

THE HIGHER COHOMOLOGIES OF E_8 LIE ALGEBRA

H. R. Karadayi and M. Gungormez
Dept.Physics, Fac. Science, Tech.Univ.Istanbul
80626, Maslak, Istanbul, Turkey

Abstract

It is well known that anomaly cancellations for D_{16} Lie algebra are at the root of the first string revolution. For E_8 Lie algebra, cancellation of anomalies is the principal fact leading to the existence of heterotic string. They are in fact nothing but the 6th order cohomologies of corresponding Lie algebras. Beyond 6th order, the calculations seem to require special care and it could be that their study will be worthwhile in the light of developments of the second string revolution.

As we have shown in a recent article, for A_N Lie algebras, there is a method which are based on the calculations of Casimir eigenvalues. This is extended to E_8 Lie algebra in the present article. In the generality of any irreducible representation of E_8 Lie algebra, we consider 8th and 12th order cohomologies while emphasizing the diversities between the two. It is seen that one can respectively define 2 and 8 basic invariant polynomials in terms of which 8th and 12th order Casimir eigenvalues are always expressed as linear superpositions. All these can be easily investigated because each one of these invariant polynomials gives us a linear equation to calculate E_8 weight multiplicities. Our results beyond order 12 are not included here because they get more complicated though share the same characteristic properties with 12th order calculations.

arXiv:hep-th/9705051v1 8 May 1997

I. INTRODUCTION

It is a clear fact that anomaly cancellations play a unique role in the construction of the way of thinking and constructing models in high energy physics since the last two decades. The ones for D_{16} Lie algebra [1] are principal for the first string revolution to begin. As it is also noted [2], the construction of heterotic string [3] is **shortly thereafter**. It is known [4] that the existence of a 10-dimensional string with a $E_8 \times E_8$ gauge symmetry relies heavily on E_8 anomaly cancellations.

On the other hand, these anomaly cancellations are in fact due to cohomology relations of corresponding Lie algebras. The cohomology for Lie algebras states non-linear relationships between elements of the center of their universal enveloping algebras [5]. The non-linearity comes from the fact that these relationships are between the elements of different orders and the non-linearly independent ones are determined by the Betti numbers [6]. A problem here is to determine the number of linearly independent elements of the same order. In two subsequent works [7], we studied this problem for A_N Lie algebras and give a method which is based on explicit construction of Casimir eigenvalues. This will be extended here to E_8 Lie algebra.

E_8 is the biggest one of finite dimensional Lie algebras and besides its own mathematical interest it plays a striking role in high energy physics. It provides a natural laboratory to study the structure of E_{10} hyperbolic Lie algebra [8] which is seen to play a key role in understanding the structure of infinite dimensional Lie algebras beyond affine Kac-Moody Lie algebras. There are so much works to show its significance in string theories and in the duality properties of supersymmetric gauge theories. This hence could give us some insight to calculate higher order cohomologies of E_8 Lie algebra. It will be seen in the following that this task is to be simplified to great extent when one uses a method based on explicit calculations of Casimir eigenvalues.

It is known that, beside degree 2, E_8 Betti numbers give us non-linearly independent Casimir elements for the degrees 8,12,14,18,20,24,30. We must therefore calculate the Casimir eigenvalues for all these degrees. In the present state of work, to give only the results for 8th and 12th orders would be more instructive. This will be possible in terms of one of the maximal subalgebras of E_8 , namely A_8 . Although our method [7] for A_N Lie algebras is previously presented, the calculations still need some special care for 8th and 12th orders. These are investigated in sections II and III. To this end, we especially emphasize our second permutational lemma to express the weights of an E_8 Weyl orbit and A_8 duality rules without which the calculations will be useless. In section IV, we show that the calculations find an end in the form of decompositions in terms of some properly chosen A_8 basis functions. The remarkable fact here is that the coefficients in these decompositions are constants and this shows us that the dependence on irreducible representations of E_8 Lie algebra are contained in these A_8 basis functions solely. For 12th order, the results of our calculations are given in three appendices because they are comparatively voluminous than 8th order calculations.

II. WEIGHT CLASSIFICATION OF E_8 WEYL ORBITS

We refer the excellent book of Humphreys [9] for technical aspects of this section though a brief account of our framework will also be given here. It is known that the weights of an irreducible representation $R(\Lambda^+)$ can be decomposed in the form of

$$R(\Lambda^+) = \Pi(\Lambda^+) + \sum m(\lambda^+ < \Lambda^+) \Pi(\lambda^+) \quad (II.1)$$

where Λ^+ is the principal dominant weight of the representation, λ^+ 's are their sub-dominant weights and $m(\lambda^+ < \Lambda^+)$'s are multiplicities of weights λ^+ within the representation $R(\Lambda^+)$. Once a convenient definition of eigenvalues is assigned to $\Pi(\lambda^+)$, it is clear that this also means for the whole $R(\Lambda^+)$ via (II.1).

In the conventional formulation, it is natural to define Casimir eigenvalues for irreducible representations which are known to have matrix representations. In ref(7), we have shown that the eigenvalue concept can be conveniently extended to Weyl orbits of A_N Lie algebras. The convenience comes from a permutational lemma governing A_N Weyl orbits. This however could not be so clear for Lie algebras other than A_N . We therefore give in the following a second permutational lemma. To this end, it is useful to decompose E_8 Weyl orbits in the form of

$$\Pi(\lambda^+) \equiv \sum_{\sigma^+ \in \Sigma(\lambda^+)} \Pi(\sigma^+) \quad (II.2).$$

where

$\Sigma(\lambda^+)$ is the set of A_8 dominant weights participating within the same E_8 Weyl orbit $\Pi(\lambda^+)$.

If one is able to determine the set $\Sigma(\lambda^+)$ completely, the weights of each particular A_8 Weyl orbit $\Pi(\sigma^+)$ and hence the whole $\Pi(\lambda^+)$ are known. We thus extend the eigenvalue concept to E_8 Weyl orbits just as in the case of A_N Lie algebras.

It is known, on the other hand, that elements of $\Sigma(\lambda^+)$ have the same square length with the E_8 dominant weight λ^+ . It is unfortunate that this remains insufficient to obtain the whole structure of the set $\Sigma(\lambda^+)$. This exposes more severe problems especially for Lie algebras having Dynkin diagrams with higher degree automorphisms, for instance affine Kac-Moody algebras. To solve this non-trivial part of this problem, we introduce 9 fundamental weights μ_I of A_8 , via scalar products

$$\kappa(\mu_I, \mu_J) \equiv \delta_{IJ} - \frac{1}{9} \quad , \quad I, J = 1, 2, \dots, 9 \quad (II.3)$$

The existence of $\kappa(.,.)$ is known to be guaranteed by A_8 Cartan matrix. The fundamental dominant weights of A_8 are now expressed by

$$\sigma_i \equiv \sum_{j=1}^i \mu_j \quad , \quad i = 1, 2, \dots, 8. \quad (II.4)$$

To prevent misconception, we list the main quantities which take place in the following discussions:

- λ^+ , Λ^+ \longrightarrow dominant weights of E_8
- λ_i \longrightarrow fundamental dominant weights of E_8 , $i=1,2, \dots, 8$
- σ^+ \longrightarrow dominant weights of A_8
- σ_i \longrightarrow fundamental dominant weights of A_8 , $i=1,2, \dots, 8$
- μ_I \longrightarrow fundamental weights of A_8 , $I=1,2, \dots, 9$

The correspondence $E_8 \leftrightarrow A_8$ is now provided by

$$\begin{aligned} \lambda_1 &= \sigma_1 + \sigma_8 \\ \lambda_2 &= \sigma_2 + 2 \sigma_8 \\ \lambda_3 &= \sigma_3 + 3 \sigma_8 \\ \lambda_4 &= \sigma_4 + 4 \sigma_8 \\ \lambda_5 &= \sigma_5 + 5 \sigma_8 \\ \lambda_6 &= \sigma_6 + 3 \sigma_8 \\ \lambda_7 &= \sigma_7 + \sigma_8 \\ \lambda_8 &= 3 \sigma_8 \end{aligned} \quad (II.5)$$

with

$$\lambda^+ \equiv \sum_{i=1}^8 r_i \lambda_i \quad , \quad r_i \in Z^+. \quad (II.6)$$

Z^+ here is the set of positive integers including zero. It is clear that this last relation turns out to be

$$\Lambda^+ \equiv \sum_{i=1}^8 q_i \sigma_i \quad , \quad q_i \in Z^+. \quad (II.7)$$

in view of (II.5) and hence $E_8 \leftrightarrow A_8$. By comparison between (II.6) and (II.7), note here that elements of $\Sigma(\lambda_1)$ are dominant weights for A_8 but not for E_8 .

It is clear that we only need here to know the weights of the sets $\Sigma(\lambda_i)$ for $i=1,2, \dots, 8$ explicitly. For instance,

$$\Sigma(\lambda_1) = (\sigma_1 + \sigma_8 , \sigma_3 , \sigma_6)$$

for which we have the decomposition

$$\Pi(\lambda_1) = \Pi(\sigma_1 + \sigma_8) \oplus \Pi(\sigma_3) \oplus \Pi(\sigma_6) \quad (II.8)$$

of 240 roots of E_8 Lie algebra. Due to permutational lemma given in ref(7), A_8 Weyl orbits here are known to have the weight structures

$$\begin{aligned} \Pi(\sigma_1 + \sigma_8) &= (\mu_{I_1} + \mu_{I_2} + \mu_{I_3} + \mu_{I_4} + \mu_{I_5} + \mu_{I_6} + \mu_{I_7} + \mu_{I_8}) \\ \Pi(\sigma_3) &= (\mu_{I_1} + \mu_{I_2} + \mu_{I_3}) \\ \Pi(\sigma_6) &= (\mu_{I_1} + \mu_{I_2} + \mu_{I_3} + \mu_{I_4} + \mu_{I_5} + \mu_{I_6}) \end{aligned} \quad (II.9)$$

where all indices are permuted over the set (1,2, .. 9) providing no two of them are equal. Note here by (II.4) that

$$\begin{aligned} \sigma_1 + \sigma_8 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 \\ \sigma_3 &= \mu_1 + \mu_2 + \mu_3 \\ \sigma_6 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \end{aligned} \quad (II.10) .$$

The formal similarity between (II.9) and (II.10) is a resume of the first permutational lemma. Now, we are ready to state our second permutational lemma:

For a dominant weight λ^+ , the set $\Sigma(\lambda^+)$ of A_8 dominant weights is specified by

$$\Sigma(\lambda^+) = \sum_{i=1}^8 r_i \Sigma(\lambda_i). \quad (II.11)$$

together with the equality of square lengths.

In addition to $\Sigma(\lambda_1)$ given above, the other 7 sets $\Sigma(\lambda_i)$ have respectively 7,15,27,35,17,5 and 11 elements for $i=2,3, .. 8$ and they are given in appendix(1). It is therefore clear that the weight decomposition of any E_8 Weyl orbit is now completely known in terms of A_8 Weyl orbits in the presence of both of our lemmas.

III. DUALITY RULES FOR A_8

In this section, we present some rules which we call A_8 **Dualities** in calculating E_8 cohomology. They are however similarly generalized for Lie algebras other than A_8 . It will be seen in the following that they are of crucial importance in calculating E_8 cohomology relations higher than degree 9.

We start by expressing an A_8 dominant weight σ^+ in the form

$$\sigma^+ \equiv \sum_{i=1}^8 k_i \mu_i \quad , \quad k_1 \geq k_2 \geq \dots \geq k_8 \geq 0 \quad . \quad (III.1)$$

To prevent repetitions, we reproduce here the main definitions and formulas of ref(7) for A_8 . The eigenvalues of a Casimir operator of degree M then are known to be defined by the aid of the formal definition

$$ch_M(\sigma^+) \equiv \sum_{\mu \in \Pi(\sigma^+)} (\mu)^M \quad (III.2)$$

for a Weyl orbit $\Pi(\sigma^+)$. Our way of calculation the right hand side of (III.2) is given in appendix(2). To this end, we need to define the following generators:

$$\mu(M) \equiv \sum_{I=1}^9 (\mu_I)^M \quad . \quad (III.3)$$

It is also convenient to define the following ones which we would like to call **K-generators**:

$$K(M) \equiv \sum_{i=1}^8 (k_i)^M . \quad (III.4)$$

We remark here by definition that $\mu(1) \equiv 0$ and hence

$$(\mu(1))^M \equiv 0 \quad , \quad M = 1, 2, \dots, 9, 10, \dots \quad (III.5)$$

It can be readily seen that (III.5) is fulfilled for $M=2,3, \dots, 9$ without any other restriction. It gives rise however to the fact that , for $M \geq 10$, all the generators $\mu(M)$ are non-linearly depend on the ones for $M=2,3, \dots, 9$. These non-linearities are clearly the reminiscents of A_8 cohomology. We therefore call them A_8 **Dualities**. It will be seen that the cohomology of E_8 Lie algebra will be provided by these A_8 dualities.

The first example is

$$\begin{aligned} \mu(10) \equiv & \frac{1}{8!} (25200 \mu(2) \mu(8) + 19200 \mu(3) \mu(7) + 16800 \mu(4) \mu(6) \\ & - 8400 \mu(2)^2 \mu(6) - 13440 \mu(2) \mu(3) \mu(5) + 8064 \mu(5)^2 + 2100 \mu(2)^3 \mu(4) \quad . \quad (III.6) \\ & - 5600 \mu(3)^2 \mu(4) - 6300 \mu(2) \mu(4)^2 + 2800 \mu(2)^2 \mu(3)^2 - 105 \mu(2)^5) \end{aligned}$$

It is seen that $\mu(10)$ consists of $p(10)=11$ monomials coming from the partitions of 10 into the set of numbers (2,3,4,5,6,7,8,9). We also have $p(8)=7$, $p(9)=8$, $p(11)=13$, $p(12)=19$ and these are the maximum numbers of monomials for corresponding degrees. We thus obtain the following expressions:

$$\begin{aligned} \mu(11) \equiv & \frac{1}{362880} (-3465 \mu(2)^4 \mu(3) + 12320 \mu(2) \mu(3)^3 + 41580 \mu(2)^2 \mu(3) \mu(4) \\ & - 41580 \mu(3) \mu(4)^2 + 16632 \mu(2)^3 \mu(5) - 44352 \mu(3)^2 \mu(5) \\ & - 99792 \mu(2) \mu(4) \mu(5) - 110880 \mu(2) \mu(3) \mu(6) + 133056 \mu(5) \mu(6) \\ & - 71280 \mu(2)^2 \mu(7) + 142560 \mu(4) \mu(7) + 166320 \mu(3) \mu(8) + 221760 \mu(2) \mu(9)) \end{aligned}$$

and

$$\begin{aligned} \mu(12) \equiv & \frac{1}{725760} (322560 \mu(3) \mu(9) + 136080 \mu(2)^2 \mu(8) + 272160 \mu(4) \mu(8) + 248832 \mu(5) \mu(7) \\ & - 60480 \mu(2)^3 \mu(6) - 80640 \mu(3)^2 \mu(6) + 120960 \mu(6)^2 - 72576 \mu(2)^2 \mu(3) \mu(5) \\ & - 145152 \mu(3) \mu(4) \mu(5) + 17010 \mu(2)^4 \mu(4) - 34020 \mu(2)^2 \mu(4)^2 \\ & - 22680 \mu(4)^3 + 20160 \mu(2)^3 \mu(3)^2 + 4480 \mu(3)^4 - 945 \mu(2)^6) \end{aligned}$$

IV.DECOMPOSITIONS OF INVARIANT POLINOMIALS IN THE A_8 BASIS

Let us start with the decomposition

$$ch_8(\Lambda^+) \equiv \sum_{\alpha=1}^7 Q_\alpha(\Lambda^+) T(\alpha) \quad (IV.1)$$

where 7 generators $T(\alpha)$ signify monomials

$$\mu(8) , \mu(2)\mu(6) , \mu(3)\mu(5) , \mu(4)^2 , \mu(4)\mu(2)^2 , \mu(3)^2\mu(2) , \mu(2)^4$$

which are known to exist because $p(8)=7$. One must stress in (IV.1) that coefficients $Q_\alpha(\Lambda^+)$ are assumed to be defined by comparison of (IV.1) with (III.2). These 7 monomials play a prominent role in expressing

eigenvalues of an 8th order Casimir operator of E_8 Lie algebra because they allow us to define the following **invariant polynomials**:

$$P_\alpha(\Lambda^+) \equiv \frac{Q_\alpha(\Lambda^+)}{Q_\alpha(\lambda_1)} \frac{\dim R(\lambda_1)}{\dim R(\Lambda^+)} P_\alpha(\lambda_1) \quad (IV.2)$$

where $\dim R(\Lambda^+)$ is the dimension of representation $R(\Lambda^+)$. An important notice is the fact that we do not need the Weyl dimension formula here. This will be provided by orbital decomposition (II.1) providing the sets $\Sigma(\lambda^+)$ are known for each particular subdominant λ^+ of Λ^+ . Let us recall from ref(7) that dimensions of A_8 Weyl orbits are calculated by counting permutations. In definition (IV.2) of invariant polynomials, the fundamental representation $R(\lambda_1)$ of E_8 is taken to be reference representation, i.e. all our expressions for Casimir eigenvalues are to be given by normalizing with respect to fundamental representation.

Explicit calculations for these 7 invariant polynomials $P_\alpha(\Lambda^+)$ show that we can find only 2 different polynomials the following one of which comes from the monomial $\mu(2)^4$:

$$P_1(8, \Lambda^+) \equiv 729 \Theta(8, \Lambda^+) - 71757069294212 . \quad (IV.3)$$

Only the following one is obtained for all other monomials:

$$\begin{aligned} P_2(8, \Lambda^+) &\equiv 68580 \Theta(8, \Lambda^+) \\ &\quad - 42672 \Theta(6, \Lambda^+) \Theta(2, \Lambda^+) \\ &\quad - 42672 \Theta(5, \Lambda^+) \Theta(3, \Lambda^+) \\ &\quad - 13335 \Theta(4, \Lambda^+)^2 \\ &\quad + 13335 \Theta(4, \Lambda^+) \Theta(2, \Lambda^+)^2 \\ &\quad + 17780 \Theta(3, \Lambda^+)^2 \Theta(2, \Lambda^+) \\ &\quad - 939 \Theta(2, \Lambda^+)^4 \\ &\quad + 385526887200 \end{aligned} \quad (IV.4)$$

The functions $\Theta(M, \Lambda^+)$ can be considered here as A_8 **basis functions** which are defined by

$$\Theta(M, \Lambda^+) \equiv \sum_{I=1}^9 (\vartheta_I(\Lambda^+))^M \quad , \quad M = 1, 2, \dots \quad (IV.5)$$

where

$$\vartheta_I = \kappa(\Lambda^+ + \rho_w, \mu_I) \quad . \quad (IV.6)$$

ρ_w here is the Weyl vector of E_8 Lie algebra. We notice that A_8 dualities are valid exactly in the same way for basis functions $\Theta(M, \Lambda^+)$ because $\Theta(1, \Lambda^+) \equiv 0$. This highly facilitates the work by allowing us to decompose all invariant polynomials $P_\alpha(\Lambda^+)$ in terms of $\Theta(M, \Lambda^+)$'s but only for $M=2,3,\dots,9$.

As in the similar way with A_8 basis functions defined above, the two polynomials P_1 and P_2 can be considered as E_8 **basis functions** in the sense that for any 8th order Casimir operator of E_8 the eigenvalues can always be expressed as linear superpositions of these E_8 basis functions. What is really significant here is the allowance of obtaining the decompositions (IV.3) and (IV.4) with coefficients which are constant for all irreducible representations of E_8 Lie algebra. In other words, beside constant coefficients, E_8 characteristic is reflected by A_8 basis functions.

E_8 cohomology manifests itself here by the fact that we have 2 polynomials P_1 and P_2 as E_8 **Basis functions** in spite of the fact that we have 7 polynomials from the beginning. As will be summarized in appendix(3), the same considerations lead us for degree 12 to 19 different polynomials which are known to exist from the beginning. It is however seen that the cohomology of E_8 dictates only 8 invariant polynomials for degree 12.

Careful reader could now raise the question that is there a way for a direct comparison of our results in presenting the E_8 basis functions

$$\begin{aligned} &P_\alpha(8, \Lambda^+) \text{ for } \alpha = 1, 2 \\ &P_\alpha(12, \Lambda^+) \text{ for } \alpha = 1, 2..8 \end{aligned} \quad .$$

A simple and might be possible way for such an investigation is due to weight multiplicity formulas which can be obtained from these polynomials. The method has been presented in another work [10] for A_N Lie algebras and it can be applied here just as in the same manner. This shows the correctness in our conclusion that any Casimir operator for E_8 can be expressed as linear superpositions of E_8 basis functions which are given in this work. An explicit comparison has been given in our previous works but only for 4th and 5th order Casimir operators of A_N Lie algebras and beyond these this does not seem to be tractable in practice.

As the final remark, one can see that the method presented in this paper are to be extended in the same manner to cases E_7 and G_2 in terms of their sub-groups A_7 and A_2 .

REFERENCES

- [1] M.B.Green and J.H.Schwarz, **Phys.Lett.** **149B** (1984) 117
- [2] J.H.Schwarz : Anomaly-Free Supersymmetric Models in Six Dimensions, hep-th/9512053
- [3] D.J.Gross, J.A.Harvey, E.Martinec and R.Rohm : **Phy.Rev.Lett.** **54** (1985) 502
- [4] J. Thierry-Mieg : **Phys.Lett.** **156B** (1985) 199
- J. Thierry-Mieg : **Phys.Lett.** **171B** (1986) 163
- [5] R. Hermann : Chapter 10, Lie Groups for Physicists, (1966) Benjamin
- [6] A. Borel and C. Chevalley : **Mem.Am.Math.Soc.** **14** (1955) 1
- Chih-Ta Yen: Sur Les Polynomes de Poincare des Groupes de Lie Exceptionnels, **Comptes Rendue Acad.Sci.** Paris (1949) 628-630
- C. Chevalley : The Betti Numbers of the Exceptional Simple Lie Groups, Proceedings of the International Congress of Mathematicians, 2 (1952) 21-24
- A. Borel : Ann.Math. **57** (1953) 115-207
- A.J. Coleman : **Can.J.Math** **10** (1958) 349-356
- [7] H.R.Karadayi and M.Gungormez : Explicit Construction of Casimir Operators and Eigenvalues:I , hep-th/9609060
- H.R.Karadayi and M.Gungormez : Explicit Construction of Casimir Operators and Eigenvalues:II , physics/9611002, to be appear in **Jour. of Math. Phys.**
- [8] V.G.Kac, R.V.Moody and M.Wakimoto ; On E_{10} , preprint
- [9] Humphreys J.E: Introduction to Lie Algebras and Representation Theory , Springer-Verlag (1972) N.Y.
- [10] H.R.Karadayi ; Non-Recursive Multiplicity Formulas for A_N Lie algebras, physics/9611008

APPENDIX.1

The Weyl orbits of E_8 fundamental dominant weights λ_i ($i=1,2, \dots, 8$) are the unions of those of the following A_8 dominant weights:

$$\Sigma(\lambda_2) \equiv (2\sigma_1 + \sigma_7, \sigma_1 + \sigma_3 + \sigma_8, \sigma_1 + \sigma_6 + \sigma_8, \\ \sigma_2 + \sigma_4, \sigma_2 + 2\sigma_8, \sigma_3 + \sigma_6, \sigma_5 + \sigma_7)$$

$$\Sigma(\lambda_3) \equiv (\sigma_3 + 3\sigma_8, \sigma_2 + \sigma_3 + 2\sigma_8, \sigma_2 + \sigma_6 + 2\sigma_8, \\ 3\sigma_1 + \sigma_6, \sigma_4 + 2\sigma_7, \sigma_1 + \sigma_3 + \sigma_6 + \sigma_8, 2\sigma_1 + \sigma_3 + \sigma_7, \\ \sigma_1 + \sigma_5 + \sigma_7 + \sigma_8, 2\sigma_1 + \sigma_6 + \sigma_7, \sigma_1 + \sigma_2 + \sigma_4 + \sigma_8, \\ \sigma_1 + 2\sigma_4, 2\sigma_2 + \sigma_5, 2\sigma_5 + \sigma_8, \sigma_2 + \sigma_4 + \sigma_6, \sigma_3 + \sigma_5 + \sigma_7)$$

$$\Sigma(\lambda_4) \equiv (\sigma_4 + 4\sigma_8, 2\sigma_3 + 3\sigma_8, \sigma_3 + \sigma_6 + 3\sigma_8, \\ \sigma_1 + \sigma_4 + 2\sigma_7 + \sigma_8, \sigma_2 + \sigma_3 + \sigma_6 + 2\sigma_8, \sigma_2 + \sigma_5 + \sigma_7 + 2\sigma_8, \\ \sigma_3 + 3\sigma_7, 2\sigma_2 + \sigma_4 + 2\sigma_8, 4\sigma_1 + \sigma_5, \\ 2\sigma_1 + \sigma_3 + \sigma_6 + \sigma_7, 2\sigma_1 + 2\sigma_4 + \sigma_8, \sigma_1 + 2\sigma_2 + \sigma_5 + \sigma_8, \\ \sigma_1 + 2\sigma_5 + 2\sigma_8, 3\sigma_2 + \sigma_6, 3\sigma_1 + \sigma_3 + \sigma_6, \\ \sigma_3 + \sigma_4 + 2\sigma_7, 2\sigma_1 + \sigma_5 + 2\sigma_7, \sigma_1 + \sigma_2 + \sigma_4 + \sigma_6 + \sigma_8, \\ 3\sigma_1 + 2\sigma_6, 2\sigma_1 + \sigma_2 + \sigma_4 + \sigma_7, \sigma_1 + \sigma_3 + \sigma_5 + \sigma_7 + \sigma_8, \\ \sigma_2 + \sigma_4 + \sigma_5 + \sigma_7, 3\sigma_4, \sigma_3 + 2\sigma_5 + \sigma_8, \sigma_1 + 2\sigma_4 + \sigma_6, 2\sigma_2 + \sigma_5 + \sigma_6, 3\sigma_5)$$

$$\Sigma(\lambda_5) \equiv (\sigma_5 + 5\sigma_8, \sigma_4 + \sigma_6 + 4\sigma_8, \sigma_2 + \sigma_4 + 2\sigma_7 + 2\sigma_8, \\ 2\sigma_3 + \sigma_6 + 3\sigma_8, \sigma_2 + 4\sigma_7, \sigma_3 + \sigma_5 + \sigma_7 + 3\sigma_8, \\ \sigma_1 + \sigma_3 + 3\sigma_7 + \sigma_8, 5\sigma_1 + \sigma_4, 2\sigma_3 + 3\sigma_7, \\ 2\sigma_1 + \sigma_4 + 3\sigma_7, 3\sigma_2 + \sigma_5 + 2\sigma_8, \sigma_2 + 2\sigma_5 + 3\sigma_8, \\ \sigma_1 + 3\sigma_2 + \sigma_6 + \sigma_8, \sigma_1 + \sigma_3 + \sigma_4 + 2\sigma_7 + \sigma_8, 2\sigma_2 + \sigma_4 + \sigma_6 + 2\sigma_8, \\ 4\sigma_2 + \sigma_7, \sigma_2 + \sigma_3 + \sigma_5 + \sigma_7 + 2\sigma_8, 3\sigma_2 + 2\sigma_6, \\ 3\sigma_1 + \sigma_3 + 2\sigma_6, 3\sigma_1 + 2\sigma_4 + \sigma_7, 2\sigma_1 + 2\sigma_2 + \sigma_5 + \sigma_7, \\ \sigma_1 + \sigma_2 + \sigma_4 + \sigma_5 + \sigma_7 + \sigma_8, 4\sigma_1 + \sigma_3 + \sigma_5, 2\sigma_1 + \sigma_2 + \sigma_4 + \sigma_6 + \sigma_7, \\ \sigma_1 + \sigma_3 + 2\sigma_5 + 2\sigma_8, 3\sigma_1 + \sigma_2 + \sigma_4 + \sigma_6, \sigma_2 + 2\sigma_4 + 2\sigma_7, \\ 2\sigma_1 + \sigma_3 + \sigma_5 + 2\sigma_7, 2\sigma_1 + 2\sigma_4 + \sigma_6 + \sigma_8, \sigma_1 + 2\sigma_2 + \sigma_5 + \sigma_6 + \sigma_8, \\ 3\sigma_4 + \sigma_6, \sigma_1 + 2\sigma_4 + \sigma_5 + \sigma_7, 2\sigma_2 + 2\sigma_5 + \sigma_7, \sigma_2 + \sigma_4 + 2\sigma_5 + \sigma_8, \sigma_3 + 3\sigma_5)$$

$$\Sigma(\lambda_6) \equiv (\sigma_6 + 3\sigma_8, \sigma_4 + \sigma_7 + 2\sigma_8, 3\sigma_7, \\ \sigma_2 + 2\sigma_7 + \sigma_8, \sigma_1 + \sigma_3 + 2\sigma_7, 2\sigma_2 + \sigma_6 + \sigma_8, \\ \sigma_1 + 2\sigma_2 + \sigma_7, \sigma_2 + \sigma_4 + \sigma_7 + \sigma_8, 3\sigma_2, 3\sigma_1 + \sigma_3, \sigma_3 + \sigma_5 + 2\sigma_8, \\ 2\sigma_1 + \sigma_2 + \sigma_5, \sigma_1 + \sigma_4 + \sigma_5 + \sigma_8, 2\sigma_1 + \sigma_4 + \sigma_6, 2\sigma_4 + \sigma_7, \\ \sigma_1 + \sigma_2 + \sigma_5 + \sigma_7, \sigma_2 + 2\sigma_5)$$

$$\Sigma(\lambda_7) \equiv (\sigma_7 + \sigma_8, \sigma_2 + \sigma_7, \sigma_1 + \sigma_2, \sigma_4 + \sigma_8, \sigma_1 + \sigma_5)$$

$$\Sigma(\lambda_8) \equiv (3\sigma_8, \sigma_3 + \sigma_7 + \sigma_8, 3\sigma_1, 2\sigma_2 + \sigma_8, \\ \sigma_5 + 2\sigma_8, \sigma_1 + 2\sigma_7, \sigma_1 + \sigma_2 + \sigma_6, \\ \sigma_1 + \sigma_4 + \sigma_7, 2\sigma_1 + \sigma_4, \sigma_2 + \sigma_5 + \sigma_8, \sigma_4 + \sigma_5)$$

As an example of (II.11), let us construct the set $\Sigma(\lambda_1 + \lambda_7)$ from $\Sigma(\lambda_1)$ and $\Sigma(\lambda_7)$ in view of our second lemma. The lemma states that elements $\sigma \in \Sigma(\lambda_1 + \lambda_7)$ are to be chosen from 15 elements of $\Sigma(\lambda_1) \oplus \Sigma(\lambda_7)$ providing the conditions

$$\kappa(\sigma, \sigma) = \kappa(\lambda_1 + \lambda_7, \lambda_1 + \lambda_7)$$

In result, one has only the following 13 elements:

$$\begin{aligned} \Sigma(\lambda_1 + \lambda_7) \equiv & (\sigma_1 + \sigma_7 + 2\sigma_8 , \\ & \sigma_1 + \sigma_2 + \sigma_7 + \sigma_8 , \\ & 2\sigma_1 + \sigma_2 + \sigma_8 , \\ & \sigma_1 + \sigma_4 + 2\sigma_8 , \\ & \sigma_6 + \sigma_7 + \sigma_8 , \\ & 2\sigma_1 + \sigma_5 + \sigma_8 , \\ & \sigma_2 + \sigma_3 + \sigma_7 , \\ & \sigma_1 + \sigma_2 + \sigma_3 , \\ & \sigma_3 + \sigma_4 + \sigma_8 , \\ & \sigma_2 + \sigma_6 + \sigma_7 , \\ & \sigma_4 + \sigma_6 + \sigma_8 , \\ & \sigma_1 + \sigma_3 + \sigma_5 , \\ & \sigma_1 + \sigma_5 + \sigma_6) . \end{aligned}$$

APPENDIX.2

Let us first borrow the following quantities from ref(7):

$$\begin{aligned} \Omega_8(\sigma^+) \equiv & 40320 K(8) \mu(8) + \\ & 20160 (\\ & 35 K(4, 4) \mu(4, 4) + 14 K(5, 3) \mu(5, 3) + 7 K(6, 2) \mu(6, 2) + 2 K(7, 1) \mu(7, 1)) + \\ & 40320 (\\ & 20 K(3, 3, 2) \mu(3, 3, 2) + 15 K(4, 2, 2) \mu(4, 2, 2) + \\ & 5 K(4, 3, 1) \mu(4, 3, 1) + 3 K(5, 2, 1) \mu(5, 2, 1) + 2 K(6, 1, 1) \mu(6, 1, 1)) + \\ & 13440 (\\ & 540 K(2, 2, 2, 2) \mu(2, 2, 2, 2) + 30 K(3, 2, 2, 1) \mu(3, 2, 2, 1) + \\ & 40 K(3, 3, 1, 1) \mu(3, 3, 1, 1) + 15 K(4, 2, 1, 1) \mu(4, 2, 1, 1) + 18 K(5, 1, 1, 1) \mu(5, 1, 1, 1)) + \\ & 483840 (\\ & 3 K(2, 2, 2, 1, 1) \mu(2, 2, 2, 1, 1) + K(3, 2, 1, 1, 1) \mu(3, 2, 1, 1, 1) + 2 K(4, 1, 1, 1, 1) \mu(4, 1, 1, 1, 1)) + \\ & 967680 (\\ & 3 K(2, 2, 1, 1, 1, 1) \mu(2, 2, 1, 1, 1, 1) + 5 K(3, 1, 1, 1, 1, 1) \mu(3, 1, 1, 1, 1, 1)) + \\ & 29030400 K(2, 1, 1, 1, 1, 1, 1) \mu(2, 1, 1, 1, 1, 1, 1) + \\ & 1625702400 K(1, 1, 1, 1, 1, 1, 1, 1) \mu(1, 1, 1, 1, 1, 1, 1, 1) \end{aligned}$$

$$\begin{aligned}
\Omega_{12}(\Lambda^+) \equiv & 40320 K(12) \mu(12) + \\
& 5040 (\\
& 1848 K(6, 6) \mu(6, 6) + 792 K(7, 5) \mu(7, 5) + 495 K(8, 4) \mu(8, 4) + \\
& 220 K(9, 3) \mu(9, 3) + 66 K(10, 2) \mu(10, 2) + 12 K(11, 1) \mu(11, 1)) + \\
& 95040 (\\
& 1575 K(4, 4, 4) \mu(4, 4, 4) + 210 K(5, 4, 3) \mu(5, 4, 3) + 252 K(5, 5, 2) \mu(5, 5, 2) + \\
& 280 K(6, 3, 3) \mu(6, 3, 3) + 105 K(6, 4, 2) \mu(6, 4, 2) + 42 K(6, 5, 1) \mu(6, 5, 1) + \\
& 60 K(7, 3, 2) \mu(7, 3, 2) + 30 K(7, 4, 1) \mu(7, 4, 1) + 45 K(8, 2, 2) \mu(8, 2, 2) + \\
& 15 K(8, 3, 1) \mu(8, 3, 1) + 5 K(9, 2, 1) \mu(9, 2, 1) + 2 K(10, 1, 1) \mu(10, 1, 1)) + \\
& 95040 (\\
& 11200 K(3, 3, 3, 3) \mu(3, 3, 3, 3) + \\
& 700 K(4, 3, 3, 2) \mu(4, 3, 3, 2) + 1050 K(4, 4, 2, 2) \mu(4, 4, 2, 2) + \\
& 350 K(4, 4, 3, 1) \mu(4, 4, 3, 1) + 420 K(5, 3, 2, 2) \mu(5, 3, 2, 2) + \\
& 280 K(5, 3, 3, 1) \mu(5, 3, 3, 1) + 105 K(5, 4, 2, 1) \mu(5, 4, 2, 1) + \\
& 168 K(5, 5, 1, 1) \mu(5, 5, 1, 1) + 630 K(6, 2, 2, 2) \mu(6, 2, 2, 2) + \\
& 70 K(6, 3, 2, 1) \mu(6, 3, 2, 1) + 70 K(6, 4, 1, 1) \mu(6, 4, 1, 1) + \\
& 60 K(7, 2, 2, 1) \mu(7, 2, 2, 1) + 40 K(7, 3, 1, 1) \mu(7, 3, 1, 1) + \\
& 15 K(8, 2, 1, 1) \mu(8, 2, 1, 1) + 10 K(9, 1, 1, 1) \mu(9, 1, 1, 1)) + \\
& 380160 (\\
& 1260 K(3, 3, 2, 2, 2) \mu(3, 3, 2, 2, 2) + \\
& 420 K(3, 3, 3, 2, 1) \mu(3, 3, 3, 2, 1) + 1890 K(4, 2, 2, 2, 2) \mu(4, 2, 2, 2, 2) + \\
& 105 K(4, 3, 2, 2, 1) \mu(4, 3, 2, 2, 1) + 140 K(4, 3, 3, 1, 1) \mu(4, 3, 3, 1, 1) + \\
& 105 K(4, 4, 2, 1, 1) \mu(4, 4, 2, 1, 1) + 189 K(5, 2, 2, 2, 1) \mu(5, 2, 2, 2, 1) + \\
& 42 K(5, 3, 2, 1, 1) \mu(5, 3, 2, 1, 1) + 63 K(5, 4, 1, 1, 1) \mu(5, 4, 1, 1, 1) + \\
& 42 K(6, 2, 2, 1, 1) \mu(6, 2, 2, 1, 1) + 42 K(6, 3, 1, 1, 1) \mu(6, 3, 1, 1, 1) + \\
& 18 K(7, 2, 1, 1, 1) \mu(7, 2, 1, 1, 1) + 18 K(8, 1, 1, 1, 1) \mu(8, 1, 1, 1, 1)) + \\
& 570240 (\\
& 56700 K(2, 2, 2, 2, 2, 2) \mu(2, 2, 2, 2, 2, 2) + \\
& 1260 K(3, 2, 2, 2, 2, 1) \mu(3, 2, 2, 2, 2, 1) + 280 K(3, 3, 2, 2, 1, 1) \mu(3, 3, 2, 2, 1, 1) + \\
& 840 K(3, 3, 3, 1, 1, 1) \mu(3, 3, 3, 1, 1, 1) + 315 K(4, 2, 2, 2, 1, 1) \mu(4, 2, 2, 2, 1, 1) + \\
& 105 K(4, 3, 2, 1, 1, 1) \mu(4, 3, 2, 1, 1, 1) + 420 K(4, 4, 1, 1, 1, 1) \mu(4, 4, 1, 1, 1, 1) + \\
& 126 K(5, 2, 2, 1, 1, 1) \mu(5, 2, 2, 1, 1, 1) + 168 K(5, 3, 1, 1, 1, 1) \mu(5, 3, 1, 1, 1, 1) + \\
& 84 K(6, 2, 1, 1, 1, 1) \mu(6, 2, 1, 1, 1, 1) + 120 K(7, 1, 1, 1, 1, 1) \mu(7, 1, 1, 1, 1, 1)) + \\
& 79833600 (\\
& 90 K(2, 2, 2, 2, 2, 1, 1) \mu(2, 2, 2, 2, 2, 1, 1) + \\
& 9 K(3, 2, 2, 2, 1, 1, 1) \mu(3, 2, 2, 2, 1, 1, 1) + 8 K(3, 3, 2, 1, 1, 1, 1) \mu(3, 3, 2, 1, 1, 1, 1) + \\
& 6 K(4, 2, 2, 1, 1, 1, 1) \mu(4, 2, 2, 1, 1, 1, 1) + 10 K(4, 3, 1, 1, 1, 1, 1) \mu(4, 3, 1, 1, 1, 1, 1) + \\
& 6 K(5, 2, 1, 1, 1, 1, 1) \mu(5, 2, 1, 1, 1, 1, 1) + 12 K(6, 1, 1, 1, 1, 1, 1) \mu(6, 1, 1, 1, 1, 1, 1)) + \\
& 479001600 (\\
& 36 K(2, 2, 2, 2, 1, 1, 1, 1) \mu(2, 2, 2, 2, 1, 1, 1, 1) + \\
& 10 K(3, 2, 2, 1, 1, 1, 1, 1) \mu(3, 2, 2, 1, 1, 1, 1, 1) + 40 K(3, 3, 1, 1, 1, 1, 1, 1) \mu(3, 3, 1, 1, 1, 1, 1, 1) + \\
& 15 K(4, 2, 1, 1, 1, 1, 1, 1) \mu(4, 2, 1, 1, 1, 1, 1, 1) + 42 K(5, 1, 1, 1, 1, 1, 1, 1) \mu(5, 1, 1, 1, 1, 1, 1, 1))
\end{aligned}$$

In all these expressions, the so-called K-generators are to be reduced to the ones defined by (III.3) for which the parameters k_i are determined via (III.1) for a dominant weight σ^+ which we prefer to suppress from K-generators. The reduction rules can be deduced from definitions given also in ref(7). The left-hand side of (III.2) can thus be calculated from

$$ch_M(\sigma^+) \equiv \frac{1}{9!} \dim \Pi(\sigma^+) \Omega_M(\sigma^+)$$

with which we obtain E_8 Weyl orbit characters. The dimension of a Weyl orbit $\Pi(\sigma^+)$ is the number of its elements and we show this number by $\dim \Pi(\sigma^+)$. Once again, we stress that both explicit forms and also the number of these weights are known due to permutational lemma given in ref(7).

APPENDIX.3

We now give the results of our 12th degree calculations. Explicit dependences on Λ^+ will be suppressed here. It will be useful to introduce the following auxiliary functions in terms of which the formal definitions of E_8 basis functions will be highly simplified:

$$\begin{aligned} \mathcal{W}_1(8) &\equiv 68580 \theta(8) - 42672 \theta(2) \theta(6) - \\ &42672 \theta(3) \theta(5) - 13335 \theta(4)^2 + \\ &13335 \theta(2)^2 \theta(4) + 17780 \theta(2) \theta(3)^2 - 939 \theta(2)^4 \\ \mathcal{W}_2(8) &\equiv 76765890960 \theta(8) - 47741514624 \theta(2) \theta(6) - \\ &47569228416 \theta(3) \theta(5) - 14950629660 \theta(4)^2 + \\ &14921466630 \theta(2)^2 \theta(4) + 19832476160 \theta(2) \theta(3)^2 - 1050561847 \theta(2)^4 \\ \mathcal{W}_1(12) &\equiv 302400 \theta(3) \theta(9) - 56700 \theta(4) \theta(8) - \\ &51840 \theta(5) \theta(7) - 158400 \theta(2) \theta(3) \theta(7) + 30240 \theta(6)^2 - \\ &168000 \theta(3)^2 \theta(6) + 33264 \theta(2) \theta(5)^2 - 80640 \theta(3) \theta(4) \theta(5) + \\ &16275 \theta(4)^3 + 92400 \theta(2) \theta(3)^2 \theta(4) + 19600 \theta(3)^4 \\ \mathcal{W}_2(12) &\equiv 42338419200 \theta(3) \theta(9) - 7938453600 \theta(4) \theta(8) - \\ &250343238600 \theta(2)^2 \theta(8) - 7258014720 \theta(5) \theta(7) - \\ &22177267200 \theta(2) \theta(3) \theta(7) + 4233841920 \theta(6)^2 - \\ &23521344000 \theta(3)^2 \theta(6) + 156357159840 \theta(2)^3 \theta(6) + \\ &4657226112 \theta(2) \theta(5)^2 - 11290245120 \theta(3) \theta(4) \theta(5) + \\ &160591001760 \theta(2)^2 \theta(3) \theta(5) + 2278630200 \theta(4)^3 + \\ &48089818350 \theta(2)^2 \theta(4)^2 + 12936739200 \theta(2) \theta(3)^2 \theta(4) - \\ &48806484300 \theta(2)^4 \theta(4) + 2744156800 \theta(3)^4 - \\ &66618900600 \theta(2)^3 \theta(3)^2 + 3440480295 \theta(2)^6 \\ \mathcal{W}_3(12) &\equiv 1976486400 \theta(3) \theta(9) - 370591200 \theta(4) \theta(8) + \\ &63622800 \theta(2)^2 \theta(8) - 338826240 \theta(5) \theta(7) - \\ &1035302400 \theta(2) \theta(3) \theta(7) + 197648640 \theta(6)^2 - \\ &1098048000 \theta(3)^2 \theta(6) - 12136320 \theta(2)^3 \theta(6) + \\ &217413504 \theta(2) \theta(5)^2 - 527063040 \theta(3) \theta(4) \theta(5) + \\ &185512320 \theta(2)^2 \theta(3) \theta(5) + 106373400 \theta(4)^3 - \\ &39822300 \theta(2)^2 \theta(4)^2 + 603926400 \theta(2) \theta(3)^2 \theta(4) + \\ &6366150 \theta(2)^4 \theta(4) + 128105600 \theta(3)^4 - \\ &63571200 \theta(2)^3 \theta(3)^2 - 274935 \theta(2)^6 \end{aligned}$$

$$\begin{aligned}
\mathcal{W}_4(12) \equiv & -1501985020838400 \theta(3) \theta(9) + 192772901311200 \theta(4) \theta(8) + \\
& 2407922770302000 \theta(2)^2 \theta(8) - 13295642434560 \theta(5) \theta(7) + \\
& 760428342950400 \theta(2) \theta(3) \theta(7) - 156516673824000 \theta(6)^2 + \\
& 33565287369600 \theta(2) \theta(4) \theta(6) + 883577458444800 \theta(3)^2 \theta(6) - \\
& 1515778400455200 \theta(2)^3 \theta(6) - 53070803904384 \theta(2) \theta(5)^2 + \\
& 579544204861440 \theta(3) \theta(4) \theta(5) - 1696086939738240 \theta(2)^2 \theta(3) \theta(5) - \\
& 47654628701400 \theta(4)^3 - 461057612469300 \theta(2)^2 \theta(4)^2 - \\
& 463327486742400 \theta(2) \theta(3)^2 \theta(4) + 472701971331450 \theta(2)^4 \theta(4) - \\
& 111245008649600 \theta(3)^4 + 684206487048000 \theta(2)^3 \theta(3)^2 - 33351005297925 \theta(2)^6
\end{aligned}$$

It is first seen that the expression (IV.4) can be cast in the form

$$P_2(8) \equiv \mathcal{W}_1(8) + 385526887200 .$$

Let us further define

$$\begin{aligned}
\Delta_{12} \equiv & (\theta(2) - 620) (-105 \theta(2)^5 + 341250 \theta(2)^4 - 443786280 \theta(2)^3 + \\
& 288672359200 \theta(2)^2 - 93922348435072 \theta(2) + 12228055880335360)
\end{aligned}$$

with the remark that the square length of E_8 Weyl vector is 620. At last, 8 basis functions of E_8 will be expressed as in the following:

$$\begin{aligned}
P_1(12) \equiv & \mathcal{W}_1(12) + \\
& \frac{105}{1392517035128} (\\
& 2327783 \mathcal{W}_2(8) \theta(2)^2 + \\
& 1641651348800 \mathcal{W}_1(8) \theta(2)^2 + \\
& 1853819288565353101504512 \theta(2)^2 - \\
& 5646385058438400 \mathcal{W}_1(8) \theta(2) - \\
& 2457714965901036308812800000 \theta(2) + \\
& 1878213525838949376 \mathcal{W}_1(8) + \\
& 474462162108792 P_{12}(0) + \\
& 814849980464400425555898009600)
\end{aligned}$$

$$\begin{aligned}
P_2(12) \equiv & \mathcal{W}_1(12) + \\
& \frac{105}{6580376} (\\
& 11 \mathcal{W}_2(8) \theta(2)^2 - \\
& 13946970 \mathcal{W}_1(8) \theta(2)^2 - \\
& 717386789108493504 \theta(2)^2 + \\
& 2185025300 \mathcal{W}_1(8) \theta(2) + \\
& 951080970408987600000 \theta(2) - \\
& 726826815792 \mathcal{W}_1(8) - \\
& 315043889595739569446400)
\end{aligned}$$

$$P_3(12) \equiv -\frac{10742925608415}{467309767} \Delta_{12} + \frac{6983349}{10867669} P_1(12) + \frac{3884320}{10867669} P_2(12)$$

$$P_4(12) \equiv -\frac{2898884687985}{487195289} \Delta_{12} + \frac{39572311}{237932583} P_1(12) + \frac{198360272}{237932583} P_2(12)$$

$$P_5(12) \equiv -\frac{511567886115}{1063875427} \Delta_{12} + \frac{2327783}{173189023} P_1(12) + \frac{170861240}{173189023} P_2(12)$$

$$P_6(12) \equiv -\frac{2557839430575}{69599327} \Delta_{12} + \frac{11638915}{11330123} P_1(12) - \frac{308792}{11330123} P_2(12)$$

$$P_7(12) \equiv -\frac{17904876014025}{1362158257} \Delta_{12} + \frac{11638915}{31678099} P_1(12) + \frac{20039184}{31678099} P_2(12)$$

$$P_8(12) \equiv -\frac{26924625585}{9942761} \Delta_{12} + \frac{2327783}{30753191} P_1(12) + \frac{28425408}{30753191} P_2(12)$$

It is seen that the generator Δ_{12} plays the role of **a kind of cohomology operators** in the sense that 6 generators $P_\alpha(12)$ (for $\alpha = 3, 4, \dots, 8$) will depend linearly on the first 2 generators $P_1(12)$ and $P_2(12)$ modulo Δ_{12} . It is therefore easy to conclude that all our 8 generators $P_\alpha(12)$ (for $\alpha = 1, 2, \dots, 8$) are linearly independent due to the fact that there are no a linear relationship among the generators $P_1(12)$ and $P_2(12)$ modulo Δ_{12} .