

Counting higher genus curves in a Calabi-Yau manifold

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We explicitly evaluate the low energy coupling F_g in a $d = 4, \mathcal{N} = 2$ compactification of the heterotic string. The holomorphic piece of this expression provides the information not encoded in the holomorphic anomaly equations, and we find that it is given by an elementary polylogarithm with index $3 - 2g$, thus generalizing in a natural way the known results for $g = 0, 1$. The heterotic model has a dual Calabi-Yau compactification of the type II string. We compare the answer with the general form expected from curve-counting formulae and find good agreement. As a corollary of this comparison we predict some numbers of higher genus curves in a specific Calabi-Yau, and extract some intersection numbers on the moduli space of genus g Riemann surfaces.

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1. Introduction

The computation of low energy effective actions in supersymmetric string compactification is an interesting problem from several points of view. First, the low energy action summarizes many important aspects of the physics of the compactified model. Second, the quantum corrections in effective actions involve interesting automorphic functions. Third, these quantum corrections often serve as generating functions for enumerative problems in geometry. Thus, the computation of a given physical quantity in two dual string descriptions often leads to striking predictions in enumerative geometry. A famous example of this is the counting of rational curves in a Calabi-Yau threefold provided by mirror symmetry. In this paper we compute a quantum correction involving higher genus curves in a Calabi-Yau threefold using heterotic/type II string duality and use the result to make some mathematical predictions.

Specifically, we consider the well-known low energy coupling F_g of low energy effective $d = 4, \mathcal{N} = 2$ supergravity, introduced and studied in [1][2][3]. Physically, this coupling enters the effective action in the schematic form

$$\int F_g(t, \bar{t}) T^{2g-2} R^2 + \dots \quad (1.1)$$

where $g \geq 1$, T is the graviphoton field strength, R is the Riemann curvature, and t, \bar{t} are vectormultiplet scalars. If one treats t, \bar{t} as independent then the Wilsonian coupling is obtained by sending $t \rightarrow \infty$ holding \bar{t} fixed [2], leaving the (anti-) holomorphic coupling $F_g^{\text{hol}}(\bar{t})$.

The mathematical formulation of F_g depends on the underlying string theory that gives rise to the low energy supergravity. In type II compactification on a Calabi-Yau 3-fold X the expression is exactly given by the string tree level result. In type IIA compactification the vectormultiplet scalars are complexified Kahler moduli and the holomorphic coupling $\bar{F}_g^{\text{hol}}(t)$ is given, roughly, by a sum over holomorphic genus g curves in X [1][2][3]. We say “roughly” because issues such as curve degeneration, multiple cover formulae, and the careful treatment of families of curves has not yet been adequately discussed. Indeed, one of the motivations of the present paper is to provide some useful information for sorting out these issues.

In heterotic compactification on $K3 \times T^2$, the effective coupling $F_g(t, \bar{t})$ has an integral representation coming from a one-loop diagram which is valid to all orders of perturbation theory [3]. Suppose such a heterotic compactification is dual to a type II compactification

on a Calabi-Yau X . Under string duality $S_{\text{heterotic}}$ is identified with a Kähler class of X . Thus, given a heterotic/type II dual we can evaluate a generating function for genus g curves on X , at one boundary of complexified Kähler moduli space. This has been done to some extent in [3]. In the present paper we extend the discussion of [3] by giving a complete evaluation of F_g . Specifically, we will consider the rank four example discussed in [4][5], and in many subsequent references. We compactify the heterotic $E_8 \times E_8$ theory on $K3 \times T^2$ and embed an $SU(2)$ bundle on each E_8 with instanton number 12. Then we Higgs completely the remaining $E_7 \times E_7$ symmetry and we obtain a model with 244 hypermultiplets and the four vector multiplets corresponding to the $U(1)^4$ gauge symmetry on the torus. In the semiclassical limit $S \rightarrow i\infty$, this is the 2-parameter $y = (T, U)$ case, with special loci corresponding to enhanced gauge symmetries. This model is dual to a type IIA model compactified on the $K3$ -fibered Calabi-Yau manifold $X = X_{24}^{1,1,2,8,12}$, which has $h_{1,1} = 3$, $h_{2,1} = 243$, therefore $\chi(X) = -480$.

The result for F_g is naturally written as a sum of two terms

$$F_g = F_g^{\text{deg}} + F_g^{\text{nondeg}} \quad (1.2)$$

given by the rather formidable expressions equations (4.21) and (4.40) below. From these expressions we can extract some interesting results. First, the effective coupling is, in contrast to other quantum corrections, continuously differentiable throughout all of moduli space, having singularities only at the locus of enhanced gauge symmetry $T = U$. This is hardly obvious from the chamber-dependent evaluation of the integral representation of F_g we will give. Second, while the expressions for F_g are formidable, the (anti-) holomorphic piece turns out to be relatively simple. It is given by an elementary polylogarithm Li_{3-2g} . The exact expression is given in equation (5.2) below. This is our main result. Using the result (5.2) we may draw some conclusions about the “number” of genus g curves in $X = X_{24}^{1,1,2,8,12}$. The variety X has a holomorphic map $\pi : X \rightarrow \mathbb{P}^1$ whose generic fibers are $K3$ surfaces. Because we must take the limit $S \rightarrow i\infty$ we can only discuss curves in the $K3$ fibers. Nevertheless, our result provides some nontrivial information. The comparison of our result for F_g^{hol} with known properties of F_g from [2] is carried out in section 6, and we find good agreement. Moreover, we make some predictions for numbers of genus 2 curves in the generic $K3$ fiber of X . Furthermore, a corollary of our discussion yields a prediction for an intersection number on the moduli space \mathcal{M}_g of genus g Riemann surfaces. Let c_{g-1}

be the Chern class of the Hodge bundle on \mathcal{M}_g (see section 6 below for a definition). Then we show that string duality predicts:

$$\int_{\mathcal{M}_g} c_{g-1}^3 = (-1)^{g-1} 2(2g-1) \frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}. \quad (1.3)$$

This intersection number could in principle be calculated using 2D topological gravity, but the computation appears to be tedious. We expect our result (5.2) to prove useful in further investigations of the role of higher genus curves in quantum cohomology.

Finally, we discuss briefly the method of our computation. The integral representation of F_g from the heterotic one-loop computation was derived in [3][6]. Such one-loop integrals have been evaluated in many papers in string theory. See, for some representative examples, [7][8][9][10][11]. The method involves lattice reduction and the ‘‘unfolding technique’’ (also known as the ‘‘Rankin-Selberg method’’ in number theory, or as ‘‘the method of orbits.’’) The most systematic discussion of such integrals was given by Borcherds in [12], generalizing the computation of [9]. Because we need the more general results, we review the notation and results of [12] in section three.

2. The integral for F_g

The F_g couplings at one-loop have been computed in [6][3], in the semiclassical limit $S \rightarrow i\infty$. It is useful to introduce a generating function for these couplings $F(\lambda, T, U) = \sum_{g=1}^{\infty} \lambda^{2g} F_g(T, U)$. Specializing to the heterotic dual of IIA compactification on $X_{24}^{1,1,2,8,12}$ the integral representation of this generating function is

$$F(\lambda, T, U) = \frac{1}{2\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{y} \left(\frac{E_4 E_6}{\eta^{24}} \right) \sum_{\Gamma^{2,2}} q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2} \left[\left(\frac{2\pi i \lambda \eta^3}{\vartheta_1(\tilde{\lambda}|\tau)} \right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{y}} \right]. \quad (2.1)$$

In this equation, \mathcal{F} denotes the fundamental domain for $SL(2, \mathbb{Z})$, $q = \exp(2\pi i\tau)$, $y = \text{Im}\tau$, and the modular form $E_4 E_6 / \eta^{24}$ has the expansion

$$\frac{E_4 E_6}{\eta^{24}} = \sum_{n=-1} c(n) q^n = \frac{1}{q} - 240 - \dots \quad (2.2)$$

Notice that $c(0) = \chi(X)/2$. The right and left moving momenta are regarded as complex numbers, given by

$$\begin{aligned} p_L &= \frac{1}{\sqrt{2T_2 U_2}} (n_1 + n_2 \bar{T} + m_2 U + m_1 \bar{T} U), \\ p_R &= \frac{1}{\sqrt{2T_2 U_2}} (n_1 + n_2 T + m_2 U + m_1 T U), \end{aligned} \quad (2.3)$$

where $T = T_1 + iT_2$, $U = U_1 + iU_2$ is the decomposition into real and imaginary parts. Finally, $\tilde{\lambda} = p_R y \lambda / \sqrt{2T_2 U_2}$, and $\vartheta_1(z|\tau)$ is the Jacobi theta function with characteristics $(1/2, 1/2)$. We have slightly changed some of the conventions in [6] in order to match the conventions of [9], since the present paper is an extension of the calculations of [9]. Notice that in writing (2.1) we have momentarily introduced a term of zeroth order in λ , as in [3]. Our goal is to give an explicit expression for the F_g terms by performing the integral over the fundamental domain. The first step will be to extract the F_g term from (2.1) by making an appropriate expansion of the modular forms. The resulting integral will involve a generalized Siegel-Narain theta function with lattice vector insertions.

The modular form involving $\vartheta_1(z|\tau)$ can be written in terms of Eisenstein series:

$$\frac{2\pi\eta^3 z}{\vartheta_1(z|\tau)} = -\exp\left[\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} E_{2k}(\tau) z^{2k}\right]. \quad (2.4)$$

It is convenient to write (2.4) as follows. We introduce the covariant Eisenstein series

$$\hat{E}_2(\tau) = E_2 - \frac{3}{\pi y}, \quad (2.5)$$

and the Schur polynomials:

$$\exp\left[\sum_{k=1}^{\infty} x_k z^k\right] = \sum_{k=0}^{\infty} \mathcal{S}_k(x_1, \dots, x_k) z^k, \quad (2.6)$$

which have the structure

$$\mathcal{S}_k(x_1, \dots, x_k) = x_k + \dots + \frac{x_1^k}{k!}. \quad (2.7)$$

One can check that

$$\left(\frac{2\pi i \tilde{\lambda} \eta^3}{\vartheta_1(\tilde{\lambda}|\tau)}\right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{y}} = \sum_{k=0}^{\infty} \tilde{\lambda}^{2k} \mathcal{P}_{2k}(\hat{G}_2, \dots, G_{2k}), \quad (2.8)$$

where we have introduced the more convenient normalized Eisenstein series

$$G_{2k} = 2\zeta(2k) E_{2k}, \quad (2.9)$$

and \mathcal{P}_{2k} is an almost holomorphic modular form of weight $(2k, 0)$ given by

$$\mathcal{P}_{2k}(\hat{G}_2, \dots, G_{2k}) = -\mathcal{S}_k\left(\hat{G}_2, \frac{1}{2}G_4, \frac{1}{3}G_6, \dots, \frac{1}{k}G_{2k}\right). \quad (2.10)$$

We have, for instance,

$$\mathcal{P}_2(\hat{G}_2) = -\hat{G}_2, \quad \mathcal{P}_4(\hat{G}_2, G_4) = -\frac{1}{2}(\hat{G}_2^2 + G_4). \quad (2.11)$$

The integral (2.1) can now be written as

$$F(\lambda, T, U) = \sum_{k=0}^{\infty} \frac{\lambda^{2(k+1)}}{2\pi^2(2T_2U_2)^k} \int_{\mathcal{F}} \frac{d^2\tau}{y} y^{2k} \frac{E_6 E_4}{\eta^{24}} \mathcal{P}_{2(k+1)} \sum_{\Gamma^{2,2}} p_R^{2k} q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2}. \quad (2.12)$$

In this expansion, $k = g - 1$. We see that the integrals to be computed are of the form

$$\mathcal{I}_k = \int_{\mathcal{F}} d^2\tau y^{2k-1} F_k(\tau) \bar{\Theta}, \quad (2.13)$$

where

$$F_k(\tau) = \frac{E_4 E_6}{\eta^{24}} \mathcal{P}_{2(k+1)}(\hat{G}_2, \dots, G_{2(k+1)}) \quad (2.14)$$

is a modular form of weight $(2k, 0)$ and

$$\Theta = \sum_{\Gamma^{2,2}} \bar{p}_R^{2k} q^{\frac{1}{2}|p_R|^2} \bar{q}^{\frac{1}{2}|p_L|^2}, \quad (2.15)$$

is a Siegel-Narain theta function of modular weight $(1 + 2k, 1)$. Notice that (2.13) is complex except for $k = 0$.

3. Review of the lattice reduction and unfolding technique

In this section we review the computation of a class of integrals called theta transforms which were systematically evaluated in [12]. The technique to compute these integrals is to perform a lattice reduction, *i.e.* to reduce the computation to a theta transform on a smaller lattice. Proceeding iteratively, one can in principle compute the integral over the fundamental domain in terms of quantities associated to the reduced lattices. The integrals (2.13) that we want to compute have precisely the structure of the theta transforms considered by Borcherds, so we will briefly review the notation and the results used in [12] in order to apply them to this particular problem.

For simplicity we will restrict ourselves to self-dual lattices, although the results in [12] apply to more general situations. Let Γ be an even, self-dual lattice of signature (b^+, b^-) , together with an isometry $P : \Gamma \otimes \mathbb{R} \rightarrow \mathbb{R}^{b^+, b^-}$. The corresponding projections on $\mathbb{R}^{b^+, 0}$, \mathbb{R}^{0, b^-} will be denoted by P_{\pm} . The inverse images of $\mathbb{R}^{b^+, 0}$, \mathbb{R}^{0, b^-} decompose

$\Gamma \otimes \mathbb{R}$ into the orthogonal sum of positive definite and negative definite subspaces. Let p be a polynomial on \mathbb{R}^{b^+, b^-} of degree m^+ in the first b^+ variables and of degree m^- in the second b^- variables, and let Δ be the (Euclidean) Laplacian in $\mathbb{R}^{b^+ + b^-}$. With these elements we construct a Siegel-Narain theta function as

$$\Theta_\Gamma(\tau; P, p) = \sum_{\lambda \in \Gamma} \exp\left[-\frac{\Delta}{8\pi y}\right] (p(P(\lambda))) \exp\left[\pi i \tau (P_+(\lambda))^2 + \pi i \bar{\tau} (P_-(\lambda))^2\right]. \quad (3.1)$$

One can include shifts in the lattice Γ to obtain a more general theta function, but we will not consider this case. Notice that $(P_-(\lambda))^2 \leq 0$.

As we have explained, the first step in computing the integral over the fundamental domain, involving a theta function with the structure (3.1), is to perform a lattice reduction. This is done as follows. Let z be a primitive vector of Γ of zero norm, choose a vector z' in Γ with $(z, z') = 1$, and let $K = (\Gamma \cap z^\perp) / \mathbb{Z}z$. This lattice, which has signature $(b^+ + 1, b^- - 1)$, is the reduced lattice. The vectors in the reduced lattice will be denoted by λ^K , in order to distinguish them from the vectors λ in the original lattice. We now define “reduced” projections \tilde{P} as follows: consider $z_\pm \equiv P_\pm(z)$, and decompose $\mathbb{R}^{b^\pm} \simeq \langle z_\pm \rangle \oplus \langle z_\pm \rangle^\perp$. The projection on the orthogonal complement $\langle z_\pm \rangle^\perp$ is the reduced projection \tilde{P}_\pm . It can be explicitly written in terms of P_\pm as

$$\tilde{P}_\pm(\lambda) = P_\pm(\lambda) - \frac{(P_\pm(\lambda), z_\pm)}{z_\pm^2} z_\pm. \quad (3.2)$$

Once this reduced projection has been constructed, we have to decompose the polynomial involved in (3.1) with respect to this projection, according to the expansion

$$p(P(\lambda)) = \sum_{h^+, h^-} (\lambda, z_+)^{h^+} (\lambda, z_-)^{h^-} p_{h^+, h^-}(\tilde{P}(\lambda)), \quad (3.3)$$

where p_{h^+, h^-} are homogeneous polynomials of degrees $(m^+ - h^+, m^- - h^-)$ on $\tilde{P}(\Gamma \otimes \mathbb{R})$.

Now we have to be more precise about the structure of the theta transform that we want to compute. Consider the modular form

$$F^\Gamma(\tau) = y^{b^+/2 + m^+} F(\tau) \quad (3.4)$$

with weight $(-b^-/2 - m^-, -b^+/2 - m^+)$, constructed from the modular form $F(\tau)$ with weights $(b^+/2 + m^+ - b^-/2 - m^-, 0)$. We will assume that $F(\tau)$ is an almost holomorphic form, *i.e.* it has the expansion

$$F(\tau) = \sum_{m \in \mathbb{Q}} \sum_{t \geq 0} c(m, t) q^m y^{-t} \quad (3.5)$$

where $c(m, t)$ are complex numbers which are zero for all but a finite number of values of t and for sufficiently small values of m . In particular, $F(\tau)$ can have a pole of finite order at cusps. The theta transform considered in [12] has three ingredients: a lattice Γ together with a projection P , a polynomial $p(P(\lambda))$, and the modular form $F(\tau)$, and it is given by the following integral over the fundamental domain.

$$\Phi_\Gamma(P, p, F^\Gamma) = \int_{\mathcal{F}} \frac{d^2\tau}{y^2} \overline{\Theta}(\tau; P, p) F^\Gamma(\tau). \quad (3.6)$$

According to the results of [12], this integral can be evaluated by reducing the lattice. The resulting expression involves two pieces. The first one is essentially another theta transform but for the reduced lattice, and is given by

$$\frac{1}{\sqrt{2z_+^2}} \sum_{h \geq 0} \left(\frac{z_+^2}{4\pi} \right)^h \Phi_K(\tilde{P}, p_{h,h}, F^K). \quad (3.7)$$

The other contribution involves a sum over the reduced lattice K . There are two different cases one has to consider when evaluating this contribution. When $\tilde{P}_+(\lambda^K)$ is different from zero, one has

$$\begin{aligned} & \sqrt{\frac{2}{z_+^2}} \sum_{h \geq 0} \sum_{h^+, h^-} \frac{h!}{(2i)^{h^+ + h^-}} \left(-\frac{z_+^2}{\pi} \right)^h \binom{h^+}{h} \binom{h^-}{h} \sum_j \sum_{\lambda^K \in K} \frac{1}{j!} \left(-\frac{\Delta}{8\pi} \right)^j \bar{p}_{h^+, h^-}(\tilde{P}(\lambda^K)) \\ & \cdot \sum_l q^{l(\lambda^K, \mu)} \sum_t 2c(\lambda^2/2, t) \left(\frac{l}{2|z_+| |\tilde{P}_+(\lambda^K)|} \right)^{h-h^+-h^- - j - t + b^+/2 + m^+ - 3/2} \\ & \cdot K_{h-h^+-h^- - j - t + b^+/2 + m^+ - 3/2} \left(\frac{2\pi l |\tilde{P}_+(\lambda^K)|}{|z_+|} \right), \end{aligned} \quad (3.8)$$

where μ is the vector in $K \otimes \mathbb{R}$ given by

$$\mu = -z' + \frac{z_+}{z_+^2} + \frac{z_-}{2z_-^2}. \quad (3.9)$$

$K_\nu(z)$ is the modified Bessel function, and it comes from an integral over the strip $y > 0$. When $\tilde{P}_+(\lambda^K) = 0$, the integral over the strip has to be regularized with a parameter ϵ . Notice that $\tilde{P}_+(\lambda^K) = 0$ in two sub-cases: when $\lambda^K = 0$, and when Γ is a lattice with $b^+ = 1$, as in this case the reduced projection will always be zero for any λ^K . For these situations the last sum in (3.8) has to be substituted by

$$\begin{aligned} & \sum_t c(\lambda^2/2, t) \left(\frac{\pi l^2}{2z_+^2} \right)^{h-h^+-h^- - \epsilon - j - t + b^+/2 + m^+ - 3/2} \\ & \cdot \Gamma(-h + h^+ + h^- + j + t - b^+/2 - m^+ + 3/2 + \epsilon). \end{aligned} \quad (3.10)$$

This expression can be analytically continued to a meromorphic function of ϵ , with a Laurent expansion at $\epsilon = 0$. The contribution to the theta transform of (3.10) is given by the constant term of this expansion. In general, the sum over l will give a Riemann ζ function after analytic continuation. In order to extract the constant term at $\epsilon = 0$ one has to be careful with possible poles in ϵ and proceed as in dimensional regularization. We will see examples of this in the computation of F_g .

An important remark concerning this result is that the theta transform will be given by the above expressions only for sufficiently small z_+^2 . For a fixed primitive vector z , the value of z_+^2 depends on the projection we choose in our lattice, which in our case will be given by the moduli (T, U) of the string compactification. This means that the answers we will obtain for the integrals will be chamber-dependent, *i.e.* they will only be valid in a region of moduli space. In general, to obtain the answer in some other chamber, we have to use wall-crossing formulae, or we must choose some other null vector \tilde{z} to perform the reduction in such a way that the value of \tilde{z}_+^2 remains small in the chamber under consideration.

We will refer to (3.7) as the contribution of the degenerate orbit, and to (3.8) as the contribution of the nondegenerate orbit. One should notice, however, that the contribution to (3.8) of the zero vector of the reduced lattice appears in the computations of [7][9][10] from the contribution of the “degenerate orbit.” Also, the zero orbit will appear in (3.7) after further reduction to the trivial lattice.

4. Computation of F_g

We will now apply this formalism to the problem of computing the integrals (2.13). In our case, the lattice has signature $(b^+, b^-) = (2, 2)$ and is given by

$$\Gamma^{2,2} = H(-1) \oplus H(1) = \langle e_1, f_1 \rangle_{\mathbf{Z}} \oplus \langle e_2, f_2 \rangle_{\mathbf{Z}}, \quad (4.1)$$

where $(e_1, f_1) = -(e_2, f_2) = -1$. The projections give isometries $P_{\pm} : \Gamma^{2,2} \otimes \mathbb{R} \rightarrow \mathbb{R}^2$. We want to construct the projections in such a way that

$$P_+(\lambda) = p_R, \quad P_-(\lambda) = p_L, \quad (4.2)$$

where we are identifying $\mathbb{R}^2 \simeq \mathbb{C}$, and $p_{R,L}$ are given as complex numbers by (2.3). Notice that $(P_-(\lambda))^2 = -|p_L|^2$. The requirement (4.2) fixes the structure of the projections. We

can obtain explicit expressions for the projections of the basis as follows. Consider for instance e_1 . As an element in the positive definite subspace of $\Gamma^{2,2} \otimes \mathbb{R}$, $P_+(e_1)$ will have the general form

$$P_+(e_1) = x_1 e_1 + y_1 f_1 + x_2 e_2 + y_2 f_2. \quad (4.3)$$

Using the fact that P_{\pm} are orthogonal projectors, namely $P_{\pm}^2 = 0$, $P_+ P_- = P_- P_+ = 0$, we obtain the equations

$$x_1 + x_2 \bar{T} + y_2 U + y_1 \bar{T} U = 0, \quad x_1 + x_2 T + y_2 U + y_1 T U = 1. \quad (4.4)$$

Solving for x_1, x_2, y_1, y_2 we obtain

$$P_+(e_1) = \frac{1}{2T_2 U_2} \left\{ -\text{Re}(TU) e_1 - f_1 + U_1 e_2 + T_1 f_2 \right\}, \quad (4.5)$$

and similarly

$$P_+(f_1) = \frac{1}{2T_2 U_2} \left\{ -|TU|^2 e_1 - \text{Re}(TU) f_1 + |U|^2 T_1 e_2 + |T|^2 U_1 f_2 \right\}. \quad (4.6)$$

Consider now the theta function in (2.14). The Laplacian in terms of the variables p_R, p_L is given by

$$\Delta = 4 \left(\frac{d}{dp_R} \frac{d}{d\bar{p}_R} + \frac{d}{dp_L} \frac{d}{d\bar{p}_L} \right), \quad (4.7)$$

therefore $\Delta(\bar{p}_R^{2k}) = 0$ and we see that the theta function involved in the one-loop integral is just the Siegel-Narain theta function for the projection given by (4.2)(4.5) and polynomial $p = (\bar{p}_R)^{2k}$, which is homogeneous of degree $2k$ in the \mathbb{R}^{b^+} variables. Therefore, $m^+ = 2k$, $m^- = 0$. For each k we have an integral (2.13) with the structure (3.6) and

$$F_k^\Gamma(\tau) = y^{2k+1} F_k(\tau), \quad (4.8)$$

where $F_k(\tau)$ is given in (2.14). We will denote the coefficients in the expansion (3.5) by $c_k(m, t)$. Notice that, in this expansion, the terms of the form y^{-t} come from the \hat{G}_2 in the Schur polynomials. Therefore, according to (2.7), the range of t is $0 \leq t \leq k + 1$.

To perform the lattice reduction, we have to choose a primitive null vector in $\Gamma^{2,2}$. A natural choice is $z = e_1, z' = -f_1$. It is easy to check that

$$(z_+, \lambda) = \sqrt{z_+^2} \text{Re}(p_R), \quad (4.9)$$

where

$$z_+^2 = \frac{1}{2T_2U_2}. \quad (4.10)$$

According to our remarks in section 3, the answer obtained with the lattice reduction will be valid for $1 \ll T_2U_2$. This choice of z is convenient for the decomposition in (3.3), as one has

$$(\bar{p}_R)^{2k} = (\operatorname{Re}(p_R) - i\operatorname{Im}(p_R))^{2k} = \sum_{h^+=0}^{2k} (z_+, \lambda)^{h^+} p_{h^+,0}(\lambda), \quad (4.11)$$

where the reduced polynomials are given by

$$p_{h^+,0}(\lambda) = \binom{2k}{h^+} \frac{(-i)^{2k-h^+}}{|z_+|^{h^+}} (\operatorname{Im}(p_R))^{2k-h^+}. \quad (4.12)$$

The reduced lattice is $K = \langle e_2, f_2 \rangle$, and the reduced projection can be easily obtained from (3.2) and (4.9):

$$\tilde{P}_+(\lambda) = \operatorname{Im}(p_R), \quad \tilde{P}_-(\lambda) = \operatorname{Im}(p_L). \quad (4.13)$$

We will now compute the integrals using the expressions (3.7)(3.8).

4.1. The degenerate orbit

First we compute the contribution of the degenerate orbit, which is now a theta transform for the lattice K , with vectors of the form

$$p_R^K = \frac{1}{\sqrt{2T_2U_2}}(n_2T + m_2U), \quad p_L^K = \frac{1}{\sqrt{2T_2U_2}}(n_2\bar{T} + m_2U). \quad (4.14)$$

Notice that the lattice K is Lorentzian, with $(b_+, b_-) = (1, 1)$. According to (3.7), the new theta transform involves the lattice K (together with the projection \tilde{P}) and the polynomial $p_{0,0}(\tilde{P}(\lambda^K)) = (-1)^k (\operatorname{Im}(p_R^K))^{2k}$ (remember from (4.12) that $h^- = 0$), which is again homogeneous of degrees $(m^+, m^-) = (2k, 0)$. The modular forms involved in this theta transform are then, according to (3.4),

$$F_k^K(\tau) = y^{2k+1/2} F_k(\tau). \quad (4.15)$$

To evaluate the theta transform for K , we have to perform a further reduction to the trivial lattice, using for instance the vector $\hat{z} = e_2$, with

$$\hat{z}_+^2 = \frac{T_2}{2U_2}. \quad (4.16)$$

The norm of this vector is computed using the reduced projection \tilde{P} . The polynomial appearing in the new theta function can be decomposed again (with respect to the new projection associated to \hat{z}) using (3.3):

$$(-1)^k (\text{Im}(p_R^K))^{2k} = \frac{(-1)^k}{|\hat{z}_+|^{2k}} (\lambda^K, \hat{z}_+)^{2k}, \quad (4.17)$$

therefore the reduced polynomial is a constant $\hat{p}_{2k,0} = (-1)^k / |\hat{z}_+|^{2k}$. For the new theta transform we apply the result of section 3 again, and we have two contributions, corresponding to degenerate and nondegenerate orbits for the reduction of K to the trivial lattice. The degenerate orbit is simply a theta transform (3.7) for the trivial lattice. Because of the structure of the reduced polynomial, only the $k = 0$ integral will give a nonzero contribution. From the definition (3.6) we then have that this theta transform is given by

$$\Phi_0(\cdot, 1, F_0) = \int_{\mathcal{F}} \frac{d^2\tau}{y^2} \frac{E_4 E_6}{\eta^{24}} \mathcal{P}_2(\hat{G}_2) = 16\pi^3, \quad (4.18)$$

where we have used that $\mathcal{P}_2(\hat{G}_2) = -\hat{G}_2$ and the general result [13]

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (\hat{G}_2(\tau))^n F(\tau) = \frac{1}{\pi(n+1)} \left[(G_2(\tau))^{n+1} F(\tau) \right]_{q^0}. \quad (4.19)$$

The theta transform for the zero lattice corresponds in fact to the contribution of the $A = 0$ orbit in the usual unfolding technique.

To evaluate the contribution of the nondegenerate orbit, we have to use the expression (3.10), as the reduced lattice is now trivial. The sum $\sum_{l>0} l^{2(1+t-k)-\epsilon}$ can be analytically continued to the Riemann zeta function $\zeta(2(1+\epsilon+t-k))$, and it is easy to check that there are no poles at $\epsilon = 0$ (recall that the only pole of $\zeta(z)$ is at $z = 1$). Taking all this into account one obtains,

$$\mathcal{I}_k^{\text{deg}} = 16\pi U_2 \delta_{k,0} + \frac{1}{2^{k-1}} \sum_{t=0}^{k+1} c_k(0, t) \frac{t!}{\pi^{t+1}} T_2 \left(\frac{T_2}{U_2} \right)^{t-k} \zeta(2(1+t-k)). \quad (4.20)$$

Including the rest of the factors, and recalling that $k = g - 1$, we obtain the contribution of the degenerate orbit to F_g :

$$F_g^{\text{deg}} = 8\pi^3 U_2 \delta_{g,1} + \frac{1}{T_2^{2g-3}} \sum_{t=0}^g c_{g-1}(0, t) \frac{t!}{2^{2(g-1)} \cdot \pi^{t+3}} \left(\frac{T_2}{U_2} \right)^t \zeta(2(2+t-g)). \quad (4.21)$$

Due to our choice of \hat{z} and to (4.16), the expression (4.20) is only valid in the chamber $T_2 < U_2$. The result in the other chamber is obtained by interchanging $T_2 \leftrightarrow U_2$ in the above formula.

4.2. The nondegenerate orbit

Let's now compute the contribution of the non-degenerate orbit. We will denote the vectors in the reduced lattice K by $r \equiv \lambda^K = ne_2 + mf_2$. We first define the following products:

$$r \cdot y = nT + mU, \quad r \hat{\cdot} y = \operatorname{Re}(nT + mU) + i|\operatorname{Im}(nT + mU)|. \quad (4.22)$$

The action of the Laplacian on the reduced polynomial $\bar{p}_{h^+,0}(r)$ in (4.12) gives the sum

$$\sum_{j=0}^{\lfloor k-h^+/2 \rfloor} \sum_{r \in K} \frac{(-1)^j i^{2k-h^+} (2k)! (\operatorname{Im}(p_R^K))^{2k-h^+-2j}}{(8\pi)^j |z_+|^{h^+} j! (2k-h^+-2j)! h^+!}. \quad (4.23)$$

We have to consider the two different cases $r = 0$, $r \neq 0$. In the first case, we have to use the expression (3.10). Notice that (4.23) vanishes when $r = 0$, unless $2k - h^+ = 2j$. This forces h^+ to be even, say $h^+ = 2s$, where $0 \leq s \leq k$, and $j = k - s$. We then find, for the contribution of the zero vector,

$$\begin{aligned} \mathcal{I}_{k,r=0}^{\text{nondeg}} &= 2 \sum_{s=0}^k \sum_{t=0}^{k+1} (-1)^s 2^{2(s-2k)+t} \frac{(2k)!}{(2s)!(k-s)!} \frac{c_k(0,t)}{\pi^{t+1/2}} |z_+|^{2(t-k)} \\ &\cdot \left(\frac{\pi}{2z_+^2} \right)^{-\epsilon} \Gamma\left(\frac{1}{2} + \epsilon + s + t - k\right) \zeta(1 + 2\epsilon + 2(t-k)), \end{aligned} \quad (4.24)$$

where the sum over l has been analytically continued again to a Riemann ζ -function. According to Borchers's formula, we have to evaluate the constant term of the Laurent expansion of this expression around $\epsilon = 0$. The only possible pole is in the Riemann ζ -function, and occurs when $t = k$. Otherwise the above expression is analytic at $\epsilon = 0$. To evaluate the constant term at $\epsilon = 0$ when $t = k$, we expand in ϵ as in dimensional regularization, using:

$$\begin{aligned} \zeta(1 + 2\epsilon) &= \frac{1}{2\epsilon} + \gamma_E + \mathcal{O}(\epsilon), \\ \Gamma\left(\frac{1}{2} + s + \epsilon\right) &= \Gamma\left(\frac{1}{2} + s\right) \left[\psi\left(\frac{1}{2} + s\right) + \epsilon \right] + \mathcal{O}(\epsilon^2), \\ \left(\frac{\pi}{2z_+^2} \right)^{-\epsilon} &= 1 - \epsilon \log\left(\frac{\pi}{2z_+^2}\right) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (4.25)$$

where γ_E is the Euler-Mascheroni constant and $\psi(z)$ is the logarithmic derivative of the Γ function. The constant term in the Laurent expansion can be easily evaluated using the above expressions. To write the final answer, one notices that some of the sums on s that

appear in the result can be explicitly computed, using $\Gamma(1/2 + n) = \pi^{1/2}2^{-n}(2n - 1)!!$ and $\Gamma(1/2 - n) = (-1)^n\pi^{1/2}2^n/(2n - 1)!!$, with $n \geq 0$. We have for instance,

$$\sum_{s=0}^k (-1)^s 2^{2s} \frac{(2k)!}{(2s)!(k-s)!} \Gamma(1/2 + s) = \pi^{1/2} \delta_{k,0}, \quad (4.26)$$

and the results turn out to be very different for $k = 0$ and $k \neq 0$. For $k = 0$ we have,

$$\mathcal{I}_{k=0,r=0}^{\text{nondeg}} = c_0(0,0) \left[-\log\left(\frac{\pi}{2z_+^2}\right) + \gamma_E - 2\log 2 \right] + c_0(0,1) \frac{2\zeta(3)}{\pi} |z_+|^2, \quad (4.27)$$

and for $k \neq 0$,

$$\begin{aligned} \mathcal{I}_{k \neq 0, r=0}^{\text{nondeg}} &= 2 \sum_{t=0}^{k-1} \frac{c_k(0,t)}{\pi^{t+1/2}} \zeta(1 + 2(t-k)) |z_+|^{2(t-k)} \\ &\cdot \sum_{s=0}^k (-1)^s 2^{2(s-2k)+t} \frac{(2k)!}{(2s)!(k-s)!} \Gamma\left(\frac{1}{2} + s + t - k\right) \\ &+ \frac{c_k(0,k)}{2^{3k} \cdot \pi^k} \sum_{s=0}^k (-1)^s \frac{(2k)!}{s!(k-s)!} \psi\left(\frac{1}{2} + s\right). \end{aligned} \quad (4.28)$$

For nonzero vectors in K , we use (3.8). Taking into account that $(\lambda, \mu) = \text{Re}(r\hat{y})$, we obtain:

$$\begin{aligned} \mathcal{I}_{k,r \neq 0}^{\text{nondeg}} &= \sqrt{\frac{2}{z_+^2}} \sum_{h^+=0}^{2k} \sum_{j=0}^{[k-h^+/2]} \sum_{r \neq 0} \frac{(-1)^{j+k+h^+}}{(8\pi)^j (2|z_+|)^{h^+}} \frac{(2k)! (\text{Im}(p_R))^{2k-h^+-2j}}{j!(2k-h^+-2j)! h^+!} \\ &\cdot \sum_{l=1}^{\infty} \sum_{t=0}^{k+1} 2c_k(r^2/2, t) \left(\frac{T_2 U_2 l}{\text{Im}(r\hat{y})} \right)^\nu l^{h^+} e^{2\pi i l \text{Re}(r\hat{y})} K_\nu(2\pi l \text{Im}(r\hat{y})), \end{aligned} \quad (4.29)$$

In this equation, $\nu = 2k - h^+ - j - t - 1/2$. Notice that this is always a half-integer, therefore we can use the explicit expression for the modified Bessel function when $s \geq 0$

$$K_{s+1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k=0}^s \frac{(s+k)!}{k!(s-k)!} \frac{1}{(2x)^k}. \quad (4.30)$$

We will give now our final expression for the contribution of the nondegenerate orbit. We define s as $s+1/2 = |\nu|$, and we take into account that $K_s(x) = K_{-s}(x)$. We also introduce the polylogarithm function,

$$\text{Li}_m(x) = \sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell^m}. \quad (4.31)$$

Notice that for $m \leq 0$ these functions are elementary: $\text{Li}_0(x) = x/(1-x)$, and

$$\text{Li}_m(x) = \left(x \frac{d}{dx}\right)^{|m|} \frac{1}{1-x} = m! \frac{x^{|m|}}{(1-x)^{|m|+1}} + \cdots, \quad m < 0 \quad (4.32)$$

We then see that (4.29) can be written as:

$$\begin{aligned} \mathcal{I}_{k,r \neq 0}^{\text{nondeg}} &= 2^{1-k} \sum_{r \neq 0} \sum_{t=0}^{k+1} \sum_{h=0}^{2k} \sum_{j=0}^{\lfloor k-h/2 \rfloor} \sum_{a=0}^s c_k(r^2/2, t) \frac{(-1)^{j+h+k}}{(4\pi)^{j+a}} \frac{(2k)!}{j!h!(2k-h-2j)!} \frac{(s+a)!}{a!(s-a)!} \\ &\cdot (\text{sgn}(\text{Im}(r \cdot y)))^h (T_2 U_2)^{k-t} (\text{Im}[r \hat{y}])^{t-j-a} \text{Li}_{1+a+j+t-2k}(e^{2\pi i r \hat{y}}). \end{aligned} \quad (4.33)$$

We will now begin to simplify the complicated expression (4.33). The nature of the sum depends strongly on the sign of $\text{Im}(r \cdot y)$. If $\text{Im}(r \cdot y) > 0$ then (4.33) *vanishes* for $g \geq 3$. The reason is that there is an unconstrained sum on h . For $g = 1$ we get:

$$\mathcal{I}_{0,r \neq 0}^{\text{nondeg}} = 2 \sum_{r \neq 0} \left[c_0(r^2/2, 0) \text{Li}_1(e^{2\pi i r \hat{y}}) + \frac{c_0(r^2/2, 1)}{T_2 U_2} \mathcal{G}(r \hat{y}) \right], \quad (4.34)$$

where the function $\mathcal{G}(x)$ is defined as:

$$\mathcal{G}(x) = \text{Im}(x) \text{Li}_2(e^{2\pi i x}) + \frac{1}{2\pi} \text{Li}_3(e^{2\pi i x}). \quad (4.35)$$

For $g = 2$ and $\text{Im}(r \cdot y) > 0$ we get:

$$\mathcal{I}_{1,r \neq 0}^{\text{nondeg}} = -\frac{c_1(r^2/2, 2)}{\pi^2 T_2 U_2} \text{Li}_3(e^{2\pi i r \hat{y}}). \quad (4.36)$$

When $\text{Im}(r \cdot y) < 0$ we get more complicated sums. For $g = 2$ we have a sum involving polylogarithms Li_k with index $-1 \leq k \leq 3$, but for $g \geq 3$ the sum starts with a rational function. The structure of (4.33) is, for $\text{Im}(r \cdot y) < 0$ and $g > 2$:

$$\sum_{t=0}^g c_{g-1}(r^2/2, t) (T_2 U_2)^{g-1-t} f_t(\text{Im}(r \hat{y}), \text{Li}_n(e^{2\pi i r \hat{y}})), \quad (4.37)$$

where the polynomials f_t are of the following form

$$f_t = \sum_{p=0}^{\Lambda} a_p^{(t)} (\text{Im}(r \hat{y}))^{t-p} \text{Li}_{t+p+3-2g}(e^{2\pi i r \hat{y}}). \quad (4.38)$$

In this equation, $a_p^{(t)}$ are some numerical constants and $\Lambda = \min\{t, 2g-t-3\}$. This implies that the index of the polylogarithms is always less than or equal to zero.

Putting all the contributions together, we find that the contribution to F_g from the nondegenerate orbit is then, for F_1

$$F_1^{\text{nondeg}} = \frac{1}{\pi^2} \sum_{r \neq 0} \left[c_0(r^2/2, 0) \text{Li}_1(e^{2\pi i r \hat{y}}) + \frac{c_0(r^2/2, 1)}{T_2 U_2} \mathcal{G}(r \hat{y}) \right] + \frac{c_0(0, 0)}{2\pi^2} \left[-\log(T_2 U_2) + \gamma_E - 2 \log 2\pi \right] + \frac{c_0(0, 1) \zeta(3)}{2\pi^2 (T_2 U_2)}. \quad (4.39)$$

It is interesting to notice that F_1 is essentially given by the integral $\tilde{\mathcal{I}}_{2,2}$ of [9]. If we explicitly compute the contribution of the degenerate orbit (4.21) for $g = 1$ (which gives a piecewise polynomial in T_2, U_2), we find that F_1 agrees with the result presented in [9], after taking into account the different normalizations. Finally, for $g > 1$ we find

$$F_{g>1}^{\text{nondeg}} = \frac{(-1)^{g-1}}{2^{2(g-1)} \cdot \pi^2} \sum_{r \neq 0} \sum_{t=0}^g \sum_{h=0}^{2g-2} \sum_{j=0}^{[g-1-h/2]} \sum_{a=0}^s c_{g-1}(r^2/2, t) \frac{(2g-2)!}{j! h! (2g-h-2j-2)!} \cdot \frac{(-1)^{j+h}}{(4\pi)^{j+a}} \frac{(s+a)!}{a!(s-a)!} (\text{sgn}(\text{Im}(r \cdot y)))^h \frac{1}{(T_2 U_2)^t} (\text{Im}[r \hat{y}])^{t-j-a} \text{Li}_{3+a+j+t-2g}(e^{2\pi i r \hat{y}}) + \frac{c_{g-1}(0, g-1)}{2^{4g-5} \cdot \pi^{g+1}} \frac{1}{(T_2 U_2)^{g-1}} \sum_{s=0}^{g-1} (-1)^s \frac{(2g-2)!}{s!(g-1-s)!} \psi\left(\frac{1}{2} + s\right) + \sum_{t=0}^{g-2} \frac{c_{g-1}(0, t)}{\pi^{t+5/2}} \frac{\zeta(3+2(t-g))}{(T_2 U_2)^t} \sum_{s=0}^{g-1} (-1)^s 2^{2(s-2g+2)} \frac{(2g-2)!}{(2s)!(g-1-s)!} \Gamma\left(\frac{3}{2} + s + t - g\right). \quad (4.40)$$

According to (2.14) and (3.5), the coefficients $c_{g-1}(m, t)$ are given by the expansion

$$\frac{E_4 E_6}{\eta^{24}} \mathcal{P}_{2g}(\hat{G}_2, \dots, G_{2g}) = \sum_{m \in \mathbb{Q}} \sum_{t \geq 0} c_{g-1}(m, t) q^m y^{-t}, \quad (4.41)$$

where the polynomials $\mathcal{P}_{2g}(\hat{G}_2, \dots, G_{2g})$ are defined in (2.10). The coefficients $c_{g-1}(-1, 0)$ and $c_{g-1}(0, 0)$ will be important in the next sections, so we will determine them. As $t = 0$ for these coefficients, they are found by looking at the holomorphic part of (2.8). Using the representation of $\vartheta_1(z|\tau)$ as an infinite product, it is easy to check that

$$-\left(\frac{2\pi\eta^3 z}{\vartheta_1(z|\tau)}\right)^2 = -\left(\frac{\pi z}{\sin \pi z}\right)^2 + 8\pi^2 z^2 q + \mathcal{O}(q^2). \quad (4.42)$$

This implies that

$$[\mathcal{P}_2(G_2)]_q = 8\pi^2, \quad [\mathcal{P}_{2g}(G_2, \dots, G_{2g})]_q = 0, \quad g > 1. \quad (4.43)$$

On the other hand, as $E_4 E_6 / \eta^{24} = q^{-1} - 240 + \dots$, the coefficient $c_{g-1}(-1, 0)$ is determined by the q^0 term of $\mathcal{P}_{2g}(G_2, \dots, G_{2g})$, and therefore can be computed using the expansion

$$\left(\frac{\pi z}{\sin \pi z} \right)^2 = - \sum_{g=0}^{\infty} \frac{(2g-1)!}{(2g)!} (-1)^g (2\pi z)^{2g} B_{2g}, \quad (4.44)$$

where B_{2g} are the Bernoulli numbers. We then obtain, using the relation between $\zeta(2g)$ and B_{2g} ,

$$c_{g-1}(-1, 0) = -2(2g-1)\zeta(2g). \quad (4.45)$$

In the same way, using (4.43), we find

$$c_{g-1}(0, 0) = c_{g-1}(-1, 0) \frac{\chi(X)}{2}, \quad g > 1, \quad (4.46)$$

where $X = X_{24}^{1,1,2,8,12}$ is the Calabi-Yau manifold on the type IIA side.

As the pole at the cusp in (4.41) is of first order, $c_{g-1}(m, t)$ will be zero for $m < -1$. This implies that, in the expression (4.40), the most negative possible value of $r^2/2$ is -1 . Therefore, the integers n, m in (4.22) giving a nonzero contribution are $n \geq 0, m \geq 0$, $n \leq 0, m \leq 0$ and the two lattice points $(n, m) = (1, -1), (n, m) = (-1, 1)$. As $T_2, U_2 > 0$, the hatted dot product that appears in (4.40) will give a chamber structure only for the two points in the lattice with $nm = -1$, and the walls will be defined by $T_2 = U_2$.

In general, when discussing theta transforms, one has to make an analytic continuation of the polylogarithms through the wall $T_2 = U_2$, and there is a nonzero wall-crossing term. An important and remarkable property of F_g that is not obvious from (4.21) and (4.40) is that the wall-crossing of the degenerate orbit will exactly cancel the wall-crossing of the nondegenerate orbit. In other words, *the whole coupling F_g is continuously differentiable on the moduli space* with coordinates (T, U) , for $g > 1$, except at the locus of enhanced gauge symmetry $T = U$ where there is a singularity. This cancellation of wall-crossing seems to require a precise knowledge of many terms in the sums (4.21) and (4.40), as well as the precise values of the coefficients $c_{g-1}(0, t)$ and $c_{g-1}(-1, t)$, and we have checked it numerically up to genus 5. Note that this is physically reasonable since there are no massless singularities at the generic point on this wall. This should be contrasted with the situation in five dimensions. Upon decompactification to five dimensions the wall $T_2 = U_2$ corresponds to the location of massless particle singularities. Considered as a $K3 \times S^1$ compactification these are well-known enhanced symmetry particles at the self-dual radius of S^1 . Considered as an M-theory compactification on a Calabi-Yau manifold the massless particles are due to wrapped M2-branes [14][15].

5. Extracting the holomorphic piece

Although the expressions just obtained are somewhat intimidating, it turns out that the antiholomorphic part of F_g has a very simple and compact expression. Notice that, with our choices for $p_{L,R}$ in (2.3), the piece of F_g which does not mix the holomorphic and antiholomorphic parts is in fact antiholomorphic. Hence, for simplicity of presentation, we will state our results for \overline{F}_g , and the holomorphic piece of this function of the moduli will be denoted by $\overline{F}_g^{\text{hol}}$. This piece should have a geometrical interpretation in terms of counting of holomorphic curves on the target space, according to [1][2]. To extract $\overline{F}_g^{\text{hol}}$, with $g \geq 2$, we take the holomorphic limit $\overline{T}, \overline{U} \rightarrow \infty$ in \overline{F}_g .

The first thing to notice is that the degenerate orbit does not give any contribution in this limit, for $g \geq 2$. We therefore consider the behavior of the nondegenerate orbit, for $g \geq 2$. In the first two lines of (4.40), one can easily see that the only surviving term in the holomorphic limit has $j = t = a = 0$, and the resulting function is then $\text{Li}_{3-2g}(x)$. In order to write a closed expression for this term, we have to be more precise about the contributions of the reduced lattice K . As $\text{Im}(r \cdot y) > 0$ for the lattice points $n \geq 0, m \geq 0$, there is no holomorphic contribution from these for $g \geq 2$. The lattice point $(1, -1)$ only contributes when $T_2 < U_2$, giving $\text{Li}_{3-2g}(q)$, where $q = \exp(2\pi i(T_1 - U_1) + 2\pi(T_2 - U_2))$. The point $(-1, 1)$ only contributes for $T_2 > U_2$, giving $\text{Li}_{3-2g}(q^{-1})$. Using (4.32) for $m < 0$, one can easily check that

$$\text{Li}_m(x) = (-1)^{|m|+1} \text{Li}_m(x^{-1}). \quad (5.1)$$

Therefore, the contributions of $(1, -1)$ and of $(-1, 1)$ add up to $\text{Li}_{3-2g}(q)$ for any value of T_2 and U_2 . Finally, the terms in the last two lines of (4.40), which correspond to the contribution of $r = 0$, vanish as $\overline{T}, \overline{U} \rightarrow \infty$, except for the term with $t = 0$ in the last line. The sum over s in the last factor can be easily evaluated to be $(-1)^{g-1} \pi^{1/2}/2$. We then obtain, taking into account (4.45) and (4.46),

$$\overline{F}_g^{\text{hol}} = \frac{(-1)^{g-1}}{\pi^2} \left[-(2g-1)\zeta(2g)\zeta(3-2g) \frac{\chi(X)}{2} + \sum_{r>0} c_{g-1}(r^2/2, 0) \text{Li}_{3-2g}(e^{2\pi i r \cdot y}) \right], \quad (5.2)$$

for $g \geq 2$. In this equation, $r > 0$ means the following possibilities: $(n, m) = (1, -1)$, $n > 0, m > 0$, $n = 0, m > 0$, or $n > 0, m = 0$. Notice that the coefficients $c_{g-1}(r^2/2, 0)$ involved in $\overline{F}_g^{\text{hol}}$ are determined by the holomorphic modular form $\mathcal{P}_{2g}(G_2, \dots, G_{2g})$.

Let us once again consider the nature of the answer at the wall $T_2 = U_2$. We note from (5.1) that there is no wall-crossing behavior at $T_2 = U_2$ for $\overline{F}_g^{\text{hol}}$, $g \geq 2$. This is of

course expected, as we are taking a particular limit of the whole F_g which does not have any wall-crossing (as we have checked for $g \leq 5$). For \bar{F}_g^{hol} , $g \geq 2$, one can indeed prove it analytically using (5.1). Viewed from the type IIA perspective the absence of wall-crossing is consistent with the fact that the Hodge numbers, and therefore the Euler character, of X are preserved under flop transitions.

In contrast to the absence of wall-crossing there is a singularity in codimension two at $T = U$. One can easily deduce the structure of the leading singularity in \bar{F}_g^{hol} using our explicit answer. The singularities on the heterotic side correspond to the appearance of extra massless states on the $T = U$ locus, and on the type II side they correspond to the conifold singularity. The enhanced gauge symmetry occurs when $p_L^2 = 2$, $p_R^2 = 0$, *i.e.* when $(n, m) = (1, -1)$, so that $r^2/2 = -1$. The leading singularity at genus g in the expression (5.2) is given by the leading pole in the polylogarithm, which has the form

$$\frac{(-1)^{g-1}}{\pi^2} c_{g-1}(-1, 0) (2g-3)! \frac{1}{(1 - e^{2\pi i r \cdot y})^{2g-2}}. \quad (5.3)$$

Notice that, for $r \cdot y \rightarrow 0$, we find the expected singular behavior $(2\pi i r \cdot y)^{-(2g-2)}$. The coefficient of this leading term can be computed explicitly: taking into account the factor $(2g-3)!$ in (5.3), and the expression for $c_{g-1}(-1, 0)$ in (4.45), we see that it is given by

$$-2^{2g} \pi^{2g-2} \chi(\mathcal{M}_g), \quad (5.4)$$

where $\chi(\mathcal{M}_g) = B_{2g}/2g(2g-2)$ is the Euler characteristic of the moduli space of genus g Riemann surfaces [16][17].

Up to a normalization factor, we recover the expansion of the free energy for the $c = 1$ string at the self-dual radius, as suggested by the conjecture of [18] and verified by [3] for a similar model. Notice that, according to our result, this pole together with the subleading poles of F_g near the conifold locus add up to the function Li_{3-2g} . It is also worth noting in this context that one can combine the result (5.2) together with (2.8)(4.32) to produce a compact expression for the sum $\sum_g \lambda^{2g} \bar{F}_g^{\text{hol}}$, essentially given by

$$\sum_{r>0} \left[q^{-r^2/2} \frac{E_4 E_6}{\eta^{24}} \left(\frac{2\pi i \tilde{\lambda} \eta^3}{\vartheta_1(\tilde{\lambda}|\tau)} \right)^2 \right]_{q^0} \text{Li}_3(x) \Big|_{x=e^{2\pi i r \cdot y}} \quad (5.5)$$

where we now take $\tilde{\lambda} = \lambda x \frac{d}{dx}$. This expression is highly reminiscent of the formula in $c = 1$ theory giving the radius dependence of the free energy [19][20].

6. Counting higher genus curves

Using string duality and the results in [1][2], the $\overline{F}_g^{\text{hol}}$ terms that we have computed should count numbers of curves of genus g on the $K3$ -fibered Calabi-Yau X , on the type IIA side. This has been checked to some extent in [21] for $\overline{F}_1^{\text{hol}}$. We will analyze here the $\overline{F}_2^{\text{hol}}$ term in some detail, and make some observations on the higher $\overline{F}_g^{\text{hol}}$.

Unfortunately, not much is known about the structure of the F_g^{IIA} in type IIA theory for $g > 1$. We do know the contribution to F_g^{IIA} coming from constant maps, which was obtained in [2] for any $g > 1$. This is the term in F_g^{IIA} that survives in the limit in which the Kähler moduli t, \bar{t} go to infinity, and it involves an integral over the moduli space of Riemann surfaces \mathcal{M}_g . Let \mathcal{H} be the Hodge bundle on \mathcal{M}_g , whose fibre over the point $m \in \mathcal{M}_g$ is given by $H^0(\Sigma_g(m), K)$, where K is the canonical bundle of the Riemann surface $\Sigma_g(m)$. Let $c_k = c_k(\mathcal{H})$ be their Chern classes. The constant term of F_g^{IIA} is then given by

$$F_g^{\text{IIA}}|_{t, \bar{t} \rightarrow \infty} = \frac{1}{2} \chi(X) \int_{\mathcal{M}_g} c_{g-1}^3, \quad (6.1)$$

where $\chi(X)$ is the Euler characteristic of the Calabi-Yau threefold X . The value of the above integral over the moduli space of Riemann surfaces is known to be $1/2880$ for $g = 2$ [22], $1/725760$ for $g = 3$ [23], and $1/43545600$ for $g = 4$ [24]. As we will see, string duality gives a very precise prediction for this coefficient for any genus $g > 1$.

In the case of F_2^{IIA} , although we don't have a general expression, explicit results have been obtained in [2] for some Calabi-Yau manifolds. These examples suggest that F_2^{IIA} has the following structure:

$$F_2^{\text{IIA}}(e^{2\pi i l \cdot t}) = \frac{\chi(X)}{5760} - \chi(\mathcal{M}_2) \sum_l d_l \frac{e^{2\pi i l \cdot t}}{(1 - e^{2\pi i l \cdot t})^2} + \sum_l D_l e^{2\pi i l \cdot t}, \quad (6.2)$$

In this equation, d_l counts the number of rational curves, and D_l counts the number of holomorphic curves of genus 2. The $t = (t_1, \dots, t_{h_{1,1}})$ denotes complexified Kähler moduli, $l = (l_1, \dots, l_{h_{1,1}})$ are integers, and

$$l \cdot t = \sum_{i=1}^{h_{1,1}} l_i t_i. \quad (6.3)$$

The first term in (6.2) is the contribution to F_2^{IIA} from constant maps (6.1), and $-\chi(\mathcal{M}_2) = 1/240$ is dictated by the structure of the leading singularity at the conifold. In order to compare the result on the heterotic side with the type IIA expression, we have to take into account that we are working in the semiclassical limit $S \rightarrow i\infty$. This corresponds on

the type IIA side to the region of the Kähler cone where the volume of the base of the $K3$ -fibration goes to infinity, *i.e.* to the limit in which the Kähler modulus dual to S goes to infinity. Therefore, our result (5.2) will take into account only holomorphic curves in the fiber of the Calabi-Yau.

The first thing to notice is that the function involved in the first member of (6.2) is precisely $\text{Li}_{-1}(e^{2\pi i l \cdot y})$. We will now match our main result (5.2) to (6.2), and extract some predictions for the numbers D_l . This can be done in a very precise way taking into account that, according to the results of [9], the number of rational curves on the $K3$ fiber on the type IIA side is counted by $-2c(r^2/2)$, where the coefficients $c(n)$ are defined by (2.2). In order to extract the genus 2 counting function, we write $\mathcal{P}_4(G_2, G_4)$ as

$$-\frac{1}{2}(G_2^2 + G_4) = -\frac{\pi^4}{15} + \frac{\pi^4}{15} \left(\frac{6 - 5E_2^2 - E_4}{6} \right), \quad (6.4)$$

where we used that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. It is convenient to define the coefficients $\tilde{c}(n)$ as follows,

$$\frac{1}{720} \frac{E_4 E_6}{\eta^{24}} (6 - 5E_2^2 - E_4) = \sum_{n=1}^{\infty} \tilde{c}(n) q^n = 6q - 1408q^2 - 856254q^3 - \dots \quad (6.5)$$

It follows from the definition of $c_{g-1}(n, 0)$ that

$$c_{g-1}(n, 0) = -\frac{\pi^4}{15} \left(c(n) - 120\tilde{c}(n) \right), \quad (6.6)$$

for $g = 2$. The holomorphic coupling $\overline{F}_2^{\text{hol}}$ can then be written, according to (5.2), as ¹

$$\overline{F}_2^{\text{hol}} = -8\pi^2 \left[-\frac{480}{5760} + \frac{1}{240} \sum_{r>0} (-2c(r^2/2)) \frac{e^{2\pi i r \cdot y}}{(1 - e^{2\pi i r \cdot y})^2} + \sum_{r>0} \sum_{k=1}^{\infty} \tilde{c}(r^2/2) k e^{2\pi i k r \cdot y} \right]. \quad (6.7)$$

The first term of this expression comes from the constant term in (5.2), and remarkably agrees with the type IIA side result in (6.2). The second term corresponds to the contribution of rational curves in F_2 , and has the structure found in [2]. The remaining piece corresponds to genus two curves on the $K3$ -fibered Calabi-Yau. We can now extract some

¹ An explicit expression for the holomorphic piece of F_2 was obtained in [25] using the anomaly equation together with target space duality. Although we have not performed a detailed comparison of the two expressions, the result presented in [25] seems to involve also the polylogarithm function $\text{Li}_{-1}(e^{2\pi i r \cdot y})$.

predictions for the genus 2 instanton numbers. The Calabi-Yau X has three Kähler moduli, denoted by t_1 , t_2 and t_3 (with the notation of [26][21]), and the heterotic weak coupling corresponds to $t_2 \rightarrow \infty$. This means that we will only be able to compute the instanton numbers $D_{l_1,0,l_3}$. The relation between the remaining Kähler moduli and the heterotic moduli is $t_1 = U$, $t_3 = T - U$ [5][21], therefore $l_1 = n + m$, $l_3 = n$. Using this explicit map, and assuming that (6.2) is in fact the right structure on the type IIA side, we find for instance (for the very first values of l_1, l_3)

$$\begin{aligned}
D_{l,0,l} &= D_{0,0,l} = D_{1,0,l} = 0, \\
D_{2,0,1} &= 6, \quad D_{3,0,1} = D_{3,0,2} = -1408, \\
D_{4,0,1} &= D_{4,0,3} = -856254, \quad D_{4,0,2} = -55723284, \\
D_{5,0,1} &= D_{5,0,4} = -55723296, \\
D_{5,0,2} &= D_{5,0,3} = -34256077056,
\end{aligned} \tag{6.8}$$

and so on. For primitive l_1, l_3 and $l_1 - l_3 \gg 1, l_3 \gg 1$ we have the asymptotic result

$$D_{l_1,0,l_3} \sim -\frac{1}{120\sqrt{2}}((l_1 - l_3)l_3)^{1/4} \exp[4\pi\sqrt{(l_1 - l_3)l_3}]. \tag{6.9}$$

Notice that all the instanton numbers will be integer numbers, due to the form of the expansion (6.5), but most of them are negative. The same phenomenon was observed in many two-parameter models [27][28], as well as in this particular Calabi-Yau [21], for the genus one instanton numbers. The reason for this is that the curves with genus $g > 1$ come in families, and what we are really computing is the Euler character of a certain vector bundle over the moduli space of curves in the family. In the context of topological sigma models, this vector bundle is associated to the antighost zero modes [29].

We can try to follow the same strategy at arbitrary genus. In particular, we should be able to recover the limiting behavior (6.1) of F_g^{IIA} by fixing the normalizations. This can be done if we consider a natural generalization of (6.7) and write F_g^{hol} as:

$$\begin{aligned}
\overline{F}_g^{\text{hol}} &= -2(2\pi)^{2g-2} \left[(-1)^{g-1} \frac{2(2g-1)\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}} \frac{\chi(X)}{2} \right. \\
&\quad \left. - \chi(\mathcal{M}_g) \sum_{r>0} (-2c(r^2/2, 0)) \frac{(e^{2\pi i r \cdot y})^{2g-3}}{(1 - e^{2\pi i r \cdot y})^{2g-2}} + \dots \right],
\end{aligned} \tag{6.10}$$

where we have decomposed $c_{g-1}(r^2/2, 0) = c_{g-1}(-1, 0)c(r^2/2) + \dots$, and written only the leading term of the polylogarithm in (4.32). We then see that the relative normalization

of F_g is given by the factor $-2(2\pi)^{2g-2}$. Matching the constant term appearing here to (6.1), we obtain

$$\int_{\mathcal{M}_g} c_{g-1}^3 = (-1)^{g-1} 2(2g-1) \frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}, \quad (6.11)$$

and one is startled to find the correct values $1/725760$ for $g = 3$ and $1/43545600$ for $g = 4$. Notice that for $g = 0$ we recover precisely the well-known constant term of the prepotential $-\zeta(3)\chi(X)/2$.

7. Conclusions

We have found an elegant expression for the holomorphic part of F_g in terms of a polylogarithm function, as well as an explicit expression for the generating function of the number of curves of genus g in terms of a modular form of weight $2g - 2$. Unfortunately, the structure of F_g on the type IIA side, for $g > 2$, has not been explored in detail, and this makes more difficult a geometrical interpretation of our expressions, as well as a precise prediction of the genus g instanton numbers. Notice that the information about the genus g curves is contained in the modular form $F_{g-1}(\tau)$ defined in (2.14), but in order to extract it one needs a previous knowledge of the different contributions to F_g , as we have seen in genus 2. It is interesting to notice that our counting function has some similarity with the modular form that counts genus g curves with g nodes on a $K3$ surface [30][31]. On the other hand, the fact that we have an exact and simple expression for F_g^{hol} should be very helpful in trying to understand the geometrical interpretation of higher genus curve counting.

We have also made two concrete predictions: first, assuming the structure of F_2 given by (6.2) (as suggested by the examples considered in [2]) we have obtained the modular form that gives the instanton numbers at genus 2 on the Calabi-Yau manifold $X_{24}^{1,1,2,8,12}$. Some of them have been written in (6.8). But we also have a prediction with a different flavor in (6.11), where a certain intersection number on the moduli space of genus g curves has been obtained from string duality. Both should be testable with other methods already available. The instanton numbers could be obtained on the type IIA side by mirror symmetry, following the original strategy of [2]. The general form of the result obtained here will certainly help in fixing the holomorphic ambiguity. As for the relation (6.11), one can perhaps make further checks using the approach to two-dimensional topological gravity presented in [32][33] or by standard methods in algebraic geometry as in [22][23][24].

Although we have only performed the computation for a single model, similar calculations could be done in other situations, such as those considered in [34]. Since the integral representation for F_g will be quite similar to that considered here, we expect that the simple form for F_g^{hol} in terms of a polylogarithm function also holds in more general situations.

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