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Comments on the Moduli Dynamics of 1/4 BPS Dyons

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We rederive the nonrelativistic Lagrangian for the low energy dynamics of 1/4 BPS dyons by considering the time dependent fluctuations around classical 1/4 BPS configurations. The relevant fluctuations are the zero modes of the underlying 1/2 BPS monopoles.

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Recently the 1/4 BPS dyonic configurations are constructed and their nature has been exploited in the $\mathcal{N}=4$ supersymmetric Yang-Mills theories [1, 2, 3, 4]. Since the supersymmetric Yang-Mills theories arise as a low energy description of parallel D3 branes in the type IIB string theory [5], the quantum 1/4 BPS states have the string interpretation as multi-pronged string [6]. In the classical field theory, the 1/4 BPS configurations can be viewed as a collection of 1/2 BPS dyons positioned with respect to each other so that a balance of the Coulomb and Higgs forces is achieved. The BPS equations satisfied by the classical 1/4 BPS configurations consist of the 1/2 BPS monopole equation and its gauge zero mode equation. The underlying 1/2 BPS configurations are uniquely determined by the moduli coordinates, which in turn determine the solution of the second BPS equation uniquely [1].

The low energy dynamics of 1/4 BPS monopoles has been explored in Ref. [7] and it is shown that a specific potential is required in addition to the kinetic terms over the moduli space. The basic ideas of the construction were as follows. In the limit where 1/4 BPS configurations are almost 1/2 BPS, it should be possible to rediscover the physics of 1/4 BPS configurations from the zero mode dynamics of 1/2 BPS configurations. Since static forces exist between 1/2 BPS solitons in the case of misaligned vacua [8], the simplest possibility is to add a potential term to the moduli space dynamics. The potential is indeed uniquely determined from the given knowledge of the electric charge and mass of the 1/4 BPS states. Here the result by Tong was particularly useful [9]. The low energy Lagrangian has a BPS bound and its BPS configuration corresponds to the 1/4 BPS field configuration [7]. For a simple case, quantum 1/4 BPS states of the corresponding supersymmetric Lagrangian have been found in Ref. [10].

However, the derivation of the low energy dynamics was in some sense indirect. Even though the presence of the potential is obvious by considering the interaction between point particle dyons, the exact structure of the potential cannot be obtained from the particle point of view. In this note, we rederive the low energy Lagrangian for 1/4 BPS configurations by the field theoretic method. The dynamical variables are the zero modes, or the moduli of the underling 1/2 BPS configurations.

We begin with the $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We choose the compact semisimple group G of the rank r. Among the six Higgs fields, only two Higgs fields a, b play the role in the BPS bound. The bosonic part of the Lagrangian is given by

$$L = \frac{1}{2}\int d^3x \; \mathrm{tr}\left\{ \mathbf{E}^2 - \mathbf{B}^2 + (D_0 a)^2 - (\mathbf{D}a)^2 + (D_0 b)^2 - (\mathbf{D}b)^2 - (-i[a,b])^2 \right\}, \tag{1}$$

where $D_0 = (\partial_0 - iA_0)$, $\mathbf{D} = \nabla - i\mathbf{A}$, and $\mathbf{E} = \partial_0 \mathbf{A} - \mathbf{D}\mathbf{A_0}$. The four vector potential $(A_0, \mathbf{A}) = (A_0^a T^a, \mathbf{A}^a T^a)$ and the group generators T^a are traceless hermitian matrices such that $\mathrm{tr}\, T^a T^b = \delta^{ab}$.

As shown in Ref. [1], there is a BPS bound on the energy functional, which is saturated when configurations satisfy

$$\mathbf{B} = \mathbf{D}b\,, \tag{2}$$
$$\mathbf{E} = \mathbf{D}a\,, \tag{3}$$
$$D_0 b - i[a,b] = 0\,, \tag{4}$$
$$D_0 a = 0\,, \tag{5}$$

together with the Gauss law,

$$\mathbf{D}\cdot\mathbf{E} - i[b, D_0 b] - i[a, D_0 a] = 0\,. \tag{6}$$

Equation (2) is the old BPS equation for 1/2 BPS monopoles and is called the primary BPS equation. Equations (3), (4), (5), and (6) can be put together into a single equation,

$$\mathbf{D}^2 a - [b,[b,a]] = 0, \tag{7}$$

which is called the secondary BPS equation. This equation is the global gauge zero mode equation for the first BPS equation. We can choose the gauge where $A_0 = -a$, in which case the configuration itself becomes static in time.

In the asymptotic region, two Higgs fields take the form

$$b \simeq \mathbf{b}\cdot\mathbf{H} - \frac{\mathbf{g}\cdot\mathbf{H}}{4\pi r}, \tag{8}$$

$$a \simeq \mathbf{a}\cdot\mathbf{H} - \frac{\mathbf{q}\cdot\mathbf{H}}{4\pi r}, \tag{9}$$

where $\mathbf{H}$ is the Cartan subalgebra. We are interested in the case where the expectation value $\mathbf{b}$ breaks the gauge group G maximally to abelian subgroups $U(1)^r$. Then, there exists a unique set of simple roots $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, ..., \boldsymbol{\beta}_r$ such that $\boldsymbol{\beta}_\alpha \cdot \mathbf{b} > 0$ [11]. The magnetic and electric charges are given by

$$\mathbf{g} = 4\pi \sum_{\alpha=1}^{r} n_\alpha \boldsymbol{\beta}_\alpha, \tag{10}$$

$$\mathbf{q} = \sum_{\alpha=1}^{r} q_\alpha \boldsymbol{\beta}_\alpha, \tag{11}$$

where integer $n_\alpha \geq 0$. Any solution to these BPS equations possesses a mass that saturates the BPS bound

$$M = Z = \mathbf{b}\cdot\mathbf{g} + \mathbf{a}\cdot\mathbf{q}, \tag{12}$$

where Z is the larger one out of two central charges in the $\mathcal{N}=4$ supersymmetric theory.

The solutions of the primary BPS equation describe the collection of 1/2 BPS monopoles. For each simple root, there exists a fundamental monopole of four zero modes. The integer n_α denotes the number of the $\boldsymbol{\beta}_\alpha$ fundamental monopoles. We consider the case where all n_α are positive so that the monopoles do not separate into mutually noninteracting subgroups. The moduli space of the 1/2 BPS configuration has the dimension of the number of zero modes, $4\sum_\alpha n_\alpha$. With the moduli space coordinates z^M, the zero modes are a linear combination of moduli coordinate dependence and a local gauge transformation. With a simple pseudo four dimensional vector $A_\mu(\mathbf{x}, z^M) = (\mathbf{A}, b)$ with $\mu = 1,2,3,4$, the zero modes will be

$$\delta_M A_\mu = \frac{\partial A_\mu}{\partial z^M} + D_\mu \epsilon_M, \tag{13}$$

where $D_\mu \epsilon_M = \partial_\mu \epsilon_M - i[A_\mu, \epsilon_M]$ with understanding $\partial_4 = 0$. The zero mode equations for the primary BPS equation are

$$\nabla \times \delta_M \mathbf{A} = \nabla \delta_M b - i[\delta_M \mathbf{A}, b], \tag{14}$$

$$D_\mu \delta_M A_\mu = 0, \tag{15}$$

where the second equation is the background field gauge fixing condition. From the field theory, there is well defined metric on the moduli space [12, 13, 14],

$$g_{MN}(z) = \int d^3x \, \mathrm{tr} \delta_M A_\mu \delta_N A_\mu. \tag{16}$$

The low energy dynamics of 1/2 BPS configurations is given by the nonrelativistic Lagrangian

$$L_{1/2} = \frac{1}{2} g_{MN}(z) \dot{z}^M \dot{z}^N . \tag{17}$$

As there are r unbroken global $U(1)$ symmetries, the corresponding electric charges should be conserved. In another word, $L_{1/2}$ should have r cyclic coordinates corresponding to these gauge transformations. For each $\boldsymbol{\beta}_\alpha$ $U(1)$ symmetry the corresponding cyclic coordinate is denoted by ψ^α with $\alpha = 1, ..., r$. Expanding the asymptotic value $\mathbf{a} = \sum_\alpha a^\alpha \boldsymbol{\lambda}_\alpha$, where $\boldsymbol{\lambda}_\alpha$'s are the fundamental weights such that $\boldsymbol{\lambda}_\alpha \cdot \boldsymbol{\beta}_\beta = \delta_{\alpha\beta}$, we notice $D_\mu a$ is the gauge zero mode

$$D_\mu a = a^\alpha K^M_\alpha \delta_M A_\mu , \tag{18}$$

where

$$K^M_\alpha \frac{\partial}{\partial z_M} = \frac{\partial}{\partial \psi_\alpha} \tag{19}$$

is the Killing vector for the $\boldsymbol{\beta}_\alpha$ $U(1)$ symmetry. If we divide the moduli coordinates z^M to ψ^α and the rest y^i, the Lagrangian (17) can be rewritten as

$$L_{1/2} = \frac{1}{2} h_{ij}(y) \dot{y}^i \dot{y}^j + \frac{1}{2} L_{\alpha\beta}(y) (\dot{\psi}^\alpha + w^\alpha_i(y) \dot{y}^i)(\dot{\psi}^\beta + w^\beta_j(y) \dot{y}^j). \tag{20}$$

Here $h_{ij} = g_{ij}$, $L_{\alpha\beta} = g_{MN} K^M_\alpha K^N_\beta$, and $w^\alpha_i = L^{\alpha\beta} g_{\beta i}$. Notice that all metric components are independent of ψ^α.

Let us now explore the low energy dynamics of 1/4 BPS configurations. The idea is to calculate the field theoretic Lagrangian for a suitable initial condition in the field theory. It needs to specify the fields and their time derivatives or their momenta. Clearly we require the initial condition to be given by a 1/4 BPS configuration when there is no real time evolution. As the momentum variables $\mathbf{E}$ and $D_0 b$ are nonzero for 1/4 BPS configurations, nontrivial time evolution will ensue only if we add additional field momenta or time derivatives to the 1/4 BPS configuration.

The moduli space dynamics of 1/2 BPS configurations is correct when the kinetic energy is much smaller than the rest mass. This means the order of the velocities, $v \sim \dot{z}^M$ is much smaller than 1. For 1/4 BPS configurations, there is a natural scale $\eta \sim |\mathbf{a}|/|\mathbf{b}|$. We will see that the limit $\eta << 1$ is the suitable region for the low energy dynamics.

Thus, let us put the initial condition to be $\mathbf{A}(\mathbf{x}, y^i), b(\mathbf{x}, y^i), a(\mathbf{x}, y^i)$ and the momentum variables, $\mathbf{D}a + \dot{z}^M \delta_M \mathbf{A}$, $i[a, b] + \dot{z}^M \delta_M b$, and $\dot{z}^M \delta_M a$. Here we have replaced the zeroth order momentum variables with the field variables by using the 1/4 BPS equations. We also choose the gauge $A_0 = -a$. $\delta_M a$ cannot be defined by the zero mode equation of the secondary BPS equation. Otherwise the asymptotic form (9) implies nonzero contribution from $\partial_0 \mathbf{q}(z)$ to $\partial_0 a$. The 1/4 BPS condition involves the field momenta and we cannot include the additional field momenta at a given point of the moduli space, maintaining the 1/4 BPS equations. Rather we choose $\dot{z}^M \delta_M a$ to be an unspecified quantity of order ηv, whose exact nature, as we will see soon, is irrelevant for the low energy dynamics. As $\delta_M \mathbf{A}$ and $\delta_M b$ satisfy the background gauge, the Gauss law is satisfied for the initial condition to order v. There is a correction of order $\eta^2 v$ due to the a field, but it is negligible to the order we are working on.

Let us now calculate the Lagrangian (1) for this initial condition. It becomes

$$L = -\mathbf{b} \cdot \mathbf{g} + \frac{1}{2} \int d^3x \, \mathrm{tr} \left\{ (\dot{z}^M \delta_M A_\mu)^2 \right\} + \int d^3x \, \mathrm{tr} \left\{ \dot{z}^M \delta_M \mathbf{A} \cdot \mathbf{D}a + \dot{z}^M \delta_M b \;\, i[a, b] \right\} , \tag{21}$$

where the first order in velocity terms appear as there were nonzero field momenta for the 1/4 BPS configurations. Here again we used 1/4 BPS equations to replace the momenta with the fields. This Lagrangian is of order v^2 or ηv. We have neglected the terms of order $v^2\eta^2$, which comes from the kinetic energy of a field. The terms linear in $\dot{z}^M$ can be rewritten as a boundary contribution by using the background gauge condition,

$$\dot{z}^M \int d^3x \,\mathrm{tr}\,\{\delta_M \mathbf{A}\cdot \mathbf{D}a + \delta_M b \;\; i[a,b]\} = \dot{z}^M \int d^3x \,\nabla\cdot \mathrm{tr}\,(a\delta_M\mathbf{A}). \tag{22}$$

Noticing that $\mathbf{D}a$ and $i[a,b]$ are a global gauge zero mode, $a^\alpha K^M_\alpha \delta_M A_\mu$, we can rewrite the non-relativistic Lagrangian as

$$L_1 = \frac{1}{2} g_{MN}\dot{z}^M\dot{z}^N + g_{MN}\dot{z}^M a^\alpha K^N_\alpha \tag{23}$$

where $-\mathbf{b}\cdot\mathbf{g}$ is omitted.

Let us introduce new moduli coordinates $\{\zeta^M\} = y^i, \chi^\alpha$ such that

$$\chi^\alpha = \psi^\alpha + a^\alpha t. \tag{24}$$

Since this transformation shifts only cyclic coordinates, the above Lagrangian becomes

$$L_{1/4} = \frac{1}{2} g_{MN}(\zeta)\dot{\zeta}^M\dot{\zeta}^N - \frac{1}{2} g_{MN}(\zeta) a^\alpha K^M_\alpha a^\beta K^N_\beta. \tag{25}$$

The kinetic term of this Lagrangian is the low energy Lagrangian (17) for 1/2 BPS configurations and there is an additional potential. In terms of the y^i, χ^α variables,

$$L_{1/4} = \frac{1}{2} h_{ij}(y)\dot{y}^i\dot{y}^j + \frac{1}{2} L_{\alpha\beta}(y)(\dot{\chi}^\alpha + w^\alpha_i(y)\dot{y}^i)(\dot{\chi}^\beta + w^\beta_j(y)\dot{y}^j) - \frac{1}{2} L_{\alpha\beta}(y) a^\alpha a^\beta. \tag{26}$$

This is exactly the low energy Lagrangian obtained in Ref. [7].

There is a couple of more points to be discussed. The exact 1/4 BPS configuration is static in y^i and ψ^α coordinates, so that $\chi^\alpha = a^\alpha t + \text{constant term}$. The velocity of χ^α coordinates is of order η, which is all right as v and η can be of the same order. When we define the Hamiltonian, the z^M coordinates are not appropriate. Again from the field theory, the energy function we have has the contribution from the momentum variables. In terms of z^M variables, the field theoretic energy functional for our initial condition becomes

$$E = \mathbf{b}\cdot\mathbf{q} + \mathbf{a}\cdot\mathbf{q} + L_1. \tag{27}$$

In terms of the $\{\zeta^M\} = \{y^i, \chi^\alpha\}$ variables, this energy becomes

$$E = \mathbf{b}\cdot\mathbf{q} + E_{1/4}, \tag{28}$$

where $E_{1/4}$ is the energy corresponding to the Lagrangian (25),

$$E_{1/4} = \frac{1}{2} g_{MN}(\zeta)\dot{\zeta}^M\dot{\zeta}^N + \frac{1}{2} g_{MN}(\zeta) a^\alpha K^M_\alpha a^\beta K^N_\beta. \tag{29}$$

Here we have used the Tong formula

$$\mathbf{a}\cdot\mathbf{q} = g_{MN}(\zeta) a^\alpha K^M_\alpha a^\beta K^N_\beta. \tag{30}$$

The energy $E_{1/4}$ has a BPS bound, which is saturated when $\dot{z}^M = 0$ or $\dot{\zeta}^M = a^\alpha K_\alpha^M$, and has the value of the electric mass $\mathbf{a} \cdot \mathbf{q}$. This nonrelativistic BPS configuration describes the field theoretic 1/4 BPS configurations. Thus, a consistent picture of the moduli space has been emerged.

As shown in the Ref. [7], the above Lagrangian can be generalized to supersymmetric case so that it describes 1/4 BPS dyons in the $N = 4$ supersymmetric Yang-Mills theory. There exists naturally quantum BPS bound on this supersymmetric Lagrangian. In Ref. [10], the quantum 1/4 BPS states are found by solving the quantum BPS conditions on the wave functions for the case of the $SU(3)$ group.

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