# Topological terms and the global symplectic geometry of the phase space in string theory

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Using an imbedding supported background tensor approach for the differential geometry of an imbedded surface in an arbitrary background, we show that the topological terms associated with the inner and outer curvature scalars of the string worldsheet, have a dramatic effect on the global symplectic geometry of the phase space of the theory. By identifying the global symplectic potential of each Lagrangian term in the string action as the argument of the corresponding pure divergence term in a variational principle, we show that those topological terms contribute explicitly to the symplectic potential of any action describing strings, without modifying the string dynamics and the phase space itself. The variation (the exterior derivative on the phase space) of the symplectic potential generates the integral kernel of a covariant and gauge invariant symplectic structure for the theory, changing thus the global symplectic geometry of the phase space. Similar results for non-Abelian gauge theories and General Relativity are briefly discussed.

#### I. INTRODUCTION

As everyone knows, within the context of the variational principle in physics, any term that can be written as a pure surface divergence does not have effects on the dynamics of the system under study, and therefore it is common to ignore completely such terms, and to focus our attention on the equations of motion and their consequences. For example, in the case of geometrical theories, it is well known that the Hilbert-Einstein action for the metric leads to a term proportional to the called Einstein tensor at the level of the field equations, plus the corresponding pure divergence term; such field equations correspond to the Einstein equations in a four-dimensional geometry, but does not give dynamics to the metric in a two-dimension worldsheet swept out by a string, since the symmetries of the curvature tensor imply that the Einstein tensor vanishes for such a geometry. Therefore, the Einstein-Hilbert action depends, in string theory, only on the topology of the worldsheet, contributing just with a pure divergence term, which is of course completely ignored in a conventional dynamics analysis.

Thus, there is no *apparently* any physical motivation for including such a topological term in any action describing strings, since the corresponding dynamics remains unaltered. For example, if we attempt to construct a (conventional) canonical formulation to quantize the Dirac-Nambu-Goto (DNG) strings from the corresponding classical dynamics, we shall obtain the same results whether we include the topological term, which turns out to be weird, at least from our particular point of view. On the other hand, it is very known also that the topological term has a global significance in the path integral formulation of string theory, weighting the different topologies in the sum over world surfaces. Thus, it is reasonable to think that a term that depends only on the global properties of the worldsheet, plays a non trivial role in such a global description of the theory.

With these preliminaries, the purpose of this work is to show that in a global description of the canonical formulation of string theory (as apposed to the *local* conventional description of the canonical formalism in terms of p's and q's widely disseminated in the literature), a topological term has effectively a global significance, such as it does in the path integral formulation of the theory. Such a global contribution of the topological term comes from the argument of the corresponding pure divergence term in a variational principle, which will be identified as a global 1-form on the covariant phase space of the theory, whose direct exterior derivation generates the integral kernel of a covariant and gauge invariant symplectic structure. As a by-product, it is shown that from a spurious total divergence term in a variational principle, one can identify physically relevant geometrical structures on the phase space.

In the next section, we summarize the basic aspects of the strongly covariant description of an imbedding given by Carter in [1], which are essential for our present aims. In Section III, we outline the definition of the covariant phase space and the exterior calculus associated with it. In Section IV, we give some remarks on the covariant canonical formulation of the DNG branes, in order to

prepare the background for the subsequent sections, where the topological terms are worked out. Specifically in Section V, the inner curvature scalar for an imbedded surface of arbitrary dimension is considered, and we show that in the particular case of a string world surface, it has not effectively any contribution to the string dynamics, but we can identify a symplectic potential for it. In Section VI, we considered again the inner curvature scalar, but now directly as a pure divergence term for a two-dimensional geometry; we find a full agreement with the previous results of Section V. In Section VII, the outer curvature scalar is considered as a Lagrangian term for string theory, and we identify for such a topological term its corresponding symplectic potential. In Section VIII, we give some concluding remarks, and we discuss some open questions for further research. In the Appendix A, we discuss the cases of the Yang-Mills theory and General Relativity and their respective topological terms. Finally in the Appendix B we summarize the basic formal aspects of symplectic geometry.

#### II. Basic differential geometry of an imbedded

In this Section we introduce the basic ideas of the imbedding supported background tensor approach developed by Carter [1], which will be used in the present treatment. In the Carter scheme, the emphasis is on the use of local coordinate patches on the background manifold for describing an imbedded p (brane world) surface in such a higher-dimensional background. The great virtue of this scheme is that avoids the (widely disseminated) use of excess mathematical baggage that obscures the simplicity and generality of laws and results on the subject, which is also manifested in the study of the symplectic geometry of the brane dynamics in [2, 3, 4].

Therefore, we outline the description given in [1] for the various kinds of curvature that are associated with a spacelike or timelike p-surface imbedded in a n-dimensional space or spacetime background with metric  $g_{\mu\nu}$ . Specifically the *internal curvature tensor* of the imbedding can be written as

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2n_{\sigma}{}^{\mu} n_{\nu}{}^{\tau} n_{[\lambda}{}^{\pi} \overline{\nabla}_{\kappa]} \rho_{\pi}{}^{\sigma}{}_{\tau} + 2\rho_{[\kappa}{}^{\mu\pi} \rho_{\lambda]\pi\nu}, \qquad (1)$$

where  $n^{\mu\nu}$  is the (first) fundamental tensor of the p-surface, that together with the complementary

orthogonal projection  $\perp^{\mu\nu}$  satisfy

$$n^{\mu}{}_{\nu} + \perp^{\mu}{}_{\nu} = g^{\mu}{}_{\nu}, \quad n^{\mu}{}_{\nu} \perp^{\nu}{}_{\rho} = 0, \tag{2}$$

and the tangential covariant differentiation operator is defined in terms of the fundamental tensor as

$$\overline{\nabla}_{\mu} = n^{\rho}{}_{\mu} \nabla_{\rho}, \tag{3}$$

where  $\nabla_{\rho}$  is the usual Riemannian covariant differentiation operator associated with  $g_{\mu\nu}$ . Additionally,  $\rho_{\lambda}{}^{\mu}{}_{\nu}$  represents the internal frame rotation (pseudo-) tensor field, or more specifically the background spacetime components of the internal frame components of the natural gauge connection for the group of p-dimensional internal frame rotations. The frame gauge dependence of this field will be crucial in order to establish our pretended results. It satisfies the properties

$$\rho_{\lambda\mu\nu} = -\rho_{\lambda\nu\mu}, \quad \perp^{\rho}{}_{\lambda} \ \rho_{\rho\mu\nu} = 0 = \perp^{\rho}{}_{\lambda} \ \rho_{\mu\rho\nu}, \tag{4}$$

whereas the internal curvature tensor (1) satisfies the usual Riemann symmetry properties and the Ricci contractions

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma}, \quad R = R_{\sigma}{}^{\sigma}, \tag{5}$$

with

$$\perp^{\sigma}{}_{\beta} R_{\sigma\lambda\mu\nu} = 0, \quad \perp^{\sigma}{}_{\beta} R_{\sigma\mu} = 0.$$
(6)

From the fundamental tensor and the Ricci contractions (5) one can define the internal adjusted Ricci tensor as

$$\widetilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2(p-1)} R n_{\mu\nu},$$
(7)

where p is the dimension of the imbedded p-surface. As pointed out in [1], for the special case p = 2 of a two-dimensional imbedded surface (that applies to string theory, for which this work is concerned), the adjusted Ricci tensor (7) vanishes identically:

$$\widetilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R \ n_{\mu\nu} = 0.$$
(8)

The identity (8) will imply, as we shall see below, that the inner curvature scalar given in (5) can not give any effective contribution in a variational principle, as already it was mentioned in the introduction. Similarly, we have the outer curvature tensor of the imbedded in terms of the external gauge connection  $\omega_{\lambda}^{\mu}{}_{\nu}$  [1],

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2 \bot_{\sigma}{}^{\mu} \bot_{\nu}{}^{\tau} n_{[\lambda}{}^{\pi}\overline{\nabla}_{\kappa]} \omega_{\pi}{}^{\sigma}{}_{\tau} + 2\omega_{[\kappa}{}^{\mu\pi} \omega_{\lambda]\pi\nu},$$

with the restricted symmetries

$$\Omega_{\mu\nu\rho\sigma} = \Omega_{[\mu\nu]} \ _{[\rho\sigma]}, \quad \Omega_{\lambda\nu} = \Omega_{\sigma\lambda}{}^{\sigma}{}_{\nu} = 0, \quad \Omega = \Omega^{\lambda}{}_{\lambda} = 0, \tag{9}$$

and thus the outer curvature tensor is purely Weyl-like, since all its traces vanish. However, as we shall see in Section VII, one can even construct a (pseudo-) scalar invariant  $\Omega$  for dimensionally restricted geometries [1], which will be related with a topological invariant, the outer analogue of the well known Gauss-Bonnet invariant associated with the inner curvature scalar.

#### III. Covariant phase space and the exterior calculus

In accordance with [5], in a given physical theory, the classical phase space is the space of solutions of the classical equations of motion, which corresponds to a manifestly covariant definition. Based on this definition, the idea of giving a covariant description of the canonical formalism consists in describing Poisson brackets of the theory in terms of a symplectic structure on such a phase space in a covariant way, instead of choosing p's and q's. Strictly speaking, a symplectic structure is a (non degenerate) closed two-form on the phase space; hence, for working in this scheme an exterior calculus associated with the phase space is fundamental. We summarize and adjust all these basic ideas about the phase space formulation given in Ref. [5] for the case of branes treated here [2].

Let Z be the phase space; any (unperturbed) background quantity such as the background and internal metrics, the projection tensors, connections, etc., will be associated with zero-forms on Z (see Appendix B). The Lagrangian deformation  $\delta$  acts as an exterior derivative on Z, taking k-forms into (k + 1)-forms, and it should satisfy the nilpotency property,

$$\delta^2 = 0, \tag{10}$$

and the Leibniz rule

$$\delta(AB) = \delta AB + (-1)^A A \ \delta B$$

A differential form A that satisfies  $\delta A = 0$ , is called *closed*. If the differential form A can be written as the exterior derivative of another form B (of lower order)  $A = \delta B$ , is called *exact*. Thus, any exact form is automatically closed, because of the nilpotency property (10). In general, the differential forms satisfy the Grassman algebra:  $AB = (-1)^{AB} BA$ .

In particular, the deformation in the coordinate field  $x^{\mu}$  of the background  $\xi^{\mu} = \delta x^{\mu}$ , is the exterior derivative of the zero-form  $x^{\mu}$ , and corresponds to an one-form on Z, and thus is an anticommutating object:  $\xi^{\mu} \xi^{\lambda} = -\xi^{\lambda} \xi^{\mu}$ . In according to (10),  $\xi^{\mu}$  will be closed,  $\delta \xi^{\mu} = \delta^2 x^{\mu} = 0$ , which is evident from the explicit form of  $\delta \xi^{\mu}$  given in [6]:

$$\delta \xi^{\mu} = -\Gamma^{\mu}_{\lambda\nu} \xi^{\lambda} \xi^{\nu} = 0, \qquad (11)$$

which vanishes because of the symmetry of the background connection  $\Gamma^{\mu}_{\lambda\nu}$  in its indices  $\lambda$  and  $\nu$ and the anticommutativity of the  $\xi^{\lambda}$  on Z. In Appendix B, we summarize other formal aspects of symplectic geometry, particularly that associated with an ordinary scalar field, with the idea of clarifying the basic scheme in the simplest case. A more complete treatment of the symplectic geometry for systems with support confined to a lower dimensional submanifold is given in [7]. However, for our purposes, the results of this section are sufficient.

# IV. Global structure of the phase space of DNG branes from a global symplectic potential

In this section, before considering the topological terms, we shall give some remarks on the covariant canonical formulation of the DNG action for an arbitrary brane developed in [2, 3], in order to prepare the background and to clarify the panorama for the subsequent inclusion of the topological terms. It is convenient to do the general treatment for branes of arbitrary dimension, and then to consider the particular case of string theory, which will show the particularities of the later with respect to the former.

The action for DNG branes in a curved embedding background is given by [6]

$$S_0 = \sigma_0 \int \sqrt{-\gamma} \, d\overline{S},\tag{12}$$

where  $\sigma_0$  is a fixed parameter,  $d\overline{S}$  is the surface element induced on the world surface by the background metric. The first order (Lagrangian) variation of  $S_0$  implies that [6]

$$\sigma_o \int \sqrt{-\gamma} \,\overline{\nabla}_\mu (n^\mu{}_\nu \,\xi^\nu) d\overline{S} - \sigma_0 \int \sqrt{-\gamma} \,\xi^\nu \,\overline{\nabla}_\mu (n^\mu{}_\nu) d\overline{S} = 0, \tag{13}$$

From Eq. (13) it follows that, modulo a total divergence, the equations of motion are

$$\overline{\nabla}_{\mu} \ n^{\mu\nu} = K^{\nu} = 0, \tag{14}$$

where  $K^{\nu}$  is the trace of the second fundamental tensor defined as  $K_{\mu\nu}{}^{\rho} = n^{\lambda}{}_{\nu} \overline{\nabla}_{\mu} n^{\rho}{}_{\lambda}$ , thus  $K^{\nu} = K^{\mu}{}_{\mu}{}^{\nu}$ .

From the equations (14) we can define the fundamental concept in the global description of the canonical formulation of the theory: the covariant phase space of DNG branes is the space of solutions of Eqs. (14), and we shall call it Z.

Following the spirit of the present work of that the pure divergence term in (13) be no ignored, in [3] we have demonstrated that the argument of such a term plays the role of a global symplectic potential on Z, in the sense that its exterior derivative on Z (identified with the deformation operator  $\delta$ , according to section III), generates the integral kernel (the symplectic current) of a (non degenerate) closed two-form on Z, which represents the symplectic structure that contains all the physical information on the Hamiltonian structure of the phase space, representing thus a starting point for the study of the symmetry and quantization aspects of the theory. Specifically the symplectic structure is given by

$$\omega = \int_{\Sigma} \delta(-\sqrt{-\gamma} \ n^{\mu}{}_{\alpha} \ \xi^{\alpha}) d\overline{\Sigma}_{\mu} = \int_{\Sigma} \sqrt{-\gamma} \ \widetilde{J}^{\mu} \ d\overline{\Sigma}_{\mu}, \tag{15}$$

with  $\sqrt{-\gamma} \tilde{J}^{\mu} = \delta(-\sqrt{-\gamma} n^{\mu}{}_{\alpha} \xi^{\alpha})$ ,  $\Sigma$  being a (spacelike) Cauchy surface for the configuration of the brane, and  $d\overline{\Sigma}_{\mu}$  is the surface measure element of  $\Sigma$ , and is normal on  $\Sigma$  and tangent to the world-surface; Eq. (15) shows that  $\omega$  is an *exact* differential form (since comes from the exterior derivative of an one-form), and in particular an identically closed two-form on Z. The closeness is equivalent to the Jacobi identity that Poisson brackets satisfy, in an usual Hamiltonian scheme. Moreover, in [3] it is proved that the symplectic current is (world surface) covariantly conserved  $(\overline{\nabla}_{\mu} \tilde{J}^{\mu} = 0)$ , which guarantees that  $\omega$  in (15) is independent on the choice of  $\Sigma$  and, in particular, is Poincaré invariant. In this manner, as a conclusion of this section, the argument of the pure divergence term in (13) represents a fundamental (gauge) field on Z, generating the (strength) geometrical structure  $\omega$  on Z, which in turns represents a complete *Hamiltonian* description of the covariant phase space of the theory.

#### V. Inner curvature scalar as a Lagrangian term

In this section, we shall consider a Hilbert term, proportional to the inner curvature scalar of the imbedded p-surface

$$\chi = \sigma_1 \int \sqrt{-\gamma} \ R \ d\overline{\Sigma},\tag{16}$$

where  $\sigma_1$  is a fixed parameter, and let us determine its contribution to the brane dynamics modulo a pure divergence term in a variational principle.

Within the covariant scheme given by Carter [6] for the deformations dynamics, it is known that

$$\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma} \ n^{\mu\nu} \ \delta \ g_{\mu\nu},\tag{17}$$

where the variation of the background metric is given by its Lie derivative with respect to the deformation vector field  $\xi^{\mu} = \delta x^{\mu}$ :

$$\delta g_{\mu\nu} = \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}.$$

In order to determine the variation of the scalar R, let us calculate first the variation of the internal curvature tensor (1), and its contractions, exploiting the frame gauge dependence of  $\rho_{\lambda}{}^{\mu}{}_{\nu}$ , which means that it can always be set equal to zero at any single chosen point by an appropriate choice of the relevant frames [1]. Therefore, if we consider a variation of  $\rho_{\lambda}{}^{\mu}{}_{\nu}$  to a new connection

$$\rho_{\lambda}{}^{\mu}{}_{\nu} \to \rho_{\lambda}{}^{\mu}{}_{\nu} + \delta \ \rho_{\lambda}{}^{\mu}{}_{\nu},$$

then this variation leads to a variation in the internal curvature tensor given by

$$\delta R_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2n_{\sigma}{}^{\mu} n_{\nu}{}^{\tau} n_{[\lambda}{}^{\pi} \overline{\nabla}_{\kappa]} \delta \rho_{\pi}{}^{\sigma}{}_{\tau}, \qquad (18)$$

and thus,

$$\delta R_{\mu\nu} = 2n_{\sigma}^{\kappa} n_{\nu}^{\tau} n_{[\mu}^{\pi} \overline{\nabla}_{\kappa]} \delta \rho_{\pi}^{\sigma}{}_{\tau}, \qquad (19)$$

and hence, we find that

$$n^{\mu\nu} \,\delta R_{\mu\nu} = \overline{\nabla}_{\mu} \,\psi^{\mu}_{top},\tag{20}$$

where

$$\psi^{\mu}_{top} = n^{\alpha\beta} \,\delta \,\rho_{\alpha}{}^{\mu}{}_{\beta} - n^{\alpha}_{\beta} \,n^{\mu\tau} \,\delta \,\rho_{\alpha}{}^{\beta}{}_{\tau}.$$
<sup>(21)</sup>

It is important to note that, although the connection  $\rho_{\alpha}{}^{\beta}{}_{\tau}$  can always be set equal to zero, its variation  $\delta \rho_{\alpha}{}^{\beta}{}_{\tau}$ , a tensor field, can no be in general gauged away.

The equations (18)–(20) may be the analogue of the very known Palatini equations in the context of general relativity, where such equations are used in order to obtain the Einstein equations from the Hilbert action.

Finally, using Eqs. (18) and (21), and considering that  $R = n^{\mu\nu} R_{\mu\nu}$ , one can calculate the variation of  $\chi$  in (16):

$$\delta \chi = \sigma_1 \int \sqrt{-\gamma} \left( \frac{1}{2} R \ n^{\mu\nu} - R^{\mu\nu} \right) \delta g_{\mu\nu} d\overline{\Sigma} + \sigma_1 \int \sqrt{-\gamma} \overline{\nabla}_{\mu} \ \psi^{\mu}_{top} d\overline{\Sigma}; \tag{22}$$

Eq. (22) gives the universal contribution of the (inner) curvature scalar as a Lagrangian term on the brane dynamics through the first term on the right hand side, and, as one can already guess at this point, its universal contribution to the symplectic potential on the phase space of the brane theory through  $\psi^{\mu}_{top}$ , in the second term. In general,  $\frac{1}{2}R n^{\mu\nu} - R^{\mu\nu}$  does not vanish for a geometry of arbitrary dimension, and therefore, in general,  $\chi$  change simultaneously the brane dynamics, the phase space, and the symplectic structure of the later (and hence this work may no make sense in such a general situation). However, as discussed in Section II (see Eq. (8)), the adjusted Ricci tensor vanishes identically for string theory, and  $\chi$  does not give dynamics to such objects (and for convenient boundary conditions, the surface term in (22) can be completely eliminated). Moreover, by defining the covariant phase space as the space of solutions of the dynamics equations, the phase space itself is unmodified by the inclusion of  $\chi$  in string theory. However, considering our present procedure for identifying the contribution of any Lagrangian term to the global symplectic potential of the theory,  $\chi$  in Eq. (22) has already modified the symplectic structure of the (unmodified) phase space of the string theory by means of  $\psi^{\mu}_{top}$ . For instance, if we consider the more general action  $S = S_0 + \chi$  for the particular case of string theory, where  $S_0$  and  $\chi$  are given in Eqs. (12) and (16), respectively, the equations of motion are, according to Eqs. (8), (13), and (22), the same equations (14) obtained just for  $S_0$ , and hence the phase space is again Z. However, the symplectic potential on Z is no longer  $\sigma_0 n^{\mu}_{\nu} \xi^{\nu}$ , but  $\sigma_0 n^{\mu}_{\nu} \xi^{\nu} + \sigma_1 \psi^{\mu}_{top}$ , and the corresponding symplectic structure is given by

$$\omega = \int_{\Sigma} \delta \left[ \sqrt{-\gamma} \left( \sigma_0 \ n_{\nu}^{\mu} \ \xi^{\nu} + \sigma_1 \ \psi_{top}^{\mu} \right) \right] d \ \overline{\Sigma}_{\mu}, \tag{23}$$

which will be evidently closed, and similarly for any action describing strings.

#### VI. The inner curvature scalar for a string worldsheet

In the previous section we have found the contribution of the Gauss-Bonnet topological term to the Hamiltonian structure of string theory considering the more general case of a brane of arbitrary dimension, which shown the particularities of the string case as opposed to the other higherdimensional objects. However, we can determine the symplectic potential for string theory directly from the expression for the inner curvature scalar for a two-dimensional worldsheet, which has the well known property of being a pure surface divergence, avoiding the general brane geometry, and exploiting the particularities of a two-dimensional geometry.

In the strong covariant scheme given in [1], it is shown that the inner curvature scalar can be written as

$$R = \overline{\nabla}_{\mu} \ (\mathcal{E}^{\mu\nu} \ \rho_{\nu}), \tag{24}$$

where the frame independent antisymmetric unit surface element tensor  $\mathcal{E}^{\mu\nu}$  is defined as

$$\mathcal{E}^{\mu\nu} = 2 \,\iota_0^{[\mu} \,\iota_1^{\nu]}; \tag{25}$$

 $\iota_0^{\mu}$  is a timelike unit vector, and  $\iota_1^{\mu}$  a spacelike one, which constitute an orthonormal tangent (to the worldsheet) frame. The rotation (co)vector  $\rho_{\mu}$  is defined in terms of the internal connection as

$$\rho_{\lambda} = \rho_{\lambda}{}^{\mu}{}_{\nu} \mathcal{E}^{\nu}{}_{\mu}, \quad \rho_{\lambda}{}^{\mu}{}_{\nu} = \frac{1}{2} \mathcal{E}^{\mu}{}_{\nu} \rho_{\lambda}.$$
(26)

In accordance with Eqs. (26), the frame gauge dependence of  $\rho_{\lambda}{}^{\mu}{}_{\nu}$  induces the same gauge dependence on  $\rho_{\mu}$  (see paragraph after Eq. (17)); therefore a variation  $\rho_{\mu} \rightarrow \rho_{\mu} + \delta \rho_{\mu}$  leads to a variation in (frame gauge dependent) R given by

$$\delta R = \overline{\nabla}_{\mu} \left( \mathcal{E}^{\mu\nu} \ \delta \ \rho_{\nu} \right). \tag{27}$$

Hence, the variation of  $\chi$  in Eq. (16) (with R given by (24) for a string) can be written simply as

$$\delta \ \chi = \sigma_1 \int \sqrt{-\gamma} \ \overline{\nabla}_\mu \ \left( \mathcal{E}^{\mu\nu} \ \delta \ \rho_\nu \right) \ d\overline{\Sigma}, \tag{28}$$

and we identify  $(\sigma_1)\mathcal{E}^{\mu\nu}\delta\rho_{\nu}$  as a symplectic potential for  $\chi$  in string theory. Considering that from Eqs. (26),  $\delta \rho_{\lambda}{}^{\mu}{}_{\nu} = \frac{1}{2}\mathcal{E}^{\mu}{}_{\nu} \delta\rho_{\lambda}$ , that  $n^{\mu\nu} = \iota_{0}^{\mu} \iota^{0\nu} + \iota_{1}^{\mu} \iota^{1\nu}$  [1], and Eq. (25), it is very easy to verify that  $\psi^{\mu}_{top}$  in Eq. (21) corresponds, for string theory, exactly to  $\mathcal{E}^{\mu\nu}\delta\rho_{\nu}$ . In this manner, there exists a full agreement between both approaches for finding out the symplectic potential for  $\chi$ . Note that in this case, there is no restriction on the dimension of the background geometry.

#### VII. The outer curvature scalar for a string worldsheet

In the case of a world sheet embedded in a four-dimensional background spacetime, using the standard fully antisymmetric four-volume measure tensor of the background  $\varepsilon^{\lambda\mu\nu\rho}$ , and the outer curvature scalar given in Section II, one can determine a scalar magnitude  $\Omega$  given by [1, see also 6, and 8]

$$\Omega = \frac{1}{2} \ \Omega_{\lambda\mu\nu\rho} \ \varepsilon^{\lambda\mu\nu\rho},$$

and a *twist* convector  $\omega_{\mu}$  (the outer analogue of  $\rho_{\mu}$ ), in the form

$$\omega_{\nu} = \frac{1}{2} \, \omega_{\nu}{}^{\mu\lambda} \, \varepsilon_{\lambda\mu\rho\sigma} \, \mathcal{E}^{\rho\sigma}, \tag{29}$$

and therefore, we can rewrite  $\Omega$  as a pure divergence as

$$\Omega = \overline{\nabla}_{\mu} \ (\mathcal{E}^{\mu\nu} \ \omega_{\nu}), \tag{30}$$

which is frame gauge dependent and is the (dimensionally restricted) outer analogue of R in Eq. (24). Thus, the world surface integral of  $\Omega$  gives a topological term expressed as

$$\chi' \equiv \sigma_2 \int \sqrt{-\gamma} \ \Omega \ d\overline{\Sigma} = \sigma_2 \int \sqrt{-\gamma} \ \overline{\nabla}_{\mu} \ (\mathcal{E}^{\mu\nu} \ \omega_{\nu}) \ d\overline{\Sigma}, \tag{31}$$

where  $\sigma_2$  is a fixed parameter. Hence,  $\chi'$  gives no an effective contribution on the string dynamics. Appealing to the frame gauge dependence of  $\omega_{\nu}$  inherited from  $\omega_{\nu}{}^{\mu\lambda}$ , the variation of  $\chi'$  is given by

$$\delta \chi' = \sigma_2 \int \sqrt{-\gamma} \,\overline{\nabla}_\mu \,\left(\mathcal{E}^{\mu\nu} \,\delta \,\omega_\nu\right) \,d\overline{\Sigma},\tag{32}$$

and we identify  $(\sigma_2)\mathcal{E}^{\mu\nu} \delta \omega_{\nu}$  as a symplectic potential for  $\chi'$  in string theory (in a four-dimensional background). Note that although  $\omega_{\nu}$  can be set equal to zero at any single point,  $\delta \omega_{\nu}$  can be no in general gauged away.

#### VIII. Remarks and prospects

#### A. Topological terms and deformation dynamics

An important conclusion from the previous sections is that the topological terms do not need modify the equations of motion of the theory for having an effective contribution on the symplectic properties of the phase space. Furthermore, the procedure followed in the present work for determining the corresponding symplectic potential for any Lagrangian term, may seem only a prescription without any solid basis (although there no either exists apparently some argument for ignoring such contributions), with the final purpose that the variation of those potentials will give an effective contribution on the integral kernel of the symplectic structure.

However, we have employed another approach for determining the contribution of any Lagrangian term on the kernel integral of the symplectic structure of the theory under study, and it consists in to construct the symplectic current from the corresponding deformation dynamics, using the concept of adjoint operators. For example, in [9] the symplectic currents originally suggested in [5] for Yang-Mills theory and General Relativity were found using the adjoint operators scheme. The corresponding current for branes in a curved background was obtained also using such a scheme in [2]; such a current corresponds exactly to that obtained using the procedure presented here, as we have outlined in Section IV. All these results suggest clearly the following: the topological terms can change drastically the deformation dynamics of the theory, without modifying the dynamics itself. With this idea, in [10] it is proved that the variation of the Einstein tensor modifies the deformation dynamics of string theory in a weakly (as opposed to the present strongly) covariant description of the theory; the symplectic current obtained from that modified deformation dynamics using the adjoint operators scheme, corresponds exactly to that obtained by variations of the symplectic potential obtained in [11] using the present procedure. Therefore, it remains a detailed study of the contributions of the topological terms considered in the present work, on the deformation dynamics of string theory. The importance of the study of a (modified) deformation dynamics goes beyond the present interest in the symplectic geometry of the phase space, for example the stability aspects of the solutions of the theory, among other.

#### B. Possible physical implications of $\psi^{\mu}_{top}$

Once we have determined explicitly the contribution of the topological terms on the global phase space formulation for string theory, it is important to discuss about the possible implications in a more physical context. As discussed in the introduction, a symplectic structure on the phase space is finally a *Hamiltonian* structure for the theory, and thus represents a starting point for the study of the symmetry and quantization aspects. For example, in [4] the Poincaré charges, the closeness of the Poincaré algebra, relevant commutation relations for DNG branes (which contains the string case as a particular case) were studied using precisely the symplectic geometry of the phase space established in [2, 3]. Does change the inclusion of the topological terms the results obtained in [4] for DNG strings, leaving the dynamics unchanged?

#### C. A new type of topological strings?

The Lagrangian terms associated with the topological terms always have been considered as corrective or additional terms of other Lagrangian terms, which whether give dynamics to the system. Apparently the fact that the topological terms leave unchanged the dynamics does no permit that one may consider only such terms "for making physics". However, the results presented here may suggest that, in spite of its null dynamics, the existence of  $\psi^{\mu}_{top}$  for a Lagrangian involving only the topological terms may imply that "the physics" for such (hypothetical) topological strings will be in other domain, different to the classical one, since finally a symplectic structure (that obtained from  $\psi^{\mu}_{top}$ ) governs the "transition" between the classical and quantum domains. However, it is important to point out that we are only speculating. It is possible that finally the answer for this open question turns out to be trivially simply.

#### **D.** Final comments

The results presented here may extend the role played by the topological terms in the context of string theory, and possibly other areas. Such results, although certainly limited, suggest a deeper research with a perspective different to that usually given in the known literature.

As we have seen in this work, the differential geometry of an imbedded developed in [1] has been crucial in order to establish our results, and various aspects considered in that reference had not been completely treated even in the pure mathematical context, as pointed out by Carter himself [1]. In this sense, following the spirit of the mathematical physics, the present work can be considered as an attempt for extracting *physics* from the Carter formalism.

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# Appendix A: Symplectic potentials and topological terms for Yang-Mills theory and General Relativity

#### A. Yang-Mills Theory

As we know, the Yang-Mills theory is a generalization of Maxwell's electromagnetic theory, and the action for this theory is

$$L = -\frac{1}{2} \int Tr(F^{\mu\nu} F_{\mu\nu}) d^4x,$$
 (A1)

where  $F_{\mu\nu}$  is the Yang-Mills curvature given by

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}], \qquad (A2)$$

and  $A_{\nu}$  is the gauge connection. The variation of the curvature (A2) is

$$\delta F_{\mu\nu} = \partial_{\mu} \ \delta \ A_{\nu} - \partial_{\nu} \ \delta \ A_{\mu} + [\delta \ A_{\mu}, A_{\nu}] + [A_{\mu}, \delta \ A_{\nu}]. \tag{A3}$$

In this manner, using the equation (A3) we can calculate the variation of the action (A1), as usual in ordinary field theory, and to obtain

$$\delta L = -\int Tr(F^{\mu\nu} \ \delta \ F_{\mu\nu})d^4x = -2\int \partial_{\mu}Tr[(\delta \ A_{\nu} \ F^{\mu\nu})]d^4x + 2\int Tr[(\partial_{\mu} \ F^{\mu\nu} + [A_{\mu}, F^{\mu\nu}])\delta \ A_{\nu}]d^4x,$$
(A4)

where we can find the equations of motion

$$\partial_{\mu} F^{\mu\nu} + [A_{\mu}, F^{\mu\nu}] = 0, \tag{A5}$$

and following the ideas of the present work, the argument of the pure divergence term in (A4) (traditionally ignored in the literature),

$$\Psi^{\mu} \equiv -Tr[\delta \ A_{\nu} \ F^{\mu\nu}],\tag{A6}$$

that does not contribute to the dynamics of the system, works as a symplectic potential on phase space.

If we take the variation of  $\Psi^{\mu}$  in equation (A6), we obtain

$$\delta \Psi^{\mu} = Tr[\delta A_{\nu} \ \delta \ F^{\mu\nu}],\tag{A7}$$

where we have considered that  $\delta$  is nilpotent and the Leibniz rule. We can see that the expression (A7) is exactly the symplectic current suggested in [5] and obtained in [9] applying the method of self-adjoint operators. Thus,  $\delta \Psi^{\mu}$  is a covariantly conserved because of the self-adjointness of the linearized theory [9], and by equation (A7) is closed because of the nilpotency of  $\delta$ . Therefore, the two-form

$$\omega = \int_{\Sigma} \delta(\Psi^{\mu}) d\Sigma_{\mu}, \tag{A8}$$

where  $\Sigma$  is a Cauchy hypersurface, is a symplectic structure for Yang-Mills theory.

Furthermore, it is well known that one can construct a topological term for the Yang-Mills theory given essentially by

$$\epsilon^{\mu\nu\sigma\rho} Tr(F_{\mu\nu} F_{\sigma\rho}), \tag{A9}$$

which is a total derivative and does not give dynamics to the gauge field, but as we already known, will contribute explicitly to the symplectic potential  $\Psi^{\mu}$  in Eq. (A6) (and to the linearized equations), without modifying the equations of motion (A5).

#### **B.** General Relativity

Let us consider the Einstein-Hilbert action

$$L = \int \sqrt{-g} R d^4x, \tag{A10}$$

where g is the determinant of the metric tensor, and R is the scalar curvature. Considering that

$$\delta \sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}, \qquad (A11)$$

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\gamma\nu} \delta g_{\alpha\gamma}, \tag{A12}$$

$$\delta R_{\mu\nu} = \nabla_{\gamma} \ \delta \ \Gamma_{\mu\nu}{}^{\gamma} - \nabla_{\nu} \ \delta \ \Gamma_{\mu\gamma}{}^{\gamma}, \tag{A13}$$

we can calculate the variation of R using the equations (A12) and (A13), obtaining

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}$$
$$= -g^{\mu\alpha} g^{\gamma\nu} \delta g_{\alpha\gamma} R_{\mu\nu} + g^{\mu\nu} [\nabla_{\gamma} \delta \Gamma_{\mu\nu}{}^{\gamma} - \nabla_{\nu} \delta \Gamma_{\mu\gamma}{}^{\gamma}].$$
(A14)

In this manner, the variation of L in equation (A10) is

$$\delta L = \int [\delta \sqrt{-g} R + \sqrt{-g} \delta R] d^4 x$$
  
= 
$$\int \sqrt{-g} (-R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R) \delta g_{\mu\nu} d^4 x + \int \sqrt{-g} \nabla_{\gamma} [g^{\mu\nu} \delta \Gamma_{\nu\mu}{}^{\gamma} - g^{\mu\gamma} \delta \Gamma_{\mu\alpha}{}^{\alpha}] d^4 x,$$
(A15)

where we can identify the very known equations of motion

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} R = 0, \qquad (A16)$$

and we identify from the pure divergence term in equation (A15), the following

$$\Psi^{\gamma} = \sqrt{-g} [g^{\mu\nu} \ \delta\Gamma_{\nu\mu}{}^{\gamma} - g^{\mu\gamma} \ \delta\Gamma_{\mu\alpha}{}^{\alpha}] \tag{A17}$$

as a symplectic potential for General Relativity, that does not contribute to the dynamics of system but will generate a symplectic structure on the phase space.

If we take the variation of (A17) we find

$$\delta \Psi^{\gamma} = \delta \sqrt{-g} [g^{\mu\nu} \ \delta \ \Gamma_{\nu\mu}{}^{\gamma} - g^{\mu\gamma} \ \delta \ \Gamma_{\mu\nu}{}^{\nu}] + \sqrt{-g} \ \delta [g^{\mu\nu} \ \delta \ \Gamma_{\nu\mu}{}^{\gamma} - g^{\mu\gamma} \ \delta \ \Gamma_{\mu\nu}{}^{\nu}]$$
  
$$= -\sqrt{-g} \ j^{\gamma}, \qquad (A18)$$

where  $j^{\gamma}$  is given by

$$j^{\gamma} = \delta \Gamma_{\nu\mu}{}^{\gamma} [\delta g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} \delta lng] - \delta \Gamma_{\mu\nu}{}^{\nu} [\delta g^{\gamma\mu} + \frac{1}{2} g^{\gamma\mu} \delta lng], \tag{A19}$$

that is the expression suggested in [5] and obtained in [9] applying the method of self-adjoint operators. In this manner  $j^{\gamma}$  is covariantly conserved because of the self-adjointness of the linearized theory [9], and by equation (A18) is closed.

Therefore

$$\omega = \int_{\Sigma} \delta(\Psi^{\gamma}) d\Sigma_{\gamma}, \tag{A20}$$

is a symplectic structure for General Relativity.

Similarly, in the case of General Relativity one can construct a topological term given, for specific numbers A, B, and C, as

$$\epsilon^{\mu\nu\alpha\beta} Tr(R_{\mu\nu} R_{\alpha\beta}) = \sqrt{-g} \left[ A R^2_{\mu\nu\alpha\beta} + B R^2_{\alpha\beta} + C R^2 \right], \tag{A21}$$

which, for convenient boundary conditions does not change the equations (A16), but it will contribute explicitly to the symplectic structure of the theory (A20).

It is well known the relevant role that the topological terms (A9), and (A21) play in the called gauge and gravitational anomalies (respectively) in the Feynman path integral formulation of the theories. Therefore, the results presented in this work may contribute to the study of the profound relationship between pure divergence terms, and topological numbers, but now in the setting of a canonical formulation of the theories, which is a subject practically unknown in the literature. We hope to extent all these subjects elsewhere.

#### Appendix B: Basic symplectic geometry

In this appendix we discuss briefly the basic elements of the symplectic geometry of a scalar field theory on spacetime (M), with the purpose of clarifying our basic ideas of Section IV of the present work. For more details about this little outline, see references [5].

If

$$\Delta \phi - V'(\phi) = 0,\tag{B1}$$

is the standard equation of motion for the scalar field  $\phi$  on spacetime, the covariant phase space Z corresponds in this case to the space of solutions of Eq. (B1). In this manner, if  $\phi \in Z$  and  $x \in M$ is a spacetime point, then we can define a function  $\hat{x}$  on Z by the mapping  $\hat{x} : Z \to R$ ,  $\hat{x}(\phi) = \phi(x)$ . The elements of the tangent vector space to Z at  $\phi$  ( $T_{\phi}Z$ ) correspond to solutions of the linearized equation

$$\Delta\delta\phi - V''(\phi)\ \delta\phi = 0,\tag{B2}$$

where  $\delta\phi$  is an infinitesimal displacement of  $\phi$ . Furthermore, we can define an one-form  $x^*$  (an element of the dual space  $T^*_{\phi}Z$  to  $T_{\phi}Z$ ), by the mapping  $T_{\phi}Z \to R$ ,  $x^*(\delta\phi) = \delta\phi(x)$ . In this manner  $\delta$  associates to the zero-form  $\hat{x}: Z \to R$ , the one-form  $\delta\hat{x} \equiv x^*: TZ \to R$ , according to the rule

$$\delta(\phi(x)) = \delta(\hat{x}(\phi)) \equiv \delta\hat{x}(\delta\phi) = x^*(\delta\phi) = \delta\phi(x).$$
(B3)

Therefore, misusing this definitions, we can denote the function  $\hat{x}(\phi)$  as  $\phi(x)$ , and the one-form  $\delta \hat{x}(\delta \phi)$  as  $\delta \phi(x)$ .

A general n-form can be written as

$$A = \int dx_1 \dots dx_n \alpha_{x_1 \dots x_n}(\phi) \delta\phi(x_1) \dots \delta\phi(x_n), \tag{B4}$$

where  $\alpha_{x_1...x_n}(\phi)$  is an arbitrary function for each *n*-tuple of the spacetime points  $x_1, ..., x_n$ . The action of  $\delta$  on this *n*-form is defined as

$$\delta A = \int dx_0 dx_1 \dots dx_n \frac{\delta \alpha_{x_1 \dots x_n}}{\delta \phi(x_0)} \delta \phi(x_0) \delta \phi(x_1) \dots \delta \phi(x_n), \tag{B5}$$

where  $\frac{\delta \alpha}{\delta \phi}$  denotes the variational derivative of  $\alpha$  with respect to  $\phi(x)$ . From this equation, it is easy to see that

$$\delta^2 A = \int dx'_0 dx_0 dx_1 \dots dx_n \frac{\delta^2 \alpha_{x_1 \dots x_n}}{\delta \phi(x'_0) \delta \phi(x_0)} \delta \phi(x'_0) \delta \phi(x_0) \delta \phi(x_1) \dots \delta \phi(x_n) = 0, \tag{B6}$$

which vanishes identically since the variational derivative in (B6) is symmetric respect to the interchange of  $\phi(x'_0)$  and  $\phi(x_0)$ , and the fact that  $\delta\phi(x'_0)$  and  $\delta\phi(x_0)$  are anticommutating objects (correspond, as seen above, to one-forms). Thus, we can establish that, in general,

$$\delta^2 = 0. \tag{B7}$$

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