

# Path integral formulation of constrained systems with singular- higher order Lagrangians

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## Abstract

Systems with singular higher order- Lagrangian are investigated by using the extended form of the canonical method. Besides, the canonical path integral formulation is generalized using the Hamilton - Jacobi formulation to investigate singular systems.

## 1 Introduction

In spite of the fact that most physical systems can be described by Lagrangians that depend at most on the first derivatives of the dynamical variables [1-4] there is a continuing interest in the so called generalized dynamics, that is, the study of physical systems described by Lagrangians containing derivatives of order higher than the first.

The generalization of Hamilton's least action principle and of the Hamiltonian formulation to non degenerate Lagrangian depending on higher-order derivatives was first achieved by Ostrogradsky. [5].

Recently a new method [6, 7] based on the Hamilton- Jacobi method [8-11] has been developed to investigate singular systems. The aim of this paper is to study the path integral quantization for singular systems with arbitrarily higher order-Lagrangian. In fact this work is a continuation of previous papers [6,7], where we have obtained the path integral for singular systems with first order Lagrangians. Our desire is to construct the path integral quantization for singular systems starting from the Hamilton -Jacobi Partial differential equations. HJPDE, which is the fundamental equation of classical mechanics. In this method the equations of motion are obtained as total differential equations in many variables which require the investigation of integrability conditions. If the system is integrable, one can construct the canonical phase space and the canonical action is obtained by this procedure. Hence, one can obtain the path integral formulation as an integration over the canonical phase space coordinates

## 2 The extended canonical path integral method

Now we will construct the canonical path integral by using the Hamilton- Jacobi method [8-11]. The starting point of this procedure is to consider a system described by a Lagrangian

$$L(q_1, \dots, q_i^{(k)}), q_i^{(l)} = \frac{d^l q_i}{dt^l},$$

$$l = 0, 1, \dots, k-1, \quad i = 1, \dots, n, \quad (1)$$

where the derivatives  $q_i^{(s)}$  ( $s = 0, 1, \dots, k-1$ ) are treated as coordinates. In Ostogrodoski's formula the momenta conjugated respectively to  $q_{(k-1)i}$  and  $q_{(m-1)i}$  ( $m = 1, \dots, k-1$ ) are defined as

$$p_{(k-1)i} = \frac{\partial L}{\partial q_i^{(k)}}, \quad (2)$$

$$p_{(m-1)i} = \frac{\partial L}{\partial q_i^{(m)}} - p_{(m)i}, \quad m = 1, 2, \dots, k-1, \quad (3)$$

using these relations one can go over from the Lagrangian description to the Hamiltonian description. The canonical Hamiltonian is defined as

$$H_0 = \sum_{s=0}^{k-1} p_{(s)i} q_i^{(s+1)} - L(q_1, \dots, q_i^{(k)}, q_i^{(l)}). \quad (4)$$

"Einstein's summation rule for repeated indices is used throughout this paper".

Now the extended Hessian matrix is defined as

$$A_{ij} = \frac{\partial^2 L}{\partial q_i^{(k)} \partial q_j^{(k)}}, \quad (5)$$

For a regular system, the Hessian has rank  $n$  and the canonical coordinates are independent. For singular Lagrangian case the Hessian has rank  $n-r$ ,  $r < n$ . In this case  $r$  of the momenta are dependent. The generalized coordinate  $q_{(k-1)i}$  are defined as

$$p_{(k-1)a} = \frac{\partial L}{\partial q_a^{(k)}}, \quad a = r+1, \dots, n, \quad (6)$$

$$p_{(k-1)\mu} = \frac{\partial L}{\partial q_\mu^{(k)}}, \quad \mu = 1, \dots, r. \quad (7)$$

Since the rank of the Hessian is  $(n-r)$  one may solve equation (6) for  $q_{(k-1)\mu}$  as functions of  $t$ ,  $q_{(s)i}$ ,  $p_{(k-1)b}$  and  $q_\mu^{(k)}$  as follows

$$q_a^{(k)} = W_{(k)a}(q_{(s)i}, p_{(k-1)b}, q_\mu^{(k)}), \quad b = r+1, \dots, n. \quad (8)$$

Now substituting equation (8) in equation (7) one has

$$p^{(k-1)\mu} = \frac{\partial L}{\partial q_\mu^{(k)}} \Big|_{q_a^{(k)} = W_{(k)a}(q_{(s)i}, p_{(k-1)b}, q_\mu^{(k)}),} \quad (9)$$

or

$$\begin{aligned} p_{(s)\mu} &= -H_{(s)\mu}(t, q_{(u)j}; p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}), \\ u, s &= 0, \dots, k-1, j = 1, \dots, n. \end{aligned} \quad (10)$$

Relapling the coordinates  $t$  as  $t_{(s)0} \equiv q_{(s)0}$  (for any value of  $s$ ); the coordinates  $q_{(s)\mu}$  will be called  $q_{(s)\mu}$  and defining  $p_{(s)0} = \frac{\partial S}{\partial t}$ , while  $H_{(s)0} = H_0$  for any value of  $s$ . In this case the canonical Hamiltonian  $H_0$  may be written as

$$\begin{aligned} H_0 &= \sum_{u=0}^{k-2} p_{(u)a} q_a^{(u+1)} + p_{(k-1)a} W_{(k)a} + \sum_{u=0}^{k-1} q_\mu^{(u+1)} p_{(u)\mu} \Big|_{p_{(s)\nu} = H_{(s)\nu}} \\ &- L(q_{(s)i} 1, \dots, q_\mu^{(k)}, q_a^{(k)} = W_{(k)a}), \\ \mu, \nu &= 1, \dots, r, a = r+1, \dots, n. \end{aligned} \quad (11)$$

Now the canonical method leads to obtain the set of Hamilton-Jacobi partial differential equations as follows

$$H'_0 = H'_{(s)0} = p_{(s)0} + H_{(s)0}(t, t_{(u)\mu}; q_{(u)a}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}) = 0, \quad (12)$$

$$\begin{aligned} H'_{(s)\mu} &= p_{(s)\mu} + H_{(s)\mu}(t_{(u)\nu}; q_{(u)a}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}) = 0, \\ u, s &= 0, \dots, k-1, \mu, \nu = 1, \dots, r, \end{aligned} \quad (13)$$

or

$$\begin{aligned} H'_{(s)\alpha} &= p_{(s)\alpha} + H_{(s)\alpha}(t_{(u)\beta}; q_{(u)a}, p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}}), \\ \alpha, \beta &= 0, 1, \dots, r. \end{aligned} \quad (14)$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$\begin{aligned} dq_{(u)i} &= \sum_{s=0}^{k-1} \frac{\partial H'_{(s)\alpha}}{\partial p_{(u)i}} dt_{(s)\alpha}; \\ i &= 1, \dots, n, \alpha = 0, 1, \dots, r, u, s = 0, 1, \dots, k-1. \end{aligned} \quad (15)$$

$$dp_{(u)c} = - \sum_{s=0}^{k-1} \frac{\partial H'_{(s)\alpha}}{\partial q_{(u)i}} dt_{(s)\alpha}; \quad (16)$$

$$c = 0, 1, \dots, r, \alpha = 0, 1, \dots, r, u, s = 0, 1, \dots, k-1.$$

$$dZ = \sum_{d=0}^{k-1} [-H_{(d)\beta} + \sum_{s=0}^{k-1} p_{(s)a} \frac{\partial H'_{(d)\beta}}{\partial p_{(s)a}}] dt_{(d)\beta}; \quad (17)$$

$$s, d = 0, \dots, k-1, \beta = 0, 1, \dots, r,$$

where  $Z = S(t_{(u)\alpha}; q_{(s)\alpha})$ . The set of equations (15-17) is integrable [8] if

$$dH'_{(s)\alpha}(t_{(u)\nu}; q_{(u)a}; p_{(u)\nu}) = \frac{\partial S}{\partial t_{(u)\nu}}; p_{(u)a} = \frac{\partial S}{\partial q_{(u)a}} = 0, \alpha = 0, 1, \dots, r, \quad (18)$$

conditions (18) considering equations (15-17), may vanish identically or give rise to new constraints. In the case of new constraints one should consider their total variations also. Repeating this procedure one may obtain a set of conditions such that all the total variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the solutions of a singular system.  $H'_{(s)\alpha}$  can be interpreted as infinitesimal generators of canonical transformations given by parameters  $t_{(s)\alpha}$  respectively. In this case as for the first-order systems, the path integral may be written as

$$D(q_i^{(u)}, t'_{(u)\alpha}; q_i^{(u)}, t_{(u)\alpha}) = \int_{q_i^{(u)}}^{q_i^{(u)'}} dq_{(u)a} dp_{(u)a} \times$$

$$\exp i \left\{ \int_{t_{(u)\alpha}}^{t'_{(u)\alpha}} \sum_{d=0}^{k-1} [-H_{(d)\beta} + \sum_{s=0}^{k-1} p_{(s)a} \frac{\partial H'_{(d)\beta}}{\partial p_{(s)a}}] dt_{(d)\beta} \right\},$$

$$u, s, d = 0, 1, \dots, k-1, \alpha, \beta = 0, 1, \dots, r,$$

$$a = r+1, \dots, n. \quad (19)$$

The path integral expression (19) is an integration over the canonical phase space coordinates  $(q_{(u)a}, p_{(u)a})$ .

### 3 Conclusion

We have obtained the canonical path integral formulation of singular higher-order systems. In this formulation, the equations of motion are obtained as total differential equations in many variables which require the investigation of integrability conditions (18). If the system is integrable then each coordinate  $q_{(s)\alpha} = t_{(s)\alpha}$  ( $\alpha = 1, \dots, r$ ) is treated as a parameter that describes the system evolution. The Hamiltonian  $H'_{(s)\alpha}$  will be the infinitesimal generators of canonical transformations given by parameters  $t_{(s)\alpha}$  respectively in the same way the Hamiltonian  $H_0$  is the generator of time evolution. For  $k = 1$  the result obtained here (equation(19)) will reduce the case of the first order path integral showed in references [6,7].

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