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N=2 Superconformal Field Theory with ADE Global Symmetry on a D3-brane Probe

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Abstract

We study mass deformations of N = 2 superconformal field theories with ADE global symmetries on a D3-brane. The N = 2 Seiberg-Witten curves with ADE symmetries are determined by the Type IIB 7-brane backgrounds which are probed by a D3-brane. The Seiberg-Witten differentials λ for these ADE theories are constructed. We show that the poles of λ with residues are located on the global sections of the bundle in an elliptic fibration. It is then clearly seen how the residues transform in an irreducible representation of the ADE groups. The explicit form of λ depends on the choice of a representation of the residues. Nevertheless the physics results are identical irrespective of the representation of λ . This is considered as the global symmetry version of the universality found in N = 2 Yang-Mills theory with local ADE gauge symmetries.

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1 Introduction

Probing the 7-brane background of Type IIB compactification on \mathbf{P}^1 by a D3-brane provides a powerful machinery to analyze the non-perturbative behavior of four-dimensional N = 2 supersymmetric gauge theories [1, 2]. In this setup, the space-time gauge symmetry is transmuted into the global symmetry in the world volume N = 2 supersymmetric gauge theory on a D3-brane. Then it is found in [3] that there exist non-trivial N = 2superconformal fixed points with exceptional global symmetries. In [4, 5], on the other hand, N = 2 fixed points with E_n global symmetries are considered as a natural extension of foregoing works [6, 7, 8, 9].

Although the N = 2 theory with exceptional symmetry does not admit the Lagrangian description, recent advances in string duality have made it possible to study the strongcoupling regime of N = 2 theory by the stringy technique. For instance, it requires a considerable amount of effort in general to analyze the properties of the BPS spectrum of N = 2 theory. The junction picture of BPS states, however, gives the simple constraint on the BPS spectrum [10, 11]. With the use of this constraint, some characteristic features of the BPS states in N = 2 theory with E_n symmetries are revealed [11].

In this paper we study mass deformations of N = 2 theories with ADE global symmetries in detail. The present work is partly motivated in our attempt to get a clearer understanding of the results obtained by Minahan and Nemeschansky [4, 5] in formulating the elliptic curves and the Seiberg-Witten (SW) differentials for E_n theories. It was found in [5] that, for a given elliptic curve, the SW differential λ is not uniquely determined, but depends on the representations (fundamental or adjoint) of the global symmetry group. It is then argued that λ in different representations lead to different physics.

In our approach we proceed along the line of the D3-brane probe picture and discuss systematically the curves and the differentials for the ADE theories. In particular we clarify a great deal the properties of the pole terms of the SW differential. Even for the case of N = 2 SU(2) QCD with $N_f \leq 3$, which is thought to be well understood, we gain a new insight. Consequently we are able to show that the representations of the ADE groups from which the SW differential is built are irrelevant to the physics results. In this regard, our conclusion is opposed to what is argued in [5]. The paper is organized as follows. In section 2, we see that the elliptic curves for N = 2 ADE theories on a D3-brane are naturally identified by examining the local geometry of singularities in the compactification of Type IIB theory on \mathbf{P}^1 with the 24 background 7-branes. In section 3 the BPS mass formula for N = 2 ADE theories is discussed in the light of the string junction lattice. In section 4 the residues of the poles of the SW differentials for our ADE theories are shown to transform in an irreducible representation of the global symmetry groups. This affords a firm foundation of somewhat empirical construction of the SW differentials in [6, 4, 5]. In section 5 the SW differentials in the fundamental as well as the adjoint representations are obtained in the A_1 , A_2 , D_4 , E_6 , E_7 and E_8 theories. In section 6 we analyze in detail how the SW differential behaves under the renormalization group flow from the E_6 theory to the D_4 theory. In section 7 it is proved that the SW periods are independent of the representations of the global symmetry which are chosen to construct the SW differential. The result in section 7 is confirmed in section 8 by further studying N = 2 SU(2) QCD with $N_f \leq 3$. Finally we conclude in section 9.

2 D3-brane probe and elliptic curves

When Type IIB theory is compactified on \mathbf{P}^1 with the 24 background 7-branes, the string coupling constant $\tau = \chi + ie^{-\phi}$, where χ is a R-R scalar field and ϕ a dilaton, is determined as the modular parameter of an elliptic curve [12]

$$y^{2} = x^{3} + f(z)x + g(z).$$
(2.1)

Here z is a complex coordinate on \mathbf{P}^1 , f and g are polynomials in z of degree 8 and 12, respectively. The 24 zeroes of the discriminant $\Delta = 4f^3 + 27g^2$ are the transverse positions of the 24 7-branes. The modular parameter τ is obtained from $j(\tau) = 4(24f)^3/\Delta$. The cubic (2.1) describes a K3 surface as an elliptic fibration over the base \mathbf{P}^1 . When the positions of some 7-branes coincide the elliptic fibration develops singularities which are well-known to follow the Kodaira classification [13]. The singularity types then have a correspondence with the ADE singularities, according to which the ADE types of gauge symmetry in Type IIB theory are identified [1, 14].

gauge	Kodaira	background	coupling
symmetry	type	7-branes	constant τ
E_8	Π^*	$A^7 B C^2$	$e^{2\pi i/3}$
E_7	III^*	$\mathbf{A}^{6}\mathbf{B}\mathbf{C}^{2}$	i
E_6	IV^*	$\mathbf{A}^{5}\mathbf{B}\mathbf{C}^{2}$	$e^{2\pi i/3}$
E_5	I_1^*	$\mathbf{A}^{4}\mathbf{B}\mathbf{C}^{2}$	∞
D_4	I_0^*	$\mathbf{A}^{4}\mathbf{BC}$	arbitrary
A_2	IV	$\mathbf{A}^{3}\mathbf{C}$	$e^{2\pi i/3}$
A_1	III	$\mathbf{A}^2\mathbf{C}$	i
$\{0\}$	II	\mathbf{AC}	$e^{2\pi i/3}$

Table 1: Symmetry and background 7-branes

The connection between the ADE gauge symmetry and the background 7-brane configurations has been established by analyzing the monodromy properties [15, 16]. In Type IIB theory there exist 7-branes which are mutually nonlocal. To distinguish them we shall refer to a 7-brane on which Type IIB (p, q) strings can end as a [p, q] 7-brane. For the purpose of describing the ADE symmetry it is sufficient to take into account [1, 0], [1, -1]and [1, 1] 7-branes which will be henceforth denoted as A-, B- and C-branes, respectively. Let $\mathbf{A}^{n}\mathbf{B}^{m}\mathbf{C}^{\ell}$ represent a set of n A-, m B- and ℓ C-branes. The E_{8} gauge symmetry, for instance, is realized at $\tau = e^{2\pi i/3}$ when a set of 7-branes $\mathbf{A}^{7}\mathbf{B}\mathbf{C}^{2}$ coalesces. Gauge symmetries and the corresponding 7-brane configurations relevant to our following discussions are summarized in Table 1. We note that $E_{5} = D_{5}$ and the brane configuration $\mathbf{A}^{4}\mathbf{B}\mathbf{C}^{2}$ is shown to be equivalent to $\mathbf{A}^{5}\mathbf{B}\mathbf{C}$ [17].

We now introduce a D3-brane which is parallel to the background 7-branes. This D3brane can probe the local geometry near the singularities which are responsible for the gauge symmetry enhancement. On the D3-brane the low-energy effective theory becomes four-dimensional N = 2 supersymmetric gauge theory. Suppose that the D3-brane probe is located near coalescing 7-branes, then N = 2 theory on the D3-brane is a fixed point theory since there are no relevant mass parameters turned on. The gauge symmetry in the bulk turns out to be the enhanced global symmetry of a fixed-point N = 2 supersymmetric theory on the brane [2].

From this point of view, let us look at Table 1. First of all, the D_4 theory on the

brane in the vicinity of the 7-branes $\mathbf{A}^4 \mathbf{BC}$ arises in N = 2 SU(2) theory with $N_f = 4$ fundamental quarks [6]. Here **B**- and **C**-branes stand for monopole and dyon singularities, and **A**-branes stand for the squark singularities in the Coulomb branch. The N = 2 SU(2)theory with $N_f = 4$ is finite and the marginal gauge coupling constant can take any values. Similarly, the A_2 , A_1 and $\{0\}$ theories also arise in the Coulomb branch of N = 2 SU(2)theory with $N_f = 3$, 2 and 1, respectively. These are non-trivial superconformal theories obtained by adjusting quark masses at particular values [8]. On the other hand, the D_5 theory describes the IR free behavior of N = 2 SU(2) theory with $N_f = 5$. The most interesting are the theories with E_n (n = 6, 7, 8) global symmetries. They are non-trivial N = 2 superconformal field theories, but do not admit the Lagrangian description. In view of the D3-brane probe approach, it is natural to place these non-trivial fixed points with exceptional symmetry in the sequence of renormalization group flows

$$E_8 \xrightarrow{\mathbf{A}} E_7 \xrightarrow{\mathbf{A}} E_6 \xrightarrow{\mathbf{A}, \mathbf{C}} D_4 \xrightarrow{\mathbf{A}, \mathbf{B}} A_2 \xrightarrow{\mathbf{A}} A_1 \xrightarrow{\mathbf{A}} \{0\}, \qquad (2.2)$$

where 7-branes indicated under the arrows are sent to infinity to generate the flows. In (2.2) only the D_4 theory is described as a local Lagrangian field theory, while the others are considered to be non-local. Note that the flows $E_6 \rightarrow D_4$ and $D_4 \rightarrow A_2$ are realized by moving away mutually non-local 7-branes simultaneously.

Starting with the D_4 theory one can also consider more familiar flows

$$D_4 \xrightarrow{\mathbf{A}} D_3 \xrightarrow{\mathbf{A}} D_2 \xrightarrow{\mathbf{A}} D_1 \xrightarrow{\mathbf{A}} D_0, \qquad (2.3)$$

where the 7-brane background for the D_n symmetry is given by $\mathbf{A}^n \mathbf{BC}$. Note that, for $n \leq 3$, the configuration $\mathbf{A}^n \mathbf{BC}$ does not fall into the Kodaira classification since it is non-collapsible [17]. On a D3-brane probing $\mathbf{A}^n \mathbf{BC}$ with $n \leq 3$, ordinary N = 2 SU(2) QCD with $N_f = n$ fundamental quarks is realized.

As mentioned previously, enhanced global symmetries at the fixed points in (2.2) are recognized in geometric terms as the ADE singularities. Thus relevant perturbations taking the system away from criticality are described in terms of versal deformations of the ADE singularities. The coupling constant τ of deformed N = 2 theories is then determined by elliptic curves in the form of (2.1) where the explicit forms of polynomials

	E_8	E_7	E_6	D_4	A_2	A_1
h	30	18	12	6	3	2
q_y	15	9	6	3	3/2	1
q_x	10	6	4	2	1	2/3
q_z	6	4	3	2	3/2	4/3

Table 2: Degree of variables

f and g are now specified by the ADE singularity types. We have

$$E_8: \quad f = w_2 z^3 + w_8 z^2 + w_{14} z + w_{20}, \quad g = z^5 + w_{12} z^3 + w_{18} z^2 + w_{24} z + w_{30}, \quad (2.4)$$

$$E_7: f = z^3 + w_8 z + w_{12}, g = w_2 z^4 + w_6 z^3 + w_{10} z^2 + w_{14} z + w_{18}, (2.5)$$

$$E_6: f = w_2 z^2 + w_5 z + w_8, g = z^4 + w_6 z^2 + w_9 z + w_{12}, (2.6)$$

$$D_4: f = z^2 + \tilde{w}_4, g = w_2 z^2 + w_4 z + w_6, (2.7)$$

$$A_2: f = w_2, g = z^2 + w_3,$$
 (2.8)

$$A_1: f = z, g = w_2,$$
 (2.9)

where the w_q are deformation parameters. Here z is understood as the gauge invariant expectation value which parametrizes the vacuum moduli of N = 2 theory. In the brane picture z is a coordinate of the position of the D3-brane probe on \mathbf{P}^1 . In the cubic (2.1) with (2.4)-(2.9) we take y^2 to be of degree h with h being the Coxeter number of G = ADE(see Table 2). Then x, y, z have the degree q_x, q_y, q_z as given in Table 2 and w_{q_i} has the degree $q_i = e_i + 1$ where e_i is the *i*-th exponent of G. Note here that $q_x + q_z = q_y + 1$ and $2q_y = h$. The value of q_z gives the scaling dimension of the expectation value z [8, 4].

Notice that only in the D_4 theory the coupling constant τ is marginal, and hence the curve may incorporate the τ -dependence. This is allowed since x and z have the same degree $q_x = q_z = 2$ which holds only for the D_4 case. In fact the Seiberg-Witten (SW) curve for the D_4 theory obtained originally in [6] depends on both τ and four bare quark masses m_1, m_2, m_3, m_4 . It is not difficult to work out how the SW curve in [6] is related to our D_4 curve (2.7). Let us write down the SW curve presented in (17.58), section 17

of $[6]^{\dagger}$

$$Y^{2} = X(X - \alpha Z)(X - \beta Z) + aX^{2} + bX + cZ + d, \qquad (2.10)$$

where we have used Z instead of u to denote the adjoint Higgs expectation value and

$$a = (\alpha - \beta)^2 u_2/4, \qquad b = -(\alpha - \beta)^2 \alpha \beta u_4/4 + i\alpha \beta (\alpha^2 - \beta^2) \widetilde{u}_4/4,$$

$$c = -i(\alpha - \beta) \alpha^2 \beta^2 \widetilde{u}_4/2, \qquad d = (\alpha - \beta)^2 \alpha^2 \beta^2 u_6/4,$$

$$\alpha = -\vartheta_3^4(\tau), \qquad \beta = -\vartheta_4^4(\tau),$$

$$\vartheta_3(\tau) = \sum_{n \in \mathbf{Z}} q^{n^2/2}, \qquad \vartheta_4(\tau) = \sum_{n \in \mathbf{Z}} (-1)^n q^{n^2/2}, \qquad q = e^{2\pi i \tau}.$$
(2.11)

Here the D_4 invariants made of quark masses are defined by

$$u_{2} = -\sum_{a} m_{a}^{2}, \qquad u_{4} = \sum_{a < b} m_{a}^{2} m_{b}^{2},$$

$$u_{6} = -\sum_{a < b < c} m_{a}^{2} m_{b}^{2} m_{c}^{2}, \qquad \tilde{u}_{4} = -2im_{1}m_{2}m_{3}m_{4}.$$
 (2.12)

Making a change of variables

$$X = -\alpha\beta x, \qquad Y = \alpha\beta(\alpha - \beta)y/2, \qquad Z = i(\alpha - \beta)z/2 - (\alpha + \beta)x/2, \qquad (2.13)$$

we see that (2.10) becomes

$$y^{2} = xz^{2} + x^{3} + u_{2}x^{2} + u_{4}x + \tilde{u}_{4}z + u_{6}$$
(2.14)

which is nothing but the standard form of deformations of the D_4 singularity. We next replace x by $x - u_2/3$ and shuffle the D_4 invariants as

$$u_{2} = -3w_{2}, u_{4} = \tilde{w}_{4} + 3w_{2}^{2}, u_{6} = w_{6} - w_{2}\tilde{w}_{4} - w_{2}^{3}, \tilde{u}_{4} = w_{4}.$$
(2.15)

Then we obtain the D_4 curve with (2.7).

[†]In writing (2.10) we have replaced m_a by $m_a/2$ in (17.58) of [6]. This is necessary to agree with section 16 of [6]. See section 17.4 of [6].

3 BPS mass formula

Having obtained the SW curve for N = 2 theory on a D3-brane probe, we next discuss the BPS mass formula. In the brane probe approach, BPS states on the D3-brane world volume are geometrically realized as Type IIB strings, or more generally string junctions obeying the BPS condition. According to [18], junctions are specified by asymptotic charges (p,q) and a weight vector of G = ADE. Denoting a junction as **J** we have [18]

$$\mathbf{J} = p\boldsymbol{\omega}^p + q\boldsymbol{\omega}^q + \sum_{i=1}^{\operatorname{rank} G} a_i \boldsymbol{\omega}_i, \qquad (3.1)$$

where $\boldsymbol{\omega}^p$ and $\boldsymbol{\omega}^q$ are junctions which are singlets under G with asymptotic charges (1,0)and (0,1) respectively, and the $\boldsymbol{\omega}_i$ with zero asymptotic charges are junctions corresponding to the fundamental weights of G. Here the a_i are the Dynkin labels representing a weight vector. The BPS condition on **J** is described as [10, 11]

$$(\mathbf{J}.\mathbf{J}) - \mathcal{GCD}(p,q) \ge -2, \tag{3.2}$$

where (.) stands for the bilinear form on the junction lattice [18].

The BPS junction with (p,q) charges can end on the D3-brane and realizes the BPS state with electric p and magnetic q charges in the world volume N = 2 theory. Sen has first figured out this and, furthermore, shown how the SW BPS mass formula in the D_4 theory is obtained from the mass formula for a (p,q) string in Type IIB theory [19]. His proof is easily extended to the general ADE case. For this, let us recapitulate the basic elements in the SW theory [20, 6]. The SW differential λ associated with an elliptic curve has to obey

$$\frac{\partial\lambda}{\partial z} = \kappa \frac{dx}{y} + d(*) \tag{3.3}$$

with a normalization constant κ . The SW periods are then given by

$$a(z) = \oint_{\alpha} \lambda, \qquad a_D(z) = \oint_{\beta} \lambda,$$
 (3.4)

where α and β are two homology cycles on a torus. The N = 2 central charge for a BPS state with charges (p, q) reads

$$Z = pa(z) + qa_D(z) + \frac{1}{\sqrt{2}} \sum_a s_a m_a,$$
(3.5)

where the m_a are the bare mass parameters and the s_a are the global abelian charges. The BPS mass is then given by

$$m = \sqrt{2}|Z|. \tag{3.6}$$

Let us now recall the standard elliptic function formula for the discriminant of the cubic

$$\Delta(z) = -2^{20} \left(\frac{\pi}{2\omega_1}\right)^{12} \eta(\tau)^{24}, \qquad (3.7)$$

where $\eta(\tau)$ is the Dedekind eta function and $2\omega_1$ is the period along the α -cycle

$$2\omega_1 = \oint_\alpha \frac{dx}{y}.$$
(3.8)

We thus verify the crucial formula from (3.7) that

$$da(z) = \kappa \pi (-1)^{\frac{1}{12}} 2^{\frac{5}{3}} \eta(\tau)^2 \Delta(z)^{-\frac{1}{12}} dz.$$
(3.9)

In Type IIB theory on \mathbf{P}^1 , on the other hand, the mass of a (p,q) string stretched along a path C is given by

$$m_{p,q} = \int_C T_{p,q} ds, \qquad (3.10)$$

where the tension of a (p, q) string reads

$$T_{p,q} = \frac{1}{\sqrt{\operatorname{Im}\tau}} |p + q\tau| \tag{3.11}$$

and the line element is given in terms of the metric

$$ds^{2} = \text{Im } \tau \left| \eta(\tau)^{2} \Delta(z)^{-\frac{1}{12}} dz \right|^{2}.$$
 (3.12)

A BPS state with a mass $m_{p,q}^{\text{BPS}}$ is obtained by choosing a curve C so that C is a geodesic. Then, following [19], one can show $m_{p,q}^{\text{BPS}} \propto m$ with the aid of (3.9).

The BPS junctions are lifted to holomorphic curves in F/M theory compactified on an elliptically fibered K3 surface. From this viewpoint, it is interesting to see that the expression (3.1) of a junction looks quite similar in form to the central charge (3.5). We may think of the α and β cycles as the projection of the ω^p and ω^q junctions on the xplane. It is obscure, however, how to understand the bare mass term in (3.5) in the light of the third term of (3.1) which consists of the junctions associated to the fundamental weights. In fact there is an important subtlety here. In massive theory, the global abelian charges s_a in (3.5) carry only "constant parts" of the physical abelian charges [21]. The periods a, a_D can also produce terms of constants multiple of bare masses [21, 22]. These terms can arise in the period integrals in massive theory since the SW differential has the poles with residues proportional to bare masses [23]. In other words, to determine the abelian charges appearing explicitly in the N = 2 central charge, one has to analyze the meromorphic properties of the SW differential carefully.

4 Residues of the Seiberg-Witten differential

In this section our purpose is to discuss some general properties of the SW differential λ associated to our *ADE* elliptic curves with (2.4)-(2.9) for the mass deformed *ADE* theories. The differential λ satisfies (3.3) where a normalization constant κ will be fixed later on. In order to find λ we first follow section 17.1 of [6]. Let X be a complex surface defined by $y^2 = W(x, z; w_i)$ as in (2.4)-(2.9). A holomorphic two-form Ω on the surface reads

$$\Omega = \kappa \frac{dx \wedge dz}{y}.$$
(4.1)

We wish to rewrite the condition (3.3) in terms of Ω . To do so, note that, for $\lambda = a(x, z)dx$, (3.3) is written as

$$\kappa \frac{dx}{y} = \frac{\partial a(x,z)}{\partial z} dx + \frac{\partial F(x,z)}{\partial x} dx, \qquad (4.2)$$

where F(x, z) has appeared from the total derivative term in (3.3). Define a one-form $\tilde{\lambda} = -a(x, z)dx + F(x, z)dz$, then (3.3) is succinctly written as

$$\Omega = d\tilde{\lambda}.\tag{4.3}$$

This means that there exists a smooth differential $\tilde{\lambda}$ obeying (4.3) if and only if the cohomology $H^2(X, \mathbb{C})$ is trivial.

Suppose now that $H^2(X, \mathbb{C})$ is non-trivial, and let the $[C_a]$ linearly span $H^2(X, \mathbb{C})$. The Poincaré dual of $[C_a]$, which is a complex curve, is a non-trivial homology cycle in X. In this case, the relation (4.3) is modified to be

$$\Omega = d\tilde{\lambda} - 2\pi i \sum_{a} \operatorname{Res}_{C_a}(\tilde{\lambda}) \cdot [C_a].$$
(4.4)

This describes the situation in which $\tilde{\lambda}$ has poles on the C_a with residues $\operatorname{Res}_{C_a}(\tilde{\lambda})$ and $[C_a]$ is a delta function supported on C_a .

There is an important relation between the period integrals of Ω and the residues [6]. We may evaluate the periods

$$\pi_a = \oint_{C_a} \Omega \tag{4.5}$$

upon compactifying X in an appropriate way. Then the cohomology class $[\Omega]$ is expanded in terms of $[C_a]$ as

$$[\Omega] = \sum_{a,b} \pi_a (M^{-1})_{ab} [C_b], \qquad (4.6)$$

where $M_{ab} = \sharp (C_a \cdot C_b)$ is the intersection matrix which is invertible. Expressing (4.4) in cohomology and comparing to (4.6) one obtains [6]

$$\operatorname{Res}_{C_a}(\tilde{\lambda}) = -\frac{1}{2\pi i} \sum_b (M^{-1})_{ab} \pi_b.$$
(4.7)

Let us further examine the periods π_a . Since the defining equation for X is $y^2 = W(x, z; w_i)$, the period integral (4.5) takes the form

$$\pi_a = \kappa \oint_{C_a} \frac{dx \wedge dz}{W(x, z; w_i)^{1/2}}.$$
(4.8)

We recall here that in the Landau-Ginzburg description of two-dimensional ADE N = 2superconformal field theories, $W(x, z; w_i)$ is identified with the superpotential [24]. Being twisted, these theories turn out to be topological ones which can couple to topological gravity. Then, exactly the same periods as (4.8) have appeared when we calculate the one-point functions in two-dimensional gravity [25]. It is shown there that the periods π_a obey the Gauss-Manin differential equation

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_{k=1}^r C_{ij}{}^k(t) \frac{\partial^2}{\partial t_k \partial t_r}\right) \pi_a(t) = 0, \qquad (4.9)$$

where $r = \operatorname{rank} G$ (G = ADE), t_i ($i = 1, \dots, r$) are the flat coordinates judiciously made of the w_i and $C_{ij}{}^k(t)$ are the three-point functions in the ADE topological Landau-Ginzburg models. It is then clear from (4.7) that $\operatorname{Res}_{C_a}(\tilde{\lambda})$ satisfy (4.9).

To find a class of solutions of the Gauss-Manin system (4.9), we introduce

$$P_G^{\mathcal{R}}(t, u_i) = \det(t - \Phi_{\mathcal{R}}).$$
(4.10)

This is the characteristic polynomial in t of degree dim \mathcal{R} where \mathcal{R} is an irreducible representation of G. Here $\Phi_{\mathcal{R}}$ is a representation matrix of \mathcal{R} and u_i $(i = 1, \dots, r)$ is the Casimir built out of $\Phi_{\mathcal{R}}$ whose degree equals $e_i + 1$ with e_i being the *i*-th exponent of G. (4.10) may be solved formally with respect to the top Casimir u_r , yielding

$$u_r = \widetilde{W}_G^{\mathcal{R}}(t, u_1, \cdots, u_{r-1}).$$
(4.11)

If we define

$$W_G^{\mathcal{R}}(t, u_1, \cdots, u_r) = \widetilde{W}_G^{\mathcal{R}}(t, u_1, \cdots, u_{r-1}) - u_r, \qquad (4.12)$$

then $W_G^{\mathcal{R}}(t, u_i)$ is the single-variable version of the Landau-Ginzburg superpotential which gives rise to the same topological field theory results with the standard ADE topological Landau-Ginzburg models equipped with the superpotential $W(x, z; w_i)$ independently of the representations \mathcal{R} [26, 27]. Upon doing these computations one figures out how the Casimirs u_i are related with the deformation parameters w_i , and hence with the flat coordinates t_i .

Let m_a $(a = 1, \dots, \dim \mathcal{R})$ be an eigenvalue of $\Phi_{\mathcal{R}}$, then (4.10) is written as

$$P_G^{\mathcal{R}}(t, u_1, \cdots, u_r) = \prod_{a=1}^{\dim \mathcal{R}} (t - m_a)$$
 (4.13)

with

$$m_a = (\lambda_a, \phi), \tag{4.14}$$

where the λ_a are the weights of \mathcal{R} and (,) stands for the inner product. Here

$$\phi = \sum_{i=1}^{r} \phi_i \alpha_i \tag{4.15}$$

with α_i being the simple roots of G. Expanding the RHS's of (4.10) and (4.13) we see how the Casimirs u_i are expressed in terms of ϕ_i .

In [28], using the technique of topological Landau-Ginzburg models, it is shown that the zeroes m_a of the characteristic polynomial for any irreducible representation of the ADE groups satisfy the Gauss-Manin system (4.9) for the ADE singularity. Therefore we are led to take

$$\operatorname{Res}_{C_a}(\lambda) = \gamma_{\mathcal{R}} \ m_a(w), \qquad a = 1, \cdots, \dim \mathcal{R}, \tag{4.16}$$

where $\gamma_{\mathcal{R}}$ is a normalization constant which may depend on \mathcal{R} . The residues of the SW differential thus transform in the representation \mathcal{R} of the global symmetry G.

Having fixed the residues we now would like to determine two-cycles C_a on which the poles are located. This is the issue to which we turn in the next section.

5 Seiberg-Witten differential

In [4, 5] the SW differentials in the cases of D_4 , E_6 , E_7 and E_8 have been constructed by exploiting the idea of [6] that y^2 in the cubic becomes a perfect square when x is at the position of the pole. It was then found that one can obtain the SW differentials for the adjoint in addition to the fundamental of the global symmetry group. We wish to demonstrate that the procedure can be formulated in a more transparent and systematic way. For this purpose, it will be shown in this section that the complex curves C_a on which the SW differential has poles are given by the global sections of the bundle in an elliptic fibration, and furthermore C_a have one-to-one correspondence with the irreducible representations of the global symmetry group G = ADE. The relations among the global sections in the elliptic fibration, characteristic polynomials and algebraic equations have been studied by Shioda in his works on the theory of Mordell-Weil lattice [29].[‡]

Let $(x_a(z), y_a(z))$ be such sections, then poles are located at $x = x_a(z)$ on the xplane. The residues of the poles are given by (4.16) where m_a are the eigenvalues of a representation matrix \mathcal{R} . Then, following Minahan and Nemeschansky [4, 5], we assume the SW differential in \mathcal{R} to take the form

$$\lambda_{\mathcal{R}} = (c_1 z + c_3 B(w)) \frac{dx}{y} + c_2 \sum_a \frac{m_a(w) y_a(z)}{x - x_a(z)} \frac{dx}{y},$$
(5.1)

where $B(w) = w_2$ for D_4 , w_2^2 for E_7 , w_2^3 for E_8 and 0 otherwise, and constants c_i will be determined up to the overall normalization in such a way that $\lambda_{\mathcal{R}}$ obeys (3.3). Note that given the degree 1 to $m_a(w)$, $\lambda_{\mathcal{R}}$ has the degree 1 which equals mass dimension of $\lambda_{\mathcal{R}}$. Since $m_a = (\lambda_a, \phi)$ as in (4.14), the ϕ_i are r (= rank G) independent mass parameters in the theory.

[‡]One of us (SKY) is indebted to K. Oguiso for informing of Shioda's works.

In the following we construct $\lambda_{\mathcal{R}}$ explicitly for the A_1 , A_2 , D_4 and E_n (n = 6, 7, 8) theories. The A_1 case is too simple to exhibit the essence of our calculations. So we start with the case of A_2 which is not only instructive but tractable by hand. In the D_4 and E_n theories we have used the Maple software on computer to carry out our calculations. The A_1 result is given at the end of this section. A full detail of how to evaluate $\partial \lambda_{\mathcal{R}}/\partial z$ is presented in Appendix A. The data of characteristic polynomials for D_4 , E_6 , E_7 and E_8 is collected in Appendix B.

5.1 The A_2 theory

The A_2 curve is written in terms of the coefficient polynomials (2.8). As a section let us assume

$$x = v, \qquad y = z. \tag{5.2}$$

with $v \in \mathbf{C}$. Substituting this into the A_2 curve it is obvious that v has to satisfy

$$v^3 + w_2 v + w_3 = 0. (5.3)$$

The LHS is in the form of the characteristic polynomial $P_{A_2}^{\mathbf{3}}(t)$ for **3** of SU(3) with two Casimirs w_2 and w_3 under the relation $t \propto v$. Thus v is determined by the three zeroes m_a of $P_{A_2}^{\mathbf{3}}(t)$. Let us set t = v/2,[§] then we have the three roots v_a of (5.3) as $v_a = 2m_a$ and

$$w_{2} = v_{1}v_{2} + v_{2}v_{3} + v_{3}v_{1} = -4(\phi_{1}^{2} + \phi_{2}^{2} - \phi_{1}\phi_{2}),$$

$$w_{3} = -v_{1}v_{2}v_{3} = -8\phi_{1}\phi_{2}(\phi_{1} - \phi_{2})$$
(5.4)

with $v_1 + v_2 + v_3 = 0$. Putting $v = v_a$ we observe that the section (5.2) belongs to **3** of SU(3).

It is quite interesting that the characteristic polynomial naturally appears when the global sections are determined. Accordingly the residues of the differential $\lambda_{\mathcal{R}}$ are fixed as was discussed before. We thus write down $\lambda_{\mathbf{3}}$ in the form

$$\lambda_{3} = c_{1} z \frac{dx}{y} + c_{2} \sum_{a=1}^{3} \frac{m_{a} y_{a}}{x - x_{a}} \frac{dx}{y}, \qquad m_{a} = \frac{v_{a}}{2}.$$
(5.5)

[§]There is no a priori reason for fixing a constant c in the relation t = cv. Our choice t = v/2 will be justified in section 7 by considering the renormalization group flows from (or to) the D_4 theory. This remark also applies to the following cases studied in this section.

Note that the sum of the residues has to vanish. This is ensured since there also exist poles with residues with opposite sign on the other sheet. These poles belong to $\overline{\mathbf{3}}$ of SU(3). Here (A.15) yields

$$\frac{\partial \lambda_3}{\partial z} = \frac{2c_1}{3} \frac{dx}{y} + (A_1 x + A_0) \frac{dx}{y^3} + d(*), \tag{5.6}$$

where

$$A_1 = \frac{1}{3}(2c_1 - 3c_2)w_2, \qquad A_0 = \frac{1}{2}(2c_1 - 3c_2)w_3$$
(5.7)

from which we get $c_1 = 3c_2/2$.

We can find another section by assuming

$$\begin{cases} x(z) = \frac{z^2}{v_1^2} + b_1 z + b_0, \\ y(z) = \frac{z^3}{v^3} + r_2 z^2 + r_1 z + r_0, \end{cases}$$
(5.8)

where $v, b_i, r_i \in \mathbf{C}$. Plugging this in the A_2 curve one obtains the relations

$$r_{0}^{2} - w_{3} - b_{0}^{3} - w_{2}b_{0} = 0,$$

$$-3b_{1}b_{0}^{2} + 2r_{1}r_{0} - w_{2}b_{1} = 0,$$

$$\left(2r_{2}r_{0} + r_{1}^{2} - 3b_{1}^{2}b_{0} - 1\right)v^{2} - 3b_{0}^{2} - w_{2} = 0,$$

$$2r_{0} + \left(2r_{2}r_{1} - b_{1}^{3}\right)v^{3} - 6b_{1}b_{0}v = 0,$$

$$-3b_{0} + 2r_{1}v - 3b_{1}^{2}v^{2} + r_{2}^{2}v^{4} = 0,$$

$$2r_{2}v - 3b_{1} = 0.$$

(5.9)

Eliminating b_i and r_i we are left with

$$64v^6 + 96w_2v^4 + 36w_2^2v^2 + 4w_2^3 + 27w_3^2 = 0, (5.10)$$

while the characteristic polynomial for 8 of SU(3) reads

$$P_{A_2}^{\mathbf{8}}(t) = t^2 \left(t^6 + \frac{3}{2} w_2 t^4 + \frac{9}{16} w_2^2 t^2 + \frac{w_2^3}{16} + \frac{27}{64} w_3^2 \right).$$
(5.11)

Thus the six roots v_a of (5.10) are identified with the generically non-vanishing zeroes of (5.11), *i.e.*

$$P_{A_2}^{\mathbf{8}}(v_a) = 0, (5.12)$$

from which we see that the section (5.8) belongs to 8 of SU(3).

For the adjoint section (5.8) the SW differential is constructed as

$$\lambda_{\mathbf{8}} = c_1 z \frac{dx}{y} + c_2 \sum_{a=1}^3 \frac{m_a y_a}{x - x_a} \frac{dx}{y}, \qquad m_a = v_a.$$
(5.13)

The non-zero weights of 8 read $\lambda_{\pm 1} = \pm (1,1)$, $\lambda_{\pm 2} = \pm (-1,2)$ and $\lambda_{\pm 3} = \pm (2,-1)$ in the Dynkin basis. We have from (4.14) and (5.12) that $v_{\pm a} = (\lambda_{\pm a}, \phi)$. Note that $v_{-a} = -v_a$ (a = 1, 2, 3) give the residues of the poles on the other sheet. In terms of this parametrization, one can find $v_a r_{1a}$ explicitly from (5.9)

$$v_1 r_{11} = -3(\phi_1 - \phi_2), \qquad v_2 r_{12} = 3\phi_1, \qquad v_3 r_{13} = -3\phi_2.$$
 (5.14)

 A_0 and A_1 in (A.16) are then evaluated to be

$$A_1 = (2c_1 - 9c_2)w_2, \qquad A_0 = -3c_2z^2 + \frac{1}{2}(2c_1 - 9c_2)w_3. \tag{5.15}$$

To manipulate the z^2 term in A_0 we note

$$z^{2} = W - \frac{1}{3}x\partial_{x}W - \frac{2}{3}xf - (g - z^{2})$$
(5.16)

which yields

$$\frac{z^2}{W^{3/2}} = \frac{1}{3\sqrt{W}} - \frac{1}{W^{3/2}} \left(\frac{2w_2}{3}x + w_3\right) + \frac{2}{3}\partial_x \left(\frac{x}{\sqrt{W}}\right).$$
(5.17)

Thus

$$\frac{\partial \lambda_8}{\partial z} = \left(\frac{2c_1}{3} - 2c_2\right) \frac{dx}{y} + \frac{2c_1 + 3c_2}{2} \left(\frac{2w_2}{3}x + w_3\right) \frac{dx}{y^3} + d(*), \tag{5.18}$$

and hence we obtain $c_1 = -3c_2/2$.

Finally it should be mentioned that the elliptic fibration (2.1) with (2.4)-(2.9) admits the section in the form of (5.8) and, as we will see, (5.8) always corresponds to the adjoint representation of G = ADE.

5.2 The D_4 theory

Taking the curve (2.7) for the D_4 theory we obtain the SW differential in parallel with the A_2 case though the computations become slightly more involved. Let us first examine the section in the form

$$\begin{cases} x(z) = \delta z + r, \\ y(z) = vz + b. \end{cases}$$
(5.19)

For $\delta = 0$, plugging (5.19) in the D_4 curve gives

$$v^{2} - r - w_{2} = 0,$$
 $2vb - w_{4} = 0,$ $b^{2} - r^{3} - r\tilde{w}_{4} - w_{6} = 0.$ (5.20)

The elimination procedure results in

$$v^{8} - 3w_{2}v^{6} + \left(\tilde{w}_{4} + 3w_{2}^{2}\right)v^{4} + \left(w_{6} - w_{2}\tilde{w}_{4} - w_{2}^{3}\right)v^{2} - w_{4}^{2}/4 = 0.$$
 (5.21)

This polynomial may be compared to the characteristic polynomial for $\mathbf{8}_{\mathbf{v}}$ (vector) of SO(8)

$$P_{D_4}^{\mathbf{8_v}}(t) = t^8 + u_2 t^6 + u_4 t^4 + u_6 t^2 - \tilde{u}_4^2 / 4.$$
(5.22)

Then (5.21) is equivalent to

$$P_{D_4}^{\mathbf{8_v}}(v_a) = 0 \tag{5.23}$$

under the relation (2.15), showing that the section with $\delta = 0$ is in the vector representation.

For $\delta = \pm i$, on the other hand, we observe

$$P_{D_4}^{\mathbf{s}_{\mathbf{s}}}(v_a/2) = 0, \qquad \text{for } \delta = +i,$$

$$P_{D_4}^{\mathbf{s}_{\mathbf{c}}}(v_a/2) = 0, \qquad \text{for } \delta = -i,$$
(5.24)

where the characteristic polynomial for $\mathbf{8}_{\mathbf{s}}$ (spinor) of SO(8) is given by

$$P_{D_4}^{\mathbf{8_s}}(t) = t^8 + u_2 t^6 + \left(\frac{3}{8}u_2^2 - \frac{u_4}{2} - \frac{3i}{2}\tilde{u}_4\right)t^4 + \left(-\frac{u_2u_4}{4} + \frac{u_2^3}{16} - \frac{i}{4}u_2\tilde{u}_4 + u_6\right)t^2 - \frac{u_2^2u_4}{32} - \frac{i}{8}\tilde{u}_4u_4 - \frac{\tilde{u}_4^2}{16} + \frac{i}{32}u_2^2\tilde{u}_4 + \frac{u_4^2}{16} + \frac{u_2^4}{256}$$
(5.25)

and that for $\mathbf{8}_{\mathbf{c}}$ (conjugate spinor) is obtained by replacing \tilde{u}_4 by $-\tilde{u}_4$. Thus the sections with $\delta = \pm i$ are in the spinorial representations.

The SW differential for the $\mathbf{8_v}$ section turns out to be

$$\lambda_{\mathbf{8}_{\mathbf{v}}} = c_1 z \frac{dx}{y} + \frac{c_1}{2} \sum_{a=1}^4 \frac{m_a^v y_a}{x - x_a} \frac{dx}{y}, \qquad m_a^v = v_a, \tag{5.26}$$

where $v_a = (\lambda_a, \phi)$ with $\lambda_1 = (1, 0, 0, 0)$, $\lambda_2 = (-1, 1, 0, 0)$, $\lambda_3 = (0, -1, 1, 1)$ and $\lambda_4 = (0, 0, -1, 1)$ in the Dynkin basis, while for the $\mathbf{8_s}$ and $\mathbf{8_c}$ sections we obtain

$$\lambda_{\mathbf{8}_{\mathbf{s}}} = c_1 \left(z + \frac{3i}{2} w_2 \right) \frac{dx}{y} - \frac{c_1}{2} \sum_{a=1}^4 \frac{m_a^s y_a}{x - x_a} \frac{dx}{y}, \qquad m_a^s = \frac{v_a}{2}, \tag{5.27}$$

where $v_a = 2(\lambda_a, \phi)$ with $\lambda_1 = (0, 0, 0, 1)$, $\lambda_2 = (0, 1, 0, -1)$, $\lambda_3 = (1, -1, 1, 0)$ and $\lambda_4 = (-1, 0, 1, 0)$, and

$$\lambda_{\mathbf{8_c}} = c_1 \left(z - \frac{3i}{2} w_2 \right) \frac{dx}{y} - \frac{c_1}{2} \sum_{a=1}^4 \frac{m_a^c y_a}{x - x_a} \frac{dx}{y}, \qquad m_a^c = \frac{v_a}{2}, \tag{5.28}$$

where $v_a = 2(\lambda_a, \phi)$ with $\lambda_1 = (0, 0, 1, 0)$, $\lambda_2 = (0, 1, -1, 0)$, $\lambda_3 = (1, -1, 0, 1)$ and $\lambda_4 = (-1, 0, 0, 1)$. These SW differentials obey

$$\frac{\partial \lambda_{\mathcal{R}}}{\partial z} = \frac{c_1}{2} \frac{dx}{y} + d(*) \tag{5.29}$$

for $\mathcal{R}=\mathbf{8_v}, \mathbf{8_s}$ and $\mathbf{8_c}.$

As in the A_2 theory, (5.8) gives the section in **28** (adjoint) of SO(8). After b_i and r_i are eliminated from the relations like (5.9), v is determined as the 24 non-zero roots $\pm v_a$ $(a = 1, \dots, 12)$ of

$$P_{D_4}^{28}(\pm v_a) = 0. (5.30)$$

Assuming the SW differential in the form

$$\lambda_{28} = (c_1 z + c_3 w_2) \frac{dx}{y} + c_2 \sum_{a=1}^{12} \frac{m_a y_a}{x - x_a} \frac{dx}{y}, \qquad m_a = v_a, \tag{5.31}$$

we find $c_1 = c_3 = 0$ and

$$\frac{\partial \lambda_{\mathbf{28}}}{\partial z} = -6c_2 \frac{dx}{y} + d(*). \tag{5.32}$$

Thus there is no holomorphic piece in λ_{28} [5].

Finally we derive the differential λ_{SW} for the original SW curve (2.10) in the D_4 theory. For this let us first take $\lambda_{\mathbf{8}_{\mathbf{v}}}$ and make a change of variables (2.13)

$$x = -\frac{X}{\alpha\beta} + \frac{u_2}{3}, \qquad y = \frac{2Y}{\alpha\beta(\alpha - \beta)}, \qquad z = \frac{2Z}{i(\alpha - \beta)} + \frac{i(\alpha + \beta)}{\alpha\beta(\alpha - \beta)}X.$$
(5.33)

Since

$$\partial_x = -\frac{\alpha + \beta}{2} \partial_Z - \alpha \beta \partial_X, \qquad (5.34)$$

one has to take care of the total derivative term in $\partial \lambda_{\mathbf{8}_{\mathbf{v}}} / \partial z$ (see (A.15)) when converting $\lambda_{\mathbf{8}_{\mathbf{v}}}$ into $\lambda_{SW}^{\mathbf{8}_{\mathbf{v}}}$. The result reads

$$\lambda_{SW}^{\mathbf{8}_{\mathbf{v}}} = c_1 \left(2Z - \frac{\alpha + \beta}{2} u_2 \right) \frac{dX}{Y} - ic_1 \sum_{a=1}^4 \frac{m_a^v Y_a^v}{X - X_a^v} \frac{dX}{Y},\tag{5.35}$$

where $X_a^v = -\alpha\beta(m_a^v)^2$ and $Y_a^v = [Y]_{X=X_a^v}$. In a similar vein we obtain from $\lambda_{\mathbf{8}_s}$ and $\lambda_{\mathbf{8}_c}$ that

$$\lambda_{SW}^{\mathbf{8}_{\mathbf{s}}} = c_1 \left(2Z + \frac{\alpha - \beta}{2} u_2 \right) \frac{dX}{Y} + ic_1 \sum_{a=1}^4 \frac{m_a^s Y_a^s}{X - X_a^s} \frac{dX}{Y},$$

$$\lambda_{SW}^{\mathbf{8}_{\mathbf{c}}} = c_1 \left(2Z - \frac{\alpha - \beta}{2} u_2 \right) \frac{dX}{Y} + ic_1 \sum_{a=1}^4 \frac{m_a^c Y_a^c}{X - X_a^c} \frac{dX}{Y}.$$
 (5.36)

These differentials obey

$$\frac{\partial \lambda_{SW}^{\mathcal{R}}}{\partial Z} = c_1 \frac{dX}{Y} + d(*) \tag{5.37}$$

for $\mathcal{R} = \mathbf{8_v}, \mathbf{8_s}$ and $\mathbf{8_c}$. Thus we set

$$c_1 = \frac{\sqrt{2}}{8\pi} \tag{5.38}$$

according to the normalization adopted in [6].

5.3 The E_6 theory

The global section which transforms in $\mathbf{27}$ of E_6 is given by

$$\begin{cases} x_a(z) = v_a z + b_a, \\ y_a(z) = z^2 + r_a z + s_a \end{cases}$$
(5.39)

with $a = 1, \dots, 27$ [29]. In fact, the elimination procedure yields

$$P_{E_6}^{27}(v_a) = 0. (5.40)$$

This reflects the well-known fact in classical algebraic geometry that the cubic surface in \mathbf{P}^3 contains exactly 27 lines [30].

The SW differential associated with the 27 section is obtained as

$$\lambda_{27} = 36c_2 z \frac{dx}{y} + c_2 \sum_{a=1}^{27} \frac{m_a y_a}{x - x_a} \frac{dx}{y}, \qquad m_a = v_a, \tag{5.41}$$

where the poles with opposite residues on the other sheet transform in the $\overline{27}$ of E_6 . Upon taking the derivative one gets

$$\frac{\partial \lambda_{27}}{\partial z} = 12c_2 \frac{dx}{y} + d(*).$$
(5.42)

In the E_6 theory too, (5.8) yields the section in **78** (adjoint) of E_6 . We see that v takes the values $\pm v_a$ ($a = 1, \dots, 36$) which correspond to the 72 non-zero roots of

$$P_{E_6}^{78}(\pm 2v_a) = 0. \tag{5.43}$$

Assuming the SW differential in the form

$$\lambda_{78} = c_1 z \frac{dx}{y} + c_2 \sum_{a=1}^{36} \frac{m_a y_a}{x - x_a} \frac{dx}{y}, \qquad m_a = 2v_a, \tag{5.44}$$

we find $c_1 = 0$ and

$$\frac{\partial\lambda_{78}}{\partial z} = -24c_2\frac{dx}{y} + d(*). \tag{5.45}$$

As in the case of D_4 the holomorphic piece is absent in λ_{78} [5].

5.4 The E_7 theory

The global section in 56 of E_7 is obtained by taking [29]

$$\begin{cases} x(z) = cz + b, \\ y(z) = vz^2 + rz + s. \end{cases}$$
(5.46)

We find after the elimination process that v is determined from the 56 non-zero roots $\pm 2v_a$ $(a = 1, \dots, 56)$ of

$$P_{E_7}^{56}(\pm 2v_a) = 0. \tag{5.47}$$

The SW differential associated with the 56 section turns out to be

$$\lambda_{56} = 48c_2(z+w_2^2)\frac{dx}{y} + c_2\sum_{a=1}^{28}\frac{m_a y_a}{x-x_a}\frac{dx}{y}, \qquad m_a = 2v_a, \qquad (5.48)$$

from which we get

$$\frac{\partial \lambda_{\mathbf{56}}}{\partial z} = 12c_2 \frac{dx}{y} + d(*). \tag{5.49}$$

The section given by (5.8) again corresponds to **133** (adjoint) of E_7 . We see that v takes the values $\pm v_a$ ($a = 1, \dots, 63$) which yield the 126 non-zero roots of

$$P_{E_7}^{133}(\pm 2v_a) = 0. \tag{5.50}$$

We obtain the SW differential as

$$\lambda_{133} = -18c_2 z \frac{dx}{y} + c_2 \sum_{a=1}^{63} \frac{m_a y_a}{x - x_a} \frac{dx}{y}, \qquad m_a = 2v_a, \tag{5.51}$$

and

$$\frac{\partial \lambda_{133}}{\partial z} = -36c_2 \frac{dx}{y} + d(*). \tag{5.52}$$

5.5 The E_8 theory

By counting degrees it is seen that there are no sections in the form of (5.39), (5.46). This distinguishes the E_8 case from E_6 and E_7 , and corresponds to the fact that the fundamental of E_8 is identical with the adjoint. It is indeed proved by the elimination procedure that the E_8 curve possesses the section as in (5.8) which transforms in **248** of E_8 [29]. As explained in [29], one can explicitly evaluate the resultant which appears in the final step of the elimination process. The result is that v takes the values $\pm v_a$ $(a = 1, \dots, 120)$ which give the 240 non-zero roots of

$$P_{E_8}^{\mathbf{248}}(\pm 2v_a) = 0. \tag{5.53}$$

The SW differential in **248** is then found to be

$$\lambda_{248} = -2c_2(60z + w_2^3)\frac{dx}{y} + c_2\sum_{a=1}^{120}\frac{m_a y_a}{x - x_a}\frac{dx}{y}, \qquad m_a = 2v_a \tag{5.54}$$

and

$$\frac{\partial \lambda_{\mathbf{248}}}{\partial z} = -60c_2 \frac{dx}{y} + d(*). \tag{5.55}$$

5.6 The A_1 theory

It is clear that the A_1 curve admits the section

$$x = 0, \qquad y = v \tag{5.56}$$

which transforms in **2** of SU(2) since $v^2 - w_2 = 0$. The SW differential in **2** is easily obtained as

$$\lambda_2 = c_2 \left(\frac{z}{3} + \frac{m_1 y_1}{x}\right) \frac{dx}{y},\tag{5.57}$$

where $m_1 = v_1/2 = \sqrt{w_2}/2$ and $y_1 = v_1$. Thus we have

$$\lambda_2 = \frac{c_2}{2} \left(\frac{2z}{3} + \frac{w_2}{x}\right) \frac{dx}{y} \tag{5.58}$$

which obeys

$$\frac{\partial \lambda_2}{\partial z} = \frac{c_2}{4} \frac{dx}{y} + d(*).$$
(5.59)

The section in **3** of SU(2) is given by

$$x = \frac{z^2}{v^2}, \qquad y = \frac{z^3}{v^3} + \frac{v}{2}$$
 (5.60)

as in (5.8). Here v satisfies $v^2 - 4w_2 = 0$ while $P_{A_1}^3(t) = t(t^2 - w_2)$, and hence **3** is realized. Correspondingly we find

$$\lambda_{3} = c_{2} \left(-\frac{5z}{6} + \frac{m_{1}y_{1}}{x - x_{1}} \right) \frac{dx}{y} = \frac{c_{2}}{2} \left(-\frac{5z}{3} + \frac{z^{3} + 8w_{2}^{2}}{4w_{2}x - z^{2}} \right) \frac{dx}{y},$$
(5.61)

where $m_1 = v_1/2 = \sqrt{w_2}$. This differential obeys

$$\frac{\partial \lambda_3}{\partial z} = -c_2 \frac{dx}{y} + d(*). \tag{5.62}$$

6 The scaling limit

According to the results in the previous section, it is inferred that one can always construct the SW differential λ in the fundamental as well as adjoint representations in general ADEcase. For D_4 , moreover, we have obtained $\lambda_{\mathbf{8}_{\bullet}}^{D_4}$ for the vector, spinor and conjugate spinor of SO(8) which are permuted under the triality automorphism of D_4 . Thus there arises a natural question whether the physics depends on representations chosen in constructing the SW differential. In order to study this problem it is important to analyze how the SW differential behaves under the renormalization group flow.

Let us analyze in great detail how the E_6 SW differential reduces to the D_4 SW differential when we move simultaneously **A**- and **C**-branes out to infinity from the E_6 seven brane background. When a **A**-brane is removed the E_6 symmetry breaks down to $SO(10) \times U(1)$. The E_6 mass parameters ϕ_i are decomposed under the $SO(10) \times U(1)$ subgroup as

$$\phi_1 = 2M_1 + b_1, \qquad \phi_2 = 4M_1 + b_2, \qquad \phi_3 = 6M_1 + b_3,$$

$$\phi_4 = 5M_1 + b_4, \qquad \phi_5 = 4M_1, \qquad \phi_6 = 3M_1 + b_5, \qquad (6.1)$$

where the b_i are the SO(10) mass parameters and M_1 is the U(1) mass [31]. Here the mass parameters are labeled as shown in the Dynkin diagrams (see Fig.1). Removing

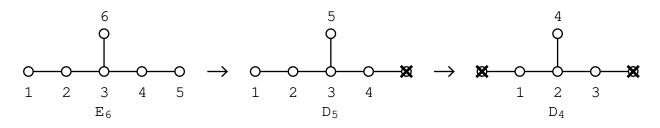


Figure 1: Dynkin diagrams

a **C**-brane induces the breaking of SO(10) to $SO(8) \times U(1)$. Under $SO(8) \times U(1)$ the SO(10) mass parameters are decomposed into the SO(8) masses c_i and U(1) mass M_2 as follows:

$$b_1 = M_2,$$
 $b_2 = M_2 + c_1,$ $b_3 = M_2 + c_2,$
 $b_4 = M_2/2 + c_3,$ $b_5 = M_2/2 + c_4.$ (6.2)

Upon sending A- and C-branes together to infinity we take the scaling limit [4]

$$M_1, M_2 \to \infty, \qquad \frac{M_1}{M_2} = -\frac{\alpha + \beta}{6(\alpha - \beta)} \quad \text{fixed},$$
 (6.3)

where the limit $M_i \to \infty$ decouples two U(1) factors and the ratio with α , β defined in (2.11) gives the value of the marginal gauge coupling constant in the D_4 theory.

In order to see that the E_6 curve reduces to the SO(8) SW curve (2.10) we first write the E_6 invariants $w_{q_i}(\phi)$ in terms of SO(8) masses c_i

$$\phi_1 = \frac{4}{3}(\alpha - 2\beta)M, \qquad \phi_2 = \frac{2}{3}(\alpha - 5\beta)M + c_1, \qquad \phi_3 = -4\beta M + c_2,$$

$$\phi_4 = -\frac{2}{3}(\alpha + 4\beta) + c_3, \qquad \phi_5 = -\frac{4}{3}(\beta + \alpha)M, \qquad \phi_6 = -2\beta M + c_4, \qquad (6.4)$$

where $M = -3M_1/(\alpha + \beta) = M_2/(2(\alpha - \beta))$ and the explicit expressions of $w_{q_i}(\phi)$ in terms of ϕ_i are given in [32]. Then making a change of variables

$$y = -iM^{3}Y,$$

$$x = M^{2} \left(-X - \frac{1}{12}(\alpha - \beta)^{2}u_{2} + \frac{1}{3}(\alpha + \beta)Z\right),$$

$$z = \frac{2}{27}(\beta - 2\alpha)(\alpha - 2\beta)(\alpha + \beta)M^{3} + M\left(-\frac{1}{2}Z + \frac{1}{24}(\alpha + \beta)u_{2}\right)$$
(6.5)

in the E_6 curve and letting $M \to \infty$, we obtain the SO(8) curve (2.10) where $m_a = m_a^v = (\lambda_a, c)$ with λ_a being a weight vector of $\mathbf{8}_v$ in section 5.2.

We next show explicitly that, in the limit (6.3), the E_6 SW differential $\lambda_{27}^{E_6}$ in 27 is reduced to the sum of the D_4 SW differentials in $\mathbf{8_v}, \mathbf{8_s}, \mathbf{8_c}$ we have constructed previously. This corresponds to the fact that the fundamental representation 27 of E_6 is decomposed under the SO(8) subgroup into

$$\mathbf{27} = \mathbf{8_v} \oplus \mathbf{8_s} \oplus \mathbf{8_c} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}. \tag{6.6}$$

Let us put (6.4), (6.5) in the E_6 differential (5.41)

$$\lambda_{27}^{E_6} = c_2 \left(36z + \sum_{a=1}^{27} \frac{m_a(\phi_i) y_a(z, \phi_i)}{x - x_a(z, \phi_i)} \right) \frac{dx}{y}.$$
 (6.7)

and let $M \to \infty$, then we obtain

$$36z\frac{1}{i}\frac{dx}{y} = \left(-\frac{3}{8}(\alpha+\beta)(2\alpha-\beta)(2\beta-\alpha)M^2 + 18u - \frac{3}{2}(\alpha+\beta)u_2(c_i)\right)\frac{dX}{Y} + O\left(\frac{1}{M}\right).$$
(6.8)

The poles in the singlets of SO(8) go to infinity in this limit. Remember that the poles appear pairwise on two sheets of the Riemann surface in such a way that the sum of residues vanish. Indeed we have

$$\sum_{a\in\mathcal{S}} \frac{1}{i} \frac{m_a y_a}{x - x_a} \frac{dx}{y}$$
$$= \left(\frac{3}{8}(\alpha + \beta)(2\alpha - \beta)(2\beta - \alpha)M^2 - 2Z + \frac{1}{6}(\alpha + \beta)u_2(c_i)\right) \frac{dX}{Y} + O\left(\frac{1}{M}\right), (6.9)$$

where S denotes a set of SO(8) singlets, and hence the divergent pieces of (6.8) and (6.9) cancel out.

The pole terms in $\mathbf{8_v}$ turn out to be

$$\frac{1}{i}\frac{m_a y_a}{x - x_a}\frac{dx}{y} = M\frac{A_1^a(Z, c_i) + A_2^a(Z, c_i)\frac{1}{M} + O\left(\frac{1}{M}\right)}{X - A_3^a(Z, c_i) + A_4^a(Z, c_i)\frac{1}{M} + O\left(\frac{1}{M}\right)}\frac{dX}{Y},$$
(6.10)

where A_i^a is a polynomial of Z and c_j . Although this seems to be divergent at first sight, the poles associated with weights λ and $-\lambda$ in $\mathbf{8}_{\mathbf{v}}$ coalesce at the same point, making these contributions finite in the limit $M \to \infty$. It is verified that the sum over terms with these weights $\pm \lambda$ of $\mathbf{8}_{\mathbf{v}}$ becomes finite,

$$M\frac{A_{1}^{a} + A_{2}^{a}\frac{1}{M} + O\left(\frac{1}{M}\right)}{X - A_{3}^{a} + A_{4}^{a}\frac{1}{M} + O\left(\frac{1}{M}\right)} \frac{dX}{Y} + M\frac{-A_{1}^{a} + A_{2}^{a}\frac{1}{M} + O\left(\frac{1}{M}\right)}{X - A_{3}^{a} - A_{4}^{a}\frac{1}{M} + O\left(\frac{1}{M}\right)} \frac{dX}{Y} = \frac{-2A_{1}^{a}A_{4}^{a}}{(X - A_{3}^{a})^{2}} \frac{dX}{Y} + \frac{2A_{2}^{a}}{X - A_{3}^{a}} \frac{dX}{Y} + O\left(\frac{1}{M}\right),$$
(6.11)

where we found that A_3^a is equal to the pole position X_a^v of $\lambda_{\mathbf{8}_v}^{D_4}$. Thus we get

$$\begin{split} \sum_{a \in \mathbf{8_v}} \frac{1}{i} \frac{m_a y_a}{x - x_a} \frac{dx}{y} &\to \sum_{a=1}^4 \left(\frac{-2A_1^a A_4^a}{(X - X_a^v)^2} \frac{dX}{Y} + \frac{2A_2^a}{X - X_a^v} \frac{dX}{Y} \right) \\ &= \sum_{a=1}^4 \left(d \left(\frac{2A_1^a A_4^a}{X - X_a^v} \frac{1}{Y} \right) + \frac{A_1^a A_4^a}{X - X_a^v} \frac{1}{Y^2} \frac{\partial Y^2}{\partial X} \frac{dX}{Y} + \frac{2A_2^a}{X - X_a^v} \frac{dX}{Y} \right) \\ &= \sum_{a=1}^4 \left(\frac{1}{X - X_a^v} \frac{A_1^a A_4^a \frac{\partial Y^2}{\partial X} + 2A_2^a Y^2}{Y^2} \frac{dX}{Y} + d \left(\frac{2A_1^a A_4^a}{X - X_a^v} \frac{1}{Y} \right) \right), \end{split}$$
(6.12)

where the sum on the RHS is taken over half of the weights of $\mathbf{8}_{\mathbf{v}}$. We can proceed further by showing that

$$A_1^a(Z,c_i)A_4^a(Z,c_i) = -\gamma_v [Y^2]_{X=X_a^v}(Z,c_i),$$
(6.13)

and

$$\gamma_v \left[\frac{\partial Y^2}{\partial X} \right]_{X=X_a^v} - 2A_2^a = 2im_a^v [Y]_{X=X_a^v}(Z,c_i), \tag{6.14}$$

where $\gamma_v = \frac{\alpha + \beta}{3\alpha\beta}$.

Thus

$$\sum_{a \in \mathbf{8_v}} \frac{1}{i} \frac{m_a y_a}{x - x_a} \frac{dx}{y} \to \sum_{a=1}^4 \left(\frac{1}{i} \frac{2m_a^v [Y]_{X = X_a^v}}{X - X_a^v} \frac{dX}{Y} - d\left(\frac{2\gamma_v [Y^2]_{X = X_a^v}}{X - X_a^v} \frac{1}{Y} \right) - R_a^v \right), \quad (6.15)$$

where

$$R_a^v = \frac{\gamma_v}{Y^2} \frac{[Y^2]_{X=X_a^v} \frac{\partial Y^2}{\partial X} - Y^2 \left[\frac{\partial Y^2}{\partial X}\right]_{X=X_a^v}}{X - X_a^v} \frac{dX}{Y}.$$
(6.16)

For the pole terms in $\mathbf{8}_{\mathbf{s}}$ of SO(8) we obtain the result as in (6.15) except that we put $\alpha \to -\alpha$ and $\beta \to \beta - \alpha$ in γ_v in (6.16) in accordance with the triality transformation and replace X_a^v and m_a^v by X_a^s and m_a^s for $\mathbf{8}_{\mathbf{s}}$ respectively. Likewise, for the pole terms in $\mathbf{8}_{\mathbf{c}}$ we let $X_a^v \to X_c^v$, $m_a^v \to m_a^c$ and $\beta \to -\beta$ and $\alpha \to \alpha - \beta$ in (6.15).

Finally we sum up the three pieces from $\mathbf{8_v}, \mathbf{8_s}, \mathbf{8_c}$. In doing so, we observe that

$$\sum_{r=v,s,c} \sum_{a=1}^{4} R_a^r = P_1(Z,c_i) \frac{dX}{Y} + 2d\left(\frac{P_1(Z,c_i)X - P_2(Z,c_i)}{Y}\right),\tag{6.17}$$

where

$$P_{1} = 4Z - \frac{1}{3}(\alpha + \beta)u_{2}(c_{i}),$$

$$P_{2} = -\frac{2}{3}(\alpha + \beta)(\alpha^{2} - \alpha\beta + \beta^{2})u_{4}(c_{i}) - 2i(\alpha - \beta)(\alpha^{2} - \alpha\beta + \beta^{2})\tilde{u}_{4}(c_{i})$$

$$+\frac{1}{12}(\alpha + \beta)^{3}u_{2}(c_{i})^{2} - \frac{4}{3}(\alpha^{2} - \alpha\beta + \beta^{2})u_{2}(c_{i})Z + \frac{4}{3}(\alpha + \beta)Z^{2}.$$
(6.18)

As a result, we find in the scaling limit that the E_6 SW differential in **27** is reduced to the SO(8) ones as

$$\lambda_{\mathbf{27}}^{E_6} \rightarrow ic_2 \left(12Z \frac{dX}{Y} - (\alpha + \beta) u_2(c_i) \frac{dX}{Y} + 2 \sum_{r=v,s,c} \sum_{a=1}^4 \frac{1}{i} \frac{m_a^r [Y]_{X=X_a^r}}{X - X_a^r} \frac{dX}{Y} \right) + d(*)$$

$$= 8\pi \sqrt{2} ic_2 \left(\lambda_{SW}^{\mathbf{s}_v} + \lambda_{SW}^{\mathbf{s}_s} + \lambda_{SW}^{\mathbf{s}_c} \right) + d(*), \qquad (6.19)$$

where $\lambda_{SW}^{\mathbf{8}_{\bullet}}$ has been normalized as in (5.38).

We encounter here a somewhat curious situation; $\lambda_{27}^{E_6}$ does not reduce to one of the $\lambda_{SW}^{\mathbf{8}_{\bullet}}$, but the sum of $\lambda_{SW}^{\mathbf{8}_{\bullet}}$. In view of (6.6) and SO(8) triality, on the one hand, (6.19) seems natural. Then one would say that picking up any one of $\lambda_{SW}^{\mathbf{8}_{\bullet}}$ is sufficient to describe the physics. Note, however, that the location of poles and their residues depend on $\mathbf{8_v}, \mathbf{8_s}, \mathbf{8_c}$, and it is not so obvious if the irrelevance of which $\mathbf{8}$ of SO(8) we choose to construct the SW differential is really due to triality invariance which is inherent in SO(8). In addition to this, the SW differential $\lambda_{78}^{E_6}$ looks totally different from $\lambda_{27}^{E_6}$. This is also the case in the D_4 theory. In what follows we will study if the representation chosen in constructing λ is relevant to the physics or not.

7 Universality of Seiberg-Witten periods

Having derived (6.19), how do we fix the normalization constant c_2 for $\lambda_{27}^{E_6}$? Let us first point out that, under the renormalization group flows (2.2), the period integrals

$$\oint \frac{\partial \lambda_{\mathcal{R}}^G}{\partial z} \tag{7.1}$$

	index	
A_1	$\ell(3) = 4$	$\ell(2) = 1$
A_2	$\ell(8) = 6$	$\ell(3) = \ell(\mathbf{\overline{3}}) = 1$
D_4	$\ell(28) = 12$	$\ell(\mathbf{8_v}) = \ell(\mathbf{8_s}) = \ell(\mathbf{8_c}) = 2$
E_6	$\ell(78) = 24$	$\ell(27) = \ell(\overline{27}) = 6$
E_7	$\ell(133) = 36$	$\ell(56) = 12$
E_8	$\ell(248) = 60$	

Table 3: Index of representations. $\ell(adjoint) = 2h$ and $\ell(1) = 0$.

exhibit the smooth limiting behavior at the generic points on the moduli space. Then we obtain from (5.42) and (6.5) that $c_2 = \frac{\sqrt{2}}{48\pi i}$ for $\lambda_{27}^{E_6}$. Eq.(6.19) is written as

$$\lambda_{27}^{E_6} \to \frac{1}{3} \left(\lambda_{SW}^{\mathbf{8_v}} + \lambda_{SW}^{\mathbf{8_s}} + \lambda_{SW}^{\mathbf{8_c}} \right) + d(*).$$
(7.2)

We also observe that the residues of the poles of $\lambda_{27}^{E_6}$ turn out to be

$$2\pi i \operatorname{Res}_{x=x_a}(\lambda_{27}^{E_6}) = \frac{1}{k_{27}} \frac{m_a}{2\sqrt{2}}$$
(7.3)

with $k_{27} = 6$. Notice that the index of 27 (or $\overline{27}$) is equal to 6. The appearance of the index of representations is not peculiar to this case. For example, in (5.49) and (5.52) we see $12 = \ell(56)$ and $36 = \ell(133)$, respectively, where $\ell(\mathcal{R})$ is the index of the representation \mathcal{R} (see Table 3).

Now examining the SW differentials obtained for various instances in section 5 and the renormalization group flows (see also [5, 6, 8]), we find that the residue should be normalized as

$$2\pi i \operatorname{Res}_{x=x_a}(\lambda_{\mathcal{R}}^G) = \frac{1}{k_{\mathcal{R}}} \frac{m_a}{2\sqrt{2}}, \qquad m_a = (\lambda_a, \phi),$$
(7.4)

where $k_{\mathcal{R}} = \ell(\mathcal{R})$, or $k_{\mathcal{R}} = \ell(\mathcal{R})/2$ if the sum of the poles is taken over half of the (nonzero) weights of \mathcal{R} , and we use $\lambda_{SW}^{\mathcal{R}}$ for the D_4 differential. Here the mass parameter ϕ is normalized so that we have $m_a^v = (\lambda_a, \phi)$ in the D_4 theory along the flows (2.2). This explains why we need to be a little careful to fix a numerical constant upon relating m_a and v_a in section 5. With this normalization of residues, it can be checked that the two-form Ω in (4.1) is invariant under the successive flows (2.2). We also see that

$$\frac{\partial \lambda_{\mathcal{R}}^G}{\partial z} = \kappa_G \frac{dx}{y},\tag{7.5}$$

where κ_G is independent of \mathcal{R} as read off from section 5.

We claim that (7.4) is the correct normalization of the residue. Taking this for granted, consider the renormalization group flow from the G theory to the G' theory. In the Gtheory, let the residues of the SW differential transform in the representation \mathcal{R} of G. If the \mathcal{R} branches to $\oplus_i \mathcal{R}'_i$ under $G \supset G'$, the pole terms of $\lambda^G_{\mathcal{R}}$ reduce to the sum of pole terms each of which transforms in \mathcal{R}'_i of G'. As observed in the flow $E_6 \to D_4$ we expect that the pole terms belonging to non-singlets of G' remain finite in the scaling limit which implements the flow $G \to G'$. If this is assumed to be the case, we obtain

$$\lambda_{\mathcal{R}}^{G} \longrightarrow \sum_{\mathcal{R}_{i}^{\prime} \neq 1} \frac{\ell(\mathcal{R}_{i}^{\prime})}{\ell(\mathcal{R})} \lambda_{\mathcal{R}_{i}^{\prime}}^{G^{\prime}} + d(*)$$
(7.6)

by matching the normalization of residues. This behavior is actually observed in (6.19).

7.1 Irrelevance of representations

For a branching $\mathcal{R} = \bigoplus_i \mathcal{R}'_i$, we recall the identity $\ell(\mathcal{R}) = \sum_i \ell(\mathcal{R}'_i)$. Then (7.6) may imply that the period integrals of $\lambda_{\mathcal{R}'_i}^{G'}$ are independent of \mathcal{R}'_i . This may sound surprising, but we now prove that the SW differentials in any representation yield the identical physics result.

Since $\lambda_{\mathcal{R}}^G$ has the poles with nonzero residues, there is ambiguity in evaluating the periods if we specify the cycles, along which $\lambda_{\mathcal{R}}^G$ is integrated, only in terms of the homology class of the SW curve. Thus we consider the SW curve as the torus with punctures at the location of the poles of $\lambda_{\mathcal{R}}^G$. The homology class of this punctured torus has a basis α , β and γ_a . Here γ_a goes around a pole at $x = x_a$ counterclockwise, and the cycles α and β will be specified later.

Given $\lambda_{\mathcal{R}_i}^G$ in the representation \mathcal{R}_j , we define

$$a_{\mathcal{R}_j}(z,\phi) = \oint_{\alpha} \lambda_{\mathcal{R}_j}^G, \qquad a_{D\mathcal{R}_j}(z,\phi) = \int_{\beta} \lambda_{\mathcal{R}_j}^G$$
(7.7)

and

 $f(z,\phi) = a_{\mathcal{R}_1} - a_{\mathcal{R}_2}, \qquad f_D(z,\phi) = a_{D\mathcal{R}_1} - a_{D\mathcal{R}_2}.$ (7.8)

[¶]This identity holds for the regular embedding since the embedding index is unity. Every embedding in the flows (2.2), (2.3) is regular.

It is an immediate consequence of (7.5) that f(z,m) = f(m) and $f_D(z,m) = f_D(m)$. When we loop around a singularity at $z = z_k$ on the z-plane, λ_R^G remains invariant but the cycles undergo the monodromy

$$\alpha \rightarrow n\alpha + m\beta + \sum_{a} l_a \gamma_a,$$

$$\beta \rightarrow n'\alpha + m'\beta + \sum_{a} l'_a \gamma_a,$$
(7.9)

where the matrix $\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}$ is conjugate to $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and l, l' are some integers which are non-zero when a cycle crosses a pole under a monodromy transformation. At the singularity $z = z_k$, therefore, we have a linear relation among $a_{\mathcal{R}_j}$, $a_{D\mathcal{R}_j}$ and $\operatorname{Res}(\lambda_{\mathcal{R}_j}^G)$. This in turn gives rise to a linear relation for f(m), $f_D(m)$ and the residues. A similar consideration at different singularity, say at $z = z_{k'}$, yields another linear relation. These two relations are linearly independent when two 7-branes at $z = z_k$ and at $z = z_{k'}$ are mutually non-local. Then we find

$$f(m) = \sum_{j=1}^{2} \sum_{a_j} c_{a_j} \operatorname{Res}_{x=x_{a_j}}(\lambda_{\mathcal{R}_j}^G), \qquad f_D(m) = \sum_{j=1}^{2} \sum_{a_j} c'_{a_j} \operatorname{Res}_{x=x_{a_j}}(\lambda_{\mathcal{R}_j}^G), \tag{7.10}$$

where c_{a_j} , c'_{a_j} are some constants. Hence we have shown that f(m) and $f_D(m)$ are linear in m_a . In fact, if f(m) were not linear in m_a , then for every z, we could have taken 1/f(m) = 0 in the codimension one subspace of the space of bare mass parameters. For a generic value of z, however, f(m) may not be divergent, and hence f(m) should be linear in m.

Let us now apply a Weyl transformations $m_a \to \widetilde{m}_a$ under which $\lambda_{R_i}^G$ is left invariant. The SW periods a(z, m) and $a_D(z, m)$, however, may exhibit a non-trivial behavior under the Weyl reflection. This occurs if the Weyl reflection moves a pole of λ_R^G on the *x*-plane across the α and/or β cycles. The SW periods, on the other hand, should be Weyl invariant as gauge invariant expectation values. We thus prescribe that the positions of the cycles α and β are fixed relatively to the poles in such a way that the relative positions of the cycles and the poles do not change under a Weyl transformation. Since it is always possible to take such α and β in the asymptotic region $z \gg m_a$ of the moduli space, we henceforth specify the cycles according to this prescription.^{||} As a consequence of this, we see that

^{||}See [23] for an explicit example in the case of N = 2 SU(2) QCD with $N_f = 2$ massive quarks.

 $f(m) = f(\widetilde{m})$ and $f_D(m) = f_D(\widetilde{m})$. Remember here the fact that there are no Weyl invariants which are linear in m_a , and hence we obtain $f(m) = f_D(m) = 0$. Therefore, we conclude that the SW periods a, a_D are independent of the choice of a representation in constructing $\lambda_{\mathcal{R}}^G$ as long as the cycles α , β are fixed properly as described above.

7.2 Numerical check

One can numerically evaluate the period integrals and check that a and a_D are independent of the representation \mathcal{R} of the residues of $\lambda_{\mathcal{R}}^G$. In the case of A_2 and D_4 , we express $a_{\mathcal{R}}$ and $a_{D\mathcal{R}}$ for $\mathcal{R} = f(\text{undamental})$ and adj(oint) in terms of standard elliptic integrals by taking two cycles α , β as we prescribed above. With the use of Maple, we then obtain, for example, $a_f = -28.99673387 + 16.74790178 i$ and $a_{adj} = -28.99673386 + 16.74790178 i$ in the A_2 theory at z = 10, $m_1 = 1 - 0.2i$ and $m_2 = -0.4 + 0.75i$. The error is indeed extremely small compared to the ratio of z to m_i . Varying the values of m_i we plot in Fig.2 the real and imaginary parts of a_f in the A_2 theory for z = 10 and $m_1 = 2x - 0.2i$, $m_2 = -0.4 + 1.5ix$. Computing the periods at various values of z and m_i , we have observed in both A_2 and D_4 theories that

$$\frac{a_f - a_{adj}}{a_f} < 10^{-8}, \qquad \frac{a_{Df} - a_{Dadj}}{a_{Df}} < 10^{-8}, \tag{7.11}$$

where the differentials (5.26), (5.31) have been utilized in the D_4 theory. Since the values of a and a_D change substantially as shown in Fig.2 upon varying parameters of the moduli space, we believe that the RHS of (7.11) are numerical errors and really mean zero.

To summarize, the SW periods will jump by a constant given by the residue of $\lambda_{\mathcal{R}}^G$ if we continuously deform the α , β cycles across a pole. Namely, the dependence of the periods on the cycles cannot be absorbed by the redefinition of the periods among themselves. Furthermore, since the positions of the poles and their residues are solely determined by the representation \mathcal{R} chosen to construct $\lambda_{\mathcal{R}}^G$, how to fix the location of the cycles relatively to the poles is a subtle issue. In spite of these, we have prescribed a way of specifying the cycles, based on which the irrelevance of representations to the SW periods is proved. To fix the BPS central charge, it remains to determine the constant piece of the global abelian charges as mentioned in section 3. This will be possible once we locate

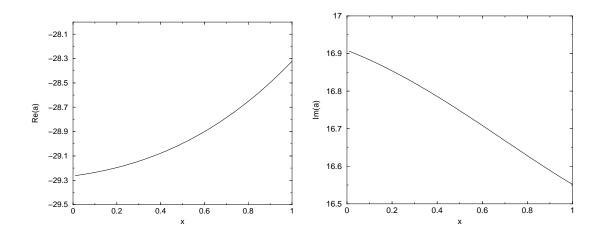


Figure 2: The period $a = \int_{\alpha} \lambda_f^{A_2}$ in the fundamental of SU(3) is plotted for z = 10, $m_1 = 2x - 0.2i$, $m_2 = -0.4 + 1.5ix$.

the cycles along which $\lambda_{\mathcal{R}}^{G}$ is integrated. It is also important to study the monodromy properties explicitly toward a full account of the BPS spectrum.

8 Flows to N = 2 SU(2) QCD with $N_f \leq 3$

It is well known that in N = 2 SU(2) QCD with N_f fundamental quarks, the global symmetry is enhanced to $SO(2N_f)$ when the quarks are massless [6]. We now analyze how the SW differentials in the $N_f = 4$ theory reduce to those in $N_f < 4$ theories.

8.1 Vector representation

Let us first take the $N_f = 4$ SW differential $\lambda_{SW}^{\mathbf{8_v}}$ in the vector representation of SO(8). Upon taking the scaling limit $\alpha\beta \to 1$, $\alpha + \beta \to -2$ and $m_4 \to \infty$ with $(\alpha - \beta)m_4 = -\Lambda_3/4$ fixed [6], we obtain the $N_f = 3$ theory. In this limit the D_4 curve (2.10), which can be rewritten as

$$Y^{2} = \alpha\beta X \left(Z - \frac{(\alpha - \beta)\alpha^{2}\beta^{2}\prod_{b=1}^{4}m_{b} + (\alpha + \beta)X^{2}}{2\alpha\beta X} \right)^{2} - \frac{(\alpha - \beta)^{2}}{4\alpha\beta X} \prod_{a=1}^{4} (X + \alpha\beta m_{a}^{2}), \quad (8.1)$$

becomes

$$Y^{2} = X\left(X + Z + \frac{m_{1}m_{2}m_{3}\Lambda_{3}}{8X}\right)^{2} - \frac{\Lambda^{2}}{64X}\prod_{a=1}^{3}(X + m_{a}^{2}).$$
(8.2)

This is shown to be equivalent to the usual $N_f = 3$ curve by setting X = X' - Z

$$Y^{2} = X^{\prime 2}(X^{\prime}-Z) - \frac{\Lambda_{3}^{2}}{64}(X^{\prime}-Z)^{2} - \frac{\Lambda_{3}^{2}}{64}(m_{1}^{2}+m_{2}^{2}+m_{3}^{2})(X^{\prime}-Z) + \frac{\Lambda_{3}}{4}m_{1}m_{2}m_{3}X^{\prime} - \frac{\Lambda_{3}^{2}}{64}(m_{1}^{2}m_{2}^{2}+m_{2}^{2}m_{3}^{2}+m_{1}^{2}m_{3}^{2}).$$
(8.3)

Turning to the differential, one can verify that $\lambda_{SW}^{\mathbf{8_v}}$ in (5.35) with $X_a^v = -\alpha\beta m_a^2$ and

$$Y_a = -i\alpha\beta m_a \left(Z + \frac{(\alpha - \beta)\alpha^2\beta^2 \prod_{a=1}^4 m_a + (\alpha + \beta)\alpha^2\beta^2 m_a^4}{2\alpha^2\beta^2 m_a^2} \right),\tag{8.4}$$

yields the $N_f = 3$ SW differential

$$\lambda_{D_3}^{\mathbf{6_v}} = \frac{\sqrt{2}}{8\pi} \left(2Z - X' - (m_1^2 + m_2^2 + m_3^2) \right) \frac{dX'}{Y} - \frac{\sqrt{2}}{8\pi} \sum_{a=1}^3 \frac{m_a^2 Z - \frac{1}{8}m_1 m_2 m_3 \Lambda_3 - m_a^4}{X' - Z + m_a^2} \frac{dX'}{Y}$$

$$\tag{8.5}$$

which corresponds to the vector representation of SO(6).

Taking here the limit $m_3 \to \infty$ with $\Lambda_3 m_3 = \Lambda_2^2$ fixed, we have the $N_f = 2$ theory with the curve

$$Y^{2} = X^{\prime 2}(X^{\prime} - Z) - \frac{\Lambda_{2}^{2}}{64}(X^{\prime} - Z) + \frac{\Lambda_{2}^{2}}{4}m_{1}m_{2}X^{\prime} - \frac{\Lambda_{2}^{4}}{64}(m_{1}^{2} + m_{2}^{2}).$$
(8.6)

The SW differential obtained from (8.5) turns out to be

$$\lambda_{D_2}^{\mathbf{4_v}} = \frac{\sqrt{2}}{8\pi} \left(2Z - 2X' - (m_1^2 + m_2^2) \right) \frac{dX'}{Y} - \frac{\sqrt{2}}{8\pi} \sum_{a=1}^2 \frac{m_a^2 Z - \frac{1}{8}m_1 m_2 \Lambda_2^2 - m_a^4}{X' - Z + m_a^2} \frac{dX'}{Y}.$$
 (8.7)

Next, in the limit $m_2 \to \infty$ with $\Lambda_2^2 m_2 = \Lambda_1^3$ fixed, we obtain the $N_f = 1$ curve from (8.6)

$$Y^{2} = X^{\prime 2}(X^{\prime} - Z) + \frac{\Lambda_{1}^{3}}{4}m_{1}X^{\prime} - \frac{\Lambda_{1}^{6}}{64}$$
(8.8)

and the differential

$$\lambda_{D_1}^{\mathbf{2_v}} = \frac{\sqrt{2}}{8\pi} \left(2Z - 3X' - m_1^2 \right) \frac{dX'}{Y} - \frac{\sqrt{2}}{8\pi} \frac{m_1^2 Z - \frac{1}{8}m_1 \Lambda_1^3 - m_1^4}{X' - Z + m_1^2} \frac{dX'}{Y}.$$
(8.9)

Finally, letting $m_1 \to \infty$ with $\Lambda_1^3 m_1 = \Lambda_0^4$ fixed, we arrive at the $N_f = 0$ theory with the curve

$$Y^{2} = X^{\prime 2}(X^{\prime} - Z) + \frac{\Lambda_{0}^{4}}{4}X^{\prime}$$
(8.10)

and the standard form of the differential

$$\lambda_{YM} = \frac{\sqrt{2}}{8\pi} \left(2Z - 4X'\right) \frac{dX'}{Y}.$$
(8.11)

Thus, under these renormalization group flows, we obtain the SW differentials in the vector representation of $SO(2N_f)$.

We see from the above that the residues of $\lambda_{D_n}^{2n_v}$ read

$$2\pi i \operatorname{Res}\left(\lambda_{D_n}^{2\mathbf{n}_{\mathbf{v}}}\right) = \frac{m_a}{2\sqrt{2}} \tag{8.12}$$

for $n \leq 4$, which agrees with (7.4) because $k_{2n_v} = 1$, but differs from (17.1) of [6].** The present result is the correct one since λ_{YM} derived through the successive flows from D_4 coincides with that obtained in [33]. In order for this to hold, it is important that $\lambda_{D_n}^{2n_v}$ obeys

$$\frac{\partial \lambda_{D_n}^{2\mathbf{n}_{\mathbf{v}}}}{\partial Z} = \frac{\sqrt{2}}{8\pi} \frac{dX'}{Y}.$$
(8.13)

Furthermore it is clear that the massless limit of $\lambda_{D_n}^{2\mathbf{n}_{\mathbf{v}}}$ is in agreement with the ones obtained in [33].

8.2 Spinor representation

One may notice that the differentials $\lambda_{D_n}^{2\mathbf{n}_v}$ do not look like those obtained in [6, 34, 23]. Our next task is to show that they are indeed derived from the $N_f = 4$ SW differentials in spinors of SO(8) and their residues transform in the spinor representation of $SO(2N_f)$ with $N_f \leq 3$.

First of all we note that the weights of $\mathbf{8}_{s}$ of SO(8) are given by

$$m_1^s = \frac{1}{2}(m_1 + m_2 + m_3 + m_4),$$

$$m_2^s = \frac{1}{2}(m_1 + m_2 - m_3 - m_4),$$

$$m_3^s = \frac{1}{2}(m_1 - m_2 + m_3 - m_4),$$

$$m_4^s = \frac{1}{2}(m_1 - m_2 - m_3 + m_4)$$
(8.14)

^{**}Our result resolves the puzzle in section 17 of [6] why one has to replace m_a by $m_a/2$ in the final form of the $N_f = 4$ curve derived from the consideration of the residues. Thus it is also required to make this replacement in (17.1) of [6], yielding the correct result as we have obtained here.

and $m_{4+a}^s = -m_a^s$. Note also $u_2 = -\sum_{a=1}^4 m_a^2 = -\sum_{a=1}^4 (m_a^s)^2$. In $\lambda_{SW}^{\mathbf{s}_s}$ (5.27), one has $X_a^s = \alpha Z - \alpha (\alpha - \beta) (\frac{1}{4}u_2 + (m_a^s)^2)$. Under the flow $D_4 \to D_3$ generated by taking the scaling limit $m_4 \to \infty$, $\mathbf{8}_s$ of SO(8) is reduced to $\mathbf{4}_s + \mathbf{4}_c$ of SO(6) where the weights of $\mathbf{4}_s$ are

$$m_1^{\mathbf{4_s}} = (m_1 + m_2 + m_3)/2, \qquad m_2^{\mathbf{4_s}} = -(m_1 + m_2 - m_3)/2,$$

$$m_3^{\mathbf{4_s}} = -(m_1 - m_2 + m_3)/2, \qquad m_4^{\mathbf{4_s}} = (m_1 - m_2 - m_3)/2 \qquad (8.15)$$

and the weights of $\mathbf{4_c}$ are $m_a^{\mathbf{4_c}} = -m_a^{\mathbf{4_s}}$. The positions X_a^s of the poles become

$$X_{a}^{s} = -Z - \frac{1}{4}m_{a}^{4_{s}}\Lambda_{3} - \frac{1}{m_{4}}\left(\frac{1}{32}m_{a}^{4_{s}}\Lambda_{3}^{2} + \frac{1}{8}Z\Lambda_{3} + \frac{1}{16}\Lambda_{3}\left(-\sum_{b=1}^{4}(m_{b}^{4_{s}})^{2} + 4(m_{a}^{4_{s}})^{2}\right)\right) + O\left(\frac{1}{m_{4}^{2}}\right),$$
(8.16)

from which we see that the poles are not sent to infinity. On the other hand, the residue is evidently divergent. This gives rise to a divergent piece in the SW periods in the scaling limit $m_4 \to \infty$. We note that this is a necessary divergence to make certain BPS states decouple. To avoid this divergent behavior, though harmless, let us alternatively take the differential $\frac{1}{2}(\lambda_{SW}^{\mathbf{s}_s} + \lambda_{SW}^{\mathbf{s}_c})$. For this combination, we can evaluate the limit as performed in the flow from E_6 to D_4 . The result is

$$\frac{1}{2}(\lambda_{SW}^{\mathbf{8}_{\mathbf{s}}} + \lambda_{SW}^{\mathbf{8}_{\mathbf{c}}}) \to \lambda_{D_3}^{\mathbf{4}_{\mathbf{s}}} + dF(X', Z, m_a^{\mathbf{4}_{\mathbf{s}}}), \tag{8.17}$$

where

$$\lambda_{D_3}^{\mathbf{4_s}} = \frac{\sqrt{2}}{8\pi} \left(2Z - X'\right) \frac{dX'}{Y} - \frac{\sqrt{2}}{8\pi} \sum_{a=1}^{4} \frac{m_a^{\mathbf{4_s}} \frac{\Lambda_3}{32} (4Z - 2\sum_{b=1}^{4} (m_b^{\mathbf{4_s}})^2 + 8(m_a^{\mathbf{4_s}})^2 + \Lambda_3 m_a^{\mathbf{4_s}})}{X' + \frac{1}{4} m_a^{\mathbf{4_s}} \Lambda_3} \frac{dX'}{Y}$$
(8.18)

and

$$F = \frac{\sqrt{2}}{256\pi} \frac{1}{Y} \left(64X'^2 - 64(Z + \Lambda_3^2)X' + 16m_1m_2m_3\Lambda_3 + 2Z\Lambda_3^2 + 64\sum_{a=1}^4 \frac{[Y]_{X'=-\frac{1}{4}m_a^{4s}\Lambda_3}}{X' + \frac{1}{4}m_a^{4s}\Lambda_3} \right).$$
(8.19)

The differential (8.18) for the $N_f = 3$ theory indeed agrees with [34] and has the poles with residues in the form of (7.4) since the index of $\mathbf{4}_{\mathbf{s}}$ of SO(6) is 1.

Next, taking the limit $m_3 \to \infty$ with $\Lambda_3 m_3 = \Lambda_2^2$ fixed to have the $N_f = 2$ theory, it is shown that

$$\lambda_{D_3}^{\mathbf{4_s}} \to \lambda_{D_2}^{\mathbf{2_L}\mathbf{2_R}} + d\left(\sqrt{2}\,\frac{4X'^2 - 4ZX' + m_1m_2\Lambda_2^2}{8\pi Y} + \frac{\sqrt{2}}{2\pi}\sum_{a=1}^2\frac{[Y]_{X'=-\frac{1}{8}\Lambda_2^2}}{X' + \frac{1}{8}\Lambda_2^2}\frac{1}{Y}\right),\tag{8.20}$$

where

$$\lambda_{D_2}^{\mathbf{2_L}\mathbf{2_R}} = \frac{\sqrt{2}}{4\pi} \left(Z - X' \right) \frac{dX'}{Y} - \frac{\sqrt{2}}{4\pi} \left(\frac{m^{\mathbf{2_L}} \frac{\Lambda_2^2}{4} m^{\mathbf{2_L}}}{X' + \frac{1}{8}\Lambda_2^2} - \frac{m^{\mathbf{2_R}} \frac{\Lambda_2^2}{4} m^{\mathbf{2_R}}}{X' - \frac{1}{8}\Lambda_2^2} \right) \frac{dX'}{Y} = \frac{-\sqrt{2}}{4\pi} \frac{Y dX'}{X'^2 - \frac{1}{64}\Lambda_2^4},$$
(8.21)

which is in agreement with the one obtained in [6]. Here $\mathbf{4}_{\mathbf{s}}$ of SO(6) is decomposed into $(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$ of $SU(2) \times SU(2) (= Spin(4))$ and the corresponding highest weights are given by $m^{\mathbf{2}_{\mathbf{R}}} = (m_1 + m_2)/2$ and $m^{\mathbf{2}_{\mathbf{L}}} = (m_1 - m_2)/2$. Thus the residues of $\lambda_{D_2}^{\mathbf{2}_{\mathbf{L}}\mathbf{2}_{\mathbf{R}}}$ read off from (8.21) become $\frac{1}{2\pi i} \frac{m_1 \pm m_2}{2\sqrt{2}}$, which is the well-known result [6].

In the limit $m_2 \to \infty$ with $\Lambda_2^2 m_2 = \Lambda_1^3$ fixed, we find the differential for the $N_f = 1$ theory

$$\lambda_{D_2}^{\mathbf{2_s}} \to \lambda_{D_1}^{\mathbf{1_s}} + d\left(\sqrt{2} \,\frac{-2X'^2 + 2ZX' - m_1\Lambda_1^3 - \Lambda_1^6/x}{8\pi Y}\right),\tag{8.22}$$

where

$$\lambda_{D_1}^{\mathbf{1}_{\mathbf{s}}} = \frac{\sqrt{2}}{8\pi} \left(2Z - 3X'\right) \frac{dX'}{Y} - \frac{\sqrt{2}}{8\pi} \frac{m_1 \Lambda_1^3}{4X'} \frac{dX'}{Y}$$
(8.23)

which again agrees with [34, 23].

Finally, we let $m_1 \to \infty$ with $\Lambda_1^3 m_1 = \Lambda_0^4$ fixed to obtain the $N_f = 0$ theory. In this limit we see that the pole at X' = 0 in (8.23) turns out to be a double pole. Then, using the $N_f = 0$ curve $\frac{1}{X'} = \frac{1}{Y^2} (X'^2 - ZX' + \frac{1}{4}\Lambda_0^4)$, we arrive at

$$\lambda_{D_1}^{\mathbf{1}_{\mathbf{s}}} \to \lambda_{YM} - \frac{\sqrt{2}}{8\pi} d\left(\frac{-4X'^2 + 4ZX' - \Lambda_0^4}{2Y}\right). \tag{8.24}$$

In this section, we have shown that the SW differentials in the $N_f \leq 3$ theories can be built from the vector as well as spinor representations of $SO(2N_f)$. According to section 7 they describe the same physics in the Coulomb branch of N = 2 SU(2) QCD with massive quarks. The SW differentials for $N_f \leq 3$ in general take the form

$$\lambda_{D_{N_f}}^{\mathcal{R}} = \frac{\sqrt{2}}{8\pi} \left(2Z - (4 - N_f)X' \right) \frac{dX'}{Y} + \text{(pole terms)}.$$
(8.25)

Note here that X'dX'/Y has double poles at infinity whose existence is characteristic of the asymptotic freedom. It is interesting that simple poles of $\lambda_{D_{N_f}}^{\mathcal{R}}$ due to a massive quark become congruent to the double poles at infinity in the scaling limit $N_f \to N_f - 1$.

9 Conclusions

In the framework of the F-theory compactification, we have written down the elliptic curves for describing the N = 2 theories with ADE global symmetries on a D3-brane in the Type IIB 7-brane background. The SW differentials λ have then been constructed for the fundamental and adjoint representations of the ADE groups. It is shown that the physics results are independent of the representation of λ . It is interesting to compare the present result with what has been known in four-dimensional N = 2 Yang-Mills theory with ADE gauge symmetries. For N = 2 ADE Yang-Mills theory the SW curves are given by the spectral curves whose form depends explicitly on the representations \mathcal{R} of ADE. However, the physics of the Coulomb branch is equally described irrespective of \mathcal{R} . In [35] this is shown in terms of the universality of the special Prym variety known in the theory of spectral curves [36]. This is seen more explicitly by analyzing the Picard-Fuchs equations for the SW periods [28]. Therefore, the universality we found here is considered as the global symmetry version of the universality in N = 2 Yang-Mills theory with local ADE gauge symmetries.

It is clear in the framework of Type II string theory that the ADE global symmetries on a D3-brane and the ADE gauge symmetries of four-dimensional Yang-Mills theory have the common origin in the ADE singularities appearing in the degeneration of a K3 surface. In fact, if we replace the top Casimir w_h by $w_h + \rho + \Lambda^{2h} / \rho$ in (2.4)-(2.9), our ADEcurves are recognized as the equations for the ADE ALE space fibered over \mathbf{P}^1 . Here ρ is a complex coordinate of the base \mathbf{P}^1 . This reflects the compactification of Type II string theory on a K3 fibered Calabi-Yau threefold. From this point of view, our calculation for the fundamental of E_6 in section 5.3 is indeed equivalent to that in [37] to obtain the SW curve for the $N = 2 E_6$ Yang-Mills theory from the fibration of the E_6 ALE space. Hence our computations in section 5 can be viewed as the determination of the SW curves in the fundamental and adjoint representations for N = 2 Yang-Mills theory with ADE gauge symmetries.

The global sections of an elliptic fibration in higher representations than the adjoint may be found by constructing the meromorphic sections.^{††} The lattice structure hidden in our explicit computations will be related to the lattice which arises in the Mordell-Weil group. It will be interesting to formulate our present results in more precise mathematical terms in view of the relation between the Mordell-Weil lattice and the *ADE* singularity theory.

Finally, it is very interesting to analyze the BPS spectrum of the E_n theories using our results. One application is to construct the junction lattice explicitly to describe the BPS states. This can be done at least numerically as has been performed in N = 2SU(2) theory [38, 39]. In the E_n theories the BPS spectrum possesses the rich structure in comparison with the $D_{n\leq 4}$ theories [11]. For instance, BPS states in arbitrary higher representations of the E_n groups are shown to exist on the basis of (3.2). Combining the SW description properly formulated in the present paper and the junction approach will be efficient to gain a deeper understanding of still mysterious four-dimensional N = 2superconformal field theories with exceptional symmetry.

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Appendix A

We explain in detail how to evaluate $\partial \lambda_{\mathcal{R}} / \partial z$. For an elliptic curve

$$y^2 = W(x, z) \tag{A.1}$$

with

$$W(x, z) = x^3 + f(z)x + g(z),$$
 (A.2)

the SW differential is assumed to be

$$\lambda_{\mathcal{R}} = (c_1 z + c_3 B(w)) \frac{dx}{y} + c_2 \sum_a \frac{v_a y_a(z)}{x - x_a(z)} \frac{dx}{y}.$$
 (A.3)

Here $(x_a(z), y_a(z))$ are the global sections of an elliptic fibration (A.1) and v_a stand for the generically non-vanishing zeroes of the characteristic polynomial for a representation \mathcal{R} of G

$$P_G^{\mathcal{R}}(v_a) = 0. \tag{A.4}$$

Taking the derivative with respect to z, we obtain

$$\frac{\partial \lambda_{\mathcal{R}}}{\partial z} = c_1 \frac{2(q_z + q_x) - h}{2q_z} \frac{dx}{\sqrt{W}} + \frac{c_1}{2q_z W^{3/2}} \mathcal{L} W dx - \frac{c_3 B(w)}{2W^{3/2}} (x \partial_z f + \partial_z g) dx
+ \frac{c_2}{2W^{3/2}} \sum_a \left(v_a (2 \partial_z y_a W - y_a \partial_z W) - v_a y_a \partial_z x_a \partial_x W \right) \frac{dx}{x - x_a}
- \partial_x \left(\frac{c_1 q_x}{q_z} \frac{x}{\sqrt{W}} + c_2 \sum_a \frac{v_a y_a \partial_z x_a}{(x - x_a)\sqrt{W}} \right) dx,$$
(A.5)

where we have defined the Euler operator

$$\mathcal{L} = \sum_{i} q_i w_{q_i} \frac{\partial}{\partial w_{q_i}} \tag{A.6}$$

in making use of the scaling equation for W

$$q_x x \partial_x W + q_z z \partial_z W + \mathcal{L}W = hW \tag{A.7}$$

to rewrite the $z\partial_z W$ term. Notice that

$$2\partial_z y_a = \partial_z W(x_a(z), z)$$

= $\partial_z x_a [\partial_x W]_{x=x_a(z)} + [\partial_z W]_{x=x_a(z)}.$ (A.8)

Then the term with $dx/(x - x_a)$ in (A.5) is expressed as

$$\frac{c_2}{2W^{3/2}} \sum_a \frac{v_a(B_1 + B_2)}{x - x_a} \frac{dx}{\sqrt{W_a}},\tag{A.9}$$

where $W_a = W(x_a(z), z)$ and

$$B_1 = \partial_z x_a \left(W[\partial_x W]_{x=x_a(z)} - W_a \partial_x W \right),$$

$$B_2 = W[\partial_z W]_{x=x_a(z)} - W_a \partial_z W$$
(A.10)

which vanish for $x = x_a(z)$. In fact, substituting (A.2) one finds

$$B_{1} = (x - x_{a})\partial_{z}x_{a}\left((x - x_{a})^{2}f - 3(x - x_{a})g + 3x_{a}^{2}x^{2} - 6x_{a}g + f^{2}\right),$$

$$B_{2} = (x - x_{a})\left(x_{a}x(x + x_{a})\partial_{z}f + (x^{2} + x_{a}x + x_{a}^{2})\partial_{z}g + f\partial_{z}g - g\partial_{z}f\right).$$
 (A.11)

Now, after some algebra, we get

$$\frac{\partial \lambda_{\mathcal{R}}}{\partial z} = c_1 \frac{2(q_z + q_x) - h}{2q_z} \frac{dx}{\sqrt{W}} + \partial_x (\cdots) dx \\
+ \left(\frac{c_1}{2q_z} (x\mathcal{L}f + \mathcal{L}g) - \frac{c_3 B(w)}{2} (x\partial_z f + \partial_z g) + c_2 (h_2 x^2 + h_1 x + h_0)\right) \frac{dx}{W^{3/2}}, \tag{A.12}$$

where

$$h_{2} = \sum_{a} v_{a} \partial_{z} y_{a},$$

$$h_{1} = \sum_{a} v_{a} \left(x_{a} \partial_{z} y_{a} - \frac{3}{2} \partial_{z} x_{a} y_{a} \right),$$

$$h_{0} = \sum_{a} v_{a} \left((x_{a}^{2} + f) \partial_{z} y_{a} - \frac{1}{2} (\partial_{z} f + 3x_{a} \partial_{z} x_{a}) y_{a} \right).$$
(A.13)

Using

$$\frac{x^2}{W^{3/2}} = \frac{1}{3W^{3/2}} (\partial_x W - f) = -\frac{f}{3W^{3/2}} - \frac{2}{3} \partial_x \left(\frac{1}{\sqrt{W}}\right), \tag{A.14}$$

we arrive at

$$\frac{\partial \lambda_{\mathcal{R}}}{\partial z} = \frac{c_1}{q_z} \frac{dx}{y} + \left(A_1(z)x + A_0(z)\right) \frac{dx}{y^3} + \partial_x F(x, z) dx,\tag{A.15}$$

where we have used $q_x + q_z = q_y - 1$, $2q_y = h$, and

$$A_{1}(z) = \frac{c_{1}}{2q_{z}} \mathcal{L}f - \frac{c_{3}}{2} B(w) \partial_{z}f + c_{2}h_{1},$$

$$A_{0}(z) = \frac{c_{1}}{2q_{z}} \mathcal{L}g - \frac{c_{3}}{2} B(w) \partial_{z}g + c_{2} \left(h_{0} - \frac{1}{3}h_{2}f\right),$$
(A.16)

$$F(x,z) = -\left(\frac{c_1 q_x}{q_z}x + c_2 \sum_a \frac{v_a y_a \partial_z x_a}{x - x_a} + \frac{2c_2 h_2}{3}\right) \frac{1}{y}.$$
 (A.17)

At this stage one has to calculate A_1 , A_0 which depend on the explicit form of the section. After tedious calculations for higher rank groups, the results are expressed in terms of the deformation parameters w_{q_i} . Imposing $A_1 = A_0 = 0$ now brings about the overdetermined system with respect to c_1 , c_2 and c_3 . It is quite impressive that we can nevertheless find the solution so as to determine c_i up to an overall normalization factor.

When we deal with the section in the adjoint representation (5.8) we need one more step of integrating by parts. This step produces an extra contribution to the term proportional to dx/y as observed in the explicit computations in the text.

Appendix B

In this appendix, we present the explicit form of characteristic polynomials $P_G^{\mathcal{R}}(t)$ for D_4 , E_6 , E_7 and E_8 from which one can read off the relation between the Casimir invariants and the deformation parameters.

First of all, the characteristic polynomial for 28 (adjoint) of D_4 reads

$$P_{D_4}^{\mathbf{28}}(t) = t^4 \Big(t^{24} - 18w_2 t^{22} + 135w_2^2 t^{20} + (12\tilde{w}_4 w_2 - 24w_6 - 552w_2^3) t^{18} \\ + (1359w_2^4 - 10\tilde{w}_4^2 - 114w_2^2\tilde{w}_4 + 30w_4^2 + 198w_2w_6) t^{16} + \cdots \Big).$$
(B.1)

Next, we give the characteristic polynomial for 27 of E_6 :

$$P_{E_6}^{27}(t) = t^{27} + 12w_2t^{25} + 60w_2^2t^{23} - 48w_5t^{22} + (168w_2^3 + 96w_6)t^{21} -336w_5w_2t^{20} + (294w_2^4 + 528w_2w_6 + 480w_8)t^{19} -(1008w_2^2w_5 + 1344w_9)t^{18} +(336w_2^5 + 1152w_2^2w_6 + 2304w_2w_8 + 144w_5^2)t^{17} -(1680w_2^3w_5 + 5568w_2w_9 + 768w_5w_6)t^{16} +(252w_2^6 + 1200w_2^3w_6 + 4768w_2^2w_8 + 608w_2w_5^2 -1248w_6^2 + 17280w_{12})t^{15} + \dots , \qquad (B.2)$$

while $P_{E_6}^{\overline{27}}(t)$ is obtained by letting $w_5 \to -w_5$ and $w_9 \to -w_9$.

For **78** (adjoint) of E_6 we have

$$P_{E_6}^{78}(t) = t^6 \Big(t^{72} + 48w_2 t^{70} + 1080w_2^2 t^{68} + (15152w_2^3 - 576w_6) t^{66} \\ + (8640w_8 + 148764w_2^4 - 22752w_6w_2) t^{64} \\ + (297216w_8w_2 - 418176w_2^2w_6 + 1087632w_2^5 + 6048w_5^2) t^{62} \\ + (-1071360w_{12} + 4749888w_2^2w_8 + 55872w_6^2 - 4760352w_2^3w_6 \\ + 187584w_5^2w_2 + 6152776w_2^6) t^{60} + \cdots \Big).$$
(B.3)

The characteristic polynomials for ${\bf 56}$ and ${\bf 133}$ (adjoint) of E_7 are given by

$$P_{E_{7}}^{56}(t) = t^{56} - 2^{2} \cdot 36w_{2}t^{54} + 2^{4} \cdot 594w_{2}^{2}t^{52} + 2^{6}(72w_{6} - 6084w_{2}^{3})t^{50} + 2^{8}(-1800w_{2}w_{6} + 60w_{8} + 43875w_{2}^{4})t^{48} + 2^{10}(21600w_{2}^{2}w_{6} - 504w_{10} - 1008w_{8}w_{2} - 238680w_{2}^{5})t^{46} + 2^{12}(-540w_{12} + 1022580w_{2}^{6} + 7008w_{2}^{2}w_{8} + 10344w_{2}w_{10} - 165600w_{6}w_{2}^{3} + 540w_{6}^{2})t^{44} + 2^{14}(910800w_{2}^{4}w_{6} - 3552120w_{2}^{7} + 7944w_{12}w_{2} - 1092w_{8}w_{6} - 100824w_{2}^{2}w_{10} - 20592w_{8}w_{2}^{3} + 3828w_{14} - 11592w_{2}w_{6}^{2})t^{42} + 2^{16}(-49284w_{2}^{2}w_{12} + 630w_{8}^{2} + 620424w_{10}w_{2}^{3} + 22716w_{2}w_{6}w_{8} - 3825360w_{6}w_{2}^{5} - 63468w_{2}w_{14} + 10212345w_{2}^{8} - 3528w_{10}w_{6} - 38808w_{2}^{4}w_{8} + 118692w_{2}^{2}w_{6}^{2})t^{40} + 2^{18}(683760w_{2}^{5}w_{8} - 12656w_{8}^{2}w_{2} - 24667500w_{2}^{9} + 1848w_{6}^{3} - 771120w_{2}^{3}w_{6}^{2} - 29496w_{18} + 489288w_{2}^{2}w_{14} - 2702280w_{2}^{4}w_{10} + 8760w_{12}w_{6} - 224040w_{2}^{2}w_{6}w_{8} + 5024w_{10}w_{8} + 12751200w_{2}^{6}w_{6} + 61824w_{2}w_{6}w_{10} + 145200w_{12}w_{2}^{3})t^{38} + \cdots ,$$

$$P_{E_7}^{133}(t) = t^7 \Big(t^{126} - 2^2 \cdot 108w_2 t^{124} + 2^4 \cdot 5616w_2^2 t^{122} + 2^6 (-144w_6 - 187200w_2^3) t^{120} \\ + 2^8 (14400w_2w_6 + 600w_8 + 4492800w_2^4) t^{118} \\ + 2^{10} (-691200w_2^2w_6 + 1008w_{10} - 54144w_8w_2 - 82667520w_2^5) t^{116} \\ + 2^{12} (16200w_{12} + 1212456960w_2^6 + 2337792w_2^2w_8 - 78144w_2w_{10} \\ + 21196800w_6w_2^3 + 5400w_6^2) t^{114} \\ + 2^{14} (-466329600w_2^4w_6 - 14549483520w_2^7 - 1345728w_{12}w_2 - 59736w_8w_6 \\ + 2809344w_2^2w_{10} - 64272384w_8w_2^3 + 71544w_{14} - 518976w_2w_6^2) t^{112} \Big]$$

$$+2^{16} (7834337280w_6w_2^5 + 145494835200w_2^8 + 53671104w_2^2w_{12} + 4816944w_2w_6w_8 - 61360128w_{10}w_2^3 + 1263144960w_2^4w_8 - 5463792w_2w_{14} + 23770368w_2^2w_6^2 + 27900w_8^2 - 210672w_{10}w_6)t^{110} + 2^{18} (2679792w_{18} + 199042752w_{14}w_2^2 - 331440w_{12}w_6 - 1368980736w_{12}w_2^3 - 339328w_{10}w_8 + 14852352w_{10}w_6w_2 + 886013952w_{10}w_2^4 - 1824128w_8^2w_2 - 184786752w_8w_6w_2^2 - 18885672960w_8w_2^5 + 252624w_6^3 - 690619392w_6^2w_2^3 - 104457830400w_6w_2^6 - 1228623052800w_2^9)t^{108} + \cdots +).$$
(B.5)

Finally, we write the characteristic polynomial

$$P_{E_8}^{\mathbf{248}}(t) = t^8 \sum_{n=0}^{240} c_n t^n$$
(B.6)

for 248 (adjoint) of E_8 . In this case, we show only eight coefficients which are sufficient to determine the relation between the Casimirs and the deformation parameters. They are given by

$$\begin{array}{lll} c_{238} &=& 2^2 \cdot 60w_2, \\ c_{232} &=& 2^8 (478170w_2^4 + 720w_8), \\ c_{228} &=& 2^{12} (47747700w_2^6 + 15120w_{12} + 1030320w_2^2w_8), \\ c_{226} &=& 2^{14} (361791144w_2^7 + 79200w_{14} + 17858880w_2^3w_8 + 753840w_2w_{12}), \\ c_{222} &=& 2^{18} (13257944700w_2^9 + 2620800w_{18} + 293378400w_2^3w_{12} + 5240640w_2w_8^2 \\ &\quad + 2277007200w_2^5w_8 + 96593280w_2^2w_{14}), \\ c_{220} &=& 2^{20} (11040480w_{20} + 65910925080w_2^{10} + 123173712w_2w_{18} + 1545977808w_2^3w_{14} \\ &\quad + 3431681424w_2^4w_{12} + 18595558800w_2^6w_8 + 128513424w_2^2w_8^2 \\ &\quad + 2492208w_8w_{12}), \\ c_{216} &=& 2^{24} (419237280w_{24} + 1153992168420w_2^{12} - 35394408w_{12}^2 + 4551984w_8^3 \\ &\quad + 11556147624w_2^2w_{20} + 42618310896w_2^3w_{18} + 168171466680w_2^5w_{14} \\ &\quad + 234127252800w_2^6w_{12} + 24236204440w_2^4w_8^2 + 2516521104w_2^2w_8w_{12} \\ &\quad + 749135368800w_2^8w_8 + 387688872w_{14}w_8w_2), \\ c_{210} &=& 2^{30} (65945880000w_{30} + 39472177353840w_2^{15} + 5508702912024w_{24}w_2^3 \\ &\quad + 15986969259936w20w_2^5 - 3209804640w_{20}w_8w_2 \\ &\quad + 28604105079744w_{18}w_2^6 + 234901945584w_{18}w_8w_2^2 \\ &\quad - 4971002400w_{18}w_{12} - 18339605640w_{14}^2w_2 + 250521815304w_{14}w_{12}w_2^2 \end{array}$$

$-422863200w_{14}w_8^2 + 1528645019808w_{14}w_8w_2^4$	
$+43713099157440w_{14}w_2^8 - 521644115232w_{12}^2w_2^3$	
$+8050693680w_{12}w_8^2w_2+3139744251456w_{12}w_8w_2^5$	
$+36016821822240w_{12}w_2^9 + 71061462976w_8^3w_2^3$	
$+10597571701120w_8^2w_2^7 + 68920453929600w_8w_2^{11}).$	(B.7)

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