Nonlinear Electrodynamics with Singularities (Modernized Born-Infeld Electrodynamics)

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Abstract. Born-Infeld nonlinear electrodynamics are considered. Main attention is given to existence of singular point at static field configuration that M.Born and L.Infeld are considered as a model of electron. It is shown that such singularities are forbidden within the framework of the Born-Infeld model. It is proposed a modernized action that make possible an existence of the singularities. It is obtained main relations in view of the singularities. In initial approximation this model gives the usual linear electrodynamics with point charged particles.

1 Introduction

In the article by M.Born and L.Infeld [1] a model for nonlinear electrodynamics was considered. This model is named now as Born-Infeld electrodynamics. The article [1] contain some arguments about the necessity for nonlinear generalization of electrodynamics and possible connection between the such theories and quantum mechanics. At the present time the interest to similar models continues. In this connection it should be noted a series of articles by A.O. Barut about connection between classical particle-like field configurations, real particles and quantum theory. For example, see the articles [2, 3] by A.O. Barut and the article [4] by A.O. Barut and A.J. Bracken. In this cited articles the authors consider a linear theory of field but the stated ideas may be used also for nonlinear models. Moreover, nonlinearity create interactions of various types between particle-like solutions when

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we use a perturbation method. Thus there is a possibility of description of real interaction of particles with help of the such models.

The Born-Infeld model has some attractive properties. In particular, a field configuration that appropriate to the electron has a finite energy in framework of Born-Infeld electrodynamics, in contrast to the case of usual linear electrodynamics. But the components of electromagnetic field of this configuration has a discontinuity at a point for Born-Infeld model, the four-vector potential being everywhere continuous. However, the field configuration at the singular point is not solution to the field equations which are in the article [1]. In the present work we derive conditions for stationarity of the Born-Infeld action in the case when the field may have singularities. It appears, that it is necessary to modernize the Born-Infeld action in order that the model allowed the existence of such singularities and, hence, could be suitable for the description of real particles. We propose a modernized action and we obtain the main relations for particles-singularities.

2 Born-Infeld Electrodynamics

Let us state the main relations about the Born-Infeld standard model and introduce some notations which we shall use. The Born-Infeld action has the following form [1, 5].

$$S_{BI} = \int \left[\sqrt{|\det(g_{\mu\nu} + \alpha\, F_{\mu\nu})|} - \sqrt{|\det(g_{\mu\nu})|} \right] (\mathrm{d}x)^4 = \int (\mathcal{L} - 1)\, \sqrt{|g|}\, (\mathrm{d}x)^4 \tag{2.1}$$

where $F_{\mu\nu} \equiv \dfrac{\partial A_\nu}{\partial x^\mu} - \dfrac{\partial A_\mu}{\partial x^\nu}$ is electromagnetic field tensor components

$g_{\mu\nu}$ is metric tensor components; the Greec indexes take value 0, 1, 2, 3

$$\mathcal{L} \equiv \sqrt{|\,1 - \alpha^2\,\mathcal{I} - \alpha^4\,\mathcal{J}^2\,|} \quad ; \quad \mathcal{I} \equiv \frac{1}{2}\, F_{\mu\nu}\, F^{\nu\mu} \quad ; \quad \mathcal{J} \equiv \sqrt{\frac{F}{|g|}} = \frac{1}{8}\,\varepsilon_{\mu\nu\sigma\rho}\, F^{\mu\nu} F^{\sigma\rho}$$

$$F \equiv \det(F_{\mu\nu}) \quad ; \quad g \equiv \det(g_{\mu\nu}) \quad ; \quad (\mathrm{d}x)^4 \equiv \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$$

$$\left.\begin{array}{lcl} \varepsilon_{\mu\nu\sigma\rho} & \equiv & \pm\sqrt{|g|} \\ \varepsilon^{\mu\nu\sigma\rho} & = & \mp\dfrac{1}{\sqrt{|g|}} \end{array}\right\}$$

here there is the $\begin{smallmatrix}\text{top}\\ \text{bottom}\end{smallmatrix}$ sign, if $\mu\nu\sigma\rho$ is $\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}$ permutation of indexes 0123

A condition for stationarity of the action (2.1) is the following Eulerian system of equations.

$$\frac{\partial}{\partial x^\mu} \sqrt{|g|}\, f^{\mu\nu} = 0 \tag{2.2}$$

where

$$f^{\mu\nu} \equiv \frac{1}{\alpha^2}\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{1}{\mathcal{L}} \left(F^{\mu\nu} - \frac{\alpha^2}{2}\,\mathcal{J}\,\varepsilon^{\mu\nu\sigma\rho}\, F_{\sigma\rho} \right) \tag{2.3}$$

The expression for the tensor components $F_{\mu\nu}$ as functions of the tensor components $f_{\mu\nu}$ has the following form.

$$F_{\mu\nu} = \frac{1}{\overline{\mathcal{L}}} \left(f_{\mu\nu} + \frac{\alpha^2}{2} \overline{\mathcal{J}} \, \varepsilon_{\mu\nu\sigma\rho} \, f^{\sigma\rho} \right) \tag{2.4}$$

where

$$\left\{ \begin{array}{lcl} \overline{\mathcal{L}} & \equiv & \sqrt{|1 - \alpha^2 \, \overline{\mathcal{I}} - \alpha^4 \, \overline{\mathcal{J}}^2|} \\ \overline{\mathcal{I}} & \equiv & \dfrac{1}{2} \, f^{\mu\nu} \, f_{\mu\nu} \\ \overline{\mathcal{J}} & \equiv & \dfrac{1}{8} \, \varepsilon_{\mu\nu\sigma\rho} \, f^{\mu\nu} \, f^{\sigma\rho} \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{lcl} \overline{\mathcal{J}} & = & \mathcal{J} \\ \overline{\mathcal{L}} \, \mathcal{L} & = & 1 + \alpha^4 \, \mathcal{J}^2 \end{array} \right. \tag{2.5}$$

In view of the electromagnetic tensor definition we have also the following relation.

$$\varepsilon^{\mu\nu\sigma\rho} \, \frac{\partial F_{\sigma\rho}}{\partial x^\nu} = 0 \tag{2.6}$$

If we introduce the notations for space vectors of the field
(the latin indexes take values $1, \, 2, \, 3$)

$$\left\{ \begin{array}{lcl} E_i & \equiv & F_{i0} \\ B^i & \equiv & -\dfrac{1}{2} \, \varepsilon^{0ijk} \, F_{jk} \\ F_{ij} & = & \varepsilon_{0ijk} \, B^k \end{array} \right. \quad ; \quad \left\{ \begin{array}{lcl} D^i & \equiv & f^{0i} \\ H_i & \equiv & \dfrac{1}{2} \, \varepsilon_{0ijk} \, f^{jk} \\ f^{ij} & = & -\varepsilon^{0ijk} \, H_k \end{array} \right. \tag{2.7}$$

then the system (2.2), (2.6) may be written in the following form.

$$\left\{ \begin{array}{rcl} \mathrm{Div}\mathbf{B} & = & 0 \\ \overline{\partial}_0 \, \mathbf{B} + \mathrm{Rot}\mathbf{E} & = & 0 \\ \mathrm{Div}\mathbf{D} & = & 0 \\ -\overline{\partial}_0 \, \mathbf{D} + \mathrm{Rot}\mathbf{H} & = & 0 \end{array} \right. \quad \text{where} \quad \left\{ \begin{array}{rcl} \overline{\partial}_\mu & \equiv & \dfrac{1}{\sqrt{|g|}} \, \dfrac{\partial}{\partial x^\mu} \, \sqrt{|g|} \\ \mathrm{Div}\mathbf{B} & \equiv & \overline{\partial}_i \, B^i \\ (\mathrm{Rot}\mathbf{E})^i & \equiv & -\varepsilon^{0ijk} \, \partial_j E_k \\ \partial_\mu & \equiv & \dfrac{\partial}{\partial x^\mu} \end{array} \right. \tag{2.8}$$

Here the definitions Div and Rot (2.8) include $\sqrt{|g|}$ for the space-time in contrast to the usual definitions div and rot for the space. It is evident that these definitions coincide for the case with $|g_{00}| = 1$ and $g_{0i} = 0$. If in addition we have $g \neq g(x^0)$ then the system of equations (2.8) has the form of the usual Maxwell's system of equations. It should be noted that it is for the definitions (2.7) that we have the form of equations system (2.8), the distinction of the top and bottom indexes being essential. The forms of the definitions (2.7) and equations system (2.8) does not depend on the form of dependence between tensor components $f^{\mu\nu}$ and $F^{\sigma\rho}$ (2.3). Thus the foregoing is right also for usual linear electrodynamics that we have through linearization the relations (2.3): $f^{\mu\nu} = F^{\mu\nu}$. The system of equations (2.8) is right for any space-time metric.

According to formulas (2.3), (2.7) the components of the vectors $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{E}$, $\mathbf{B}$ are interconnected by means of the following formulas.

$$\left\{ \begin{array}{lcl} D^i & = & \dfrac{1}{\mathcal{L}} \left[\left(g^{0j}\, g^{i0} - g^{00}\, g^{ij} \right) E_j + g^{0j}\, g^{ip}\, \varepsilon_{0jpk}\, B^k + \alpha^2\, \mathcal{J}\, B^i \right] \\ H_i & = & \dfrac{1}{\mathcal{L}} \left[\dfrac{1}{2}\, \varepsilon_{0ijk}\, g^{jl}\, g^{kp}\, \varepsilon_{0lpm}\, B^m + \dfrac{1}{2}\, \varepsilon_{0ijk} \left(g^{jl}\, g^{k0} - g^{j0}\, g^{kl} \right) E_l + \alpha^2\, \mathcal{J}\, E_i \right] \end{array} \right. \tag{2.9}$$

If we substitute $\alpha = 0$, $\mathcal{L} = 1$ in (2.9) we have the appropriate expressions for the case of the linear electrodynamics.

The electromagnetic invariant are expressed through the field space vectors components with the following formulas.

$$\begin{array}{lcl} \mathcal{I} & = & \left(g^{k0}\, g^{0l} - g^{kl}\, g^{00} \right) E_k\, E_l + \dfrac{1}{2}\, \varepsilon_{0ijk}\, g^{jm}\, g^{in}\, \varepsilon_{0mnl}\, B^k\, B^l - \\ & & \qquad - \varepsilon_{0ijk} \left[\dfrac{1}{2} \left(g^{jl}\, g^{i0} - g^{j0}\, g^{il} \right) - g^{li}\, g^{0j} \right] E_l\, B^k \\ \mathcal{J} & = & E_i\, B^i \end{array} \tag{2.10}$$

If $|g^{00}| = 1$ and $g^{0j} = 0$ we have the simple relations interconnecting the vectors $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{E}$, $\mathbf{B}$.

$$\left\{ \begin{array}{lcl} \mathbf{D} & = & \dfrac{1}{\mathcal{L}} (\mathbf{E} + \alpha^2\, \mathcal{J} \mathbf{B}) \\ \mathbf{H} & = & \dfrac{1}{\mathcal{L}} (\mathbf{B} - \alpha^2\, \mathcal{J} \mathbf{E}) \end{array} \right. \quad ; \quad \left\{ \begin{array}{lcl} \mathbf{E} & = & \dfrac{1}{\overline{\mathcal{L}}} (\mathbf{D} - \alpha^2\, \overline{\mathcal{J}} \mathbf{H}) \\ \mathbf{B} & = & \dfrac{1}{\overline{\mathcal{L}}} (\mathbf{H} + \alpha^2\, \overline{\mathcal{J}} \mathbf{D}) \end{array} \right. \tag{2.11}$$

In this case the expressions for the electromagnetic invariants have the following form.

$$\left\{ \begin{array}{lcl} \mathcal{I} & = & \mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B} \\ \mathcal{J} & = & \mathbf{E} \cdot \mathbf{B} \end{array} \right. \quad ; \quad \left\{ \begin{array}{lcl} \overline{\mathcal{I}} & = & \mathbf{H} \cdot \mathbf{H} - \mathbf{D} \cdot \mathbf{D} \\ \overline{\mathcal{J}} & = & \mathbf{H} \cdot \mathbf{D} \end{array} \right. \tag{2.12}$$

(Expressions for $\mathcal{J}$ and $\overline{\mathcal{J}}$ in the case of general metric are the same).

A little more about the linear electrodynamics. If we take a variational principle as the basis of the model, then the usual relations for the linear electrodynamics $\mathbf{D} = \mathbf{E}$, $\mathbf{H} = \mathbf{B}$ are right provided we have $|g_{00}| = 1$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$. In the case of general metric we must take the relations (2.9) (with $\alpha = 0$, $\mathcal{L} = 1$ in this case).

In the article [1] it is proposed the following spherically symmetrical field configuration as a solution to the model that appropriate to electron.

$$E_r = \frac{e}{\sqrt{\alpha^2\, e^2 + r^4}} \quad ; \quad D_r = \frac{e}{r^2} \tag{2.13}$$

where r is the radial coordinate, and the index r denote the radial component of vector.

As we see, the vector $\mathbf{E}$ has discontinuity in the beginning of coordinates. But the function $A_0(x)$ is continuous everywhere and the action is finite. It is obvious that (2.13) is a solution to the system of equations (2.8) everywhere except the coordinate center. In the beginning of coordinates $\mathrm{div}\mathbf{D}$ is infinity for the field configuration (2.13).

With Lorentz transformations we have the appropriate to (2.13) moved field configuration, that has the following form in a cartesian system of coordinates.

$$F_{\mu\nu} = \left(L^i_{.\mu} L^0_{.\nu} - L^0_{.\mu} L^i_{.\nu}\right) E_i(r) \quad ; \quad r = \sqrt{y^i y_i} \quad ; \quad y^i = L^i_{.j}\left(x^j - V^j x^0\right) \tag{2.14}$$

Here E_i is the cartesian components appropriate to (2.13), $\{y^i\}$ is own cartesian coordinate system of the singularity moving with it together, V^j is components of velocity of the singularity, $||L_{\mu\nu}||$ is matrix of Lorentz transformations appropriate to boost:

$$L^0_{.0} = \frac{1}{\sqrt{1-\mathbf{V}^2}} \quad ; \quad L^0_{.i} = L_{i0} = -\frac{V_i}{\sqrt{1-\mathbf{V}^2}} \quad ; \quad L^i_{.j} = \delta^i_j + \left(\frac{1}{\sqrt{1-\mathbf{V}^2}} - 1\right)\frac{V^i V_j}{\mathbf{V}^2} \tag{2.15}$$

We use a metric with the signature $\{-+++\}$.

Because $\mathcal{J} = 0$ for the field configuration (2.14), it satisfies (out of the singularity) also a more simple system of equation then (2.8), (2.9) or (2.8), (2.11). We obtain this system if substitute $\mathcal{J} = 0$ into relations (2.9), (2.11) and take $\mathcal{L} = \sqrt{|\,1-\alpha^2\,\mathcal{I}\,|}$, $\overline{\mathcal{L}} = \sqrt{|\,1-\alpha^2\,\overline{\mathcal{I}}\,|}$. An action has the following form [1] for this case.

$$S'_{BI} = \int\left[\sqrt{|\det(g_{\mu\nu} + \alpha\,F_{\mu\nu}) - \det(\alpha\,F_{\mu\nu})|} - \sqrt{|\det(g_{\mu\nu})|}\right](\mathrm{d}x)^4 \tag{2.16}$$

3 Conditions for Stationarity of the Action with Singular Points of Field

Let the functions $A_\mu(x^\nu)$ that give stationarity to the action functional (2.1) or (2.16) be continuous and have N singular points as discontinuity of derivatives on the coordinates. We assume that the singularities exist for all time. We do not consider here a possible process of birth-destruction of the singularities. Thus in any moment of time each n-th singularity has certain space coordinates. We shall use the notation $\overset{n}{a}{}^i(x^0)$ for it. Let us enclose each singularity in a small closed surface moving with the singularity together. Subsequently we shall contract these surfaces to the points. Let us consider the following action.

$$\overline{S}_{BI} = \int\limits_{-\infty}^{+\infty}\mathrm{d}x^0\int\limits_{\overline{\Omega}}(\mathcal{L} - 1)\sqrt{|g|}\,(\mathrm{d}x)^3 \tag{3.1}$$

where $(\mathrm{d}x)^3 \equiv \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$, $\overline{\Omega}$ is the all three-dimensional space with the excluded regions bounded by the surfaces which enclose the singular points.

We shall make a variation of the functional (3.1) through variations of the field functions $A_\mu(x)$ and trajectories of the singularities. Here we have a moving boundary variational problem with free variation on the boundary [6]. As it is usual, we shall replace $A_\mu(x) \to A_\mu(x) + \varepsilon\,\delta A_\mu(x)$ and $\overset{n}{a}{}^i(x^0) \to \overset{n}{a}{}^i(x^0) + \varepsilon\,\delta\overset{n}{a}{}^i(x^0)$ in the action (3.1), considering that $\delta A_\mu(x) = 0$ and $\delta\overset{n}{a}{}^i(x^0) = 0$ at infinity. We differentiate the action on ε and take $\varepsilon = 0$. Then we make the partial integration and obtain the following variation of the action (3.1).

$$\begin{aligned}\delta\overline{S}_{BI} \quad = \quad & \int\limits_{-\infty}^{+\infty} \mathrm{d}x^0 \left\{ -\alpha^2 \int\limits_{\overline{\Omega}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|}\, f^{\mu\nu}\right) \delta A_\nu\, (\mathrm{d}x)^3 \; - \right. \qquad (3.2)\\ & \left. - \sum_{n=1}^{N} \left[\alpha^2 \left(\int\limits_{\overset{n}{\sigma}} f^{i\nu}\, \delta A_\nu\, \mathrm{d}\,\overset{n}{\sigma}_i \; - \int\limits_{\overset{n}{\sigma}} f^{0\nu}\, \delta A_\nu\, \overset{n}{V}{}^i\, \mathrm{d}\,\overset{n}{\sigma}_i \right) + \int\limits_{\overset{n}{\sigma}} (\mathcal{L} - 1)\, \delta\overset{n}{a}{}^i\, \mathrm{d}\,\overset{n}{\sigma}_i \right] \right\}\end{aligned}$$

Here $\overset{n}{\sigma}$ is closed surface enclosing the n-th singularity, it make the integration on external (relative to the singular point) side of this surface

$$\left.\begin{aligned} \mathrm{d}\overset{n}{\sigma}_1 &= \pm\sqrt{|g|}\,\mathrm{d}x^2\mathrm{d}x^3 \\ \mathrm{d}\overset{n}{\sigma}_2 &= \pm\sqrt{|g|}\,\mathrm{d}x^1\mathrm{d}x^3 \\ \mathrm{d}\overset{n}{\sigma}_3 &= \pm\sqrt{|g|}\,\mathrm{d}x^1\mathrm{d}x^2 \end{aligned}\right\} \qquad (3.3)$$

where it take the sign "+", if we have an acute angle between i-th coordinate axis and external normal to the surface $\overset{n}{\sigma}$ and it take sign "−", if the angle is obtuse

with the differential form concept the following expression can be written

$$\mathrm{d}\overset{n}{\sigma}_i = \frac{1}{2}\,\varepsilon_{0ijk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k \qquad (3.4)$$

$$\overset{n}{V}{}^i \equiv \frac{\mathrm{d}\overset{n}{a}{}^i}{\mathrm{d}x^0} \qquad (3.5)$$

We have the last integral on the surfaces in (3.2) with differentiation of location of the integration region border by ε [6].

When we contract the all surfaces $\overset{n}{\sigma}$ to the points we have $\overline{\Omega} \to \dot{\Omega}$, where $\dot{\Omega}$ is the whole three-dimensional space without the singular points. Replacement of Ω in $\dot{\Omega}$ is similar to partition of argument value interval of some function having discontinuity of derivative on two intervals: at the right and at the left from the singularity. Exception of the singular points means only that the infinite values of derivatives of the functions $f^{\mu\nu}(x)$ are not included in the integral. Thus it is evident that if $\overline{\Omega} \to \dot{\Omega}$ then $\overline{S}_{BI} \to S$ and $\delta\overline{S}_{BI} \to \delta S_{BI}$.

The functions $\delta A_\nu(x)$ are continuous everywhere, hence, we can factor out these functions from the integral on the surfaces $\overset{n}{\sigma}$ when we contract these surfaces to the points. Let us introduce the following designation for charge and current of the singularity.

$$\overset{n}{e} \equiv \frac{1}{4\pi} \lim_{\overset{n}{\sigma}\to 0} \int\limits_{\overset{n}{\sigma}} f^{0i}\,\mathrm{d}\overset{n}{\sigma}_i \quad ; \qquad \overset{n}{J}{}^l \equiv \frac{1}{4\pi} \lim_{\overset{n}{\sigma}\to 0} \int\limits_{\overset{n}{\sigma}} \left(f^{li}\,\mathrm{d}\overset{n}{\sigma}_i - f^{l0}\,\overset{n}{V}{}^i\,\mathrm{d}\overset{n}{\sigma}_i \right) \qquad (3.6)$$

According to the notations (2.7) we have the following expressions.

$$\overset{n}{e} \equiv \frac{1}{4\pi} \lim_{\overset{n}{\sigma}\to 0} \int\limits_{\overset{n}{\sigma}} \mathbf{D}\cdot \mathrm{d}\overset{n}{\boldsymbol{\sigma}} \quad ; \qquad \overset{n}{\mathbf{J}} \equiv \frac{1}{4\pi} \lim_{\overset{n}{\sigma}\to 0} \int\limits_{\overset{n}{\sigma}} \left[\mathbf{D}\left(\overset{n}{\mathbf{V}}\cdot \mathrm{d}\overset{n}{\boldsymbol{\sigma}}\right) - \mathbf{H}\times \mathrm{d}\overset{n}{\boldsymbol{\sigma}}\right] \tag{3.7}$$

where $(\mathbf{H}\times \mathrm{d}\overset{n}{\boldsymbol{\sigma}})_i \equiv \varepsilon_{0ijk}\, H^j\, \mathrm{d}\overset{n}{\sigma}{}^k \quad ; \qquad (\mathbf{H}\times \mathrm{d}\overset{n}{\boldsymbol{\sigma}})^i = -\,\varepsilon^{0ijk}\, H_j\, \mathrm{d}\overset{n}{\sigma}_k$
It is clear that if we take a non-singular as n-th point then $\overset{n}{e} = 0$, $\overset{n}{\mathbf{J}} = 0$.

As result we have variation of the action (2.1) in the following form.

$$\begin{aligned} \delta S_{BI} \;=\; & \int\limits_{-\infty}^{+\infty} \mathrm{d}x^0 \left\{ -\alpha^2 \int\limits_{\dot{\Omega}} \frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\, f^{\mu\nu}\right) \delta A_\nu\, (\mathrm{d}x)^3 \;+ \right. \\ & \left. + \sum_{n=1}^{N} \left[\alpha^2\, 4\pi \left(\overset{n}{e}\, \delta A_0(x^0,\overset{n}{\mathbf{a}}) + \overset{n}{J}{}^i\, \delta A_i(x^0,\overset{n}{\mathbf{a}}) \right) - \lim_{\overset{n}{\sigma}\to 0} \int\limits_{\overset{n}{\sigma}} \mathcal{L}\, \delta \overset{n}{a}{}^i\, \mathrm{d}\overset{n}{\sigma}_i \right] \right\} \end{aligned} \tag{3.8}$$

According to the general principles of calculus of variations [6] we can take $\delta A_\nu(x^0,\overset{n}{\mathbf{a}}) = 0$ and $\delta\overset{n}{a}{}^i = 0$ at first. Thus if $\delta S = 0$ then the second term in (3.2) is zero. Hence, the field satisfies the homogeneous system of equations (2.2) outside of the singular points. Now we take $\delta\overset{n}{a}{}^i \neq 0$. As result we have the following condition for each singularity.

$$\lim_{\overset{n}{\sigma}\to 0} \int\limits_{\overset{n}{\sigma}} \mathcal{L}\, \mathrm{d}\overset{n}{\boldsymbol{\sigma}} = 0 \tag{3.9}$$

As we see the field configuration (2.13), (2.14) satisfies this condition.

Now if we take $\delta A_\nu(x^0,\overset{n}{\mathbf{a}}) \neq 0$ for any one n-th singularity then we have $\overset{n}{e} = \overset{n}{J}{}^i = 0$ that are boundary natural conditions for this variational problem. It is evident that these conditions forbid existence of the solution (2.13), (2.14) for which $\overset{n}{e} \neq 0$, $\overset{n}{J}{}^i \neq 0$.

The same conclusion can be obtained if we consider $\overline{\partial}_\mu f^{\mu\nu}(x)$ as distributions or generalized functions. As it is known, a partial integration is allowable for the class of generalized functions. Thus if we have the stationary action (2.1) then the system of equations (2.2) should be satisfied everywhere. On the other hand the field configuration (2.13), (2.14) satisfies the system of equations (2.2) only outside of the singular point.

4 Modernized Action, Field Equations and Charge Conservation

We can modernize the action (2.1) for the field configuration (2.13), (2.14) could be the solution of model equations. Let us add to the action (2.1) the terms that compensate the terms with charges and currents of the singularities in the expression (3.8). Thus a modernized action has the following form.

$$S = \int (\mathcal{L} - 1)\, \sqrt{|g|}\, (\mathrm{d}x)^4 - 4\pi\, \alpha^2 \sum_{n=1}^{N} \int_{-\infty}^{+\infty} \left(\overset{n}{e}\, \overset{|n}{A_0} + \overset{n}{J^i}\, \overset{|n}{A_i} \right) \mathrm{d}x^0 \tag{4.1}$$

Here we introduce the following notation.

$$\overset{|n}{A_\nu} \equiv A_\nu(x^0, \overset{n}{\mathbf{a}}) \tag{4.2}$$

We can use Dirac δ-function and write the action (4.1) in the following form.

$$S = \int \left(\mathcal{L} - 1 - 4\pi\, \alpha^2\, A_\mu\, \jmath^\mu \right) \sqrt{|g|}\, (\mathrm{d}x)^4 \tag{4.3}$$

where

$$\jmath^0 \equiv \frac{1}{\sqrt{|g|}} \sum_{n=1}^{N} \overset{n}{e}\; \delta(\mathbf{x} - \overset{n}{\mathbf{a}}) \quad ; \qquad \jmath^i \equiv \frac{1}{\sqrt{|g|}} \sum_{n=1}^{N} \overset{n}{J^i}\, \delta(\mathbf{x} - \overset{n}{\mathbf{a}}) \tag{4.4}$$

We use the following definition for three-dimensional δ-function.

$$\int_{\overset{n}{\Omega}} f(\mathbf{x})\, \delta(\mathbf{x} - \overset{n}{\mathbf{a}})\, (\mathrm{d}x)^3 \equiv \lim_{\overset{n}{\sigma} \to 0} \left\{ \frac{1}{|\,\overset{n}{\sigma}\,|} \int_{\overset{n}{\sigma}} f(\mathbf{x})\, \mathrm{d}\overset{n}{\sigma} \right\} \equiv \Big\langle f(\mathbf{x}) \Big\rangle_n \quad ; \qquad |\,\overset{n}{\sigma}\,| \equiv \int_{\overset{n}{\sigma}} \mathrm{d}\overset{n}{\sigma} \tag{4.5}$$

Here $\overset{n}{\Omega}$ is a region of the space Ω including the point $\mathbf{x} = \overset{n}{\mathbf{a}}$, $\mathrm{d}\overset{n}{\sigma}$ is an area element of the surface $\overset{n}{\sigma}$, $|\,\overset{n}{\sigma}\,|$ is an area of the whole surface $\overset{n}{\sigma}$. We suppose that function $f(\mathbf{x})$ may have discontinuity at the point $\mathbf{x} = \overset{n}{\mathbf{a}}$.

First we can consider $\overset{n}{e}$, $\overset{n}{J^i}$ and $\overset{n}{a^i}$ as some given functions of time. Then making a variation of the field functions in the action (4.3) we obtain the following field equation.

$$\overline{\partial}_\mu\, f^{\nu\mu} = 4\pi\, \jmath^\nu \tag{4.6}$$

Then the system of equations for the field vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$ has the following form.

$$\left\{ \begin{array}{rcl} \mathrm{Div}\mathbf{B} &=& 0 \\ \overline{\partial}_0\, \mathbf{B} + \mathrm{Rot}\mathbf{E} &=& 0 \end{array} \right. \quad ; \qquad \left\{ \begin{array}{rcl} \mathrm{Div}\mathbf{D} &=& 4\pi\, \jmath^0 \\ -\overline{\partial}_0\, \mathbf{D} + \mathrm{Rot}\mathbf{H} &=& 4\pi\, \boldsymbol{\jmath} \end{array} \right. \tag{4.7}$$

Let us act to the system of equations (4.6) by the operator $\overline{\partial}_\nu$ with convolution on the index ν: $\overline{\partial}_\nu \overline{\partial}_\mu\, f^{\nu\mu} = 4\pi\, \overline{\partial}_\nu\, \jmath^\nu$. Here left part is zero because $f^{\mu\nu} = -\, f^{\nu\mu}$ and it is possible for distribution to change an order of differentiation by different coordinates. Thus we have the following conservation law.

$$\overline{\partial}_\nu\, \jmath^\nu = 0 \tag{4.8}$$

Substituting definition of $\jmath^\nu$ (4.4) into (4.8) we obtain the following relation.

$$\sum_{n=1}^{N} \left[\left(\frac{\mathrm{d}\, \overset{n}{e}}{\mathrm{d}x^0} \right) \delta(\mathbf{x} - \overset{n}{\mathbf{a}}) + \left(\overset{n}{J^i} - \overset{n}{e}\; \overset{n}{V^i} \right) \frac{\partial}{\partial x^i}\, \delta(\mathbf{x} - \overset{n}{\mathbf{a}}) \right] = 0 \tag{4.9}$$

Let us integrate this relation on a region of the space including only one n-th singularity in some any moment of time. As result we obtain that the charge of an individual singularity is conserved.

$$\frac{\mathrm{d}\overset{n}{e}}{\mathrm{d}x^0} = 0 \quad \Longrightarrow \quad \overset{n}{e} = \mathrm{const} \tag{4.10}$$

Substituting (4.10) into (4.9) we obtain

$$\overset{n}{\mathbf{J}} = \overset{n}{e}\,\overset{n}{\mathbf{V}} \tag{4.11}$$

Let us rewrite the action (4.1) in the following form using (4.10), (4.11).

$$S = \int (\mathcal{L} - 1)\,\sqrt{|g|}\,(\mathrm{d}x)^4 - 4\pi\,\alpha^2 \sum_{n=1}^{N} \overset{n}{e} \int\limits_{-\infty}^{+\infty} \left(\overset{|n}{A_0} + \frac{\mathrm{d}\overset{n}{a}{}^i}{\mathrm{d}x^0}\,\overset{|n}{A_i} \right) \mathrm{d}x^0 \tag{4.12}$$

Introducing an integration on the trajectories of the singularities we can write the action (4.12) in the following form.

$$S = \int (\mathcal{L} - 1)\,\sqrt{|g|}\,(\mathrm{d}x)^4 - 4\pi\,\alpha^2 \sum_{n=1}^{N} \overset{n}{e} \int\limits_{\overset{n}{\tau}} A_\mu\,\mathrm{d}x^\mu \tag{4.13}$$

where $\overset{n}{\tau}$ is trajectory of n-th singularity.

5 Motion Equations of the Singularities

Let us make a variation of the action (4.3) on the functions $\overset{n}{a}{}^i(x^0)$. We suppose that the condition (3.9) and the equation (4.6) are satisfied. Taking into account the definition (4.4) and relation (4.11) we obtain the following condition.

$$F_{i0}\,\jmath^0 + F_{ij}\,\jmath^j = 0 \tag{5.1}$$

Let us integrate this condition on a region of the space including only one n-th singularity in some any moment of time. As result we have the following conditions for each singularity.

$$\Big\langle F_{i0} \Big\rangle_n + \Big\langle F_{ij} \Big\rangle_n \overset{n}{V}{}^j = 0 \tag{5.2}$$

Here the angle brackets $\langle ... \rangle_n$ is the averaging of function near the n-th singular point (see definition of δ-function (4.5)). Remember that the functions $F_{\mu\nu}$ have not single value at the singular points. The condition (5.2) may be obtained also without using the δ-function. We can substitute the definition of δ-function (4.5) into action (4.3) and derive the condition (5.2) directly. From (5.2), it is evident that

$$\overset{n}{V}{}^i \Big\langle F_{i0} \Big\rangle_n = 0 \tag{5.3}$$

Using again the definitions of δ-function (4.5) and current density (4.4), (4.11) we can pool (5.2) and (5.3) to the following formula.

$$F_{\mu\nu}\, \jmath^{\nu} = 0 \tag{5.4}$$

Using the definition (2.7), we can write the condition (5.2) in the following form.

$$\Big\langle \mathbf{E} \Big\rangle_n + \overset{n}{\mathbf{V}} \times \Big\langle \mathbf{B} \Big\rangle_n = 0 \tag{5.5}$$

We can search a solution with N singularity by a perturbation method. As initial approximation we shall take a sum of the solutions with one singularity, the free parameters of which being dependent from time. The components of velocity $\overset{n}{V}{}^{i}$ are free parameters for the solution of type (2.14). So, the initial approximation has the following form.

$$A_\mu(x) = \sum_{n=1}^{N} \overset{n)}{A}_\mu(x) \tag{5.6}$$

where $\overset{n)}{A}_\mu$ is the solution with one singularity (not to be confused with $\overset{|n}{A}_\mu$ (4.2)).

If $\overset{n)}{A}_\mu$ correspond to the solutions of type (2.14) then

$$\overset{n)}{A}_\mu = \overset{n}{L}{}^{0}_{.\mu}(x^0)\ \overset{n}{\phi}\,(\overset{n}{r}) \tag{5.7}$$

where $\overset{n}{\phi}\,(\overset{n}{r})$ is zero component of the electromagnetic potential for the solution with one rest singularity (2.13), $\overset{n}{r} = \sqrt{\overset{n}{y}{}^{i}\, \overset{n}{y}{}_{i}}$

$$\overset{n}{y}{}^{i} = \overset{n}{L}{}^{i}_{.j}\left[x^j - \overset{n}{a}{}^{j}(x^0)\right] \quad ; \qquad \overset{n}{a}{}^{j}(x^0) = \overset{n}{a}{}^{j}(0) + \int_0^{x^0} \overset{n}{V}{}^{j}(x'^0)\, \mathrm{d}x'^0 \tag{5.8}$$

$$\Longrightarrow \qquad \frac{\partial\, \overset{n}{y}{}^{i}}{\partial x^j} = \overset{n}{L}{}^{i}_{.j} \quad ; \qquad \frac{\partial\, \overset{n}{y}{}^{i}}{\partial x^0} = \overset{n}{L}{}^{i}_{.0} + \frac{\mathrm{d}\overset{n}{L}{}^{i}_{.j}}{\mathrm{d}x^0}\, \overset{n}{L}{}^{-1\,j}_{\ \ .p}\ \overset{n}{y}{}^{p} \tag{5.9}$$

$\overset{n}{L}{}^{\mu}_{.\nu}$ is the Lorentz transformation matrix components (2.15) that include the velocity of n-th singularity $\overset{n}{V}{}^{j} = \overset{n}{V}{}^{j}(x^0)$.

Thus we have the following expression for initial approximation.

$$F_{\mu\nu} = \sum_{n=1}^{N} \left[\left(\overset{n}{L}{}^{0}_{.\nu} \frac{\partial\, \overset{n}{y}{}^{i}}{\partial x^\mu} - \overset{n}{L}{}^{0}_{.\mu} \frac{\partial\, \overset{n}{y}{}^{i}}{\partial x^\nu} \right) \overset{n}{F}_{i0} + \left(\delta^0_\mu\, \delta^i_\nu - \delta^0_\nu\, \delta^i_\mu \right) \frac{\mathrm{d}\overset{n}{L}{}^{0}_{.i}}{\mathrm{d}x^0}\ \overset{n}{\phi} \right] \tag{5.10}$$

where
$$\overset{n}{F}_{i0} \equiv \overset{n}{E}_i = \frac{\partial\, \overset{n}{\phi}}{\partial\, \overset{n}{y}{}^{i}} \tag{5.11}$$

Let us consider a motion of one k-th singularity ($n = k$). Let us designate a field connecting with all other singularities ($n \neq k$) as $\tilde{F}_{\mu\nu}$. Thus we have

$$F_{\mu\nu} = \left(\overset{k}{L}{}^0_{.\nu} \frac{\partial \overset{k}{y}{}^i}{\partial x^\mu} - \overset{k}{L}{}^0_{.\mu} \frac{\partial \overset{k}{y}{}^i}{\partial x^\nu} \right) \overset{k}{E}_i + \left(\delta^0_\mu \delta^i_\nu - \delta^0_\nu \delta^i_\mu \right) \frac{\mathrm{d}\overset{k}{L}{}^0_{.i}}{\mathrm{d}x^0} \overset{k}{\phi} + \tilde{F}_{\mu\nu} \tag{5.12}$$

Let us average these functions $F_{\mu\nu}$ near the k-th singular point. It is evident that

$$\left\langle \frac{\partial \overset{k}{y}{}^i}{\partial x^\mu} \overset{k}{E}_i \right\rangle_k = 0 \quad ; \qquad \left\langle \overset{k}{\phi}(\overset{k}{r}) \right\rangle_k = \overset{k}{\phi}(0) \quad ; \qquad \left\langle \tilde{F}_{\mu\nu} \right\rangle_k = \tilde{F}_{\mu\nu}(x^0, \overset{k}{\mathbf{a}}) \tag{5.13}$$

As can be shown (see also [1]), for used here system of designation we have the following equality for the solution (2.13).

$$\overset{k}{\phi}(0) = -\beta \frac{\overset{k}{e}}{\sqrt{|\alpha \overset{k}{e}|}} \qquad \text{where} \qquad \beta \equiv \int\limits_0^\infty \frac{\mathrm{d}r}{\sqrt{1+r^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4\sqrt{\pi}} \approx 1.8541 \tag{5.14}$$

Let us enter the following designation for rest mass of the singularity.

$$\overset{k}{m} \equiv -\overset{k}{e} \overset{k}{\phi}(0) \qquad \left[= \beta \frac{\overset{k}{e}{}^2}{\sqrt{|\alpha \overset{k}{e}|}} \quad \text{for the solution (2.13)} \right] \tag{5.15}$$

Then we can write

$$\left\langle F_{\mu\nu} \right\rangle_k = -\left(\delta^0_\mu \delta^i_\nu - \delta^0_\nu \delta^i_\mu \right) \frac{\overset{k}{m}}{\overset{k}{e}} \frac{\mathrm{d}\overset{k}{L}{}^0_{.i}}{\mathrm{d}x^0} + \tilde{F}_{\mu\nu}(x^0, \overset{k}{\mathbf{a}}) \tag{5.16}$$

Substituting (5.16) into the condition (5.5) with $n = k$ we obtain the known Lorentz equation for motion of k-th particle-singularity in initial approximation.

$$\overset{k}{m} \frac{\mathrm{d}}{\mathrm{d}x^0} \frac{\overset{k}{\mathbf{V}}}{\sqrt{1 - \overset{k}{\mathbf{V}}{}^2}} = \overset{k}{e} \left(\tilde{\mathbf{E}} + \overset{k}{\mathbf{V}} \times \tilde{\mathbf{B}} \right) \tag{5.17}$$

At the singular point we have $\mathcal{L} = 0$ and $f^{\mu\nu} = \infty$ for the solution (2.13), (2.14). But, as we can show, if we take $\left(\mathrm{d}\overset{k}{L}{}^0_{.i}/\mathrm{d}x^0 \right) = 0$ in (5.12) then $\mathcal{L} = 0$, $f^{\mu\nu} = \infty$ for filed $F_{\mu\nu}$ (5.12) on a singular surface near to the point $\mathbf{x} = \overset{k}{\mathbf{a}}$. The term with derivative on time from $\overset{k}{L}{}^0_{.i}$ in (5.12) compensates the field $\tilde{F}_{\mu\nu}$ near to the point $\mathbf{x} = \overset{k}{\mathbf{a}}$ according to (5.17), so we have the singular point but not surface.

If other singularities (with $n \neq k$) are sufficiently distant from k-th one then the field $\tilde{\mathbf{E}}$, $\tilde{\mathbf{B}}$ satisfies to the linear Maxwell's system of equations with $n \neq k$ singularities as sources. Thus we have the standard linear electrodynamics as initial approximation.

The motion equation (5.17) can be also obtained with somewhat different way. Let us substitute $A_\mu = \overset{k)}{A_\mu} + \tilde{A}_\mu$ in the action (4.13) and make the trajectory variation of the k-th singularity only. Thus, using (5.7), (2.15), (5.15), we obtain the following variational principle that well known as one for motion of a point charge particle in electromagnetic field.

$$\begin{aligned} \overset{k}{e}\,\delta \int\limits_{\overset{k}{\tau}} \left(\overset{k)}{A_\mu} + \tilde{A}_\mu\right) \mathrm{d}x^\mu &= \delta\Big(- \overset{k}{m} \int\limits_{-\infty}^{\infty} \sqrt{1-\overset{k}{\mathbf{V}}{}^2}\,\mathrm{d}x^0 + \overset{k}{e} \int\limits_{\overset{k}{\tau}} \tilde{A}_\mu\,\mathrm{d}x^\mu\Big) \\ &= \delta \int\limits_{\overset{k}{\tau}} \Big(- \overset{k}{m}\,\mathrm{d}s + \overset{k}{e}\,\tilde{A}_\mu\,\mathrm{d}x^\mu\Big) = 0 \end{aligned} \tag{5.18}$$

where $\mathrm{d}s = \sqrt{|\mathrm{d}x_\nu\,\mathrm{d}x^\nu|}$.

The approach that present here may be applied to any form of regular part of Lagrangian $(\mathcal{L}-1)$. In particular we can take the Lagrangian for linear theory $\alpha^2\mathcal{I}$. It is well known that in the case of the linear electrodynamics we have a singular solution with $|\overset{k}{\phi}(0)| = \infty$. Hence, in this case $\overset{k}{m} = \infty$ and there is not an interaction between single-singular solutions according to equation (5.17), that full conforms with superposition property of solutions for linear theory. Thus we can consider the linear electrodynamics only as initial approximation in a nonlinear theory pursuant to above-stated.

Because all observable charges in Nature multiple to the electron charge, it would appear reasonable that $\overset{k}{e} = \pm|\text{the electron charge}|$. We can make an any value for the mass (5.15) of the singularity with help of the model constant α. But we assume that this mass is not equal to the electron mass because a possible field configuration coincide to electron should be a more complex then the solution (2.13). Thus we have the two model constants α and $e \equiv |\text{the electron charge}|$. But they are the dimensional constants and, hence, we can make $\alpha = e = 1$ for suitable dimensional system. Thus qualitative characteristics of the model does not depend from sizes of these constants.

6 Conservation Laws for Energy-Impulse and Angular Momentum

Here we shall use a cartesian system of coordinates with components of metric $h_{\mu\nu}$:
$-h_{00} = h_{11} = h_{22} = h_{33} = 1$, $h_{\mu\nu} = 0$ for $\mu \neq \nu$.

Using the field equation (4.6), the current conservation (4.8) and the condition (5.4), we can show that there is the following differential conservation law for canonical energy-impulse tensor.

$$\frac{\partial T^\mu_{.\nu}}{\partial x^\mu} = 0 \quad \text{where} \qquad T^\mu_{.\nu} = f^{\mu\rho}\,\frac{\partial A_\rho}{\partial x^\nu} - \frac{1}{\alpha^2}\,(\mathcal{L}-1)\; h^\mu_\nu + 4\pi\, \jmath^\mu\, A_\nu \tag{6.1}$$

Then we have a full canonical energy-impulse in the following form.

$$P^{\nu} \equiv \frac{1}{4\pi} \int\limits_{\Omega} T^{0\nu}\,(\mathrm{d}x)^3 = \frac{1}{4\pi} \int\limits_{\Omega} \left[f^{0\rho}\,\frac{\partial A_\rho}{\partial x_\nu} + \frac{h^{0\nu}}{\alpha^2}\,(\mathcal{L}-1) \right] (\mathrm{d}x)^3 + \sum_{n=1}^{N} \overset{n}{e}\ \overset{|n}{A}{}^{\nu} \tag{6.2}$$

As we see, the full canonical energy-impulse is divided into two parts. The first part has a regular density and the second part has a singular density. This looks like a field and point particles in the standard electrodynamics. According to relations (5.7), (2.15), (5.15) we have an energy-impulse of n-th particle-singularity for initial approximation in the following form.

$$\overset{n}{P}{}^0 = \frac{\overset{n}{m}}{\sqrt{1-\overset{n}{\mathbf{V}}{}^2}} \quad ; \qquad \overset{n}{P}{}^i = \frac{\overset{n}{m}\ \overset{n}{V}{}^i}{\sqrt{1-\overset{n}{\mathbf{V}}{}^2}} \tag{6.3}$$

This expressions coincide with energy-impulse of point particle.

With direct verification we can show that there is the following differential conservation law for angular momentum tensor.

$$\frac{\partial M^{\mu}_{.\nu\rho}}{\partial x^\mu} = 0 \quad \text{where} \qquad M^{\mu}_{.\nu\rho} \equiv T^{\mu}_{.\nu}\,x_\rho - T^{\mu}_{.\rho}\,x_\nu - f^{\mu}_{.\nu}\,A_\rho + f^{\mu}_{.\rho}\,A_\nu \tag{6.4}$$

Using the definition (6.1) we can write the following differential conservation law for metric energy-impulse tensor.

$$\frac{\partial \overline{T}{}^{\mu}_{.\nu}}{\partial x^\mu} = 0 \quad \text{where} \quad \overline{T}{}^{\mu}_{.\nu} \ \equiv \ T^{\mu}_{.\nu} - \partial_\rho\,\left(f^{\mu\rho}\,A_\nu\right) \ = f^{\mu\rho}\,F_{\nu\rho} - \frac{1}{\alpha^2}\,\left(\mathcal{L}-1\right)\,h^{\mu}_{\nu} \tag{6.5}$$

In this case we have the full metric energy-impulse in the following form.

$$\overline{P}{}^{\nu} \equiv \frac{1}{4\pi} \int\limits_{\Omega} \overline{T}{}^{0\nu}\,(\mathrm{d}x)^3 = \frac{1}{4\pi} \int\limits_{\Omega} \left[f^{0\rho}\,F^{\nu}_{.\rho} - \frac{h^{0\nu}}{\alpha^2}\,(\mathcal{L}-1) \right] (\mathrm{d}x)^3 \tag{6.6}$$

As we see the full metric energy-impulse is expressed by a regular density in this case.

7 Conclusion

Thus we have presented an initial theory of nonlinear electrodynamics with singularities. The main obtained relations do not depend on a form of relations between the tensor components $f^{\mu\nu}$ and $F^{\mu\nu}$. Hence, the results are sufficiently general.

The existence of the singularities in this theory is connected with addition the integrals on the trajectories of the singularities (4.13) to the action (2.1) that provide also the stability of the singular particle-like solutions. The singularity may be as positive as negative depending on whether the sign before the k-th trajectory integral is minus ($\overset{k}{e} \ = e > 0$) or plus ($\overset{k}{e} \ = \ -e < 0$).

Within the framework of the presented theory the canonical energy-impulse density is naturally divided on two part: the regular part ("field") and the singular part ("particles"). The singular part of the energy for rest singularity is equal to the her rest mass that included in Lorentz equation of motion of the singularity that we have in initial approximation. However the metric energy-impulse density is regular.

In initial approximation this theory gives the usual linear electrodynamics with point charged particles.

The obtained results can be used for search of the possible field configurations that could describe the real particles.

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