Weak-QES extensions of the Calogero model

Y. Brihaye[◇] and P. Kosinski[†]

[◇]Dep. of Mathematical Physics, University of Mons-Hainaut, Mons, Belgium

[†]Dep. of Theoretical Physics, University of Lodz, Lodz, Poland

Abstract

We construct families of Hamiltonians extending the Calogero model and such that a finite number of eigenvectors can be computed algebraically.

1 Introduction

The notion of quasi exactly solvable equations has many different meanings. Initially [1, 2] it was invented to qualify operators (typically quantum Hamiltonian ones) which, after suitable change of variables and(or) "gauge rotation", are equivalent to an element of the envelopping algebra of some Lie algebra, represented by differential operators. In the following we refer to the above property as to algebraic QES property. In opposition some operators can possess a series of values of the coupling constants for which an explicit eigenvector is available without beeing related to any representation of a Lie algebra. We will qualify this property as "weak-QES". In the recent few years the Calogero model [3] (and several of its extensions [4]) have received a considerable new impetus of interest. One of the new results was the construction [5] of a set of variables (the so called τ-variables) in which the N-body Calogero model can be written as an element of the enveloping algebra of $sl(N)$. The complete integrability of the model was then directly related to the finite representations of a Lie algebra, following very closely one of the ideas underlying the notion of quasi exact solvability [1]. Some algebraic QES generalisations of the Calogero models were proposed soon after the basic result of [5]. Unfortunately these extentions are all based of an $sl(2)$ algebra. Basically, only one of the coordinates, for instance τ_2, is involved into the additional piece of the potential. The purpose of this note is to exhibit series of weak-QES hamiltonians, generalizing the Calogero hamiltonian, some of them depending generically of all the τ's.

2 The Hamiltonian

We are interested in hamiltonians of the form

$$H = H_c + V \tag{1}$$

$$H_c = \frac{1}{2}(-\Delta + \omega^2 r^2) + \sum_{j<k=1}^{N} \frac{\nu(\nu-1)}{(x_j - x_k)^2} \tag{2}$$

with

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \quad , \quad r^2 = \sum_{i=1}^{N} x_i^2 \tag{3}$$

and such that the eigenvalue equation

$$H\psi = E\psi \tag{4}$$

admits eigenvectors of the form

$$\psi(x) = (\beta(x))^\nu e^{-\frac{\omega}{2}r^2} e^{P(x)} p(x) \quad , \quad \beta(x) = \Pi_{j<k}(x_j - x_k) \quad , \tag{5}$$

The function $\beta(x)$ is the Vandermonde determinant of the matrix $M_{ij} = (x_i)^j$. Moreover we restrict ourselves to polynomial forms of $V(x), P(x), p(x)$ in $x \equiv (x_1, x_2, \ldots, x_N)$. The operator acting on $p(x)$ will be denoted $\tilde{H}$:

$$\tilde{H}p = Ep \tag{6}$$

In the following we use the variables $\sigma_1, \tau_2, \tau_3, \cdots, \tau_N$ introduced in [5]. In these variables, the laplacian (3) reads [5]

$$\Delta(\tau) = N\frac{\partial^2}{\partial\sigma_1^2} + \sum_{j,k=2}^{N} A_{jk}\frac{\partial^2}{\partial\tau_j\partial\tau_k} + \sum_{j=2}^{N} B_j\frac{\partial}{\partial\tau_j} \tag{7}$$

$$\begin{aligned} A_{jk} &= \frac{(N-j+1)(k-1)}{N}\tau_{j-1}\tau_{k-1} \\ &+ \sum_{l\geq\max(1,k-j)}(k-j-2l)\tau_{j+l-1}\tau_{k-l-1} \end{aligned} \tag{8}$$

$$B_j = -\frac{(N-i+2)(N-i+i)}{N}\tau_{i-2} \tag{9}$$

For the manipulation of a generic (translation-invariant) change of basis, the relevant formula reads

$$\sum_k \frac{\partial w}{\partial x_k}\frac{\partial}{\partial x_k} = \sum_{j,k=2}^{N} A_{jk}\frac{\partial w}{\partial\tau_j}\frac{\partial}{\partial\tau_k} \tag{10}$$

The following identities are also useful :

$$\beta^{-\nu}(-\frac{1}{2}\Delta(x)+\sum_{j<k}\frac{\nu(\nu-1)}{(x_j-x_k)^2})\beta^{\nu} = -\frac{1}{2}\Delta(\tau)+\frac{\nu}{2}\sum_{j=2}^{N}(N-j+2)(N-j+1)\tau_{j-2}\frac{\partial}{\partial\tau_j} \tag{11}$$

$$e^{\frac{\omega}{2}r^2}(-\frac{1}{2}\Delta(x) + \frac{\omega^2}{2}r^2)e^{-\frac{\omega}{2}r^2} = -\frac{1}{2}\Delta(\tau) + \omega\sum_{j-2}^{N} j\tau_j\frac{\partial}{\partial\tau_j} + \frac{N}{2}\omega \quad ; \tag{12}$$

they allow to handle respectively the repulsive and the harmonic parts of the potential in (1).

We have attempted to construct the polynomial potentials $V(\tau_2, \tau_3, \cdots, \tau_N)$ such that the operator (1) is equivalent (after a change of basis and with the variables τ) to an operator preserving one vector space of the form

$$\mathcal{P}(N,n) = \text{span}\{\tau_2^{n_2}\tau_3^{n_3}\cdots\tau_N^{n_N} \mid n_2+n_3+\cdots+n_N \leq n\} \tag{13}$$

Our calculations indicate that, for $N = 2,3,4,5$, the Calogero hamiltonian (corresponding to $V = 0$ in (1)) is the only one to possess such a property. After this negative result, we investigate alternative possibilities of algebraic solutions by imposing weaker requirements. Four types of situation have been considered which presented in the next section.

3 Weak-QES Hamiltonians

3.1 Type 1

We set $N = 3$; as previously stated V, P, p are then polynomials in τ_2, τ_3. Use of the identities of Sec. 2 leads to the operator $\tilde{H}$ acting on $p(\tau_2, \tau_3)$:

$$\begin{aligned}
\tilde{H} &= \tau_2 \frac{\partial^2}{\partial \tau_2^2} + 3\tau_3 \frac{\partial^2}{\partial \tau_2 \partial \tau_3} - \frac{1}{3}\tau_2^2 \frac{\partial}{\partial \tau_3^2} + \frac{\partial}{\partial \tau_2} \\
&+ (\frac{\partial P}{\partial \tau_2})(2\tau_2 \frac{\partial}{\partial \tau_2} + 3\tau_3 \frac{\partial}{\partial \tau_3}) \\
&+ (\frac{\partial P}{\partial \tau_3})(3\tau_3 \frac{\partial}{\partial \tau_2} - \frac{2}{3}\tau_2^2 \frac{\partial}{\partial \tau_3}) \\
&+ \omega(2\tau_2 \frac{\partial}{\partial \tau_2} + 3\tau_3 \frac{\partial}{\partial \tau_3}) \\
&+ 3\nu \frac{\partial}{\partial \tau_2} + V_{eff}
\end{aligned} \tag{14}$$

where we define

$$V_{eff} \equiv V - \frac{1}{2} e^{-P} \Delta e^{P} \tag{15}$$

We have constructed the solutions of this equation for particular values of the degrees of P and of p.

P is of degree four

We assume $P(x)$ to be at most quartic in x, i.e.

$$P = \frac{c}{2}\tau_2^2 + b\tau_2 + d\tau_3 \tag{16}$$

Choosing for p a polynomial of global degree n in τ_2, τ_3, a careful power counting in (6), (14), reveals that polynomial solutions can exist only if V_{eff} is the form

$$V_{eff} = v_1 \tau_2 + v_0 \tag{17}$$

(v_0 accounts for the eigenvalue of $\tilde{H}$, i.e. $v_0 \equiv -E$). Moreover there are generically $\frac{(n+2)(n+3)}{2} - 1$ algebraic equations to be solved and $\frac{(n+1)(n+2)}{2} + 1$ parameters (including the parameters of p and the two defining V_{eff}). We solved explicitely these equations for the first few values of n.

Let $n=1$, then

$$p = \tau_2 + c_1\tau_3 + c_0 \tag{18}$$

Two types of solutions are found after some algebra.

Solution (a)

$$\begin{aligned} c_1 &= d = 0 \quad , \quad v_1 = -2c \\ c_0 &= \frac{2b + 2\omega + v_0}{2c} \\ v_0 &= -b - w \pm \sqrt{\omega^2 - 2c - 6c\nu + 2b\omega + b^2} \end{aligned} \tag{19}$$

This solution does not depend on τ_3 and is therefore not generic. It depends only on τ_2 and it has two possible of eigenvalues of the energy ($E \equiv -v_0$). It is a particular case of the QES extension of the Calogero model constructed in [6]. The potential depends on four parameters ω, ν, b, c. Such solutions can be constructed for any values of n. The corresponding Hamiltonian is an element of the enveloping algebra of sl(2) realized on the space of polynomials of degree at most n in τ_2.

Solution (b)

$$\begin{aligned} c_1 &= \frac{-3c}{2d} \quad , \quad c_0 = \frac{2d^2 - \omega c - bc}{3c^2} \\ v_1 &= -3c \quad , \quad v_0 = \frac{1}{c}(2d^2 - 3\omega c - 3bc) \end{aligned} \tag{20}$$

which has to be supplemented by one condition on ω, ν, b, c, d:

$$3b^2c^2 + 6bc^2\omega - 8bcd^2 + 9c^3\nu + 3c^3 + 3c^2\omega^2 - 8cd^2\omega + 4d^4 = 0 \tag{21}$$

Unlike the "solution (a)", this solution non trivially depends on τ_2 and τ_3. The potential is parametrized by ω, ν, b, c, d constrained by (21). One can in principle solve (21) with respect to one of the constants and obtain a finite number of potentials each admitting one algebraic eingenvector. This result is similar in spirit to a QES-type II equation [1].

Repeating the above calculation for $n=2$, i.e.

$$p = \tau_2^2 + c_1\tau_2\tau_3 + c_3\tau_3^2 + c_2\tau_2 + c_4\tau_3 + c_5 \tag{22}$$

As a counterpart of solution (b), we find the following expressions

$$v_1 = -6c \quad , \quad v_0 = 2\frac{2d^2 - 3b\ c - 3c\ \omega}{c} \tag{23}$$

$$c_1 = -3\frac{c}{d} \quad , \quad c_2 = \frac{32d^4 - 16\ b\ c\ d^2 - 9c^3 - 16c\ d^2\omega}{24c^2\ d^2} \tag{24}$$

$$c_3 = \frac{9c^2}{4d^2} \quad , \quad c_4 = \frac{b\ c + c\ \omega - 2d^2}{c\ d} \tag{25}$$

$$\begin{aligned} c_5 &= (16b^2\ c^2\ d^2 + 9b\ c^4 + 32b\ c^2\ d^2\ \omega - 48b\ c\ d^4 + 9c^4\omega \\ &+ 36c^3d^2\ \nu + 15c^3\ d^2 + 16c^2\ d^2\omega^2 - 48c\ d^4\omega + 32d^6)/(36c^4d^2) \end{aligned} \tag{26}$$

which fix the parameters of p and V_{eff} in terms of the coupling constant ω, ν, b, c, d. Two supplementary relations, analog to (21), have to be imposed of these coupling constants

$$0 = 24b^2c^2 + 48bc^2\omega - 64bcd^2 + 72c^3\nu + 105c^3 + 24c^2\omega^2 - 64cd^2\omega + 32d^4 \tag{27}$$

$$\begin{aligned} 0 &= -192b^3c^3d^2 - 108b^2c^5 - 576b^2c^3d^2\omega + 704b^2c^2d^4 \\ &- 216bc^5\omega - 576bc^4d^2\nu - 156bc^4d^2 - 576bc^3d^2\omega^2 \\ &+ 1408bc^2d^4\omega - 768bcd^6 - 81c^6\ \nu - 27c^6 - 108c^5\ \omega^2 - 576c^4\ d^2\ \omega\ \nu \\ &- 156c^4d^2\omega + 576c^3\ d^4\nu + 216c^3\ d^4 - 192c^3\ d^2\omega^3 + 704c^2\ d^4\omega^2 \\ &- 768c\ d^6\omega + 256d^8 \end{aligned} \tag{28}$$

So that we end up with a three-parameters family of weak-QES potentials.

P is of degree six

We also considered the case of a polynomial P of degree at most six in x, i.e.

$$P = \frac{c}{2}\tau_2^2 + b\tau_3 + d\tau_2 + \frac{\tilde{c}}{3}\tau_2^3 + \frac{\tilde{b}}{2}\tau_3^2 + \tilde{d}\tau_2\tau_3 \tag{29}$$

and found that V_{eff} has to be of the form

$$V_{eff} = v_0 + v_1\tau_2 + v_2\tau_3 + v_3\tau_2^2 \tag{30}$$

i.e. at most quartic in the variables x_i. Solving the equations (6) for $n = 3$, we checked that a four-parameter family of weak-QES potentials exists. The form of the constraints on the parameters rapidly becomes cumbersome when the degree of P increases.

3.2 Type 2

When we demand the functions V, P, p in (1)-(5) to depend on the variable τ_2 only, the operator $\tilde{H}$ takes the form

$$\begin{aligned}\tilde{H} &= \tau_2\frac{\partial^2}{\partial\tau_2^2} + (\tau_2 P' + 2\omega\tau_2 + \frac{N-1}{2}(1+N\nu))\frac{\partial}{\partial\tau_2} \\ &+ (2\omega\tau_2 + \frac{N-1}{2}(1+N\nu))P' + \tau_2(P'' + (P')^2) + V(\tau_2)\end{aligned} \tag{31}$$

where $P' = \frac{dP}{d\tau_2}$, etc.

The case $P = -\frac{a}{2}\tau_2^2 - b\tau_2$ corresponds to the QES extension of [6]; the potential V is then of third degree in τ_2. However weak-QES potentials of higher degree can be constructed. Let indeed V be of degree $2\delta + 1$, then a power counting reveals that P has to be of degree $\delta + 1$ and that the algebraic equation (6) can be fulfilled with a polynomial $p(\tau_2)$ provided a number of δ conditions among the $2\delta + 1$ coupling constants entering in the potential are satisfied. This results into families of weak-QES potentials depending of $\delta + 1$ parameters.

3.3 Type 3

We have reconsidered the QES Hamiltonian of [7] and tried to generalize it by following the approach presented in Sec.2. The hamiltonian has the form [7]

$$H = -\Delta + V + g\sum_{i,j=1,i\neq j}^{N}(\frac{1}{(x_i - x_j)^2} - \frac{1}{(x_i + x_j)^2}) \quad . \tag{32}$$

We look for the most general change of function e^P and potential V such that

$$\tilde{H} = e^P H e^{-P}|_{\xi_i} \qquad , < qquad \xi_i \equiv x_i^2 \tag{33}$$

is an element of the enveloping algebra of the Lie algebra $sl(N)$ in the representation given by Eq.(16) of [5]. After an algebra, we find a four parameters family of potentials with

$$e^{-P} = (\Pi_{i,j=1,i<j}^N (\xi_i - \xi_j))^\alpha (\Pi_{j=1}^N \xi_j)^\beta \exp\{-\frac{a}{4}\sum_{j=1}^N \xi_j^2 - \frac{b}{2}\sum_{j=1}^N \xi_j\} \tag{34}$$

and

$$V = \sum_{i=1}^N \{\frac{g'}{4\xi_i} + (b^2 - a(4n + 4\alpha(N-1) + 4\beta + 3))\xi_i + 2ab\xi_i^2 + a^2\xi_i^3\} \tag{35}$$

with $g = \alpha(\alpha - 1), g' = 2\beta(2\beta - 1)$. This is slightly more general than the potential given in [7] since the coupling constants g and g' are here independent.

3.4 Type 4

Finally we present a result which directly generalizes to N dimensions the famous one-dimensional sextic QES potential [1, 2]. In this purpose we consider

$$H = -\frac{1}{2}\Delta + V_6(x) \tag{36}$$

and assume that the even function $V_6(x)$ contain powers of degree at most six in x_j. The most general change of function e^P such that the operator

$$\tilde{H} = e^P H e^{-P}|_{t_i} \quad , \quad t_i = x_i^2 \tag{37}$$

is an element of the enveloping algebra of $sl(N)$ (in the representation given by Eq.(16) of [5]) is given by the $2N+1$-parameters function

$$P = \alpha T^2 + \sum_{a=1}^N p_a t_a + \sum_{a=1}^N q_a \log t_a \quad , \quad T = \sum_{i=1}^N t_i \tag{38}$$

where α, p_a, q_a are constants. Correspondingly the most general QES potential reads

$$V_6 = 4\sum_i t_i (\frac{\partial P}{\partial t_i})^2 + 4\sum_i t_i \frac{\partial^2 P}{\partial t_i^2} + 2\sum_i \frac{\partial P}{\partial t_i} - 16\alpha n T \tag{39}$$

For $N = 2$, the 2-body polynomial potential of [2, 8] is then recovered as the special case $q_1 = q_2 = 0$.

Summarizing, we have analyzed the rational Calogero model from the point of view of the notion of quasi exact solvability. We found, at least for the small number of particles, that it is unique in the following sense : it is exactly solvable but has no exactly or quasi-exactly solvable translational-invariant extension except those following from an sl(2) structure. By relaxing the notion of quasi-exact solvability (weak QES) we were able to find a number of extensions of the Calogero model. They are characterized by the existence of analytical expressions for some levels without hidden symmetry behind.

Acknowledgments. This work was supported by the Belgian F.N.R.S. and carried out under the grant n^0 2 P03B 076 10 of the Polish Government.

References

[1] A.V. Turbiner, Comm. Math. Phys. **119**, 467 (1988).

[2] A.G. Ushveridze, *Quasi exact solvability in quantum mechanics* (Institute of Physics Publishing, Bristol and Philadelphia, 1993).

[3] F. Calogero, Journ. Math. Phys. **12**, 419 (1971).

[4] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. C 94, 313 (1983).

[5] W. Ruhl and A. Turbiner, Mod. Phys. Lett. **A10** (1995) 2213.

[6] A. Minzoni, M. Rosenbaum and A. Turbiner, Mod. Phys. Lett. **A11** (1996) 1977.

[7] X. Hou and M. Shifman, Int. Journ. Mod. Phys. **A 14** (1999) 2993.

[8] M. Shifman and A. Turbiner, Commun. Math. Phys. **126** (1989) 347.