

# Elliptic analog of the Toda lattice

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## Abstract

The action-angle variables for  $N$ -particle Hamiltonian system with the Hamiltonian  $H = \sum_{n=0}^{N-1} \ln sh^{-2}(p_n/2) + \ln(\wp(x_n - x_{n-1}) - \wp(x_n + x_{n-1}))$ ,  $x_N = x_0$ , are constructed, and the system is solved in terms of the Riemann  $\theta$ -functions. It is shown that this system describes pole dynamics of the elliptic solutions of 2D Toda lattice corresponding to spectral curves defined by the equation  $w^2 - P_N^{el}(z)w + \Lambda^{2N} = 0$ , where  $P_N^{el}(z)$  is an elliptic function with pole of order  $N$  at the point  $z = 0$ .

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# 1 Introduction

The main goal of this paper is to construct the action-angle variables for a finite dimensional Hamiltonian system of equations

$$\ddot{x}_n = (\dot{x}_n^2 - 1)(V(x_n, x_{n+1}) + V(x_n, x_{n-1})), \quad x_{n+N} = x_n, \quad (1.1)$$

where

$$V(u, v) = \zeta(u - v) + \zeta(u + v) - \zeta(2u) = -\frac{1}{2} \frac{\wp'(u - v) - \wp'(u + v)}{\wp(u - v) - \wp(u + v)}, \quad (1.2)$$

and to identify it as *an elliptic analog of  $N$ -periodic Toda lattice*. Here  $\wp(x) = \wp(x|2\omega, 2\omega')$  and  $\zeta(x) = \zeta(x|2\omega, 2\omega')$  are classical Weierstrass functions.

Recently, finite-dimensional integrable soliton systems have attracted very special interest due to their unexpected relations to the theory of supersymmetric gauge models. The celebrated Seiberg-Witten ansatz ([1, 2]) identifies moduli space of physically non-equivalent vacua of the model with moduli space of a certain family of algebraic curves. In [3, 4] it was shown that the family of curves corresponding to four-dimensional  $N = 2$  supersymmetric  $SU(N_c)$  theory is defined by the equation

$$w^2 - wP_{N_c}(E) + \Lambda^{2N_c} = 0, \quad P_{N_c}(E) = E^{N_c} + \sum_{i=0}^{N_c-1} u_i E^i. \quad (1.3)$$

In [5] it was noted that this family can be identified with the family of spectral curves of  $N_c$ -periodic Toda lattice, and the Seiberg-Witten ansatz was linked with the Whitham perturbation theory of finite-gap solutions of soliton equations proposed in [6, 7]. Integrable systems related to various gauge models coupled with matter hypermultiplets in various representations were considered in [8]-[33], where more complete list of references can be found.

In [32, 33]  $N_c$ -periodic spin chain related to  $XYZ$  model was proposed as soliton counterpart of  $N = 1$  supersymmetric  $SU(N_c)$  theory in six dimensions compactified in two directions, and coupled with  $N_f = 2N_c$  matter hypermultiplets. Spectral curves of  $N_c$ -periodic homogeneous  $XYZ$  spin chain have the form

$$w^2 - wP_{N_c}^{el}(z) + Q_{2N_c}^{el}(z) = 0, \quad (1.4)$$

where  $P_{N_c}^{el}(z)$  and  $Q_{2N_c}^{el}(z)$  are elliptic polynomials, i.e. elliptic functions with poles of order  $N_c$  and  $2N_c$  at the point  $z = 0$ . Note, that (1.4) is an elliptic deformation of the family of curves found in [34] for four-dimensional  $N = 2$  supersymmetric  $SU(N_c)$  model coupled with matter hypermultiplets.

A particular case of (1.4), when  $Q_{2N_c}^{el}(z)$  is a constant,  $Q_{2N_c}^{el}(z) = \Lambda^{2N_c}$  can be seen as an elliptic deformation of (1.3). The corresponding family of curves depends on  $N_c$  parameters which can be chosen as  $\Lambda$  and the coefficients  $u_i$  of the representation of  $P_N^{el}(z)$  in the form:

$$P_N^{el}(z) = \frac{(-1)^N}{(N-1)!} \partial_z^{N-2} \wp(z) + \sum_{i=1}^{N-2} u_i \partial_z^{i-1} \wp(z) + u_0, \quad (1.5)$$

An attempt to find a soliton system corresponding to the family of spectral curves defined by the equation

$$w^2 - wP_N^{el}(z) + \Lambda^{2N} = 0, \quad (1.6)$$

led us to (1.1). After the system was found it turned out that, by itself, it is not new. Up to a change of variables  $q_n = \wp(x_n)$ , it coincides with one of the systems listed in [35], where the classification of all Toda type chains which have Toda type symmetries was obtained. The new results obtained in this work are: isomorphism of (1.1) with a pole system corresponding to elliptic solutions of 2D Toda lattice, the construction of action-angle variables, and explicit solution of the system in terms of the theta-functions.

In [16, 17] it was shown that a wide class of solutions of the Seiberg-Witten ansatz can be described in terms of a special foliation on the moduli space of curves with punctures. That allows one to consider such systems as reductions of 2D soliton equations. Following this approach, let us note, that (1.6) defines an algebraic curve  $\Gamma$  as two-sheeted cover of the elliptic curve  $\Gamma_0$  with periods  $2\omega, 2\omega'$ . Let  $P_{\pm}$  be preimages on  $\Gamma$  of  $z = 0$ . According to the construction of [36], any algebraic curve with two punctures generates a family of algebro-geometric solutions of 2D Toda lattice

$$(\partial_{tt}^2 - \partial_{xx}^2)\varphi_n = 4(e^{\varphi_{n+1}-\varphi_n} - e^{\varphi_n-\varphi_{n-1}}), \quad (1.7)$$

parameterized by points of the Jacobian  $J(\Gamma)$  of the curve.

In the next section we show that algebro-geometric solutions  $\varphi_n(x, t)$  corresponding to  $\Gamma$  defined by (1.6) are periodic in  $n$  up to the shift,  $\varphi_n = \varphi_{n+N} + 2N \ln \Lambda$ , and have the form

$$\varphi_n(x, t) = \alpha_n(t) + \ln \frac{\sigma(x - x_{n+1}(t) + a)\sigma(x + x_{n+1}(t) + a)}{\sigma(x - x_n(t) + a)\sigma(x + x_n(t) + a)}. \quad (1.8)$$

Substitution of (1.8) into (1.7) leads to equations (1.1) for  $x_n(t)$ .

In section 3 we construct a new Lax representation for (1.1) and show that the spectral curve defined by the Lax operator has the form (1.6). We prove also, that if  $x_n(t)$  is a solution of (1.1), then there exist functions  $\alpha_n(t)$  (unique up to the transformation  $\alpha_n(t) \rightarrow \alpha_n(t) + c_1 t + c_2$ ,  $c_i = const$ ), such that the functions  $\varphi_n(x, t)$  of the form (1.8) satisfy (1.7).

The last section is devoted to the Hamiltonian theory of system (1.1). Equations (1.1) are generated by the Hamiltonian

$$H = \sum_{n=0}^{N-1} \ln sh^{-2}(p_n/2) + \ln(\wp(x_n - x_{n-1}) - \wp(x_n + x_{n-1})), \quad (1.9)$$

and the canonical Poisson brackets  $\{p_m, x_n\} = \delta_{nm}$ . We would like to emphasize that though this Hamiltonian structure can be easily checked directly, it was found by the author using the algebro-geometric approach to Hamiltonian theory of the Lax equations proposed in [16, 17], and developed in [39]. The main advantage of this approach is that it allows us to find simultaneously the action-angle variables and a generating differential which defines low-energy effective prepotential.

Note, that from the relation of system (1.1) to 2D Toda lattice it is clear that degeneration of the elliptic curve  $\Gamma_0$  corresponds to a degeneration of this system to the Toda lattice.

It would be very interesting to consider this degeneration explicitly on the level of the Hamiltonian structure. We will consider this problem elsewhere.

## 2 Elliptic solutions of 2D Toda lattice

Algebro-geometric solutions of 2D Toda lattice were constructed in [36]. Let  $\Gamma$  be a smooth genus  $g$  algebraic curve with fixed local coordinates  $z_{\pm}(Q)$  in neighborhoods of two punctures  $P_{\pm} \in \Gamma$ ,  $z_{\pm}(P_{\pm}) = 0$ . Then, for any set of  $g$  points  $\gamma_1, \dots, \gamma_g$  in general position there exists a unique function  $\psi_n(x, t, Q)$  such that:

1<sup>0</sup>.  $\psi_n(x, t, Q)$ , as a function of the variable  $Q \in \Gamma$ , is meromorphic on  $\Gamma$  outside the punctures  $P_{\pm}$  and has at most simple poles at the points  $\gamma_s$  (if all of them are distinct);

2<sup>0</sup>. in the neighborhoods of the punctures the function  $\psi_n$  has the form

$$\psi_n = z_{\pm}^{\mp N} e^{(x \pm t)z^{-1}} \left( \sum_{s=0}^{\infty} \xi_s^{\pm}(x, t) z_{\pm}^s \right), \quad \xi_0^+ = 1. \quad (2.1)$$

Uniqueness of  $\psi_n$  implies that it satisfies the following system of linear equations

$$(\partial_t + \partial_x)\psi_n(x, t, Q) = 2\psi_{n+1}(x, t, Q) + v_n(x, t)\psi_n(x, t, Q), \quad (2.2)$$

$$(\partial_t - \partial_x)\psi_n(x, t, Q) = 2c_n(x, t)\psi_{n-1}(x, t, Q), \quad (2.3)$$

where the coefficients are defined by the leading coefficient  $\xi_0^-$  of expansion (2.1) with the help of the formulae:

$$v_n = (\partial_t + \partial_x)\varphi_n(x, t), \quad c_n = e^{\varphi_n(x, t) - \varphi_{n-1}(x, t)}, \quad \varphi_n(x, t) = \ln \xi_0^-(x, t). \quad (2.4)$$

Compatibility of (2.2) and (2.3) implies that  $\varphi_n(x, t)$  is a solution of 2D Toda lattice (1.7).

The function  $\psi_n(x, t, Q)$  is called the Baker-Akhiezer function and can be explicitly expressed in terms of the Riemann theta-function associated with a matrix of  $b$ -periods of holomorphic differentials on  $\Gamma$ . The corresponding formula for  $\varphi_n$  is as follows.

Let us fix a basis of cycles  $a_i, b_i$ ,  $i = 1, \dots, g$ , on  $\Gamma$  with the canonical matrix of intersections:  $a_i \circ a_j = b_i \circ b_j = 0$ ,  $a_i \circ b_j = \delta_{ij}$ . The basis of normalized holomorphic differentials  $d\Omega_j^h(Q)$ ,  $j = 1, \dots, g$ , is defined by conditions  $\oint_{a_i} d\Omega_j^h = \delta_{ij}$ . The  $b$ -periods of these differentials define the Riemann matrix  $B_{kj} = \oint_{b_j} d\Omega_k^h$ . The basic vectors  $e_k$  of  $C^g$  and the vectors  $B_k$ , which are columns of the matrix  $B$ , generate a lattice  $\mathcal{B}$  in  $C^g$ . The  $g$ -dimensional complex torus

$$J(\Gamma) = C^g / \mathcal{B}, \quad \mathcal{B} = \sum n_k e_k + m_k B_k, \quad n_k, m_k \in \mathbb{Z}, \quad (2.5)$$

is called the Jacobian variety of  $\Gamma$ . A vector with the coordinates

$$A_k(Q) = \int_{P_+}^Q d\Omega_k^h \quad (2.6)$$

defines the Abel map  $A : \Gamma \rightarrow J(\Gamma)$ .

The Riemann matrix has a positive-definite imaginary part. The entire function of  $g$  variables  $z = (z_1, \dots, z_g)$

$$\theta(z) = \theta(z|B) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i(z,m) + \pi i(Bm,m)},$$

is called the Riemann theta-function. It has the following monodromy properties

$$\theta(z + e_k) = \theta(z), \quad \theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}} \theta(z). \quad (2.7)$$

The function  $\theta(A(Q) + Z)$  is a multi-valued function of  $Q$ . But according to (2.7), the zeros of this function are well-defined. For  $Z$  in a general position the equation  $\theta(A(Q) + Z) = 0$  has  $g$  zeros  $\gamma_1, \dots, \gamma_g$ . The vector  $Z$  and the divisor of these zeros are connected by the relation  $Z = -\sum_s A(\gamma_s) + \mathcal{K}$ , where  $\mathcal{K}$  is the vector of Riemann constants.

Let us introduce normalized Abelian differentials  $d\Omega^{(x)}$  and  $d\Omega^{(t)}$  of the second kind. They are holomorphic on  $\Gamma$  except at the punctures  $P_{\pm}$ . In the neighborhoods of  $P_{\pm}$  they have the form

$$d\Omega^{(x)} = d(z_{\pm}^{-1} + O(1)), \quad d\Omega^{(t)} = d(\pm z_{\pm}^{-1} + O(1)).$$

Normalized means that they have zero  $a$ -periods. The vectors of  $b$ -periods of these differentials are denoted by  $2\pi iV$  and  $2\pi iW$ , i.e. the coordinates of the vectors  $V$  and  $W$  are equal to

$$V_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega^{(x)}, \quad W_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega^{(t)}. \quad (2.8)$$

Let  $d\Omega^{(n)}$  be a normalized abelian differential of the third kind with simple poles at the punctures  $P_{\pm}$  with residues  $\mp 1$ . From the Riemann bilinear relations it follows that the vector of its  $b$ -periods satisfies the relation

$$U_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega^{(n)} = A(P_-) - A(P_+). \quad (2.9)$$

If we choose a branch of the Abelian integral  $\Omega^{(n)}$  near  $P_+$  such that  $\Omega^{(n)} = -\ln z_+ + O(z_+)$ , then near  $P_-$  it has the form

$$\Omega^{(n)} = \ln z_- + I_0 + O(z_-).$$

**Theorem 2.1** ([36]) *The Baker-Akhiezer function is equal to*

$$\psi_n(x, t, Q) = \frac{\theta(A(Q) + nU + xV + tW + Z)\theta(Z)}{\theta(nU + xV + tW + Z)\theta(A(Q) + Z)} \exp\left(n\Omega^{(n)} + x\Omega^{(x)} + t\Omega^{(t)}\right). \quad (2.10)$$

The function  $\varphi_n(x, t)$  given by the formula

$$\varphi_n(x, t) = nI_0 + \ln \frac{\theta((n+1)U + xV + tW + Z)}{\theta(nU + xV + tW + Z)} \quad (2.11)$$

is a solution of 2D Toda lattice.

For a generic set of the algebro-geometric data the function  $\varphi_n(x, t)$  given by (2.11) is a quasi-periodic meromorphic function of all the variables  $(n, x, t)$ . In [37] the solutions of 2D Toda lattice which are *elliptic in the discrete variable  $n$*  were considered. It was found that dynamics of its poles coincides with the elliptic Ruijsenaars-Schneider system [38]. In this paper we consider solutions that are elliptic in the variable  $x$  and are periodic in  $n$ .

The condition that  $\varphi_n$  is elliptic in one of the variables is equivalent to the property that the complex linear subspace in  $J(\Gamma)$  spanned by the corresponding directional vector is *compact*, i.e. it is an elliptic curve  $\Gamma_0$ . In the case of  $x$ -variable it means that the vectors  $2\omega_\alpha V, \alpha = 1, 2$ , belong to the lattice  $\mathcal{B}$  defined by (2.5):

$$2\omega_\alpha V = \sum_k n_k^\alpha e_k + m_k^\alpha B_k, \quad n_k^\alpha, m_k^\alpha \in Z. \quad (2.12)$$

Here and below  $\omega_1 = \omega, \omega_2 = \omega'$  are half-periods of the elliptic curve  $\Gamma_0$ .

**Theorem 2.2** *Let  $\Gamma$  be a smooth curve defined by equation (1.6) and let  $P_\pm$  be preimages on  $\Gamma$  of the point  $z = 0 \in \Gamma_0$  with local coordinates in their neighborhoods defined by the local coordinate  $z$  on  $\Gamma_0$ . Then the corresponding algebro-geometric solutions given by formula (2.11) satisfy the relation*

$$\varphi_{n+N}(x, t) = \varphi_n(x, t) + 2N \ln \Lambda, \quad (2.13)$$

and have the form (1.8), i.e.

$$\varphi_n(x, t) = \alpha_n(t) + \ln \frac{\sigma(x - x_{n+1}(t) + a)\sigma(x + x_{n+1}(t) + a)}{\sigma(x - x_n(t) + a)\sigma(x + x_n(t) + a)}.$$

The functions  $x_n(t)$  defined by this representation enjoy equations (1.1). The functions  $\alpha_n$  satisfy the relation

$$4e^{\alpha_{n-1}(t) - \alpha_n(t)} = (1 - \dot{x}_n^2(t)) W(x_n, x_{n+1}) W(x_n, x_{n-1}), \quad (2.14)$$

where

$$W(u, v) = \frac{\sigma(u - v)\sigma(u + v)}{\sigma(2u)}. \quad (2.15)$$

*Proof.* The first statement of the theorem is a direct corollary of the uniqueness of the Baker-Akhiezer function. The projection  $Q = (w, z) \in \Gamma \rightarrow w$  defines  $w = w(Q)$  as a function on the curve. This function is holomorphic on  $\Gamma$  outside the puncture  $P_+$ , where it has the pole of order  $N$ ,  $w = z^{-N}(1 + O(z))$ . At the point  $P_-$  it has zero of order  $N$ ,  $w = \Lambda^{2N} z^N(1 + O(z))$ . Therefore, we have the equality

$$\psi_{n+N}(x, t, Q) = w(Q)\psi_n(x, t, Q), \quad (2.16)$$

because the functions defined by its left- and right-hand sides have the same analytical properties.

Let us consider the functions

$$T_\alpha(z) = e^{2\zeta(z)\omega_\alpha - 2\eta_\alpha z}, \quad \eta_\alpha = \zeta(\omega_\alpha). \quad (2.17)$$

They are double-periodic and holomorphic on  $\Gamma_0$  except at  $z = 0$ . Again, comparison of analytical properties of the left- and right-hand sides proves the equality

$$\psi_n(x + 2\omega_\alpha, t, Q) = T_\alpha(z)\psi_n(x, t, Q), \quad Q = (w, z). \quad (2.18)$$

The function  $e^{\varphi_n}$  is defined as a ratio of the leading coefficients of an expansions of  $\psi_n$  on two sheets of  $\Gamma$ . Therefore, it does not change under the shifts  $x \rightarrow x + 2\omega_\alpha$ , and consequently, it is an elliptic function of the variable  $x$ . From (2.11) it follows that if we denote roots of the equation  $\theta(nU + xV + tW + Z) = 0$  in the fundamental domain of  $\Gamma_0$  by  $x_n^j(t)$ ,  $j = 1, \dots, D$ , then

$$e^{\varphi_n(x,t)} = e^{\alpha_n(t)} \prod_{j=1}^D \frac{\sigma(x - x_{n+1}^j(t))}{\sigma(x - x_n^j(t))}. \quad (2.19)$$

Our next step is to show that  $e^{\varphi_n}$  has only two poles and zeros in  $\Gamma_0$ .

**Lemma 2.1** *The function  $\theta(xV + \xi)$  corresponding to a smooth algebraic curve  $\Gamma$  defined by (1.6), as a function of the variable  $x$  is an elliptic theta-function of weight 2, i.e. it can be represented in the form*

$$\theta(xV + \xi) = r(\xi)\sigma(x - x^1(\xi))\sigma(x - x^2(\xi)). \quad (2.20)$$

*Proof.* Let us find the coefficients of expansion (2.12). The branching points  $z_i^\pm$  of  $\Gamma$  over  $\Gamma_0$  are roots of the equations  $P_N^{el}(z) = \pm\Lambda^N$ . In a generic case, when they are distinct, the curve  $\Gamma$  is smooth. The Riemann-Hurwitz formula  $2g - 2 = \nu$  which connects genus  $g$  of branching cover of an elliptic curve with a number  $\nu$  of branching points, implies that  $\Gamma$  has genus  $N + 1$ . We choose  $a_i, b_i$  cycles on it as follows:

$a_i$ ,  $i = 1, \dots, N - 1$ , are cycles around cuts between branching points  $z_i^+, z_i^-$  and  $a_N$  and  $a_{N+1}$  are two preimages of a-cycle on  $\Gamma_0$ . (We assume that  $a$  and  $b$ -cycles on  $\Gamma_0$  correspond to the periods  $2\omega$  and  $2\omega'$ , respectively.)

From the definition of the differential  $d\Omega^{(x)}$  it follows that

$$d\Omega^{(x)} = d\left(\zeta(z) - \frac{\eta}{\omega}z\right). \quad (2.21)$$

Therefore, the coordinates of the vector  $V$  defined by (2.8) are equal to

$$V_i = 0, \quad i = 1, \dots, N - 1, \quad V_N = V_{N+1} = \frac{1}{\pi i} \left( \eta' - \frac{\eta}{\omega} \omega' \right) = -\frac{1}{2\omega}. \quad (2.22)$$

Comparing the vector of  $b$ -periods of  $d\Omega^{(x)}$  with the vector  $(0, \dots, 0, 2\omega', 2\omega')$  of  $b$ -periods of the differential  $dz$ , considered as a differential on  $\Gamma$ , we get

$$\oint_{b_i} d\Omega^{(x)} = -\frac{\pi i}{2\omega\omega'} \oint_{b_i} dz, \quad i = 1, \dots, N + 1. \quad (2.23)$$

The  $a$ -periods of  $dz$  are equal  $(0, \dots, 0, 2\omega, 2\omega)$ . Therefore,

$$dz = 2\omega(d\Omega_N^h + d\Omega_{N+1}^h),$$

where  $d\Omega_i^h$  are normalized holomorphic differentials. From (2.23) we finally obtain that

$$2\omega'V = -B_N - B_{N+1}, \quad (2.24)$$

where  $B_i$  are the vector of  $b$ -periods of  $d\Omega_i^h$ . The monodromy properties of  $\theta$ -function imply

$$\theta((x + 2\omega)V + Z) = \theta(xV + Z), \quad \theta((x + 2\omega')V + Z) = e^{l(x)}\theta(xV + Z), \quad (2.25)$$

where

$$l(x) = \pi i (2x(V_N + V_{N+1}) + B_{N+1,N+1} + B_{N,N} - B_{N,N+1} - B_{N+1,N} + 2Z_{N+1} + 2Z_N)$$

Using (2.22) we obtain

$$dl(x) = -\frac{2\pi i}{\omega}dx. \quad (2.26)$$

The number  $D$  of zeros of the function  $\theta(xV + \xi)$  in the fundamental domain can be found by integrating of the logarithmic derivative of this function over the boundary of the domain. From (2.25) and (2.26) it follows that

$$D = \frac{1}{2\pi i} \oint_{\partial G_0} d \ln \theta(xV + Z) = 2. \quad (2.27)$$

The equality (2.20) is proved. It implies that the index  $j$  in (2.19) takes values  $j = 1, 2$ . The sums of zeros and poles of an elliptic function are equal to each other (modulo periods of  $\Gamma_0$ ). Hence,  $x_n^j(t)$  can be represented in the form

$$x_n^1 = x_n(t) + a(t), \quad x_n^2(t) = -x_n(t) + a(t). \quad (2.28)$$

In order to complete a proof of (1.8) we need only to show that  $a(t)$  does not depend on  $t$ .

Let us substitute (2.19) into (1.7). A priori the difference of the left- and right-hand sides of (1.7) is an elliptic function of  $x$  with poles of degree 2 at the points  $x_n^j(t)$  and  $x_{n+1}^j(t)$ . Vanishing of the pole of degree 2 at  $x_n^i$  implies that

$$\left(\dot{x}_n^i\right)^2 - 1 = F_n^i(x_n^i), \quad (2.29)$$

where

$$F_n^i(x) = r_n \frac{\prod_j \sigma(x - x_{n+1}^j) \sigma(x - x_{n-1}^j)}{\prod_{j \neq i} \sigma^2(x - x_n^j)}, \quad r_n = -4e^{\alpha_n - \alpha_{n-1}}. \quad (2.30)$$

Vanishing of the pole of degree 1 at  $x_n^i$  implies

$$\begin{aligned} \ddot{x}_n^i &= \partial_x F_n^i(x_n^i) = F_n^i(x_n^i) \left( \partial_x \ln F_n^i(x_n^i) \right) = \\ &= \left( \left(\dot{x}_n^i\right)^2 - 1 \right) \left( \sum_j \zeta(x_n^i - x_{n+1}^j) + \zeta(x_n^i - x_{n-1}^j) - 2 \sum_{j \neq i} \zeta(x_n^i - x_n^j) \right). \end{aligned} \quad (2.31)$$

Substitution of (2.28) in (2.30) shows that

$$F_n^1(x_n^1) = F_n^2(x_n^2) = r_n(t)W(x_n, x_{n+1})W(x_n, x_{n-1}).$$

Hence, we obtain the equality  $(\dot{x}_n^1)^2 = (\dot{x}_n^2)^2$ , which implies that  $\dot{a} = 0$ . Equalities (1.8) and (2.14) are proved. At the same time substitution of (2.28) into (2.31) gives equations (1.1).

### 3 Generating problem and Lax representation

In this section we construct the Lax representation for (1.1) following an approach proposed in [40], and developed in [37, 41, 42] (see their summary in [39]). According to this approach pole dynamics can be obtained simultaneously with its Lax representation from a specific inverse problem for a *linear* operator with elliptic coefficients.

In the most general form this inverse problem is to find linear operators with elliptic coefficients that have *sufficiently enough* double-Bloch solutions. A *meromorphic* function  $f(x)$  is called double-Bloch if it has the following monodromy properties:

$$f(x + 2\omega_\alpha) = B_\alpha f(x), \quad \alpha = 1, 2. \quad (3.1)$$

The complex numbers  $B_\alpha$  are called *Bloch multipliers*. (In other words,  $f$  is a meromorphic section of a vector bundle over the elliptic curve.) It turns out that existence of the double-Bloch solutions is so restrictive that only in exceptional cases such solutions do exist.

The basis in the space of double-Bloch functions can be written in terms of the fundamental function  $\Phi(x, z)$  defined by the formula

$$\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}. \quad (3.2)$$

From the monodromy properties of the Weierstrass functions it follows that  $\Phi$ , considered as a function of  $z$ , is double-periodic:  $\Phi(x, z + 2\omega_\alpha) = \Phi(x, z)$ , though it is not elliptic in the classical sense due to essential singularity at  $z = 0$  for  $x \neq 0$ . As a function of  $x$ , the function  $\Phi(x, z)$  is double-Bloch function, i.e.

$$\Phi(x + 2\omega_\alpha, z) = T_\alpha(z)\Phi(x, z),$$

where  $T_\alpha(z)$  are given by (2.17). In the fundamental domain of the lattice defined by  $2\omega_\alpha$  the function  $\Phi(x, z)$  has a unique pole at the point  $x = 0$ :

$$\Phi(x, z) = x^{-1} + O(x). \quad (3.3)$$

Let  $f(x)$  be double-Bloch function with simple poles  $x_i$  in the fundamental domain and with Bloch multipliers  $B_\alpha$  (such that at least one of them is not equal to 1). Then it can be represented in the form:

$$f(x) = \sum_{i=1}^N c_i \Phi(x - x_i, z) e^{kx}, \quad (3.4)$$

where  $c_i$  is the residue of  $f$  at  $x_i$  and  $(z, k)$  are parameters such that  $B_\alpha = T_\alpha(z) \exp(2\omega_\alpha k)$ .

Now we are in position to present the generating problem for (1.1).

**Theorem 3.1** *The equation*

$$(\partial_t + \partial_x)\Psi_n = 2\Psi_{n+1} + v_n(x, t)\Psi_n \quad (3.5)$$

with an elliptic coefficient of the form

$$v_n(x, t) = \gamma_n(t) + \sum_{i=1}^2 \left[ h_n^i(t) \zeta(x - x_n^i(t)) - h_{n+1}^i(t) \zeta(x - x_{n+1}^i(t)) \right], \quad (3.6)$$

where

$$x_n^1(t) = x_n(t) + a, \quad x_n^2(t) = -x_n(t) + a, \quad a = \text{const}, \quad (3.7)$$

has two linear independent double-Bloch solutions with Bloch multipliers  $T_\alpha(z)$  (for some  $z$ ), i.e. solutions of the form

$$\Psi_n(x, t) = \sum_{i=1}^2 c_n^i \Phi(x - x_n^i(t), z) \quad (3.8)$$

if and only if the functions  $x_n(t)$  satisfy equation (1.1).

If equation (3.5) has two linear independent solutions of the form (3.8) for some  $z$ , then they exist for all values of  $z$ .

*Proof.* Let us substitute (3.8) into (3.5). The both sides of the equation are double-Bloch functions with the same Bloch multipliers and with the pole of order 2 at  $x_n^i$ , and the simple pole at  $x_{n+1}^i$ . They coincide iff the coefficients of their singular parts at these points are equal to each other. The equality of the coefficients at  $(x - x_n^i)^{-2}$  implies

$$h_n^i = \dot{x}_n^i - 1. \quad (3.9)$$

The equality of residues at  $x_{n+1}^i$  is equivalent to the equation

$$c_{n+1}^i = 2^{-1} h_{n+1}^i \sum_j \Phi(x_{n+1}^i - x_n^j) c_n^j. \quad (3.10)$$

The equality of residues at  $x_n^i$  is equivalent to the equation

$$\partial_t c_n^i = M_n^i c_n^i + h_n^i \sum_{j \neq i} \Phi(x_{n+1}^i - x_n^j) c_n^j, \quad (3.11)$$

where

$$M_n^i = \gamma_n - \sum_j h_{n+1}^j \zeta(x_n^i - x_{n+1}^j) + \sum_{j \neq i} h_n^j \zeta(x_n^i - x_n^j). \quad (3.12)$$

Equations (3.10) and (3.11) are linear equations for  $c_n^i$ . Their compatibility is just a system of the equations:

$$\begin{aligned} \partial_t (\ln h_{n+1}^i) \Phi(x_{n+1}^i - x_n^j) + (\dot{x}_{n+1}^i - \dot{x}_n^j) \Phi'(x_{n+1}^i - x_n^j) &= (M_{n+1}^i - M_n^j) \Phi(x_{n+1}^i - x_n^j) + \\ &+ \sum_{k \neq i} \Phi(x_{n+1}^i - x_{n+1}^k) h_{n+1}^k \Phi(x_{n+1}^k - x_n^j) - \sum_{k \neq j} \Phi(x_{n+1}^i - x_n^k) h_n^k \Phi(x_n^k - x_n^j), \end{aligned} \quad (3.13)$$

which can be written in the matrix form:

$$\partial_t L_n = M_{n+1} L_n - L_n M_n, \quad (3.14)$$

where  $L_n$  and  $M_n$  are matrices defined by the right-hand sides of (3.10) and (3.11). Equations (3.14) are necessary and sufficient conditions for the existence of solutions of (3.5) which have the form (3.8). Therefore, the following statement completes a proof of the theorem.

**Lemma 3.1** Let  $L_n = (L_n^{ij}(t, z))$  and  $M_n = (M_n^{ij}(t, z))$  be defined by the formulae

$$L_n^{ij} = 2^{-1} h_{n+1}^i \Phi(x_{n+1}^i - x_n^j, z), \quad M_n^{ii} = M_n^i, \quad M_{ij} = h_n^i \Phi(x_n^i - x_n^j, z), \quad i \neq j, \quad (3.15)$$

where  $x_n^1 = x_n$ ,  $x_n^2 = -x_n$ ,  $h_n^i = \dot{x}_n^i - 1$ , and  $M_n^i$  is given by (3.12) with  $\gamma_n$  such that

$$\gamma_n - \gamma_{n-1} = dt \ln \left( \frac{(\dot{x}_n^2 - 1) \sigma^2(2x_n)}{\sigma(x_n - x_{n+1}) \sigma(x_n + x_{n+1}) \sigma(x_n - x_{n-1}) \sigma(x_n + x_{n-1})} \right). \quad (3.16)$$

Then they satisfy equation (3.14) if and only if the functions  $x_n(t)$  solve equations (1.1).

Note, that (3.16) defines  $\gamma_n(t)$  up to a constant shift  $\gamma_n(t) \rightarrow \gamma_n(t) + g(t)$ , which corresponds to the gauge transformation  $\Psi_n \rightarrow e^g \Psi_n$  of equation (3.5), and which does not effect equations for  $x_n$ .

*Proof.* The right- and left-hand sides of (3.13) are double-periodic functions of  $z$  that are holomorphic except at  $z = 0$ , where they have the form  $O(z^{-2}) \exp((x_{n+1}^i - x_n^j) \zeta(z))$ . Such functions are equal if and only if the corresponding coefficients at  $z^{-2}$  and  $z^{-1}$  are equal. The equality of the coefficients at  $z^{-2}$  gives

$$(\dot{x}_{n+1}^i - \dot{x}_n^j) = h_{n+1}^i - h_n^j + \sum_k (h_n^k - h_{n+1}^k) = h_{n+1}^i - h_n^j, \quad (3.17)$$

which is fulfilled due to (3.9) (the second equality in (3.17) holds because  $v(x, t)$  is an elliptic function of  $x$  and, therefore, a sum of its residues is equal to zero).

The equality of the coefficients at  $z^{-1}$  in the expansion of (3.13) at  $z = 0$  gives

$$\begin{aligned} & \partial_t (\ln h_{n+1}^i) - (\dot{x}_{n+1}^i - \dot{x}_n^j) \zeta(x_{n+1}^i - x_n^j) = M_{n+1}^i - M_n^j + \\ & + \sum_{k \neq i} h_{n+1}^k \left[ \zeta(x_{n+1}^i - x_{n+1}^k) + \zeta(x_{n+1}^k - x_n^j) \right] - \sum_{k \neq j} h_n^k \left[ \zeta(x_{n+1}^i - x_n^k) + \zeta(x_n^k - x_n^j) \right]. \end{aligned} \quad (3.18)$$

The second line in (3.18) is equal up to the sign to the sum of residues at  $x_n^k$ ,  $k \neq j$ , and at  $x_{n+1}^k$ ,  $k \neq i$ , of the elliptic function

$$\tilde{v}_n(x, t) = v_n(x, t) \left[ \zeta(x_{n+1}^i - x) + \zeta(x - x_n^j) \right].$$

Therefore, it equals to the sum of residues of this function at  $x_{n+1}^i$  and  $x_n^j$ . We have

$$\begin{aligned} \text{res}_{x_{n+1}^i} \tilde{v}_n(x, t) + \text{res}_{x_n^j} \tilde{v}_n(x, t) &= (h_n^j - h_{n+1}^i) \zeta(x_{n+1}^i - x_n^j) + M_n^j - \\ &- \gamma_n - \sum_k h_n^k \zeta(x_{n+1}^i - x_n^k) + \sum_{k \neq i} h_{n+1}^k \zeta(x_{n+1}^i - x_{n+1}^k). \end{aligned} \quad (3.19)$$

Substitution of the right-hand side of the last equality into (3.18) implies (after the shift  $n + 1 \rightarrow n$ )

$$\frac{\dot{h}_n^i}{h_n^i} = \gamma_n - \gamma_{n-1} + \sum_{k \neq i} 2h_n^k \zeta(x_n^i - x_n^k) - \sum_k \left[ h_{n+1}^k \zeta(x_n^i - x_{n+1}^k) + h_{n-1}^k \zeta(x_n^i - x_{n-1}^k) \right]. \quad (3.20)$$

From (3.9) it follows, that (3.20) can be rewritten in the form

$$\frac{\ddot{x}_n^i}{\dot{x}_n^i - 1} = \partial_x G_n^i(x_n^i) + \partial_t G_n^i(x_n^i), \quad (3.21)$$

where the function

$$G_n^i(x) = a_n + \ln \left( \frac{\prod_k \sigma(x - x_{n+1}^k) \sigma(x - x_{n-1}^k)}{\prod_{k \neq i} \sigma^2(x - x_n^k)} \right), \quad \partial_t a_n = \gamma_n - \gamma_{n-1}, \quad (3.22)$$

depends on  $t$  through the dependence on  $t$  of  $x_n^i$  and  $a_n$ , only. By chain rule we have

$$d_t \left( G_n^i(x_n^i) \right) = \dot{x}_n^i \partial_x G_n^i(x_n^i) + \partial_t G_n^i(x_n^i), \quad d_t = \frac{d}{dt}. \quad (3.23)$$

Therefore,

$$\frac{\ddot{x}_n^i}{\dot{x}_n^i - 1} = (1 - \dot{x}_n^i) \partial_x G_n^i(x_n^i) + d_t \left( G_n^i(x_n^i) \right), \quad (3.24)$$

From (3.7) it follows that

$$G_n^1(x_n^1) = G_n^2(x_n^2) = G_n(x_n), \quad \partial_x G_n^1(x_n^1) = -\partial_x G_n^2(x_n^2) = \partial_x G_n(x_n),$$

where

$$G_n(x) = a_n + \ln \left( \frac{\sigma(x - x_{n+1}) \sigma(x + x_{n+1}) \sigma(x - x_{n-1}) \sigma(x + x_{n-1})}{\sigma^2(x + x_n)} \right). \quad (3.25)$$

Therefore, equations (3.24) for  $i = 1, 2$ , have the form

$$\frac{\ddot{x}_n}{\dot{x}_n - 1} = d_t (G_n(x_n)) - (\dot{x}_n - 1) \partial_x G_n(x_n), \quad (3.26)$$

$$\frac{\ddot{x}_n}{\dot{x}_n + 1} = d_t (G_n(x_n)) - (\dot{x}_n + 1) \partial_x G_n(x_n). \quad (3.27)$$

Equations (3.26) and (3.27) are equivalent to the equations:

$$\ddot{x}_n = (\dot{x}_n^2 - 1) \partial_x G_n(x_n), \quad d_t (G_n(x_n)) = d_t \ln (\dot{x}_n^2 - 1). \quad (3.28)$$

The first among them, coincides with equations (1.1) for  $x_n$ , and the second one (compare it with (2.14)) is equivalent to the definition of  $\gamma_n$  by (3.16). Lemma and Theorem are proved.

## 4 Direct problem. Spectral curves.

In this section we consider periodic in  $n$  solutions of equations (1.1).

**Lemma 4.1** *Let  $x_n(t) = x_{n+N}(t)$  be a solution of (1.1). Then*

$$I = \prod_{n=1}^N \left( \frac{\sigma(x_n - x_{n+1})\sigma(x_n + x_{n+1})\sigma(x_n - x_{n-1})\sigma(x_n + x_{n-1})}{(\dot{x}_n^2 - 1)\sigma^2(2x_n)} \right), \quad (4.1)$$

*is an integral of motion,  $I = \text{const}$ , and the monodromy matrix*

$$T(t, z) = \prod_{n=0}^{N-1} L_n(t, z) \quad (4.2)$$

*satisfies the Lax equation*

$$\partial_t T = [M_0, T]. \quad (4.3)$$

*Proof.* If  $x_n(t)$  is periodic in  $n$ , then the corresponding matrix functions  $L_n(t, z)$  and  $M_n(t, z)$  defined by (3.15) satisfy the relations

$$L_{n+N} = L_n, \quad M_{n+N} = M_n - d_t(\ln I). \quad (4.4)$$

Therefore, equation (3.14),  $\partial_t L_n = M_{n+1}L_n - L_nM_n$ , implies that

$$\partial_t T = -d_t(\ln I) T + [M_0, T]. \quad (4.5)$$

Note, that if  $\partial_t I = 0$  then (4.5) coincides with (4.3), and therefore, the second statement of the Lemma follows from the first one.

Equation (4.5) implies that the function

$$P(z) = I(t)(\text{tr } T(t, z)) \quad (4.6)$$

*is time-independent.*

Matrix entries of  $L_n$  are double-periodic functions that are holomorphic on  $\Gamma_0$  except at  $z = 0$ . Therefore,  $(\text{tr } T)$  is also double-periodic and holomorphic on  $\Gamma_0$  outside  $z = 0$ . In order to prove that this function is meromorphic on  $\Gamma_0$ , it is enough to note that  $L_n$  has the form

$$L_n(t, z) = g_{n+1} \tilde{L}_n g_n^{-1}, \quad (4.7)$$

where

$$g_n = \begin{pmatrix} e^{x_n \zeta(z)} & 0 \\ 0 & e^{-x_n \zeta(z)} \end{pmatrix}.$$

From (3.2) it follows that in the neighborhood of  $z = 0$

$$\tilde{L}_n = (z)^{-1} \tilde{L}_n^0 + \tilde{L}_n^1 + O(z), \quad (4.8)$$

where

$$\tilde{L}_n^0 = \frac{1}{2} \begin{pmatrix} 1 - \dot{x}_{n+1} & 1 - \dot{x}_{n+1} \\ 1 + \dot{x}_{n+1} & 1 + \dot{x}_{n+1} \end{pmatrix} \quad (4.9)$$

and

$$\tilde{L}_n^1 = \frac{1}{2} \begin{pmatrix} 1 - \dot{x}_{n+1} & 0 \\ 0 & 1 + \dot{x}_{n+1} \end{pmatrix} \begin{pmatrix} -\zeta(x_{n+1} - x_n) & -\zeta(x_{n+1} + x_n) \\ \zeta(x_{n+1} + x_n) & \zeta(x_{n+1} - x_n) \end{pmatrix}. \quad (4.10)$$

Therefore,

$$\mathrm{tr} T = \mathrm{tr} \left( \prod_{n=0}^{N-1} \tilde{L}_n(t, z) \right) = z^{-N}(1 + 0(z)). \quad (4.11)$$

The last equality shows that  $(\mathrm{tr} T)$  is a monic elliptic polynomial  $P_N^{\mathrm{el}}(z)$ . Therefore, at  $z = 0$  we have  $P(z) = I(t)z^{-N}(1 + 0(z))$ . Hence,  $I(t)$  is an integral of (1.1) because  $P(z)$  does not depend on  $z$ . Lemma is proved.

Due to (4.3) the spectral curve  $\Gamma$  defined by the characteristic equation

$$R(w, z) \equiv \det(w - T(t, z)) = w^2 - (\mathrm{tr} T)w + \det T = 0 \quad (4.12)$$

is *time-independent*.

**Lemma 4.2** *The characteristic equation (4.12) has the form (1.3).*

*Proof.* We have already proved that  $(\mathrm{tr} T)$  has the form (1.5). The relation  $\Phi(x, z)\Phi(-x, z) = \wp(z) - \wp(x)$ , which is equivalent to the addition formula for the Weierstrass  $\sigma$ -function, implies

$$\det L_n(t, z) = 2^{-2}(\dot{x}_{n+1}^2 - 1) [\wp(x_{n+1} - x_n) - \wp(x_{n+1} + x_n)]. \quad (4.13)$$

Therefore, though  $L_n(t, z)$  depends on  $z$ , its determinant does not depend on  $z$ . Hence,  $(\det T)$  is also  $z$ -independent. As it does not depend on  $t$ , we identify  $\Lambda^{2N}$  in (1.6) with

$$\Lambda^{2N} = \det T(t, z) = 2^{-2N} \prod_{n=0}^{N-1} (\dot{x}_n^2 - 1) (\wp(x_n - x_{n-1}) - \wp(x_n + x_{n-1})) = 2^{-2N} e^H, \quad (4.14)$$

where  $H$  is the Hamiltonian of system (1.1). Lemma is proved.

For a generic point  $Q$  of the spectral curve  $\Gamma$ , i.e. for a pair  $(w, z)$  that satisfies (4.12) there exists a unique solution  $C_n = (c_n^i(t, Q))$  of the equations

$$C_{n+1}(t, Q) = L_n(t, z)C_n(t, Q), \quad \partial_t C_n(t, Q) = M_n(t, z), \quad (4.15)$$

such that

$$C_{n+N}(t, Q) = wC_n(t, Q), \quad (4.16)$$

and normalized by the condition

$$c_0^1(0, Q)\Phi(-x_0(0), z) + c_0^2(0, Q)\Phi(x_0(0), z) = 1. \quad (4.17)$$

*Remark.* Normalization (4.17) corresponds to a usual normalization  $\Psi_0(0, 0, Q) = 1$  of the solution  $\Psi_n(x, t, Q)$  of (3.5) defined by (3.8).

**Theorem 4.1** *The coordinates  $c_n^i(t, Q)$  of the vector-valued function  $C_n(t, Q)$  are meromorphic functions on  $\Gamma$  except at the preimages  $P_{\pm}$  of  $z = 0$ . Their poles  $\gamma_1, \dots, \gamma_{N+1}$  do not depend on  $n$  and  $t$ . The projections  $z(\gamma_s)$  of these poles on  $\Gamma_0$  satisfy the constraint*

$$\sum_{s=1}^{N+1} z(\gamma_s) = 0. \quad (4.18)$$

In the neighborhoods of  $P_{\pm}$  the coordinates of  $C_n(t, Q)$  have the form

$$c_n^1(t, Q) = z^{\mp n} \chi_{n,\pm}^1(t, z) e^{(\pm t + x_n(t))z^{-1}}, \quad (4.19)$$

$$c_n^2(t, Q) = z^{\mp n} \chi_{n,\pm}^2(t, z) e^{(\pm t - x_n(t))z^{-1}}, \quad (4.20)$$

where  $\chi_{n,\pm}^i(t, z)$  are regular functions of  $z$ :

$$\chi_{n,+}^i(t, z) = z \chi_{n,+}^i(t) + O(z^2), \quad \chi_{n,-}^i(t, z) = \chi_{n,-}^i(t) + z \chi_{n,-}^{i,1}(t) + O(z^2), \quad (4.21)$$

such that the leading coefficients of their expansions have the form:

$$\chi_{n,+}^1(t) = c(t)(1 - \dot{x}_n), \quad \chi_{n,+}^2(t) = c(t)(1 + \dot{x}_n), \quad c(0) = 1, \quad (4.22)$$

$$\chi_{n,-}^1(t) = s_n(t), \quad \chi_{n,-}^2(t) = -s_n(t), \quad (4.23)$$

where functions  $s_n$  satisfy the relation

$$s_{n+1} = 2^{-2}(\dot{x}_{n+1}^2 - 1) [\wp(x_{n+1} - x_n) - \wp(x_{n+1} + x_n)] s_n. \quad (4.24)$$

*Proof.* Vector-columns  $S_n^{(1)}$  and  $S_n^{(2)}$  of the matrix-function

$$S_0^{ij} = \delta_{ij}, \quad S_n(t, z) = \prod_{m=0}^{n-1} L_m(t, z), \quad n > 0, \quad (4.25)$$

are holomorphic functions on  $\Gamma_0$  except at  $z = 0$ . They satisfy the equation  $S_{n+1}^{(i)} = L_n S_n^{(i)}$ . Therefore, the Bloch solution  $C_n$  of (4.15) has the form

$$C_n(t, Q) = h_1(Q) S_n^{(1)}(t, z) + h_2(Q) S_n^{(2)}(t, z), \quad (4.26)$$

where  $h_i(Q)$ ,  $Q = (w, z) \in \Gamma$ , are the coordinates of the normalized eigenvector of the monodromy matrix  $T(z)$ , corresponding to the eigenvalue  $w$ . They are equal to

$$h_1(Q) = \frac{1}{r(Q)} T^{12}(z), \quad h_2(Q) = \frac{1}{r(Q)} (w - T^{11}(z)), \quad (4.27)$$

where  $T^{ij}(z)$  are entries of the monodromy matrix and the normalization constant  $r(Q)$  equals

$$r(Q) = T^{12}(z) \Phi(-x_0(0), z) + (w - T^{11}(z)) \Phi(x_0(0), z). \quad (4.28)$$

The function  $r(Q)$  has the pole of degree  $N + 1$  at  $P_+$  and the pole of degree  $N$  at  $P_-$ . Therefore, it has  $2N + 1$  zeros.

Let us show that  $N$  of these zeros are situated over roots of the equation  $T^{12}(z) = 0$  on one of the sheets of  $\Gamma$ . Indeed, if  $T^{12}(z) = 0$ , then eigenvalues  $w(z)$  of the monodromy matrix are equal to  $T^{11}(z)$  or  $T^{22}(z)$ . Therefore,  $r = 0$  at the points  $Q = (T^{11}(z), z)$ . Equations (4.27) imply that  $C_n$  has no poles at these points. The poles  $\gamma_s$  of  $C_n(t, Q)$  on  $\Gamma$  outside the punctures  $P_{\pm}$  are the other zeros of  $r(Q)$  and do not depend on  $n$  and  $t$ . Let us prove now that they satisfy (4.18).

The function  $r^*(z) = r(Q)r(Q^\sigma)$  with  $\sigma : Q \rightarrow Q^\sigma$  as permutation of sheets of  $\Gamma$ , is a well-defined function on  $\Gamma_0$  with the pole of degree  $2N + 1$  at  $z = 0$ . As it was shown above it is divisible by  $T^{12}(z)$ . Therefore, the ratio  $r^*(z)/T^{12}(z)$  is an elliptic function with the pole of degree  $N + 1$  at  $z = 0$  and zeros at the points  $z(\gamma_s)$ . Divisors of zeros and poles of an elliptic function are equivalent. Therefore, (4.18) is proved.

From (4.7) it follows that the vector-function  $\tilde{C}_n = g_n^{-1}C_n$  is a Bloch solution of the equation  $\tilde{C}_{n+1} = \tilde{L}_n\tilde{C}_n$ . Let us first consider the neighborhood of the puncture  $P_+$ , which corresponds to the branch  $w = z^{-N}(1 + O(z))$  of the eigenvalue of the monodromy matrix.

The vector-function  $X_n(t)$  with the coordinates given by (4.22) satisfies the equation  $X_{n+1} = \tilde{L}_n^0 X_n$ , where  $\tilde{L}_n^0$  is defined in (4.8). That implies that in the neighborhood of  $P_+$  the vector-function  $C_n(t, Q)$  has the form stated in the theorem up to a time-dependent factor  $f_+(t, z)$ . Substitution of (4.19, 4.20) into the equation  $\partial_t C_n = M_n C_n$  shows that  $\partial_t f = O(z)$ . Therefore, the analytical properties of  $C_n$  near  $P_+$  are established.

Now we are going to prove by induction that at  $P_-$  equalities (4.19, 4.20), and (4.23) hold. For  $n = 0$  they are fulfilled by the normalization conditions. Let us prove first, that if (4.19, 4.20), and (4.23) hold for  $n' \leq n$ , then

$$2\kappa_n = (\zeta(x_{n+1} + x_n) - \zeta(x_{n+1} - x_n))s_n + \chi_{n,-}^{1,1} + \chi_{n,-}^{1,2} = 0. \quad (4.29)$$

Indeed, the equation  $C_{n+1} = L_n C_n$  implies that  $\tilde{C}_{n+1}$  at  $P_-$  has the form

$$\tilde{C}_{n+1} = z^n \begin{pmatrix} (1 - \dot{x}_{n+1})\kappa_n \\ (1 + \dot{x}_{n+1})\kappa_n \end{pmatrix} + O(z^{n+1}). \quad (4.30)$$

Hence,

$$\tilde{C}_N = \left( \prod_{m=n+1}^{N-1} L_m \right) \tilde{C}_{n+1} = z^{2n-N-1} \begin{pmatrix} (1 - \dot{x}_0)\kappa_n \\ (1 + \dot{x}_0)\kappa_n \end{pmatrix} + O(z^{2n-N}). \quad (4.31)$$

If  $\kappa_n \neq 0$ , then the last equality contradicts the monodromy property  $\tilde{C}_N = w\tilde{C}_0 = O(z^N)$ . Therefore,  $\kappa_n = 0$ , and then (4.30) shows that  $\tilde{C}_{n+1}$  has zero of order  $n+1$  at  $P_-$ . Therefore, a step of induction for equalities (4.19, 4.20) is proved. The same arguments show that if (4.23) does not hold then the vector  $\tilde{C}_N$  has zero of order  $(2n - N)$  which again contradicts the relation  $\tilde{C}_N = O(z^N)$ .

Equalities (4.19, 4.20) near  $P_-$  are proved, possibly up to a time-dependent factor  $f_-(t, z)$ . Their substitution into the equation  $\partial_t C_n = M_n C_n$  shows that  $\partial_t f_- = O(z)$  and completes a proof of (4.19, 4.20), and (4.23).

Let  $\mathcal{C}_n(z)$  be a matrix formed by the vectors  $C_n(t, Q_i(z))$ , corresponding to two different sheets  $Q_i(z) = (w_i(z), z)$  of  $\Gamma$ . This matrix is defined up to permutation of sheets. From (4.19-4.23) it follows that in the neighborhood of  $z = 0$

$$\mathcal{C}_n(z) = \begin{pmatrix} e^{x_n \zeta(z)} & 0 \\ 0 & e^{-x_n \zeta(z)} \end{pmatrix} \left[ \begin{pmatrix} (1 - \dot{x}_n)c & s_n \\ (1 + \dot{x}_n)c & -s_n \end{pmatrix} + O(z) \right] \begin{pmatrix} z^{-n+1} e^{t\zeta(z)} & 0 \\ 0 & z^n e^{-t\zeta(z)} \end{pmatrix} \quad (4.32)$$

Therefore,

$$\det \mathcal{C}_n = -2cs_n z + O(z^2), \quad (4.33)$$

and from the definition of  $\mathcal{C}_n$  we have  $\mathcal{C}_{n+1} = L_n \mathcal{C}_n$ . Hence,

$$s_{n+1} = s_n \det L_n, \quad (4.34)$$

which coincides with (4.24). Theorem is proved.

The correspondence which assigns to each solution  $x_n(t) = x_{n+N}(t)$  of (1.1) a set of algebro-geometric data  $\{\Gamma, D\}$ , is a direct spectral transform. The following statement shows that the results of Section 2, can be seen as the inverse spectral transform.

**Corollary 4.1** *The solution*

$$\Psi_n(x, t, Q) = c_n^1(t, Q)\Phi(x - x_n(t), t) + c_n^2(t, Q)\Phi(x + x_n(t), t) \quad (4.35)$$

of equation (3.5) is equal to  $\Psi_n(x, t, Q) = c(t)\psi_n(x, t, Q)$ , where  $\psi_n(x, t, Q)$  is the Baker-Akhiezer function corresponding to  $\Gamma$  and the divisor  $D$  of the poles of  $C_n$ ; the factor  $c(t)$  is defined in (4.22).

All the solutions  $x_n(t)$  of (1.1) have the form  $x_n = \frac{1}{2}(x_n^1 - x_n^2)$ , where  $x_n^i(t)$  are roots of the equation

$$\theta(nU + x_n^i(t)V + Wt + Z) = 0. \quad (4.36)$$

Here  $\theta(z)$  is the Riemann theta-function corresponding to  $\Gamma$ ; vectors  $U, V, W$  are defined by (2.8, 2.9); vector  $Z$  corresponds to the divisor  $D$  via the Abel transform.

As follows from the Theorem 4.1, the function  $\Psi_n$  defined by (4.35) has the same analytical properties on  $\Gamma$  as the function  $c(t)\psi_n$ . Therefore, they coincide. Equation (4.36) immediately follows from the formula (2.10) for  $\psi_n$ .

## 5 Action-angle variables

Until now we have not used the Hamiltonian structure of equations (1.1). Moreover, a priori it is not clear why a system that has arisen as a pole system of elliptic solutions of the 2D Toda lattice is Hamiltonian. The general algebro-geometric approach which allows to derive a Hamiltonian structure starting from the Lax representation was proposed and developed in [16, 17, 39].

The main goal of this section is to construct action-angle variables for (1.1). First of all, let us summarize necessary results of the previous sections. A point  $(p_n, x_n)$  of the phase space  $\mathcal{M}$  of the system defines a matrix function  $L_n(z)$  with the help of the formulae

$$L_n^{ij} = 2^{-1}h_{n+1}^i\Phi(x_{n+1}^i - x_n^j, z), \quad (5.1)$$

$$x_n^1 = x_n, \quad x_n^2 = -x_n, \quad h_n^1 = h_n - 1, \quad h_n^2 = -h_n - 1, \quad h_n = \frac{1 + e^{p_n}}{1 - e^{p_n}}. \quad (5.2)$$

This function defines the spectral curve  $\Gamma$  (with the help of (4.12)), and the divisor  $D$  of poles  $\gamma_1, \dots, \gamma_{N+1}$  of the Baker-Akhiezer function  $C_n(Q) = (c_n^1(Q), c_n^2(Q))$

$$C_{n+1}(Q) = L_n(z)C_n(Q), \quad C_N(Q) = wC_0(Q), \quad Q = (w, z) \in \Gamma, \quad (5.3)$$

normalized by the condition

$$c_0^1(0, Q)\Phi(-x_0, z) + c_0^2(Q)\Phi(x_0, z) = 1. \quad (5.4)$$

The divisor  $D$  satisfies (4.18), i.e. defines a point of an odd part of the Jacobian  $J^{Pr}(\Gamma) \in J(\Gamma)$ , which is defined as a fiber of the projection

$$\{\gamma_1, \dots, \gamma_{N+1}\} \in J(\Gamma) \mapsto 2\omega\phi_+ = \sum_{s=1} z(\gamma_s) \in \Gamma_0, \quad (5.5)$$

corresponding to  $\phi_+ = 0$ . All the fibers are equivalent and can be identified with the Prym variety of  $\Gamma$ . Note, that a shift of  $\phi_+$  corresponds to the shift  $x \rightarrow x + a$  for the solution (1.8) of 2D Toda lattice.

The correspondence

$$(p_n, x_n) \in \mathcal{M} \mapsto \{\Gamma, D \in J^{Pr}(\Gamma)\} \quad (5.6)$$

is an isomorphism. The coefficients  $(u_i, \Lambda)$  of equation (1.6) are integrals of the Hamiltonian system (1.1). Equations (1.1) on a fiber over  $\Gamma$  of the map (5.6) are linearized by the Abel transform (2.6).

The main goal of this section is to construct the action variables that are canonically conjugated to the coordinates  $\phi_1, \dots, \phi_{N-1}, \phi_-$ :

$$\phi_k = \sum_{s=1}^{N+1} A_k(\gamma_s), \quad \phi_- = \phi_N - \phi_{N+1}, \quad (5.7)$$

on the Prymian  $J^{Pr}(\Gamma)$ . Note, that  $\phi_+ = \phi_N + \phi_{N+1}$ .

**Theorem 5.1** *The transformation*

$$(x_n, p_n) \mapsto (\phi_1, \dots, \phi_{N-1}, \phi_-; I_1, \dots, I_N) \quad (5.8)$$

where  $I_k$  are  $a$ -periods of the differential  $dS = \ln(\Lambda^{-N}w)dz$ :

$$I_k = \oint_{a_k} \ln(\Lambda^{-N}w)dz, \quad (5.9)$$

is a canonical transformation, i.e.

$$\sum_{n=1}^N dp_n \wedge dx_n = \sum_{k=1}^{N-1} (\delta I_k \wedge \delta \phi_k) + \delta I_N \wedge \delta \phi_-. \quad (5.10)$$

*Proof.* First of all, following the approach proposed in [16], we define a symplectic structure on  $\mathcal{M}$  in terms of the Lax operator and its eigenfunctions. After that we will calculate it in two different ways which immediately imply (5.10).

The external differential  $\delta L_n(z)$  can be seen as an operator-valued one-form on  $\mathcal{M}$ . Canonically normalized eigenfunction  $C_n(Q)$  of  $L_n(z)$  is the vector-valued function on  $\mathcal{M}$ .

Hence, its differential is a vector-valued one-form. Let us define a two-form  $\omega$  on  $\mathcal{M}$  by the formula

$$\omega = \frac{1}{2} \left( \text{res}_{P_+} \Omega + \text{res}_{P_-} \Omega \right), \quad (5.11)$$

where

$$\Omega = \langle C_{n+1}^*(Q) \delta L_n(z) \wedge \delta C_n(Q) \rangle dz. \quad (5.12)$$

Here and below  $\langle \cdot \rangle$  stands for the sum over a period of a periodic in  $n$  function, i.e.

$$\langle f_n \rangle = \sum_{n=0}^{N-1} f_n;$$

$C_n^*(Q)$  is the dual Baker-Akhiezer function, which is defined as a co-vector (row-vector) solution of the equation

$$C_{n+1}^*(Q) L_n(z) = C_n^*(Q), \quad C_N^*(Q) = w^{-1} C_0^*(Q), \quad (5.13)$$

normalized by the condition

$$C_0^*(Q) C_0(Q) = 1. \quad (5.14)$$

The form  $\omega$  can be rewritten as

$$\omega = \frac{1}{2} \text{res}_0 \text{Tr} \langle \left( C_{n+1}^{-1}(z) \delta L_n(z) \wedge \delta C_n(z) \right) \rangle dz, \quad (5.15)$$

where  $C_n(z)$  is a matrix with the columns  $C_n(Q_j(z))$ ,  $Q_j(z) = (z, w_j)$  corresponding to different sheets of  $\Gamma$ .

Note, that  $C_n^*(Q)$  are rows of the matrix  $C_n^{-1}(z)$ . That implies that  $C_n^*(Q)$  as a function on the spectral curve is: meromorphic outside the punctures; has poles at the branching points of the spectral curve, and zeros at the poles  $\gamma_s$  of  $C_n(Q)$ . These analytical properties are used in the proof of the following lemma.

**Lemma 5.1** *The two-form  $\omega$  equals*

$$\omega = \sum_{s=1}^{N+1} \delta z(\gamma_s) \wedge \delta \ln w(\gamma_s). \quad (5.16)$$

The meaning of the right-hand side of this formula is as follows. The spectral curve by definition arises with the meromorphic function  $w(Q)$  and multi-valued holomorphic function  $z(Q)$ . Their evaluations  $w(\gamma_s)$ ,  $z(\gamma_s)$  at the points  $\gamma_s$  define functions on  $\mathcal{M}$ , and the wedge product of their external differentials is a two-form on  $\mathcal{M}$ .

*Proof.* The differential  $\Omega$ , defined by (5.12) is a meromorphic differential on the spectral curve (the essential singularities of the factors cancel each other at the punctures). Therefore, the sum of its residues at the punctures is equal to the sum of other residues with negative sign. There are poles of two types.

First of all,  $\Omega$  has poles at the poles  $\gamma_s$  of  $C_n$ . Note, that  $\delta C_n$  has the pole of the second order at  $\gamma_s$ . Taking into account that  $C_n^*$  has zero at  $\gamma_s$  we obtain

$$\text{res}_{\gamma_s} \Omega = \langle C_{n+1}^* \delta L_n C_n \rangle \wedge \delta z(\gamma_s). \quad (5.17)$$

From (5.3) and (5.13) it follows that

$$\langle C_{n+1}^* \delta L_n C_n \rangle = \langle C_N^* \left( \prod_{m=n+1}^{N-1} L_m \right) \delta L_n \left( \prod_{m=0}^{n-1} L_m \right) C_0 \rangle = (C_N^* \delta T C_0), \quad (5.18)$$

where  $T$  is the monodromy matrix. Using the standard formula for the variation of the eigenvalue of an operator:  $\delta w = C_0^* \delta T C_0$ , we obtain that

$$\text{res}_{\gamma_s} \Omega = \delta \ln w(\gamma_s) \wedge \delta z(\gamma_s). \quad (5.19)$$

The second set of poles of  $\Omega$  is a set of branching points  $q_i$  of the cover. The pole of  $C_n^*$  at  $q_i$  cancels with the zero of the differential  $dz$ ,  $dz(q_i) = 0$ , considered as differential on  $\Gamma$ . The vector-function  $C_n$  is holomorphic at  $q_i$ . If we take an expansion of  $C_n$  in the local coordinate  $(z - z(q_i))^{1/2}$  (in general position when the branching point is simple), and consider its variation, we get that

$$\delta C_n = -\frac{dC_n}{dz} \delta z(q_i) + O(1). \quad (5.20)$$

Therefore,  $\delta C_n$  has a simple pole at  $q_i$ . In the similar way, we obtain

$$\delta w = -\frac{dw}{dz} \delta z(q_i). \quad (5.21)$$

Equalities (5.20) and (5.21) imply that

$$\text{res}_{q_i} \Omega = \text{res}_{q_i} \left[ \langle C_{n+1}^* \delta L_n dC_n \rangle \wedge \frac{\delta w dz}{dw} \right]. \quad (5.22)$$

At  $q_i$  we have  $dL_n(q_i) = 0$ . Therefore, in the way similar to (5.18), we get

$$\text{res}_{q_i} \Omega = \text{res}_{q_i} \left[ (C_N^* \delta T dC_0) \wedge \frac{\delta w dz}{dw} \right]. \quad (5.23)$$

Due to skew-symmetry of the wedge product we may replace  $\delta T$  in (5.23) by  $(\delta T - \delta w)$ . Then using identities  $C_N^* (\delta T - \delta w) = \delta C_N^* (w - T)$  and  $(w - T) dC_0 = (dT - dw) C_0$  we obtain

$$\text{res}_{q_i} \Omega = -\text{res}_{q_i} (\delta C_N^* C_0) \wedge \delta w dz = \text{res}_{q_i} (C_N^* \delta C_0) \wedge \delta w dz. \quad (5.24)$$

Note, that the  $dT$  does not contribute to the residue, because  $dT(q_i) = 0$ .

Expansions (4.19, 4.20) near the punctures imply that

$$\text{res}_{P_{\pm}} (C_N^* \delta C_0) \wedge \delta w dz = 0. \quad (5.25)$$

Therefore,

$$\sum_{q_i} \text{res}_{q_i} (C_N^* \delta C_0) \wedge \delta w dz = -\sum_{s=1}^{N+1} \text{res}_{\gamma_s} (C_N^* \delta C_0) \wedge \delta w dz = \sum_{s=1}^{N+1} \delta \ln w(\gamma_s) \wedge \delta z(\gamma_s). \quad (5.26)$$

The sum of (5.19) and (5.26) gives (5.16), because

$$2\omega = -\sum_{s=1}^{N+1} \text{res}_{\gamma_s} \Omega - \sum_{q_i} \text{res}_{q_i} \Omega. \quad (5.27)$$

Our next goal is to prove the following statement.

**Lemma 5.2** *The symplectic form given by (5.11) coincides with the canonical symplectic structure*

$$\omega = \sum_{n=0}^{N+1} \delta p_n \wedge \delta x_n. \quad (5.28)$$

*Proof.* Using the gauge transformation (4.7)

$$L_n = g_{n+1} \tilde{L}_n g_n^{-1}, \quad \mathcal{C}_n = g_n \tilde{\mathcal{C}}, \quad g_n = \begin{pmatrix} e^{x_n \zeta(z)} & 0 \\ 0 & e^{-x_n \zeta(z)} \end{pmatrix},$$

we obtain

$$\begin{aligned} \omega &= \frac{1}{2} \text{res}_0 \text{Tr} \left\langle \tilde{\mathcal{C}}_{n+1}^{-1}(z) \delta \tilde{L}_n(z) \wedge \delta \tilde{\mathcal{C}}_n(z) + \tilde{\mathcal{C}}_{n+1}^{-1} \delta \tilde{L}_n \wedge \delta f_n \tilde{\mathcal{C}}_n \right. \\ &\quad \left. - \tilde{\mathcal{C}}_{n+1}^{-1} \left( \delta f_{n+1} \wedge \delta \tilde{L}_n + \delta f_{n+1} \wedge \tilde{L}_n \delta f_n \right) \tilde{\mathcal{C}}_n \right\rangle dz, \end{aligned} \quad (5.29)$$

where  $\delta f_n = \delta g_n g_n^{-1}$ . From (4.32), using the equality

$$(\zeta(x_{n+1} + x_n) - \zeta(x_{n+1} - x_n)) \delta s_n + \delta(\chi_{n,-}^{1,1} + \chi_{n,-}^{1,2}) = s_n \delta(\zeta(x_{n+1} - x_n) - \zeta(x_{n+1} + x_n)),$$

which follows from (4.28), we obtain that the first term in (5.29)

$$J_1 = \text{res}_0 \left\langle \text{Tr} \left( \tilde{\mathcal{C}}_{n+1}^{-1}(z) \delta \tilde{L}_n(z) \wedge \delta \tilde{\mathcal{C}}_n(z) \right) \right\rangle dz$$

is equal to

$$J_1 = \left\langle \frac{s_n}{2s_{n+1}} \delta h_{n+1} \wedge \delta(\zeta(x_n - x_{n+1}) + \zeta(x_{n+1} + x_n)) \right\rangle.$$

Equation (4.34) implies

$$J_1 = \left\langle \frac{2\delta h_{n+1} \wedge \delta x_{n+1}}{h_{n+1}^2 - 1} - \frac{2\delta h_{n+1} \wedge \delta x_n}{h_{n+1}^2 - 1} \left( \frac{\wp(x_n - x_{n+1}) + \wp(x_n + x_{n+1})}{\wp(x_n - x_{n+1}) - \wp(x_n + x_{n+1})} \right) \right\rangle. \quad (5.30)$$

The second term in (5.29) equals

$$J_2 = \left\langle \text{res}_0 \text{Tr} \left( \tilde{\mathcal{C}}_{n+1}^{-1} \delta \tilde{L}_n \wedge \delta f_n \tilde{\mathcal{C}}_n \right) \right\rangle dz = \left\langle \text{res}_0 \text{Tr} \left( \tilde{L}_n^{-1} \delta \tilde{L}_n \wedge \delta f_n \right) \right\rangle dz$$

From definition (5.1) of  $L_n$  by direct calculations we obtain that

$$J_2 = \left\langle \frac{2\delta h_{n+1} \wedge \delta x_n}{h_{n+1}^2 - 1} \left( \frac{\wp(x_n - x_{n+1}) + \wp(x_n + x_{n+1})}{\wp(x_n - x_{n+1}) - \wp(x_n + x_{n+1})} \right) \right\rangle. \quad (5.31)$$

At last, the third term in (5.29) is equal to

$$J_3 = - \left\langle \text{res}_0 \text{Tr} \left( \tilde{\mathcal{C}}_{n+1}^{-1} \delta f_{n+1} \wedge \delta \tilde{L}_n \tilde{\mathcal{C}}_n \right) \right\rangle dz = \left\langle \text{res}_0 \text{Tr} \left( (\delta \tilde{L}_n) \tilde{L}_n^{-1} \wedge \delta f_{n+1} \right) \right\rangle dz, \quad (5.32)$$

because

$$J_4 = \left\langle \text{res}_0 \text{Tr} \left( \tilde{\mathcal{C}}_{n+1}^{-1} \delta f_{n+1} \wedge \tilde{L}_n \delta f_n \right) \tilde{\mathcal{C}}_n \right\rangle dz = 0. \quad (5.33)$$

In order to prove (5.33), let us note that at  $z = 0$

$$f_n(z) = z^{-1}f_n^0 + O(z^3), \quad f_n^0 = \begin{pmatrix} x_n & 0 \\ 0 & -x_n \end{pmatrix}. \quad (5.34)$$

Therefore,

$$\begin{aligned} J_4 &= \langle \text{res}_0 \text{Tr} \left( \tilde{\mathcal{C}}_{n+1}^{-1} \delta f_{n+1}^0 \wedge \tilde{L}_n \delta f_n^0 \right) \tilde{\mathcal{C}}_n \rangle \wp(z) dz \\ &= \langle \text{res}_0 \text{Tr} \left( \mathcal{C}_{n+1}^{-1} \delta f_{n+1}^0 \wedge L_n \delta f_n^0 \right) \mathcal{C}_n \rangle \wp(z) dz. \end{aligned} \quad (5.35)$$

The last term in (5.35) is equal to the sum of residues at the punctures  $P_{\pm}$  of the differential

$$\langle \mathcal{C}_{n+1}^* \delta f_{n+1}^0 \wedge L_n \delta f_n^0 \mathcal{C}_n \rangle \wp(z) dz,$$

which is holomorphic on  $\Gamma$  outside the punctures. Hence  $J_4 = 0$ .

From (5.32) by direct calculations we obtain, that

$$J_3 = \left\langle \frac{2\delta h_{n+1} \wedge \delta x_{n+1}}{h_{n+1}^2 - 1} \right\rangle. \quad (5.36)$$

Equations (5.30,5.31), and (5.36) imply (5.28). Lemma is proved.

Now we are ready to complete the proof of the Theorem. Equations (5.16) and (2.14) imply that

$$\omega = -\delta\alpha, \quad \alpha = \sum_{s=1}^{N+1} \int_{P_+}^{\gamma_s} \delta \ln(\Lambda^{-N} w) dz. \quad (5.37)$$

Indeed, we have

$$\delta\alpha = \sum_{s=1}^{N+1} \delta \ln w(\gamma_s) \wedge \delta z(\gamma_s) - N \delta \ln \Lambda \wedge \delta \sum_{s=1}^{N+1} z(\gamma_s). \quad (5.38)$$

The last term in (5.38) equals zero on the fibers  $\phi_+ = \text{const}$  of the map (5.5).

The differential  $dS = \ln(\Lambda^{-N} w) dz$  is multi-valued on  $\Gamma$  but, following the arguments of [16], one can show that its derivatives with respect to  $I_k$ ,  $k = 1, \dots, N$  (which can be considered as coordinates on a space of curves given by (1.6)), are holomorphic differentials.  $dS$  is an odd differential with respect to the permutation of sheets of  $\Gamma$ . Therefore,  $I_{N+1} = -I_N$  and the definition of  $I_k$  implies that

$$\frac{\partial}{\partial I_k} dS = d\Omega_k^h, \quad k = 1, \dots, N-1, \quad \frac{\partial}{\partial I_N} dS = d\Omega_N^h - d\Omega_{N+1}^h. \quad (5.39)$$

Equations (5.7) and (2.6) imply that

$$\alpha = \sum_{k=1}^{N-1} (\phi_k \delta I_k) + \phi_- \delta I_N, \quad (5.40)$$

and using (5.37) we finally obtain (5.10). Theorem is proved.

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