, we generally expect that the true physical fixed point mixes with non-local ones with similar behaviour for the truncated action but with different anomalous dimensions. To pick up the local solution among the non-local ones, one should try to find the reminiscence of the line of fixed points: a marginal redundant operator. Its presence would signal that our scheme is truly approximating the local fixed point behaviour and not something else. This would probably fixed some, or perhaps even all, of the scheme dependences. This analysis has not been performed.

Summarizing, we have presented a fermionic RG equation and an example of its application. It seems that, although technically harder to work with, there emerges the same patterns as in the bosonic case. In particular the annoying issue of spurious solutions is also present. However, it does seem that with sufficiently accurate work and restricting oneself to a derivative expansion without further truncations reliable non-trivial results should come out.

Acknowledgments

References

1. For a brief summary, see Yu. Kubyshin, these Proceedings.
2. M. Bonini, these Proceedings; and references therein.

3. J. Comellas, Yu. Kubyshin and E. Moreno, *Exact Renormalization Group study of fermionic theories*, `hep-th/9512086`.
4. D. Gross and A. Neveu, *Phys. Rev.* D **10** 3235 (1974).
5. J. Polchinski, *Nucl. Phys.* B **231** 269 (1984).
6. For a review, see T.R. Morris, these Proceedings.
7. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Oxford Science Publications, Oxford,1989).
8. K.G. Wilson and J. Kogut, *Phys. Rep.* **12** 75 (1974).
9. P.M. Stevenson, *Phys. Rev.* D **23** 2916 (1981).
10. R. Dashen and Y. Frishman, *Phys. Lett.* B **46** 439 (1973); *Phys. Rev.* D **11** 2781 (1975).
11. T.R. Morris, *Phys. Lett.* B **334** 355 (1994).
12. T.L. Bell and K.G. Wilson, *Phys. Rev.* B **11** 3431 (1975).