

Casimir-Polder energy and dilute dielectric ball : nondispersive case

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Abstract

We apply general formalism of quantum field theory and addition theorem for Bessel functions to derive formula for the Casimir-Polder energy of interaction between a polarizable particle and a dilute dielectric ball and Casimir energy of a dilute dielectric ball. The correspondence between the Casimir-Polder formula and Casimir energy of a dilute dielectric ball is shown. Different approaches to the problem of Casimir energy of a dielectric ball are reviewed and analysed by use of addition theorem for Bessel functions.

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1 Introduction

The study of spherical geometry in Casimir effect meets a lot of technical problems, the most difficult one is the problem of divergencies, which appear in many expressions. The special interest to the subject arised after the series of articles by Julian Schwinger where he had proposed a connection between the Casimir effect and sonoluminescence [1, 2]. The main reason for drastically different results in the calculations on the topic is the problem of divergent expressions and their regularization. In the present paper we show how the calculations can be performed to obtain finite results for the case of a dilute dielectric ball. For a discussion of possible divergencies and their regularization in this case see also [3].

We study a dielectric nonmagnetic ball of radius a and permittivity ε , surrounded by a vacuum. The ball is dilute, i.e. all final expressions are obtained under the assumption $\varepsilon - 1 \ll 1$. The permittivity ε for simplicity is a constant.

In the present paper we follow the formalism which was developed by E.Lifshitz et.al.[4] and K.Milton et.al.[5]. We start from a short overview of known facts. Then we derive the Casimir-Polder energy between a dielectric ball and a particle of constant polarizability α , which is placed at the distance r from the centre of the ball. The addition theorem for Bessel functions is used. In the limiting case $r \gg a$ the Casimir-Polder formula for two polarizable particles [6] can be simply obtained from this expression. Also we show how Casimir energy of a dilute dielectric ball can be derived analytically with no divergencies in intermediate calculations by use of proper analytic continuation. The value of Casimir energy of a dilute dielectric ball has been obtained in [7, 8, 9, 10, 11, 12] using various methods, an overview and analysis of different approaches to this problem by use of addition theorem for Bessel functions are given in the Conclusions section.

We put $\hbar = c = 1$. Heaviside-Lorentz units are used.

2 Energy calculation

The change in the ground state energy E of the system under the infinitesimal variation of ε is

$$\delta E = \frac{1}{2} \int d^3 \mathbf{x} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \delta\varepsilon(\mathbf{x}, \omega) \langle \mathbf{E}^2(\mathbf{x}, \mathbf{x}, \omega) \rangle. \quad (1)$$

Here

$$\langle \mathbf{E}^2(\mathbf{r}, \mathbf{r}', \omega) \rangle = \sum_{i=1}^3 \langle \mathbf{E}_i(\mathbf{r}), \mathbf{E}_i(\mathbf{r}') \rangle(\omega) \quad (2)$$

is a Fourier component of electric field propagator trace. The system of equations for this Green's function was discussed extensively in [4, 5]. The solution of this system for spherical geometry with standard boundary conditions classically imposed at $r = a$ can be written as in [13] (δ -functions are omitted since we are interested in the limit $\mathbf{r} \rightarrow \mathbf{r}'$):

$$\begin{aligned} \langle \mathbf{E}_i(\mathbf{r}), \mathbf{E}_j(\mathbf{r}') \rangle(\omega) = & \frac{1}{i} \sum_{l=1}^{\infty} \sum_{m=-l}^l (\omega^2 F_l(r, r') X_{ilm}(\Omega) X_{jlm}^*(\Omega') + \\ & + \frac{1}{\varepsilon} \text{rot}_{\mathbf{r}} \text{rot}_{\mathbf{r}'} G_l(r, r') X_{ilm}(\Omega) X_{jlm}^*(\Omega')). \end{aligned} \quad (3)$$

Here we have used the following notations ($X_{ilm}(\Omega)$ are vector spherical harmonics; $j_l(r), h_l^{(1)}(r)$ are spherical Bessel functions, $\tilde{e}_l(r) = r h_l^{(1)}(r)$, $\tilde{s}_l(r) = r j_l(r)$ are Riccati-Bessel functions [14]):

$$X_{ilm}(\Omega) = \frac{1}{\sqrt{l(l+1)}} (\mathbf{L}Y_{lm}(\Omega))_i \quad (4)$$

$$F_l, G_l = \begin{cases} ikj_l(kr_<)[h_l^{(1)}(kr_>) - A_{F,G}j_l(kr_>)], k = |\omega|\sqrt{\varepsilon}, & r, r' < a, \\ i|\omega|h_l^{(1)}(|\omega|r_>)[j_l(|\omega|r_<) - B_{F,G}h_l^{(1)}(|\omega|r_<)], & r, r' > a, \end{cases} \quad (5)$$

$$A_F = \frac{\tilde{e}_l(|\omega|\sqrt{\varepsilon}a)\tilde{e}'_l(|\omega|a) - \sqrt{\varepsilon}\tilde{e}_l(|\omega|a)\tilde{e}'_l(|\omega|\sqrt{\varepsilon}a)}{\Delta_l}, \quad (6)$$

$$B_F = \frac{\tilde{s}_l(|\omega|\sqrt{\varepsilon}a)\tilde{s}'_l(|\omega|a) - \sqrt{\varepsilon}\tilde{s}_l(|\omega|a)\tilde{s}'_l(|\omega|\sqrt{\varepsilon}a)}{\Delta_l}, \quad (7)$$

$$A_G = \frac{\sqrt{\varepsilon}\tilde{e}_l(|\omega|\sqrt{\varepsilon}a)\tilde{e}'_l(|\omega|a) - \tilde{e}_l(|\omega|a)\tilde{e}'_l(|\omega|\sqrt{\varepsilon}a)}{\tilde{\Delta}_l}, \quad (8)$$

$$B_G = \frac{\sqrt{\varepsilon}\tilde{s}_l(|\omega|\sqrt{\varepsilon}a)\tilde{s}'_l(|\omega|a) - \tilde{s}_l(|\omega|a)\tilde{s}'_l(|\omega|\sqrt{\varepsilon}a)}{\tilde{\Delta}_l}, \quad (9)$$

$$\Delta_l = \tilde{s}_l(|\omega|\sqrt{\varepsilon}a)\tilde{e}'_l(|\omega|a) - \sqrt{\varepsilon}\tilde{s}'_l(|\omega|\sqrt{\varepsilon}a)\tilde{e}_l(|\omega|a), \quad (10)$$

$$\tilde{\Delta}_l = \sqrt{\varepsilon}\tilde{s}_l(|\omega|\sqrt{\varepsilon}a)\tilde{e}'_l(|\omega|a) - \tilde{s}'_l(|\omega|\sqrt{\varepsilon}a)\tilde{e}_l(|\omega|a), \quad (11)$$

differentiation is taken over the whole argument.

When we insert point particle of constant polarizability α into the point \mathbf{r} , $|\mathbf{r}| > a$ from the centre of the ball, the energy change is given by (1) with $\delta\epsilon = 4\pi\alpha\delta^3(\mathbf{r} - \mathbf{x})$. However, we have to subtract contact terms - the volume vacuum contribution, i.e. when we calculate physical quantities in the region $r, r' > a$ we have to subtract the volume vacuum term $i|\omega|h_l^{(1)}(|\omega|r_>)j_l(|\omega|r_<)$ from (5) (for $r, r' < a$ in full analogy the term $ikj_l(kr_<)h_l^{(1)}(kr_>)$ should be subtracted from (5)). Doing so, we have to substitute \tilde{F}_l, \tilde{G}_l instead of F_l, G_l in all the expressions, where

$$\tilde{F}_l, \tilde{G}_l = \begin{cases} -iA_{F,G}k j_l(kr_<)j_l(kr_>), k = |\omega|\sqrt{\epsilon}, & r, r' < a, \\ -iB_{F,G}|\omega|h_l^{(1)}(|\omega|r_>)h_l^{(1)}(|\omega|r_<), & r, r' > a. \end{cases} \quad (12)$$

The Casimir-Polder energy of this configuration is

$$E_1(r, a) = \alpha \int_{-\infty}^{+\infty} d\omega \langle \tilde{\mathbf{E}}^2(r, r, \omega) \rangle = \frac{\alpha}{i} \int_{-\infty}^{+\infty} d\omega \sum_{l=1}^{\infty} \frac{2l+1}{4\pi} \times \\ \times \left(\omega^2 \tilde{F}_l(r, r') + l(l+1) \frac{\tilde{G}_l(r, r')}{rr'} + \frac{1}{rr'} \frac{\partial}{\partial r} r \frac{\partial}{\partial r'} (r' \tilde{G}_l(r, r')) \right) \Big|_{r' \rightarrow r}. \quad (13)$$

We perform a Euclidean rotation then: $\omega \rightarrow i\omega$,

$$\tilde{s}_l(x) \rightarrow s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x), \tilde{e}_l(x) \rightarrow e_l(x) = \sqrt{\frac{2x}{\pi}} K_{l+1/2}(x). \quad (14)$$

Let $x = \omega a$. For $E_1(r, a)$ we obtain

$$E_1(r, a) = \frac{2\alpha}{a} \int_0^{+\infty} dx \sum_{l=1}^{+\infty} \frac{2l+1}{4\pi} \left[\frac{x}{ar^2} e_l^2(xr/a) B_F - \right. \\ \left. - \frac{l(l+1)a}{r^4 x} e_l^2(xr/a) B_G - \frac{x}{r^2 a} (e_l'(xr/a))^2 B_G \right]. \quad (15)$$

This expression can be transformed to a simple formula in the limit $\epsilon - 1 \ll 1$. The functions B_F and B_G are proportional to $(\epsilon - 1)$ in this limit. To proceed, the following addition theorem for Bessel functions [15] is useful:

$$u(p, k, x, \rho) \equiv \sum_{l=0}^{+\infty} (2l+1) s_l(xp) e_l(xk) P_l(\cos \theta) = \frac{x e^{-x\rho} p k}{\rho}, \quad (16)$$

$$\rho = \sqrt{p^2 + k^2 - 2pk \cos \theta}. \quad (17)$$

To our knowledge this formula was first used in Casimir effect calculations in [16], where it was applied to analytic calculation of Casimir energy of perfectly conducting spherical shell and dilute dielectric ball satisfying $\varepsilon\mu = 1$.

In our case it can be applied as follows. The simple identity holds (we assume $k > p > 0$ for definiteness):

$$\begin{aligned} \int dx \sum_{l=0}^{+\infty} (2l+1) f(x) s_l(xp) e_l(xk) s_l(xp) e_l(xk) &= \\ &= \frac{1}{2} \int_{k-p}^{k+p} \frac{d\rho \rho}{pk} \int dx f(x) u(p, k, x, \rho) u(p, k, x, \rho), \end{aligned} \quad (18)$$

where we have used

$$\int_{-1}^1 d(\cos \theta) P_l(\cos \theta) P_m(\cos \theta) = \frac{2}{2l+1} \delta_{lm}, \quad (19)$$

$$\int_{-1}^{+1} d(\cos \theta) \dots = \int_{k-p}^{k+p} \frac{d\rho \rho}{pk} \dots \quad (20)$$

We only need the first order $\sim (\varepsilon - 1)$ in E_1 . We put $k = r/a$, $p = 1$ and use (18) and its obvious generalizations in (15) to calculate E_1 . Finally we get

$$E_1(r, a) = -\frac{23}{15} \alpha \frac{\varepsilon - 1}{4\pi} \frac{a^3(5r^2 + a^2)}{r(r+a)^4(r-a)^4}, r > a. \quad (21)$$

Substitution $\varepsilon - 1 = 4\pi N_{mol} \alpha_{ball}$ in the limit $r \gg a$ yields

$$E_1(r, a) \Big|_{r \gg a} = N_{mol} \left(\frac{4\pi a^3}{3} \right) \frac{-23\alpha\alpha_{ball}}{4\pi r^7} = N_{mol} \left(\frac{4\pi a^3}{3} \right) E_{Cas-Pol}. \quad (22)$$

Thus in this limit the famous Casimir-Polder energy of interaction between two polarizable particles $E_{Cas-Pol}$ [6] can be obtained directly from (22).

Imagine now that there is a bubble of radius a in a dielectric of permittivity ε and we insert dielectric into the point \mathbf{r} , $|\mathbf{r}| < a$ inside the bubble so that the change in dielectric permittivity is equal to $\delta\varepsilon = (\varepsilon - 1)\delta^3(\mathbf{r} - \mathbf{x})$. The energy change is given by (1) again. We omit details of calculations because of obvious similarity with discussion above, note only that formula for energy change $E_2(r, a)$ in the order $(\varepsilon - 1)^2$ can be obtained from formula for $E_1(r, a)$ in the order $(\varepsilon - 1)^2$ written in terms of Bessel functions by a simple interchange $s_l \leftrightarrow e_l$, $\alpha \rightarrow (\varepsilon - 1)/(4\pi)$ and adding an overall minus sign. The result for E_2 can be written as follows :

$$E_2(r, a) = \frac{(\varepsilon - 1)^2}{16\pi^2 a^4} \frac{23}{60} \frac{(d^4 - 10d^2 - 15)}{(1+d)^4(1-d)^4}, d = \frac{r}{a}, r < a. \quad (23)$$

To obtain Casimir energy of a dilute dielectric ball we calculate the energy change δE_{Cas} using formula (1) and symmetry in interchange $\varepsilon \leftrightarrow 1$ in the order $(\varepsilon - 1)^2$, so that δE_{Cas} can be written via previously obtained $E_1(\alpha = (\varepsilon - 1)/(4\pi), r, a)$ and $E_2(r, a)$:

$$\begin{aligned} \delta E_{Cas} &= \frac{1}{2} \left[E_1(r, a)(4\pi a^2) da - E_2(r, a)(4\pi a^2) da \right] \Big|_{r=a} = \\ &= -\frac{(\varepsilon - 1)^2}{8\pi a} \frac{23}{60} \frac{\frac{r}{a} + 4}{r(1 + \frac{r}{a})^4} \Big|_{r=a} da = -\frac{(\varepsilon - 1)^2}{\pi a^2} \frac{23}{1536} da. \end{aligned} \quad (24)$$

(Strictly speaking, we had to substitute $r \leftrightarrow a$ in $E_2(r, a)$ first and only after that perform the subtraction. We would have got two divergent terms then and a finite term that is general for all approaches - see Conclusions section for an overview. However, there is a possibility to derive the finite term directly - to use analytic continuation from $r < a$ to $r > a$ in E_2 and then perform the subtraction as in (24) - a simple way to cancel divergent terms.) If we take into consideration only the finite term (for an extensive discussion of this issue see [8]), the Casimir energy of a dilute dielectric ball and surface force on a unit area are given by

$$E_{Cas} = \frac{23}{1536} \frac{(\varepsilon - 1)^2}{\pi a}, \quad F_{surf} = -\frac{1}{4\pi a^2} \frac{\partial}{\partial a} E_{Cas}. \quad (25)$$

3 Conclusions

We derive formulas for Casimir-Polder type energy between a polarizable particle and a dilute dielectric ball (21,22) and Casimir surface force on a dilute dielectric ball (25) analytically using quantum field theory approach and addition theorem for Bessel functions. Our approach directly shows correspondence of this force and Casimir-Polder potential. Proper analytic continuation has been used to obtain the final result (25).

It is of interest to give here an overview of different approaches which have been used to derive the result (25) and show how formula (18) can effectively be used in other approaches.

First attempts to solve the problem of finding the Casimir energy of a dielectric ball gave a lot of different answers, as it is described in [10]. The mathematical reason for these differences was found in the work [7], where by use of Debye expansion for Bessel functions and ζ -function the correct limits on Casimir surface force in the order $(\varepsilon - 1)^2$ were established. Later in the article [8] the value of Casimir energy of a dilute dielectric ball was calculated numerically with high accuracy, which made it possible to establish

equivalence of Casimir effect and retarded van der Waals energy for the case of nondispersive dielectric ball.

The retarded van der Waals energy for two distant molecules or the Casimir- Polder potential is $E_{Cas-Pol} = -23\alpha_1\alpha_2/4\pi r^7$. The mutual van der Waals energy for molecules inside the compact sphere (ball) was calculated by Milton and Ng [9], and its finite part after regularization in terms of gamma functions was first given by (25) in the work [9].

In the article [8] the Casimir energy of a dilute dielectric ball was studied via the formula which has the following form for nondispersive case:

$$E_C = -\frac{(\varepsilon - 1)^2}{8\pi a} \sum_{l=1}^{\infty} (2l + 1) \int_0^{\infty} dx x \frac{d}{dx} F_l(x), \quad (26)$$

where

$$F_l(x) = -\frac{1}{4} \left(\frac{d}{dx}(e_l s_l) \right)^2 - x^2 (s'^2 - s s'') (e'^2 - e e''), \quad (27)$$

and Riccati-Bessel functions are assumed to depend on argument x . This formula possesses an interesting symmetry : $s_l \leftrightarrow e_l$. From our derivation of formula (25) and discussion before formula (23) it follows that this symmetry is equivalent to the symmetry $\varepsilon \leftrightarrow 1$, so our derivation clarifies this point.

The formula (26) can be studied using formula (18) as well. We put $k = 1, p = 1 - m$, where we are interested in the limit $m \rightarrow 0$. For E_C we find

$$E_C = \lim_{m \rightarrow 0} \frac{(\varepsilon - 1)^2}{\pi a} \left(\frac{23}{1536} + O\left(\frac{1}{m}\right) \right). \quad (28)$$

Recently the formula (26) has been obtained from the mode summation method [12], there divergent terms being analysed and the finite result (25) being derived also by making use of the addition theorem for the Bessel functions.

Another approach based on quantum mechanical perturbation theory was suggested in the work [10]. The Casimir energy was obtained there in the form:

$$E = -(\varepsilon - 1) \frac{3}{2\pi^2} \frac{V}{\lambda^4} + (\varepsilon - 1)^2 \left(-\frac{3}{128\pi^2} \frac{V}{\lambda^4} + \frac{7}{360\pi^3} \frac{S}{\lambda^3} - \frac{1}{20\pi^2} \frac{1}{\lambda} + \frac{23}{1536\pi} \frac{1}{a} \right). \quad (29)$$

Here V is the volume and S the surface area, $1/\lambda$ is an exponential cutoff on wavenumbers. In this approach the contact terms haven't been subtracted,

this is why the term proportional to $\varepsilon - 1$ is present in (29). The cutoff independent term is essentially the same in all approaches.

The approach based on quantum statistical mechanics was developed in [11]. This work led to similar results.

The theory of QED in a dielectric background was studied in [3] by path integral and ζ -function methods. ζ -function can be written as in [3]:

$$\zeta(s) = \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} (2l+1) \int_0^{\infty} d\omega \omega^{-2s} \frac{\partial}{\partial \omega} \ln(\Delta_l \tilde{\Delta}_l). \quad (30)$$

For a dilute ball it is possible to expand the logarithm in powers of $\varepsilon - 1$, here we analyse only the order $(\varepsilon - 1)^2$ of this expansion:

$$\zeta(s) = \frac{(\varepsilon - 1)^2 \sin \pi s}{4\pi} \sum_{l=1}^{\infty} (2l+1) \int_0^{\infty} d\omega \omega^{-2s} \frac{\partial}{\partial \omega} F_l(\omega a), \quad (31)$$

where $F_l(x)$ is defined in (27). We assume $s > 1$ and calculate expression (31) using formula (18) again, where now we put $p = k = 1$. The result is

$$\zeta(s) = -(\varepsilon - 1)^2 \frac{\sin(\pi s)}{\pi s} a^{2s} 2^{4s-7} \frac{(s^2 - 3s + 4)\Gamma(-2s + 2)}{(s - 1)}, \quad (32)$$

with no poles in this expression for $s < 1$. After analytic continuation to $s = -1/2$, for Casimir energy we find

$$E_{Cas} = \frac{\zeta(-1/2)}{2} = \frac{23}{1536} \frac{(\varepsilon - 1)^2}{\pi a}, \quad (33)$$

ζ -function method makes it possible to avoid divergent terms in calculations.

Most difficulties in Casimir effect problems result from divergent structure of different expressions near boundaries, though the divergent behaviour of these expressions is general for all Casimir effect calculations. When dispersion is neglected, divergent behaviour is the same as in (21) or (23) when $r \rightarrow a$ (the simplest example is the Casimir-Polder energy between the polarizable particle and perfectly conducting plate, see e.g. [17] and a discussion there). Divergencies are usually present in Casimir effect calculations as the reminders of short-distance behaviour of Casimir-Polder type and van der Waals type potentials between interacting particles, the example presented in this work serves as a confirmation of this statement. It would be illuminating to overcome the problem of divergencies and understand how to deal with van der Waals and Casimir-Polder type potentials in Casimir related problems by methods of quantum field theory, though probably this is not only the problem of Casimir effect itself but rather quantum field theory in general.

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