

Massive-Scalar Effective Actions on Anti-de Sitter Spacetime

M. Kamela and C.P. Burgess

Physics Department, McGill University

3600 University St., Montréal, Québec, Canada, H3A 2T8.

Abstract

Closed forms are derived for the effective actions for free, massive spinless fields in anti-de Sitter spacetimes in arbitrary dimensions. The results have simple expressions in terms of elementary functions (for odd dimensions) or multiple Gamma functions (for even dimensions). We use these to argue against the quantum validity of a recently-proposed duality relating such theories with differing masses and cosmological constants.

1. Introduction

In this note we give explicit expressions for the effective actions for free, massive scalar fields propagating within anti-de Sitter (AdS) spacetimes of arbitrary dimension. Besides their intrinsic interest as exact expressions for quantum systems interacting with nontrivial gravitational fields, or as the first terms in a derivative expansion for more complicated backgrounds, these actions may also have applications to the calculation of quantum effects within cosmologically-interesting spacetimes. Remarkably, their supersymmetric extensions in five-dimensions may prove useful for study of large- N corrections to nonabelian gauge theories, in view of the recently-proposed duality between these theories and AdS supergravity in five dimensions [1].

Our calculations extend a number of similar calculations which have been performed by others in the past. Much of the early interest was motivated by the questions of principle which arise when quantizing fields in these spacetimes [2], [3], and by vacuum-stability [3], [4] and divergence [5] issues associated with the appearance of AdS spacetimes as supersymmetric vacua in extended-supergravity models. Starting very early, the maximal symmetry of these spacetimes was harnessed to perform explicit effective-action calculations for scalar fields in both de Sitter [6], [7], and anti-de Sitter [7], [8], [9], [10], as well as calculations of the functional determinants which arise in higher-spin calculations [11], [12]. The main advantage of our expressions over those in the literature is their validity for general spacetime dimension. For odd dimensions the results may be expressed in closed form using elementary functions. For even dimensions we also obtain closed-form results in terms of a class of special functions — the multiple gamma functions, $\{G_n\}$ — whose properties have been extensively studied.

Although we had performed the calculations we describe here for other applications in mind, one of our motivations for reporting the results now is the recent claim [13] for the existence of a duality relating scalar field theories of mass $m^2 = 0$ and $m^2 = R$ in two-dimensional anti-de Sitter spacetimes with Ricci scalar R . We believe our calculations provide evidence against this duality existing as a quantum symmetry.

Our presentation is organized in the following way. In §1 we briefly review some properties of anti-de Sitter spaces which are useful for obtaining the effective action. §2 contains our main result: the derivation of the scalar-field effective action in an AdS space-time of arbitrary dimension, n . §3 specializes this result to various cases of particular interest. For even dimensions we display results for $n = 2$ and $n = 4$, where we reproduce previous calculations. (For $n = 4$ we also give, in passing, an expression of the results for general spin in terms of the multiple Gamma functions.) We also present the odd-dimensional cases $n = 3, 5$ and 7 , which have not been previously calculated. Since our results are valid for arbitrary scalar masses and cosmological constants, they bear on the issue of the existence of duality transformations relating different values of these parameters. The duality analysis is the topic of §4. Finally, we gather some useful definitions and properties of the multiple Gamma functions in an appendix.

2. Scalar Fields on Anti-de Sitter Spacetime

An n -dimensional spacetime which admits $\frac{1}{2}n(n+1)$ Killing vectors is said to be maximally symmetric [14], [15]. The Riemann curvature tensor for any such spacetime may be written in the following way:¹

$$R_{\lambda\rho\sigma\nu} = K(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}) \quad R = -n(n-1)K, \quad (1)$$

where K a real constant. The possible maximally-symmetric spaces which can be entertained may be characterized by the signatures of their metrics as well as the sign of their Ricci scalar R (or, K). In our conventions anti-de Sitter space is the pseudo-Riemannian space for which $R > 0$, and so for which $K = -\lambda^2 < 0$.

Quantization of scalar field theory on AdS spacetime involves additional complications over those which arise for flat Minkowski space. Besides unrolling the compact time direction and working on the Universal Covering Space, the tricky feature about field quantization on AdS is connected with this spacetime not being globally hyperbolic [2],

¹ Our conventions are those of ref. [15].

[3]. (That is, in order for the scalar-field equations to formulate a well-posed boundary-value problem, boundary information is required on a time-like surface at spatial infinity in addition to the usual initial conditions which would have been sufficient in Minkowski space.) This complications lead to the existence of more than one Fock vacuum for the quantum field theory. As a consequence, different physical situations can lead to different boundary conditions, and so to different quantum field theories.

Given a scalar quantum field on AdS spacetime, our goal is to compute the scalar-field contribution, Σ , to the effective action. This is given by the following path integral:

$$\begin{aligned} e^{i\Sigma(g_{\mu\nu}, m^2)} &= \int [DX]_{g_{\mu\nu}} \exp \left[-\frac{i}{2} \int d^n x \sqrt{-g} X (-\square_g + m^2) X \right] \\ &= [\det' (-\square_g + m^2)]^{-1/2}, \end{aligned} \quad (2)$$

where $\square_g := \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu)$ is the usual Laplacian operator acting on scalar fields, and the prime in the second equality indicates the omission of any zero modes. Rather than using eq. (2) directly in what follows, we instead use its derivative with respect to m^2 , which implies:

$$\frac{d\Sigma}{dm^2} = \frac{i}{2} \text{Tr}' \left(\frac{1}{-\square_g + m^2} \right) = \frac{i}{2} \int d^n x \lim_{x' \rightarrow x} G(x, x'), \quad (3)$$

where $G(x, x')$ is the scalar Feynman propagator:

$$(-\square_g + m^2)_x G(x, x') = \frac{\delta^n(x, x')}{\sqrt{-g}}. \quad (4)$$

To obtain the effective action we integrate eq. (3) with respect to m^2 :

$$\Sigma(g_{\mu\nu}, m^2) - \Sigma(g_{\mu\nu}, m_0^2) = \int_{m_0^2}^{m^2} dm^2 \left\{ \frac{i}{2} \int d^n x \sqrt{-g} G(x, x) \right\}. \quad (5)$$

The result will equal the desired effective action up to terms independent of the mass m^2 . The quantity m_0 is a reference mass, for which we imagine the functional determinant to have been explicitly evaluated using other means. Convenient choices for which this is often possible are $m_0 = 0$ or $m_0 \rightarrow \infty$.

In this way the problem reduces to the construction of the scalar-field Feynman propagator on AdS spacetime, whose form in n dimensions has long been known [8].

3. The n -dimensional Effective Action

It only remains to evaluate the previous expression using the explicit expression for the Feynman propagator. To do so requires a choice of vacuum state. We work with the propagator which satisfies the energy-conserving boundary conditions on anti-de Sitter space [8], which is given in terms of standard hypergeometric functions, $F(a, b; c; x)$ [16], by:

$$-\frac{i}{2} G_F(z) = \frac{C_{F,n}}{2 z^\beta} F\left(\frac{\beta}{2}, \frac{\beta+1}{2}; \beta - \frac{n}{2} + \frac{3}{2}; z^{-2}\right), \quad (6)$$

where $z = 1 + \lambda^2 \sigma(x, x')$ and $\sigma(x, x')$ is the square of the geodesic distance between the points x and x' , and β denotes the expression

$$\beta = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} + \frac{m^2}{\lambda^2}}. \quad (7)$$

Finally, the coefficient $C_{F,n}$ is a known constant, defined in equation (9) of ref. [8]:

$$C_{F,n} = \frac{\lambda^{(n-2)} \Gamma(\beta)}{2^{\beta+1} \pi^{n/2-1/2} \Gamma\left(\beta - \frac{1}{2}(n-3)\right)} \quad (8)$$

We require the coincidence limit ($\sigma \rightarrow 0$) of eq. (6), and so take $z \rightarrow 1$. Using the corresponding limit for the hypergeometric function:

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}, \quad (9)$$

and simplifying further the Γ functions in the denominator, the propagator's coincidence limit takes the form:

$$-\frac{i}{2} G_F(1) = \frac{C_{F,n} 2^{\beta-n} \Gamma\left(\beta - \frac{n}{2} + \frac{3}{2}\right) \Gamma\left(1 - \frac{n}{2}\right)}{\sqrt{\pi} \Gamma(\beta - n + 2)}. \quad (10)$$

When n is a positive, even integer, this expression suffers from the usual divergences that are associated with the coincidence limit of the Feynman propagator. We regularize these by temporarily imagining the spacetime dimension, n , to be complex, with n taken to the physical dimension of spacetime only at the end of the calculation.

Combining all of these expressions² we find the coincidence limit of the scalar Feynman propagator to be

$$-\frac{i}{2} G_F(1) = \frac{\Gamma\left(\frac{n}{2} - \frac{1}{2} + \sqrt{\frac{(n-1)^2}{4} + \frac{m^2}{\lambda^2}}\right) \Gamma\left(1 - \frac{n}{2}\right) \lambda^{(n-2)}}{2^{n+1} \pi^{n/2} \Gamma\left(-\frac{n}{2} + \frac{3}{2} + \sqrt{\frac{(n-1)^2}{4} + \frac{m^2}{\lambda^2}}\right)}. \quad (11)$$

To proceed, we now integrate eq. (11) with respect to m^2 . The limit $n \rightarrow D$ of eq. (11), when D is an odd integer, is well-defined and so may be taken directly, and the result integrated with respect to m^2 . When D is even, however, the pole from the Γ -function in the numerator gives a divergent result, which we may isolate by performing a Laurent series in powers of $(n - D)$. It is generally useful to perform this expansion first, and reserving until last the integration over m^2 .

4. Applications to Specific Dimensions

We now perform the limit $n \rightarrow D$ of eq. (11) for several choices of positive integer D .

4.1) The Case $D = 2$

Specializing to $D = 2$, the Laurent expansion of the scalar propagator becomes (neglecting terms which are $O(n - 2)$):

$$\frac{i}{2} G_F(1) = \frac{1}{4\pi(n-2)} - \frac{1}{8\pi} \left[\ln\left(\frac{4\pi\Lambda^2}{\lambda^2}\right) - \gamma - 2\Psi\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4m^2}{\lambda^2}}\right) \right], \quad (12)$$

² We correct here a typo in the coincidence limit of ref. [8].

where $\Psi(x) := d \ln \Gamma(x)/dx$, γ is the Euler-Mascherelli constant and Λ is the usual arbitrary scale which enters when dimensions are continued to complex values.

Integrating eq. (12) with respect to mass, we obtain the effective action as the integral over an effective lagrangian density: $\Sigma = - \int d^2x \sqrt{-g} V_{\text{eff}}(\lambda^2, m^2)$, with

$$\begin{aligned}
V_{\text{eff}}(\lambda^2, m^2) = & V_{\text{eff}}(\lambda^2, 0) - \left[-\frac{1}{4\pi(n-2)} + \frac{1}{8\pi} \left(-\gamma + \ln \left(\frac{4\pi\Lambda^2}{\lambda^2} \right) - 2 \right) \right] m^2 \\
& + \frac{\lambda^2}{8\pi} \left[2 \ln G_1 \left(\frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda^2}} + \frac{1}{2} \right) + 4 \ln G_2 \left(\frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda^2}} + \frac{1}{2} \right) \right. \\
& \left. + \left(1 - \sqrt{1 + \frac{4m^2}{\lambda^2}} \right) \ln(2\pi) \right]. \quad (13)
\end{aligned}$$

Here $G_n(x)$ denote the multiple Gamma functions, which are defined to satisfy the following Gamma-function-like properties:

$$\begin{aligned}
(1) \quad & G_n(z+1) = G_{n-1}(z)G_n(z), \\
(2) \quad & G_n(1) = 1, \\
(3) \quad & \frac{d^{n+1}}{dz^{n+1}} \log G_n(z+1) \geq 0 \quad \text{for } z \geq 0, \\
(4) \quad & G_0(z) = z
\end{aligned} \quad (14)$$

It is a theorem [17] that the solutions to these conditions are unique. Furthermore the first few functions are old friends: $G_0(z) = z$ and $G_1(z) = \Gamma(z)$. Some useful properties of these functions are summarized in the Appendix.

Notice, in two dimensions, that the massless reference point is useful because the functional integral for massless scalars is known to give the Liouville action:

$$\Sigma(g_{\mu\nu}, 0) = -\frac{1}{96\pi} \int d^2x \sqrt{-g} R \left(\frac{1}{\square} \right) R, \quad (15)$$

where $\square^{-1}R$ denotes the convolution of R with the Feynman propagator of eq. (4): $\int d^2y \sqrt{-g} G(x, y) R(y)$.

Using the asymptotic expansions of the G_n which are given in the Appendix, the small curvature limit ($\lambda^2 \ll m^2$) of eq. (13) is found to be:

$$V_{\text{eff}}(\lambda^2, m^2) \sim V_{\text{eff}}(\lambda^2, 0) - \frac{m^2}{8\pi} \left[\frac{2}{(n-2)} - \ln \left(\frac{4\pi \Lambda^2}{m^2} \right) + \gamma - 1 \right] - \frac{\lambda^2}{24\pi} \left[\ln \left(\frac{\lambda^2}{8\pi^3 m^2} \right) + \frac{3}{2} - 12 \zeta'(1) \right] + \frac{\lambda^4}{120\pi m^2} + O(\lambda^6), \quad (16)$$

where $\zeta(x)$ denotes the usual Riemann zeta function.

4.2) The Case $D = 4$

Evaluating eq. (11) for $n \rightarrow D = 4$ dimensions permits a comparison of this expression with previous work.

- *Spinless Particles:*

The expansion of eq. (6) about $n = 4$ produces the following coincidence limit:

$$\frac{i}{2} G_F = -\frac{2\lambda^2 + m^2}{16\pi^2(n-4)} + \frac{m^2}{32\pi^2} + \left(\frac{2\lambda^2 + m^2}{32\pi^2} \right) \left[\ln \left(\frac{4\pi \Lambda^2}{\lambda^2} \right) - \gamma - 2 \Psi \left(\frac{1}{2} + \sqrt{\frac{9}{4} + \frac{m^2}{\lambda^2}} \right) \right] + O(n-4). \quad (17)$$

Integrating with respect to mass then gives:

$$V_{\text{eff}}(\lambda^2, m^2) = V_{\text{eff}}(\lambda^2, 0) - \frac{\lambda^4}{64\pi^2} \left\{ \left(-\frac{2}{n-4} + \ln \left(\frac{4\pi \Lambda^2}{\lambda^2} \right) - \gamma + \frac{1}{3} \right) \left(b^2 - \frac{9}{4} \right) \left(b^2 + \frac{7}{4} \right) + \left[(6 + 8C_2) \left(\frac{1}{2} + b \right) - 9 + 24C_3 + 8C_2 \right] \left(b^2 - \frac{9}{4} \right) + (24C_2 + 11 + 48C_3 + 48C_4) \left(-\frac{3}{2} + b \right) - 72 \ln G_3 \left(\frac{1}{2} + b \right) - 24 \ln G_2 \left(\frac{1}{2} + b \right) - 48 \ln G_4 \left(\frac{1}{2} + b \right) \right\}, \quad (18)$$

where $b^2 := \frac{9}{4} + \frac{m^2}{\lambda^2}$, and the C_n are as defined in the Appendix.

This expression can be compared with earlier calculations. These have been computed in terms of the integral over m^2 in ref. [8] (using the same methods as used here) and ref. [9] (using ζ -function methods). The result of ref. [9] is:

$$V_{\text{eff}} = -\mathcal{L}_{\text{eff}} = -\frac{\lambda^4}{64\pi^2} \left[\left(b^4 - \frac{1}{2} b^2 - \frac{17}{240} \right) \ln \left(\frac{\nu^2}{\lambda^2} \right) + b^4 + \frac{1}{6} b^2 + 8c \right] \\ + \frac{\lambda^4}{16\pi^2} \int_{1/2}^{1/2+b} x(x-1)(2x-1)\Psi(x) dx \quad (19)$$

where ν is the arbitrary scale which arises in ζ -function regularization, and the constant c is given by³ [18]:

$$c = \int_0^\infty \frac{2u(u^2 + 1/4) \ln u}{e^{2\pi u} + 1} du \\ = -\frac{\ln 2}{160} - \frac{17}{960} \ln \pi + \frac{137}{5760} - \frac{17}{960} \gamma + \frac{21}{32} \frac{\zeta'(4)}{\pi^4} + \frac{1}{16} \frac{\zeta'(2)}{\pi^2} \\ = -0.01744158583... \quad (20)$$

If we evaluate the integrals in eq. (19) in terms of the multiple Gamma functions, and subtract the result for $m = 0$ limit, we find agreement with eq. (18), provided the arbitrary scales ν and Λ are related in the following way:

$$\Lambda = \nu \exp \left[\frac{(12b^2 + 21)(\gamma - \ln(4\pi)) + 56}{6(4b^2 + 7)} \right] \quad (21)$$

- *Higher Spins for $D = 4$ Anti-de Sitter Space:*

Some results are also available in four dimensions for higher-spin particles. It is often possible to express the one-loop functional determinants for higher-spin fields in the form

$$\det \left(-\square_s + X \right) \quad (22)$$

where \square_s is the Laplacian operator acting on various constrained tensor and/or spinor fields. (For instance, for spin-1 particles the relevant field is a divergenceless vector field.)

³ We correct here a typo in ref. [9], where the value for the constant c is incorrect by $-137/360$

The functional determinants for these fields have been evaluated for dS spacetimes in ref. [11], and for AdS spacetimes in ref. [12], using ζ -function regularization. Following these references, we label these fields by the corresponding spin, s , where s is an integer for tensors and a half-odd integer for spinors. For tensor fields ($s = \text{integer}$) on AdS with $D = 4$ ref. [12] gives the following result (with the overall sign chosen for bose statistics):

$$V_{\text{eff}}^s = -g(s) \frac{\lambda^4}{64\pi^2} \left\{ \left[b^4 - \left(s + \frac{1}{2} \right)^2 \left(2b^2 + \frac{1}{6} \right) - \frac{7}{240} \right] \ln \left(\frac{\nu^2}{\lambda^2} \right) + b^4 + \frac{1}{6} b^2 + 8c_+ \right\} \\ - g(s) \frac{\lambda^4}{8\pi^2} \int_0^b \left[\left(s + \frac{1}{2} \right)^2 - t^2 \right] \Psi \left(t + \frac{1}{2} \right) t dt, \quad (23)$$

with $g(s) = 2s + 1$. The quantity b is given in refs. [12] and [11], and depends on both m^2/λ^2 and s . For the special case $s = 0$ we have $b^2 = \frac{9}{4} + \frac{m^2}{\lambda^2}$, while for $s = 1$, $b^2 = \frac{1}{4} + \frac{m^2}{\lambda^2}$. The constant c_+ is given by [18],

$$c_+ = \int_0^\infty \frac{2u \left[u^2 + \left(s + \frac{1}{2} \right)^2 \right] \ln u}{e^{2\pi u} + 1} du \\ = \frac{s(s+1)}{24} \left(-\ln \pi + 1 - \gamma + \frac{6\zeta'(2)}{\pi^2} \right) - \frac{\ln 2}{160} - \frac{17 \ln \pi}{960} \\ + \frac{137}{5760} - \frac{17\gamma}{960} + \frac{21\zeta'(4)}{32\pi^4} + \frac{\zeta'(2)}{16\pi^2}. \quad (24)$$

Evaluating the integrals in eq. (23) we find the effective Lagrangian produced by (con-

strained) tensor fields on AdS expressed in terms of the multiple Gamma functions:

$$\begin{aligned}
V_{\text{eff}} = g(s) \frac{\lambda^4}{64\pi^2} & \left\{ \left[\ln \left(\frac{\lambda^2}{\nu^2} \right) - \frac{1}{3} \right] b^4 - (8 C_2 + 6) b^3 \right. \\
& + \left[-2s(s+1) \left(1 + \ln \left(\frac{\lambda^2}{\nu^2} \right) \right) - 24 C_3 + \frac{3}{2} - 12 C_2 - \frac{1}{2} \ln \left(\frac{\lambda^2}{\nu^2} \right) \right] b^2 \\
& + \left[2s(s+1) (4 C_2 + 1) - 48 C_3 + \frac{5}{2} - 48 C_4 - 6 C_2 \right] b \\
& + \left[-\frac{1}{6} \ln \left(\frac{\lambda^2}{\nu^2} \right) + 4 \ln G_1 \left(\frac{1}{2} \right) + 8 \ln G_2 \left(\frac{1}{2} \right) - 4 \ln G_1 \left(\frac{1}{2} + b \right) \right. \\
& \left. - 8 \ln G_2 \left(\frac{1}{2} + b \right) \right] s(s+1) + 24 \ln G_2 \left(\frac{1}{2} + b \right) + 72 \ln G_3 \left(\frac{1}{2} + b \right) - 8 c_+ \\
& - 24 \ln G_2 \left(\frac{1}{2} \right) - \frac{17}{240} \ln \left(\frac{\lambda^2}{\nu^2} \right) - 48 \ln G_4 \left(\frac{1}{2} \right) - 72 \ln G_3 \left(\frac{1}{2} \right) \\
& \left. + 48 \ln G_4 \left(\frac{1}{2} + b \right) \right\} \quad \text{for } s = \text{integer.}
\end{aligned} \tag{25}$$

A similar result may be derived for (constrained) spinor fields. Ref. [12] gives the following expression (assuming fermi statistics):

$$\begin{aligned}
V_{\text{eff}}^s = g(s) \frac{\lambda^4}{64\pi^2} & \left\{ \left[b^4 - \left(s + \frac{1}{2} \right)^2 \left(2b^2 - \frac{1}{3} \right) + \frac{1}{30} \right] \ln \left(\frac{\nu^2}{\lambda^2} \right) \right. \\
& \left. + b^4 - \frac{4b^3}{3} - \frac{b^2}{3} + 4 \left(s + \frac{1}{2} \right)^2 b - 8 c_- \right\} \\
& + g(s) \frac{\lambda^4}{8\pi^2} \int_0^b \left[\left(s + \frac{1}{2} \right)^2 - t^2 \right] \Psi(t) t dt,
\end{aligned} \tag{26}$$

where b is again spin dependent, equal to $b^2 = \frac{m^2}{\lambda^2}$ for $s = \frac{1}{2}$. The constant c_- is [18]:

$$\begin{aligned}
c_- &= \int_0^\infty \frac{2u \left[u^2 + \left(s + \frac{1}{2} \right)^2 \right] \ln u}{e^{2\pi u} - 1} du \\
&= -\frac{7 \ln 2}{240} - \frac{7 \ln \pi}{240} + \frac{13}{360} - \frac{7 \gamma}{240} + \frac{3 \zeta'(4)}{4\pi^4} \\
&\quad + \frac{s(s+1)}{12} \left[-\ln(2\pi) + 1 - \gamma + \frac{6 \zeta'(2)}{\pi^2} \right] + \frac{1}{8} \frac{\zeta'(2)}{\pi^2}.
\end{aligned} \tag{27}$$

Combining expressions we find the following form for the spinor effective Lagrangian on AdS:

$$\begin{aligned}
V_{\text{eff}}^s = g(s) \frac{\lambda^4}{64\pi^2} & \left\{ \left[-\ln\left(\frac{\lambda^2}{\nu^2}\right) - \frac{13}{3} \right] b^4 + \left(64 C_2 + \frac{124}{3} \right) b^3 \right. \\
& + \left[\left(16 + 2 \ln\left(\frac{\lambda^2}{\nu^2}\right) \right) s(s+1) + \frac{1}{2} \ln\left(\frac{\lambda^2}{\nu^2}\right) - \frac{101}{3} + 96 C_2 + 192 C_3 \right] b^2 \\
& + \left[(-64 C_2 - 28)s(s+1) - 39 + 384 C_3 + 384 C_4 + 48 C_2 \right] b \\
& + \left[64 \ln G_2(b) - \frac{1}{3} \ln\left(\frac{\lambda^2}{\nu^2}\right) + 64 \ln G_1(b) \right] s(s+1) \\
& - 768 \ln G_3(b) - \frac{7}{60} \ln\left(\frac{\lambda^2}{\nu^2}\right) - 8 c_- - 432 \ln G_2(b) \\
& \left. - 48 \ln G_1(b) - 384 \ln G_4(b) \right\} \quad \text{for } s = \text{half-integer.} \tag{28}
\end{aligned}$$

The following technical point bears notice. When evaluated for massless, spin $\frac{1}{2}$ fermions ($b = 0$), eq. (28) superficially appears to be ill-defined, due to the appearance of the divergent quantities $\ln G_2(0)$, $\ln G_3(0)$ and $\ln G_4(0)$. It happens that these divergences cancel in eq. (28), leaving a well-defined massless limit.

4.3) Scalar Fields in Odd Dimensions

We now turn to the effective action for massive scalar fields in odd-dimensional anti-de Sitter spacetimes. As is usually the case for dimensionally-regularized one-loop quantities, the resulting expressions are easier to evaluate due to the absence in odd dimensions of logarithmic divergences at one loop.

We simply quote here the final results for the effective lagrangian for the lowest odd dimensions.

- $D = 3$:

For 3-dimensional AdS spacetimes the massive scalar effective lagrangian density becomes:

$$V_{\text{eff}}(K, m) - V_{\text{eff}}(K, 0) = -\frac{\lambda^3}{12\pi} \left[\left(\frac{\lambda^2 + m^2}{\lambda^2} \right)^{3/2} - 1 \right]. \tag{29}$$

- $D = 5$:

The corresponding result for 5-dimensional AdS spacetimes is:

$$V_{\text{eff}}(K, m) - V_{\text{eff}}(K, 0) = \frac{\lambda^5}{360 \pi^2} \left[\left(\frac{4\lambda^2 + m^2}{\lambda^2} \right)^{3/2} \left(\frac{7\lambda^2 + 3m^2}{\lambda^2} \right) - 56 \right]. \quad (30)$$

- $D = 7$:

For $D = 7$ we have:

$$V_{\text{eff}}(K, m) - V_{\text{eff}}(K, 0) = - \frac{\lambda^7}{5,040 \pi^3} \left[\left(\frac{9\lambda^2 + m^2}{\lambda^2} \right)^{3/2} \left(\frac{82\lambda^2 + 33\lambda^2 m^2 + 3m^4}{\lambda^4} \right) - 2,214 \right]. \quad (31)$$

- $D = 9$:

For $D = 9$:

$$V_{\text{eff}}(K, m) - V_{\text{eff}}(K, 0) = \frac{\lambda^9}{151,200 \pi^4} \left[\left(\frac{16\lambda^2 + m^2}{\lambda^2} \right)^{3/2} \times \left(\frac{3,956\lambda^6 + 1401\lambda^4 m^2 + 150\lambda^2 m^4 + 5m^6}{\lambda^6} \right) - 253,184 \right]. \quad (32)$$

- $D = 11$:

Finally, the 11-dimensional expression is:

$$V_{\text{eff}}(K, m) - V_{\text{eff}}(K, 0) = - \frac{\lambda^{11}}{1,995,840 \pi^5} \left[\left(\frac{25\lambda^2 + m^2}{\lambda^2} \right)^{3/2} \times \left(\frac{128,536\lambda^8 + 40,188\lambda^6 m^2 + 4,287\lambda^4 m^4 + 190\lambda^2 m^6 + 3m^8}{\lambda^8} \right) - 16,067,000 \right]. \quad (33)$$

5. Duality

Recently, Cruz [13] has proposed the classical equivalence of two types of free scalar fields in two-dimensional AdS spacetime. The proposed equivalence relates a massless, minimally-coupled scalar with a massive scalar having mass $m^2 = R = 2\lambda^2$. He argues for this equivalence by constructing a time-dependent canonical transformation which maps one system into the other.

In this section we wish to argue against the existence of this equivalence at the quantum level. Of course, the absence of a quantum symmetry need not preclude the existence of a classical symmetry. The failure of a canonical transformation to survive promotion to the quantum theory is similar to what happens for the Liouville action, which is canonically equivalent to a free field theory — and so is integrable [19] — but is nonetheless quantum mechanically distinct from it (see, ref. [20], and references therein).

In defense of our point of view we use the calculations of the previous section to see if duality is maintained at the quantum level. One would expect equivalence to imply the equality of the effective actions Σ computed for the two types of scalars. This amounts to the vanishing of expression (13), which gives the difference between the massive and massless effective potentials. Since the arguments of ref. [13] apply for any $\lambda^2 > 0$, eq. (13) should vanish for all such λ^2 . We find:

$$\begin{aligned}
 V_{\text{eff}}(\lambda^2, m^2) - V_{\text{eff}}(\lambda^2, 0) = & - \left[C + \frac{1}{8\pi} \ln \left(\frac{\Lambda^2}{\lambda^2} \right) \right] m^2 + \frac{\lambda^2}{8\pi} \left[\left(1 - \sqrt{1 + \frac{4m^2}{\lambda^2}} \right) \ln(2\pi) \right. \\
 & \left. + 2 \ln G_1 \left(\frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda^2}} + \frac{1}{2} \right) + 4 \ln G_2 \left(\frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda^2}} + \frac{1}{2} \right) \right], \tag{34}
 \end{aligned}$$

where C is the contribution of any counterterms. Besides cancelling the divergence of eq. (13) as $n \rightarrow 2$, these depend on Λ in just such a way as to ensure the Λ -independence of V_{eff} . Evaluating this expression for $m^2 = 2\lambda^2$ we find

$$V_{\text{eff}}(\lambda^2, m^2 = 2\lambda^2) - V_{\text{eff}}(\lambda^2, 0) = - \left[2C + \frac{1}{4\pi} \ln \left(\frac{2\pi \Lambda^2}{\lambda^2} \right) \right] \lambda^2, \tag{35}$$

where we have used $G_1(2) = G_2(2) = 1$.

Clearly, so long as C may depend arbitrarily on λ^2 and m^2 , we are always free to choose C to ensure the vanishing of eq. (35). C may certainly depend on λ^2 , since the counterterms can involve powers of the curvature, R .

(The reader might wonder why we entertain here the possibility of curvature-dependent counterterms when, for the noninteracting scalar on a fixed gravitational background under consideration, we have seen that no λ^2 dependence is required to cancel divergences in two dimensions. We do so because more complicated counterterms *are* required once interactions are included, and if the gravitational field is also treated as a quantum field. Moreover, we must consider the possibility that duality at the quantum level may require special choices for finite counterterms, even if these are not required to cancel divergences.)

We now come to the main point. There are now two ways to proceed, depending on how much λ^2 dependence we are prepared to entertain.

- *Option 1: Arbitrary λ^2 Dependence:*

One way to proceed is to damn the torpedoes and to permit C to depend arbitrarily on λ^2 . This might be reasonable if we regarded the metric strictly as a background field, and permitted the addition to the classical action of an arbitrary metric-dependent functional which is independent of our scalar field, ϕ . In this case, in the interest of enforcing a quantum duality, we choose C to ensure the vanishing of eq. (35) for all λ^2 . With this choice, eq. (34) becomes:

$$V_{\text{eff}}(\lambda^2, m^2) - V_{\text{eff}}(\lambda^2, 0) = \frac{m^2}{8\pi} \ln(2\pi) + \frac{\lambda^2}{8\pi} \left[\left(1 - \sqrt{1 + \frac{4m^2}{\lambda^2}} \right) \ln(2\pi) \right. \\ \left. + 2 \ln G_1 \left(\frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda^2}} + \frac{1}{2} \right) + 4 \ln G_2 \left(\frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda^2}} + \frac{1}{2} \right) \right]. \quad (36)$$

Eq. (36) is plotted in Figure 1, using the variables $y = [V_{\text{eff}}(\lambda^2, m^2) - V_{\text{eff}}(\lambda^2, 0)]/\lambda^2$ vs. $x = m^2/2\lambda^2$. The following points emerge from an inspection of this plot.

1. By construction $y(0) = y(1) = 0$ indicating the equivalence of V_{eff} when evaluated at $m^2 = 0$ and $m^2 = 2\lambda^2$. But the construction just given shows that there is nothing

special about the choice $m^2 = 2\lambda^2$, since we could have equally well renormalized to ensure $y = 0$ for some other value of m^2 .

2. Because $y(x)$ is not monotonically increasing or decreasing, there are many pairs $\{x_1, x_2\}$ which satisfy $y(x_1) = y(x_2)$, and so many pairs $\{m_1^2, m_2^2\}$ for which V_{eff} takes the same value. What is less obvious from the plot, but nevertheless true, is that the slope, $\partial V_{\text{eff}}/\partial\lambda^2$, is not the same for both members of these pairs. Since these slopes are related to the expectation $\langle T^\mu{}_\mu \rangle$ for the scalar field stress-energy tensor, this quantity must differ for m_1 and m_2 even though V_{eff} takes the same value for these two masses.

We conclude that duality is not a property of the quantum theory.

- *Option 2: Polynomial λ^2 Dependence:*

A more reasonable requirement on C , in our opinion, is to require it to be at most a polynomial in λ^2 (to any fixed order in perturbation theory). Physically, counterterms arise once higher-energy physics is integrated out, and so they should be interpreted in an effective-lagrangian sense. That is, they should be treated as perturbations in a low-energy derivative expansion. If so, to any fixed order in this expansion, they must be generally-covariant powers of the fields ϕ and $g_{\mu\nu}$ and their derivatives, restricting C to be a polynomial in λ^2 .

If so, it is no longer possible to choose C to ensure the vanishing of eq. (35) because cancellation would require C to depend logarithmically on λ^2 . Once again we are led to conclude that duality does not survive quantization.

6. Acknowledgements

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Appendix A. G_n : the Multiple Gamma function

In this appendix we state some principal formulae pertaining to the multiple gamma function. We also derive an integral representation for these functions, and use it to obtain closed forms for the integral moments of the $\Psi(x) = d \ln \Gamma(x)/dx$ function.

- *Defining Properties :*

In 1900, Barnes [22] introduced a generalization of the Γ function, denoted $G(x)$, which satisfies:

$$G(z+1) = (2\pi)^{1/2z} e^{-1/2z(z+1)-1/2\gamma z^2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z-1/2\frac{z^2}{n}} \quad (37)$$

and which satisfies the properties $G(z+1) = \Gamma(z)G(z)$ and $G(1) = 1$.

This was further generalized by Vignéras [17] in 1979, who introduced a hierarchy of Multiple Gamma functions, $\{G_n\}$, for $n = 0, 1, 2, \dots$. These functions may be defined using the following theorem.

Theorem [17]: There exists a unique hierarchy of functions which satisfy

$$\begin{aligned} (1) \quad & G_n(z+1) = G_{n-1}(z)G_n(z), \\ (2) \quad & G_n(1) = 1, \\ (3) \quad & \frac{d^{n+1}}{dz^{n+1}} \log G_n(z+1) \geq 0 \quad \text{for } z \geq 0, \\ (4) \quad & G_0(z) = z \end{aligned} \quad (38)$$

The first three elements of this sequence of functions are then $G_0(z) = z$, $G_1(z) = \Gamma(z)$ and $G_2(z) = G(z)$, with $G(z)$ as defined in eq. (37).

- *'Stirling' Formulae:*

Vignéras [17] derived a Weistrass product representation for the multiple gammas. Another infinite product representation is derived by Ueno and Nishizawa in [21]. They

also derive asymptotic expansions for general G_n , which are the analogues of the Sterling formula for the Γ function. We quote [21] for some of these results for low values of n .

In the case $n = 1$, we have the usual Stirling formula for large z :

$$\begin{aligned} \log G_1(z+1) &= \log \Gamma(z+1) \\ &\sim \left(z + \frac{1}{2}\right) \log(z+1) - (z+1) - \zeta'(0) + \sum_{r=1}^{\infty} \frac{B_{2r}}{[2r]_2} \left(\frac{1}{z+1}\right)^{2r-1}, \end{aligned} \quad (39)$$

where $[2r]_n$ stands for $\Gamma(2r+1)/\Gamma(2r-n+1)$. The generalization to $n = 2$, first derived by Barnes [22], is:

$$\begin{aligned} \log G_2(z+1) &\sim \left(\frac{z^2}{2} - \frac{1}{12}\right) \log(z+1) - \frac{3}{4}z^2 - \frac{z}{2} + \frac{1}{4} - z\zeta'(0) + \zeta'(-1) \\ &\quad - \frac{1}{12(z+1)} + \sum_{r=2}^{\infty} \frac{B_{2r}}{[2r]_3} \frac{(z-2r+1)}{(z+1)^{2r-1}} \end{aligned} \quad (40)$$

For $n = 3$ and $n = 4$, the asymptotic expansions are as follows:

$$\begin{aligned} \log G_3(z+1) &\sim \left(\frac{z^3}{6} - \frac{z^2}{4} + \frac{1}{24}\right) \log(z+1) - \frac{11}{36}z^3 + \frac{5}{24}z^2 + \frac{z}{3} - \frac{13}{72} \\ &\quad - \frac{z(z-1)}{2} \zeta'(0) + \frac{2z-1}{2} \zeta'(-1) - \frac{1}{2} \zeta'(-2) \\ &\quad + \frac{1}{12(z+1)} + \sum_{r=2}^{\infty} \frac{B_{2r}}{[2r]_4} \frac{[z^2 - (6r-11)z + (4r^2 - 16r + 16)]}{(z+1)^{2r-1}} \\ \log G_4(z+1) &\sim \left(\frac{z^4}{24} - \frac{z^3}{6} + \frac{z^2}{6} - \frac{19}{720}\right) \log(z+1) - \frac{4}{72}z^4 + \frac{2}{9}z^3 + \frac{z^2}{8} - \frac{11}{36}z + \frac{31}{144} \\ &\quad - \frac{z^3 - 3z^2 + 2z}{6} \zeta'(0) + \frac{3z^2 - 6z + 2}{6} \zeta'(-1) - \frac{z-1}{2} \zeta'(-2) + \frac{1}{6} \zeta'(-3) \\ &\quad - \frac{1}{12(z+1)} + \frac{1}{720(z+1)^3} \left(6z^2 + \frac{13z}{2} + \frac{5}{2}\right) + \sum_{r=3}^{\infty} \frac{B_{2r}}{[2r]_5} \frac{N(z)}{(z+1)^{2r-1}}, \end{aligned}$$

$$\text{where } N(z) := [z^3 - (12r-27)z^2 + (20r^2 - 94r + 111)z - (8r^3 - 56r^2 + 134r - 109)]. \quad (41)$$

- *Integral Representations:*

Next, we prove the following line integral representation of the logarithm of the multiple Gammas.

Theorem:

$$\ln G_n(z+1) = \int_0^\infty dt \frac{e^{-t}}{t} (-1)^n \left[\frac{1 - e^{-zt}}{(1 - e^{-t})^n} + \sum_{m=1}^n \frac{(-1)^m}{(1 - e^{-t})^{n-m}} \binom{z}{m} \right] \quad (42)$$

Proof: We show explicitly that the defining conditions in (38) are satisfied. The proof follows by induction on n and from the uniqueness of the hierarchy of $\{G_n\}$ (38).

i) $\ln G_n(z+2) = \ln G_{n-1}(z+1) + \ln G_n(z+1)$ follows from the binomial relation:

$$\binom{z+1}{m} = \binom{z}{m-1} + \binom{z}{m} \quad (43)$$

The integrand splits up as follows:

$$\begin{aligned} & (-1)^n \left(\frac{1 - e^{-zt}e^{-t}}{(1 - e^{-t})^n} + \sum_{m=1}^n \frac{(-1)^m}{(1 - e^{-t})^{n-m}} \binom{z+1}{m} \right) \\ &= (-1)^n \left(\frac{1 - e^{-zt}}{(1 - e^{-t})^n} + \sum_{m=1}^n \frac{(-1)^m}{(1 - e^{-t})^{n-m}} \binom{z}{m} \right) \\ &+ (-1)^{n-1} \left(\frac{1 - e^{-zt}}{(1 - e^{-t})^{n-1}} + \sum_{m=1}^{n-1} \frac{(-1)^m}{(1 - e^{-t})^{n-m-1}} \binom{z}{m} \right) \end{aligned} \quad (44)$$

where the index on the second sum has been shifted to bring it to the standard form.

ii) $\ln G_n(1) = 0$ follows from the vanishing integrand in the limit $z \rightarrow 0$;

iii) $(d/dz)^{n+1} \ln G_n(z+1) \geq 0$ follows from the absolute positivity of the integrand:

$$\int_0^\infty e^{-t} \left[\frac{-(-t)^n + 1}{(1 - e^{-t})^n} \right] \frac{dt}{t} \geq 0 \quad (45)$$

iv) Setting $n \rightarrow 0$ reduces to an integral representation of $\ln(z+1)$ and $n \rightarrow 1$ to a standard representation of the logarithm of the Γ function, thereby completing the proof by induction on n .

Corollary: Using the integral representation of G_n we derive the following tower of relations among the logarithmic derivatives $\psi_n(z+1) := d \ln G_n(z+1)/dz$:

$$\begin{aligned} \psi_2(z+1) - z \psi_1(z+1) &= C_2 - \frac{z}{2} \\ \psi_3(z+1) - z \psi_2(z+1) + \frac{z(z+1)}{2!} \psi_1(z+1) &= C_3 + \frac{3z}{4} + \frac{z^2}{4} \end{aligned} \quad (46)$$

and

$$\begin{aligned} \psi_4(z+1) - z \psi_3(z+1) + \frac{z(z+1)}{2!} \psi_2(z+1) - \frac{z(z+1)(z+2)}{3!} \psi_1(z+1) \\ = C_4 - \frac{11z}{18} - \frac{z^2}{3} - \frac{z^3}{18}. \end{aligned} \quad (47)$$

where $C_2 := -\zeta'(0) - \frac{1}{2} = \frac{1}{2}[\ln(2\pi) - 1] = 0.4189385\dots$, $C_3 := -.3332237448\dots$, $C_4 := .2786248832\dots$, *etc.*.

Corollary: Substituting lower order relations in the higher order ones, and integrating with respect to z , we find

$$\begin{aligned} \int^a z \psi_1(z+1) dz &= \ln G_2(a+1) - a C_2 + \frac{a^2}{4} \\ \int^a \frac{1}{2!} z(z-1) \psi_1(z+1) dz &= \ln G_3(a+1) + \frac{a^3}{12} - \left(\frac{C_2}{2} + \frac{3}{8}\right) a^2 - a C_3 \\ \int^a \frac{1}{3!} z(z-1)(z-2) \psi_1(z+1) dz &= \ln G_4(a+1) + \frac{a^4}{72} - \left(\frac{C_2}{6} + \frac{2}{9}\right) a^3 \\ &\quad - \left(\frac{C_3}{2} - \frac{11}{36} - \frac{C_2}{4}\right) a^2 - a C_4 \end{aligned} \quad (48)$$

The integrals (48) may be rewritten as follows:

$$\int^a z^n \psi(z+1) dz = \begin{cases} n=0: & \ln G_1(a+1) \\ n=1: & \ln G_2(a+1) - a C_2 + \frac{1}{4} a^2 \\ n=2: & \frac{1}{6} a^3 + \left(-\frac{1}{2} - C_2\right) a^2 + (-C_2 - 2 C_3) a + 2 \ln G_3(a+1) + \\ & \ln G_2(a+1) \\ n=3: & \frac{1}{12} a^4 + \left(-C_2 - \frac{5}{6}\right) a^3 + \left(-\frac{1}{6} - \frac{3}{2} C_2 - 3 C_3\right) a^2 + \\ & (-6 C_3 - C_2 - 6 C_4) a + 6 \ln G_4(a+1) + \\ & 6 \ln G_3(a+1) + \ln G_2(a+1) \end{cases} \quad (49)$$

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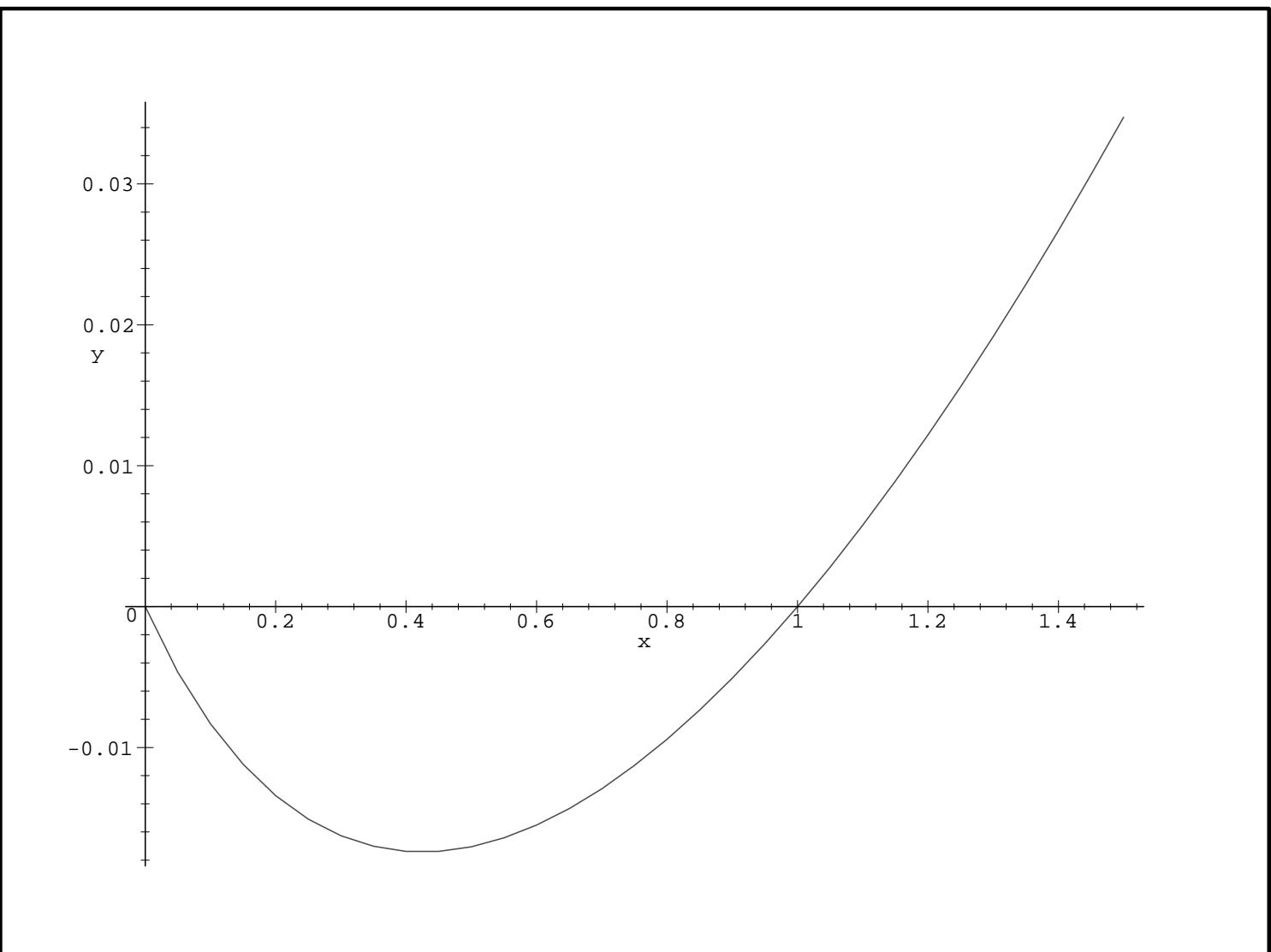


Figure 1: A plot of $y = [V_{\text{eff}}(\lambda^2, m^2) - V_{\text{eff}}(\lambda^2, 0)]/\lambda^2$ vs. $x = m^2/2\lambda^2$.