

# Remarks on sufficient conditions for conservativity of minimal quantum dynamical semigroups

Changsoo Bahn

Natural Science Research Institute, Yonsei University, Seoul 120-749, Korea  
e-mail: bahn@yonsei.ac.kr

Chul Ki Ko

Natural Science Research Institute, Yonsei University, Seoul 120-749, Korea  
e-mail: kochulki@hotmail.com

Yong Moon Park

Department of Mathematics, Yonsei University, Seoul 120-749, Korea  
e-mail: ympark@yonsei.ac.kr

## Abstract

We obtain sufficient conditions for conservativity of minimal quantum dynamical semigroup by modifying and extending the method used in [1]. Our criterion for conservativity can be considered as a complement to Chebotarev and Fagnola's conditions [1]. In order to show that our conditions are useful, we apply our results to a concrete example ( a model of heavy ion collision).

*Keywords* : Quantum dynamical semigroups; criterion for conservativity; Lindblad operators.

## 1 Introduction

In this paper we are looking for any possible extension of Chebotarev and Fagnola's sufficient conditions[1] of conservativity of minimal quantum dynamical semigroup. By modifying and extending the method employed in [1], we obtain sufficient conditions for conservativity which extend the previous one in some directions. In order to show that our conditions are useful, we apply our results to a concrete example ( a model of heavy ion collision).

The concept of quantum dynamical semigroup(q.d.s.) has become a fundamental notion in study of irreversible evolutions in quantum mechanics [2, 3], open system [4] and quantum probability theory [5 - 7]. The theory of q.d.s. has been intensively studied in recent years laying special emphasis to the minimal q.d.s. as well as to sufficient conditions to ensure its conservativity (markovianity) [1, 8 - 13]. It is worthy to mention that there has been attention on the existence of stationary states for a given conservative q.d.s. and faithfulness of the stationary states[14, 15].

A q.d.s.  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  in  $\mathcal{B}(\mathfrak{h})$ , the Banach space of bounded operators in a Hilbert space  $\mathfrak{h}$ , is a (ultraweakly continuous) semigroup of completely positive linear maps on  $\mathcal{B}(\mathfrak{h})$ . A q.d.s.  $\mathcal{T}$  is *conservative* if  $\mathcal{T}_t(I) = I$  where  $I$  is the identity operator on  $\mathfrak{h}$ . In rather general cases, the infinitesimal generator  $\mathcal{L}$  can be written (formally) as

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}XM + \sum_{l=1}^{\infty} L_l^* X L_l - \frac{1}{2}MX, \quad X \in B(\mathfrak{h}) \quad (1.1)$$

where  $M = \sum_{l=1}^{\infty} L_l^* L_l$ ,  $L_l$  is densely defined and  $H$  a symmetric operator on  $\mathfrak{h}$ [16, 7]. However, for unbounded generator  $\mathcal{L}$  in (1.1) with (unbounded) coefficients  $H$  and  $L_l$ , the solution  $\mathcal{T}$  of the quantum master Markov equation

$$\frac{d}{dt}\mathcal{T}_t(X) = \mathcal{L}(\mathcal{T}_t(X)), \quad \mathcal{T}_0(X) = X, \quad (1.2)$$

may not be unique and conservative [8, 17]. Under suitable conditions, the above equation (1.2) has a minimal solution known as the minimal q.d.s.(see Sec. 2). Moreover if the minimal q.d.s. is conservative, it is the unique solution of the above equation. Also the study of conservativity conditions is important in quantum probability because they play a key role in the proof of uniqueness and unitarity of solutions of an Hudson-Parthasarathy quantum stochastic differential equation [18 - 20].

Chebotaev gave necessary and sufficient conditions for conservativity [8]. Some of the conditions, however, are impossible to check practically in many interesting examples. Simplified forms of sufficient conditions were developed in [1, 9, 10]. Especially the form

of sufficient conditions in [1] can be written as follows: there exists a positive self-adjoint operator  $C$  bounded from below by  $M$  satisfying a form inequality

$$\mathcal{L}(C) \leq bC \tag{1.3}$$

where  $b$  is a constant.

The main aim of this work is to improve the inequality (1.3). Our form of sufficient conditions for conservativity is as follows: there exists a positive self-adjoint operator  $C$  bounded from below by  $\delta M$  for some positive  $\delta > 0$  such that for all  $\epsilon \in (0, 1)$ , two inequalities

$$\mathcal{L}(C) \leq \epsilon C^2 + bC + a\epsilon^{-p}I, \tag{1.4}$$

$$i[H, C] + C^2 - \frac{1}{2}(MC + CM) \leq \epsilon C^2 + bC + a\epsilon^{-p}I \tag{1.5}$$

hold for some constants  $p \in (0, 1)$ ,  $b \geq 0$  and  $a \geq 0$ . For details, see Theorem 3.1.

In case the positive self-adjoint operator  $C$  satisfy (1.5), the inequality (1.4) improves (1.3) obviously. Let us mention that if we choose  $M$  for  $C$ ,

$$i[H, M] \leq \epsilon M^2 + bM + a\epsilon^{-p}I$$

is equivalent to (1.5). In order to explain our conditions (1.4) and (1.5) are useful in practical sense, we give some relative bounds(Lemma 4.1) and apply our result to a concrete q.d.s. associated to a quantum system with dissipative heavy ion collisions(Example 4.1). The conservativity of this example has been already considered in [1]. However, applying our criterion, we are able to control local singularities of (derivatives of ) coefficients of the infinitesimal generator (see Remark 4.2).

The paper is organized as follows. In Sec. 2, we give a brief review on the theory of minimal q.d.s. and criteria for conservativity. In Sec. 3, we first list our sufficient conditions for conservativity and then produce the proof of our result. In Sec. 4, we give some relative bounds to apply the results of Sec. 3 to a concrete q.d.s..

## 2 The minimal quantum dynamical semigroup

Let  $\mathfrak{h}$  be a complex separable Hilbert space with the scalar product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . Let  $\mathcal{B}(\mathfrak{h})$  denote the Banach space of bounded linear operators on  $\mathfrak{h}$ . The uniform norm in  $\mathcal{B}(\mathfrak{h})$  is denoted by  $\|\cdot\|_\infty$  and the identity in  $\mathfrak{h}$  is denoted by  $I$ . We denote by  $D(G)$  the domain of operator  $G$  in  $\mathfrak{h}$ .

**Definition 2.1** *A quantum dynamical semigroup (q.d.s.) on  $\mathcal{B}(\mathfrak{h})$  is a family  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  of operators in  $\mathcal{B}(\mathfrak{h})$  with the following properties:*

- (i)  $\mathcal{T}_0(X) = X$ , for all  $X \in \mathcal{B}(\mathfrak{h})$ ,
- (ii)  $\mathcal{T}_{t+s}(X) = \mathcal{T}_t(\mathcal{T}_s(X))$ , for all  $s, t \geq 0$  and all  $X \in \mathcal{B}(\mathfrak{h})$ ,
- (iii)  $\mathcal{T}_t(I) \leq I$ , for all  $t \geq 0$ ,
- (iv) (completely positivity) for all  $t \geq 0$ , all integer  $n$  and all finite sequences  $(X_j)_{j=1}^n, (Y_l)_{l=1}^n$  of elements of  $\mathcal{B}(\mathfrak{h})$ , we have

$$\sum_{j,l=1}^n Y_l^* \mathcal{T}_t(X_l^* X_j) Y_j \geq 0,$$

- (v) (normality) for every sequence  $(X_n)_{n \geq 1}$  of  $\mathcal{B}(\mathfrak{h})$  converging weakly to an element  $X$  of  $\mathcal{B}(\mathfrak{h})$  the sequence  $(\mathcal{T}_t(X_n))_{n \geq 1}$  converges weakly to an element  $\mathcal{T}_t(X)$  for all  $t \geq 0$ ,
- (vi) (ultraweak continuity) for all trace class operator  $\rho$  on  $\mathfrak{h}$  and all  $X \in \mathcal{B}(\mathfrak{h})$  we have

$$\lim_{t \rightarrow 0^+} \text{Tr}(\rho \mathcal{T}_t(X)) = \text{Tr}(\rho X).$$

We recall that as a consequence of properties (iii), (iv) for each  $t \geq 0$  and  $X \in \mathcal{B}(\mathfrak{h})$ ,  $\mathcal{T}_t$  is a contraction, i.e.,

$$\|\mathcal{T}_t(X)\|_\infty \leq \|X\|_\infty. \tag{2.1}$$

Also recall that as a consequence of properties (iv), (vi), for all  $X \in \mathcal{B}(\mathfrak{h})$ , the map  $t \mapsto \mathcal{T}_t(X)$  is strongly continuous.

**Definition 2.2** A q.d.s.  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  on  $\mathcal{B}(\mathfrak{h})$  is called to be conservative if  $\mathcal{T}_t(I) = I$  for all  $t \geq 0$ .

As mentioned in Introduction, the natural generator of q.d.s. would be the Lindblad type generator [16, 7]. Letting

$$G = -iH - \frac{1}{2}M, \quad \text{where } M = \sum_{l=1}^{\infty} L_l^* L_l, \quad (2.2)$$

the infinitesimal generator in (1.1) can be formally written by

$$\mathcal{L}(X) = XG + G^*X + \sum_{l=1}^{\infty} L_l^* X L_l.$$

A very large class of q.d.s. was constructed by Davies [21] under the following assumption.

**A.** The operator  $G$  is the infinitesimal generator of a strongly continuous contraction semigroup  $P = (P(t))_{t \geq 0}$  in  $\mathfrak{h}$ . The domain of the operators  $(L_l)_{l=1}^{\infty}$  contains the domain  $D(G)$  of the operator  $G$ . For all  $v, u \in D(G)$ , we have

$$\langle v, Gu \rangle + \langle Gv, u \rangle + \sum_{l=1}^{\infty} \langle L_l v, L_l u \rangle = 0. \quad (2.3)$$

As a result of Proposition 2.5 of [10], we can assume only that the domain of the operators  $L_l$  contains a subspace  $D$  which is a core for  $G$  and (2.3) holds for all  $v, u \in D$ .

For all  $X \in \mathcal{B}(\mathfrak{h})$ , consider the sesquilinear form  $\mathcal{L}(X)$  on  $\mathfrak{h}$  with domain  $D(G) \times D(G)$  given by

$$\langle v, \mathcal{L}(X)u \rangle = \langle v, XGu \rangle + \langle Gv, Xu \rangle + \sum_{l=1}^{\infty} \langle L_l v, X L_l u \rangle. \quad (2.4)$$

Under the assumption **A**, one can construct a q.d.s.  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  satisfying the equation

$$\langle v, \mathcal{T}_t(X)u \rangle = \langle v, Xu \rangle + \int_0^t \langle v, \mathcal{L}(\mathcal{T}_s(X))u \rangle ds \quad (2.5)$$

for all  $v, u \in D(G)$  and all  $X \in \mathcal{B}(\mathfrak{h})$ . For a strongly continuous family  $(\mathcal{T}_t(X))_{t \geq 0}$  of elements of  $\mathcal{B}(\mathfrak{h})$  satisfying (2.1), the followings are equivalent:

(i) equation (2.5) holds for all  $v, u \in D(G)$ ,

(ii) for all  $v, u \in D(G)$  we have

$$\begin{aligned} \langle v, \mathcal{T}_t(X)u \rangle &= \langle P(t)v, XP(t)u \rangle \\ &+ \sum_{l=1}^{\infty} \int_0^t \langle L_l P(t-s)v, \mathcal{T}_s(X)L_l P(t-s)u \rangle ds. \end{aligned} \quad (2.6)$$

We refer to the proof of Proposition 2.3 in [1]. A solution of the equation (2.6) is obtained by the iterations

$$\begin{aligned} \langle u, \mathcal{T}_t^{(0)}(X)u \rangle &:= \langle P(t)u, XP(t)u \rangle, \\ \langle u, \mathcal{T}_t^{(n+1)}(X)u \rangle &:= \langle P(t)u, XP(t)u \rangle \\ &+ \sum_{l=1}^{\infty} \int_0^t \langle L_l P(t-s)u, \mathcal{T}_s^{(n)}(X)L_l P(t-s)u \rangle ds \end{aligned} \quad (2.7)$$

for all  $u \in D(G)$ . In fact, for all positive elements  $X \in \mathcal{B}(\mathfrak{h})$  and all  $t \geq 0$ , the sequence of operators  $(\mathcal{T}_t^{(n)}(X))_{n \geq 0}$  is non-decreasing. Therefore it is strongly convergent and its limits for  $X \in \mathcal{B}(\mathfrak{h})$  and  $t \geq 0$  define the *minimal solution*  $\mathcal{T}^{(min)}$  of (2.6) in the sense that, given another solution  $(\mathcal{T}'_t)_{t \geq 0}$  of (2.5), one can easily check that

$$\mathcal{T}_t^{(min)}(X) \leq \mathcal{T}'_t(X) \leq \|X\|_{\infty} I$$

for any positive element  $X$  and all  $t \geq 0$ . For details, we refer to [8, 11].

We recall here a necessary and sufficient condition for conservativity of minimal q.d.s. obtained by Chebotarev. Let us consider the linear monotone maps  $\mathcal{P}_{\lambda} : \mathcal{B}(\mathfrak{h}) \rightarrow \mathcal{B}(\mathfrak{h})$  and  $\mathcal{Q}_{\lambda} : \mathcal{B}(\mathfrak{h}) \rightarrow \mathcal{B}(\mathfrak{h})$  defined by

$$\langle v, \mathcal{P}_{\lambda}(X)u \rangle = \int_0^{\infty} e^{-\lambda s} \langle P(s)v, XP(s)u \rangle ds, \quad (2.8)$$

$$\langle v, \mathcal{Q}_{\lambda}(X)u \rangle = \sum_{l=1}^{\infty} \int_0^{\infty} e^{-\lambda s} \langle L_l P(s)v, XL_l P(s)u \rangle ds \quad (2.9)$$

for all  $\lambda > 0$  and  $X \in \mathcal{B}(\mathfrak{h})$ ,  $v, u \in D(G)$ . It is easy to check that both  $\mathcal{P}_\lambda$  and  $\mathcal{Q}_\lambda$  are completely positive, and also both  $\lambda\mathcal{P}_\lambda$  and  $\mathcal{Q}_\lambda$  are normal contractions in  $\mathcal{B}(\mathfrak{h})$  (see Sec. 2 of [10]).

The resolvent of the minimal q.d.s.  $(\mathcal{R}_\lambda^{(min)})_{\lambda>0}$  defined by

$$\langle v, \mathcal{R}_\lambda^{(min)}(X)u \rangle = \int_0^\infty e^{-\lambda s} \langle v, \mathcal{T}_s^{(min)}(X)u \rangle ds$$

(with  $X \in \mathcal{B}(\mathfrak{h})$  and  $v, u \in \mathfrak{h}$ ) can be represented as

$$\mathcal{R}_\lambda^{(min)}(X) = \sum_{k=0}^{\infty} \mathcal{Q}_\lambda^k(\mathcal{P}_\lambda(X)), \quad (2.10)$$

the series being convergent for the strong operator topology (see Theorem 3.1 of [1]).

**Proposition 2.1** *Suppose that the condition **A** holds and fix  $\lambda > 0$ . Then the sequence of positive operators  $(\mathcal{Q}_\lambda^k(I))_{k \geq 0}$  is non-increasing. Moreover the following conditions are equivalent:*

(i) *the minimal q.d.s.  $\mathcal{T}^{(min)}$  is conservative,*

(ii)  *$s\text{-}\lim_{k \rightarrow \infty} \mathcal{Q}_\lambda^k(I) = 0$ .*

The above proposition has been proved in [1, 10]. Due to Proposition 2.1, the minimal q.d.s. is conservative whenever, for a fixed  $\lambda > 0$ , the series

$$\sum_{k=0}^{\infty} \langle u, \mathcal{Q}_\lambda^k(I)u \rangle \quad (2.11)$$

is convergent for all  $u$  in a dense subspace of  $\mathfrak{h}$ . In fact in this case, the condition (ii) of Proposition 2.1 holds because the sequence of positive operators  $(\mathcal{Q}_\lambda^k(I))_{k \geq 0}$  is non-increasing.

Employing the above facts, Chebotarev and Fagnola have obtained a criteria to verify the conservativity of minimal q.d.s. ( see Sec. 4 in [1]). Here we give their result (Theorem 4.4 in [1]):

**Theorem 2.1** *Under the assumption **A** suppose that there exists a positive self-adjoint operator  $C$  in  $\mathfrak{h}$  with the following properties:*

- (a) *the domain  $D(G)$  of  $G$  is contained in the domain of the positive square root  $C^{1/2}$  and  $D(G)$  is a core for  $C^{1/2}$ ,*
- (b) *the linear manifolds  $L_l(D(G^2))$ ,  $l \geq 1$ , are contained in the domain of  $C^{1/2}$ ,*
- (c) *there exists a self-adjoint operator  $\Phi$ , with  $D(G) \subset D(\Phi^{1/2})$  and  $D(C) \subset D(\Phi)$ , such that, for all  $u \in D(G)$ , we have*

$$-2\operatorname{Re}\langle u, Gu \rangle = \sum_{l=1}^{\infty} \|L_l u\|^2 = \|\Phi^{1/2} u\|^2,$$

- (d) *for all  $u \in D(C)$  we have  $\|\Phi^{1/2} u\| \leq \|C^{1/2} u\|$ ,*

- (e) *for all  $u \in D(G^2)$  there exists a positive constant  $b$  depending only on  $G, C, L_l$*

$$2\operatorname{Re}\langle C^{1/2} u, C^{1/2} Gu \rangle + \sum_{l=1}^{\infty} \|C^{1/2} L_l u\|^2 \leq b \|C^{1/2} u\|^2. \quad (2.12)$$

*Then the minimal q.d.s. is conservative.*

We will call the conditions in Theorem 2.1 *C-F sufficient condition*.

### 3 Sufficient condition for conservativity

In this section we extend more or less C-F sufficient condition for conservativity of the minimal q.d.s.. First we introduce our assumption.

**C.** *There exists a positive self-adjoint operator  $C$  such that*

- (a) *the domain of its positive square root  $C^{1/2}$  contains the domain  $D(G)$  of  $G$  and  $D(G)$  is a core of  $C^{1/2}$ . Also the domain of  $C$  contains the domain of  $G^2$ .*

- (b) *the linear manifolds  $L_l(D(G^2))$ ,  $l \geq 1$ , are contained in the domain of  $C^{1/2}$ ,*



(c) there exist  $p \in (0, 1)$ ,  $b \geq 0$  and  $a \geq 0$  such that for any  $\varepsilon \in (0, 1)$  two inequalities

$$2\operatorname{Re}\langle Cu, Gu \rangle \leq -(1 - \varepsilon)\|Cu\|^2 + b\|C^{1/2}u\|^2 + a\varepsilon^{-p}\|u\|^2 \quad (3.1)$$

and

$$\begin{aligned} 2\operatorname{Re}\langle Cu, Gu \rangle + \sum_{l=1}^{\infty} \|C^{1/2}L_l u\|^2 \\ \leq \varepsilon\|Cu\|^2 + b\|C^{1/2}u\|^2 + a\varepsilon^{-p}\|u\|^2 \end{aligned} \quad (3.2)$$

hold for all  $u \in D(G^2)$ .

The following is our main result:

**Theorem 3.1** *Suppose that assumptions **A** and **C** hold for some positive self-adjoint operator  $C$  and there exists a positive self-adjoint operator  $\Phi$  in  $\mathfrak{h}$  such that:*

(a) *the domain of the positive square root  $\Phi^{1/2}$  contains the domain of  $G$  and, for every  $u \in D(G)$ , we have*

$$-2\operatorname{Re}\langle u, Gu \rangle = \sum_{l=1}^{\infty} \langle L_l u, L_l u \rangle = \langle \Phi^{1/2}u, \Phi^{1/2}u \rangle,$$

(b) *the domain of  $C$  is contained in the domain  $\Phi$  and, for some  $\delta > 0$ , we have*

$$\delta\langle \Phi^{1/2}u, \Phi^{1/2}u \rangle \leq \langle C^{1/2}u, C^{1/2}u \rangle, \quad \forall u \in D(C).$$

*Then the minimal q.d.s. is conservative.*

Before proceeding the proof of the above theorem, it may be worth to give some remarks on the assumption **C**.

**Remark 3.1** (a) *If we choose the operator  $C$  satisfying (3.1), the inequality (3.2) evidently improves (2.12) in C-F sufficient condition.*

(b) As mentioned in Introduction, the inequality (3.1) can be written formally by

$$i[H, C] + C^2 - \frac{1}{2}(MC + CM) \leq \varepsilon C^2 + bC + a\varepsilon^{-p}I.$$

If we choose  $C = M (= \sum_{i=1}^{\infty} L_i^* L_i)$ , then (3.1) is equivalent to the following condition

$$i\langle u, [H, M]u \rangle \leq \varepsilon \|Mu\|^2 + b\|M^{1/2}u\|^2 + a\varepsilon^{-p}\|u\|^2.$$

Thus, in many cases the condition (3.1) is easier to check than (3.2).

(c) As Kato's relative bounds[22] control local singularities of potentials in the Schrödinger operator, we believe that the bounds in (3.1) and (3.2) will be able to control local singularities of (derivatives of) the coefficients of generators of q.d.s..

In the rest of this section we produce the proof of Theorem 3.1. The following is an extension of the condition that the series (2.11) converges.

**Lemma 3.1** *Suppose that for fixed  $\lambda > 0$ , the series*

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \langle u, \mathcal{Q}_{\lambda}^k(I)u \rangle \tag{3.3}$$

*is convergent for all  $u$  in a dense subspace of  $\mathfrak{h}$ . Then we have  $s\text{-}\lim_{k \rightarrow \infty} \mathcal{Q}_{\lambda}^k(I) = 0$ .*

*Proof:* Notice that  $(\mathcal{Q}_{\lambda}^k(I))_{k \geq 0}$  is a positive and non-increasing sequence. Therefore it is strongly convergent to a positive operator  $Y$ , i.e.,

$$Y := s\text{-}\lim_{k \rightarrow \infty} \mathcal{Q}_{\lambda}^k(I) \geq 0.$$

Suppose that  $Y$  is not zero. Then there exists a non-zero vector  $u \in \mathfrak{h}$  such that  $\langle u, Yu \rangle > 0$ . This implies that

$$0 < \langle u, Yu \rangle \leq \langle u, \mathcal{Q}_{\lambda}^k(I)u \rangle \quad \text{for all } k \geq 0,$$

and also

$$\langle u, Yu \rangle \sum_{k=0}^n \frac{1}{k+1} \leq \sum_{k=0}^n \frac{1}{k+1} \langle u, \mathcal{Q}_\lambda^k(I)u \rangle$$

for any nonnegative integer  $n$ . Thus the series (3.3) is divergent, which is contrary to the assumption. Thus  $Y$  must be zero.  $\square$

By Proposition 2.1 and Lemma 3.1, the minimal q.d.s. is conservative whenever, for a fixed  $\lambda > 0$ , the series

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \langle u, \mathcal{Q}_\lambda^k(I)u \rangle$$

converges for all  $u$  in a dense subspace of  $\mathfrak{h}$ . By Monotone Convergence Theorem, we have

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \langle u, \mathcal{Q}_\lambda^k(I)u \rangle = \int_0^1 \left( \sum_{k=0}^{\infty} x^k \langle u, \mathcal{Q}_\lambda^k(I)u \rangle \right) dx. \quad (3.4)$$

Fix  $x \in (0, 1)$ . For all  $u \in D(G)$  and  $X \in \mathcal{B}(\mathfrak{h})$ , let  $\mathcal{T}_{t,x}^{(min)}(X)$  be the solution obtained by the iterations

$$\begin{aligned} \langle u, \mathcal{T}_{t,x}^{(0)}(X)u \rangle &= \langle P(t)u, XP(t)u \rangle, \\ \langle u, \mathcal{T}_{t,x}^{(n+1)}(X)u \rangle &= \langle P(t)u, XP(t)u \rangle \\ &\quad + x \sum_{l=1}^{\infty} \int_0^t \langle L_l P(t-s)u, \mathcal{T}_{s,x}^{(n)}(X) L_l P(t-s)u \rangle ds. \end{aligned} \quad (3.5)$$

For all  $u \in \mathfrak{h}$  and  $X \in \mathcal{B}(\mathfrak{h})$ , and for  $\lambda > 0$ , let

$$\begin{aligned} \langle u, \mathcal{R}_{\lambda,x}^{(n)}(X)u \rangle &= \int_0^{\infty} e^{-\lambda t} \langle u, \mathcal{T}_{t,x}^{(n)}(X)u \rangle dt, \\ \langle u, \mathcal{R}_{\lambda,x}^{(min)}(X)u \rangle &= \int_0^{\infty} e^{-\lambda t} \langle u, \mathcal{T}_{t,x}^{(min)}(X)u \rangle dt. \end{aligned} \quad (3.6)$$

Clearly (2.1) guarantees that  $\mathcal{R}_{\lambda,x}^{(n)}(X)$  and  $\mathcal{R}_{\lambda,x}^{(min)}(X)$  are well defined. We can also obtain the relation corresponding to (2.10).

**Proposition 3.1** *For any  $x \in (0, 1)$ ,  $\lambda > 0$  and  $X \in \mathcal{B}(\mathfrak{h})$  we have*

$$\mathcal{R}_{\lambda,x}^{(min)}(X) = \sum_{k=0}^{\infty} x^k \mathcal{Q}_\lambda^k(\mathcal{P}_\lambda(X)) \quad (3.7)$$

the series being convergent for the strong operator topology.

*Proof:* For any positive element  $X$  of  $\mathcal{B}(\mathfrak{h})$ , the sequence  $(\mathcal{R}_{\lambda,x}^{(n)}(X))_{n \geq 0}$  is non-decreasing. Therefore by (3.6), for all  $u \in \mathfrak{h}$  we have

$$\begin{aligned} \langle u, \mathcal{R}_{\lambda,x}^{(min)}(X)u \rangle &= \int_0^\infty e^{-\lambda t} \langle u, \mathcal{T}_{t,x}^{(min)}(X)u \rangle dt \\ &= \sup_{n \geq 0} \langle u, \mathcal{R}_{\lambda,x}^{(n)}(X)u \rangle. \end{aligned}$$

The second equation (3.5) yields

$$\begin{aligned} \langle u, \mathcal{R}_{\lambda,x}^{(n+1)}(X)u \rangle &= \int_0^\infty e^{-\lambda t} \langle P(t)u, XP(t)u \rangle dt \\ &\quad + x \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \int_0^t \langle L_l P(t-s)u, \mathcal{T}_{s,x}^{(n)}(X)L_l P(t-s)u \rangle ds dt \end{aligned} \quad (3.8)$$

for all  $u \in D(G)$ . By the change of variables in the above double integral and (2.8) we have

$$\begin{aligned} \langle u, \mathcal{R}_{\lambda,x}^{(n+1)}(X)u \rangle &= \langle u, \mathcal{P}_\lambda(X)u \rangle \\ &\quad + x \sum_{l=1}^\infty \int_0^\infty e^{-\lambda r} \int_0^\infty e^{-\lambda s} \langle L_l P(r)u, \mathcal{T}_{s,x}^{(n)}(X)L_l P(r)u \rangle ds dr. \end{aligned} \quad (3.9)$$

Thus we obtain the recursion relation

$$\mathcal{R}_{\lambda,x}^{(n+1)}(X) = \mathcal{P}_\lambda(X) + x \mathcal{Q}_\lambda(\mathcal{R}_{\lambda,x}^{(n)}(X)).$$

Iterating  $n$  times, we have

$$\mathcal{R}_{\lambda,x}^{(n+1)}(X) = \sum_{k=0}^{n+1} x^k \mathcal{Q}_\lambda^k(\mathcal{P}_\lambda(X)) \quad (3.10)$$

and (3.7) follows from letting  $n$  tend to  $\infty$ . Since any bounded operator can be written as a linear combination of four positive self-adjoint operators (3.7) also holds for an arbitrary element of  $\mathcal{B}(\mathfrak{h})$ .  $\square$

**Lemma 3.2** *Condition C implies that, for each  $u \in D(G^2)$ , the function  $t \mapsto \|C^{1/2}P(t)u\|^2$  is differentiable and*

$$\frac{d}{dt}\|C^{1/2}P(t)u\|^2 = 2\operatorname{Re}\langle C^{1/2}P(t)u, C^{1/2}GP(t)u \rangle.$$

*Proof:* For each  $u \in D(G)$  and each  $\lambda > 0$ , let  $v = \lambda(\lambda - G)^{-1}u := \lambda R(\lambda, G)u$ . Obviously  $v \in D(G^2)$ . The inequality (3.1) yields

$$\begin{aligned} \|C^{1/2}u\|^2 &= \frac{1}{\lambda^2}\langle C^{1/2}(\lambda - G)v, C^{1/2}(\lambda - G)v \rangle \\ &= \|C^{1/2}v\|^2 - 2\lambda^{-1}\operatorname{Re}\langle Cv, Gv \rangle + \lambda^{-2}\|C^{1/2}Gv\|^2 \\ &\geq (1 - \lambda^{-1}b)\|C^{1/2}v\|^2 - a\lambda^{-1}\varepsilon^{-p}\|v\|^2. \end{aligned} \quad (3.11)$$

Note that  $\|u\|^2 \geq \|\lambda R(\lambda, G)u\|^2$ . Let  $\beta = \max\{b, a\varepsilon^{-p}\}$ . It follows from (3.11) that the inequality

$$\begin{aligned} \|C^{1/2}u\|^2 + \|u\|^2 &\geq (1 - \lambda^{-1}b)\|C^{1/2}\lambda R(\lambda, G)u\|^2 + (1 - \lambda^{-1}a\varepsilon^{-p})\|\lambda R(\lambda, G)u\|^2 \\ &\geq (1 - \lambda^{-1}\beta)\left(\|C^{1/2}\lambda R(\lambda, G)u\|^2 + \|\lambda R(\lambda, G)u\|^2\right). \end{aligned} \quad (3.12)$$

The above inequality also holds for  $u \in D(C^{1/2})$  since  $D(G)$  is a core for  $C^{1/2}$ .

Note  $D(C^{1/2})$  is a Hilbert space endowed with the graph norm. Let  $\tilde{G} : D(C^{1/2}) \rightarrow D(C^{1/2})$  be given by  $D(\tilde{G}) = \{u \in D(G) : Gu \in D(C^{1/2})\}$  and  $\tilde{G}u = Gu$ , for all  $u \in D(\tilde{G})$ . It is easily checked that  $\tilde{G}$  is closed. Since  $D(G^2)$  is a core for  $G$  and  $D(G)$  is a core for  $C^{1/2}$ ,  $D(G^2)$  is a core for  $C^{1/2}$  (see Lemma 2.5 of [23]). Thus  $\tilde{G}$  is densely defined in the Hilbert space  $D(C^{1/2})$ . Let us check  $R(\lambda, \tilde{G})u = R(\lambda, G)u$  for all  $u \in D(C^{1/2})$ . If  $(\lambda - G)u \in D(C^{1/2})$  for  $u \in D(G)$ , then  $Gu \in D(C^{1/2})$  and we have  $u \in D(\tilde{G})$ . Since  $\lambda - G$  is a bijection from  $D(G)$  to  $\mathfrak{h}$ , the range of  $\lambda - \tilde{G}$  is  $D(C^{1/2})$ . Thus  $\lambda - \tilde{G}$  is invertible on  $D(C^{1/2})$  and  $R(\lambda, \tilde{G})$  is the restriction of  $R(\lambda, G)$  to  $D(C^{1/2})$ . Therefore the inequality (3.12) implies that  $\tilde{G}$  is the infinitesimal generator of a strongly continuous semigroup on the Hilbert space  $D(C^{1/2})$  endowed with the graph norm. See Sec. 1 Corollary 3.8

in [24]. This semigroup is obtained by restricting the operators  $P(t)$  to  $D(C^{1/2})$ . Since  $D(G^2) \subset D(\tilde{G})$ , the claimed differentiation formula follows.  $\square$

Under assumption **C** we can obtain a useful estimate of  $\mathcal{R}_{\lambda,x}^{(min)}(C_\epsilon)$  where  $(C_\epsilon)_{\epsilon>0}$  is the family of bounded regularization  $C_\epsilon = C(I + \epsilon C)^{-1}$ .

**Proposition 3.2** *Suppose that the conditions **A** and **C** hold. Then, for any  $x \in (0, 1)$ ,  $\lambda > \max(b, 1)$  and any  $u \in D(G^2)$ , the bound*

$$(\lambda - b) \sup_{\epsilon>0} \langle u, \mathcal{R}_{\lambda,x}^{(min)}(C_\epsilon)u \rangle \leq \|C^{1/2}u\|^2 + 2a(1-x)^{-p}\|u\|^2 \quad (3.13)$$

holds.

*Proof:* Let  $(\mathcal{R}_{\lambda,x}^{(n)})_{n \geq 0}$  be the sequence of monotone linear maps on  $\mathcal{B}(\mathfrak{h})$  defined in (3.6). Clearly it suffices to show that for all  $n \geq 0$ ,  $\lambda > \max(b, 1)$ ,  $x \in (0, 1)$  and  $u \in D(G^2)$ , the operator  $\mathcal{R}_{\lambda,x}^{(n)}(C_\epsilon)$  satisfies

$$(\lambda - b) \sup_{\epsilon>0} \langle u, \mathcal{R}_{\lambda,x}^{(n)}(C_\epsilon)u \rangle \leq \|C^{1/2}u\|^2 + 2a(1-x)^{-p}\|u\|^2. \quad (3.14)$$

For  $n = 0$ , integrating by parts, we have

$$\begin{aligned} \lambda \langle u, R_{\lambda,x}^{(0)}(C_\epsilon)u \rangle &= \lambda \langle u, \mathcal{P}_\lambda(C_\epsilon)u \rangle \\ &= \lambda \int_0^\infty e^{-\lambda t} \|C_\epsilon^{1/2}P(t)u\|^2 dt \\ &\leq \lambda \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt \\ &= \|C^{1/2}u\|^2 + 2\operatorname{Re} \int_0^\infty e^{-\lambda t} \langle CP(t)u, GP(t)u \rangle dt. \end{aligned} \quad (3.15)$$

Two inequalities (3.1) and (3.15) yield

$$\begin{aligned}
\lambda \langle u, \mathcal{R}_{\lambda, x}^{(0)}(C_\epsilon)u \rangle &\leq \|C^{1/2}u\|^2 - (1 - \epsilon) \int_0^\infty e^{-\lambda t} \|CP(t)u\|^2 dt \\
&\quad + b \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt + a\epsilon^{-p} \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt \\
&= \|C^{1/2}u\|^2 - (1 - \epsilon) \int_0^\infty e^{-\lambda t} \|CP(t)u\|^2 dt \\
&\quad + b \sup_{\epsilon > 0} \langle u, \mathcal{R}_{\lambda, x}^{(0)}(C_\epsilon)u \rangle + a\epsilon^{-p} \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt.
\end{aligned} \tag{3.16}$$

Notice that

$$\int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt \leq \frac{1}{\lambda} \|u\|^2. \tag{3.17}$$

Choose  $\epsilon = 1 - x$  in (3.16). Then for  $\lambda > 1/2$ , (3.14) holds for  $n = 0$ .

By induction, we assume that (3.14) holds for an integer  $n$ . It follows from (3.9) and (3.14) that

$$\begin{aligned}
\langle u, \mathcal{R}_{\lambda, x}^{(n+1)}(C_\epsilon)u \rangle &= \langle u, \mathcal{P}_\lambda(C_\epsilon)u \rangle \\
&\quad + x \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \langle L_l P(t)u, \mathcal{R}_{\lambda, x}^{(n)}(C_\epsilon)L_l P(t)u \rangle dt \\
&\leq \langle u, \mathcal{P}_\lambda(C_\epsilon)u \rangle + x \frac{1}{\lambda - b} \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|C^{1/2}L_l P(t)u\|^2 dt \\
&\quad + x \frac{1}{\lambda - b} 2a(1 - x)^{-p} \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|L_l P(t)u\|^2 dt.
\end{aligned} \tag{3.18}$$

By (2.3), we have

$$\begin{aligned}
\sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|L_l P(t)u\|^2 dt &= \int_0^\infty e^{-\lambda t} \left( -\frac{d}{dt} \|P(t)u\|^2 \right) dt \\
&= \|u\|^2 - \lambda \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt.
\end{aligned} \tag{3.19}$$

By (3.15), we also have

$$\begin{aligned}
& \langle u, \mathcal{P}_\lambda(C_\varepsilon)u \rangle \\
& \leq \frac{\lambda}{\lambda-b} \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt - \frac{b}{\lambda-b} \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt \\
& = \frac{1}{\lambda-b} \left( \|C^{1/2}u\|^2 + 2\operatorname{Re} \int_0^\infty e^{-\lambda t} \langle CP(t)u, GP(t)u \rangle dt \right) \\
& \quad - \frac{b}{\lambda-b} \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt
\end{aligned} \tag{3.20}$$

We combine (3.18), (3.19) and (3.20) to conclude that

$$\begin{aligned}
& (\lambda-b) \sup_{\varepsilon>0} \langle u, \mathcal{R}_{\lambda,x}^{(n+1)}(C_\varepsilon)u \rangle \leq \|C^{1/2}u\|^2 \\
& + 2\operatorname{Re} \int_0^\infty e^{-\lambda t} \langle CP(t)u, GP(t)u \rangle dt - b \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt \\
& + x \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|C^{1/2}L_l P(t)u\|^2 dt + 2a(1-x)^{-p} \left( \|u\|^2 - \lambda \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt \right).
\end{aligned} \tag{3.21}$$

Next, we use (3.2) with  $\varepsilon = (1-x)/2$  to obtain

$$\begin{aligned}
& 2x\operatorname{Re} \int_0^\infty e^{-\lambda t} \langle CP(t)u, GP(t)u \rangle dt + x \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|C^{1/2}L_l P(t)u\|^2 dt \\
& \leq \frac{x(1-x)}{2} \int_0^\infty e^{-\lambda t} \|CP(t)u\|^2 dt + xb \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt \\
& \quad + 2ax(1-x)^{-p} \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt.
\end{aligned} \tag{3.22}$$

On the other hand it follows from (3.1) with  $\varepsilon = 1/2$  that

$$\begin{aligned}
& 2(1-x)\operatorname{Re} \int_0^\infty e^{-\lambda t} \langle CP(t)u, GP(t)u \rangle dt \\
& \leq -\frac{(1-x)}{2} \int_0^\infty e^{-\lambda t} \|CP(t)u\|^2 dt \\
& \quad + (1-x)b \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt + 2a(1-x) \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt.
\end{aligned} \tag{3.23}$$



Summing (3.22) and (3.23) yields

$$\begin{aligned}
& 2\operatorname{Re} \int_0^\infty e^{-\lambda t} \langle CP(t)u, GP(t)u \rangle dt + x \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|C^{1/2}L_l P(t)u\|^2 dt \\
& \leq b \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt + 2a(x(1-x)^{-p} + (1-x)) \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt \\
& \leq b \int_0^\infty e^{-\lambda t} \|C^{1/2}P(t)u\|^2 dt + 2a(1-x)^{-p} \int_0^\infty e^{-\lambda t} \|P(t)u\|^2 dt. \tag{3.24}
\end{aligned}$$

For  $\lambda > \max(b, 1)$ , substituting (3.24) into (3.21), we obtain that

$$(\lambda - b) \sup_{\epsilon > 0} \langle u, \mathcal{R}_{\lambda, x}^{(n+1)}(C_\epsilon)u \rangle \leq \|C^{1/2}u\|^2 + 2a(1-x)^{-p}\|u\|^2.$$

This completes the proof of the Proposition.  $\square$

*Proof of Theorem 3.1:* Let  $\lambda > \max(b, 1)$ . Recall that for  $\epsilon > 0$ ,  $C_\epsilon = C(I + \epsilon C)^{-1}$ . For  $u \in D(G)$ , we have

$$\begin{aligned}
\sup_{\epsilon > 0} \langle u, \mathcal{P}_\lambda(\Phi_\epsilon)u \rangle &= \int_0^\infty e^{-\lambda t} \|\Phi^{1/2}P(t)u\|^2 dt \\
&= \sum_{l=1}^\infty \int_0^\infty e^{-\lambda t} \|L_l P(t)u\|^2 dt = \langle u, \mathcal{Q}_\lambda(I)u \rangle.
\end{aligned}$$

This implies that the non-decreasing family of operators  $(\mathcal{P}_\lambda(\Phi_\epsilon))_{\epsilon > 0}$  is uniformly bounded and since  $D(G)$  is dense in  $\mathfrak{h}$ , it follows that it converges strongly to  $\mathcal{Q}_\lambda(I)$  as  $\epsilon$  goes to 0. By the normality of the maps  $\mathcal{Q}_\lambda^k$  and the equation (3.7), for any  $x \in (0, 1)$ , we have

$$\begin{aligned}
\sum_{k=0}^\infty x^k \langle u, \mathcal{Q}_\lambda^{k+1}(I)u \rangle &= \sup_{\epsilon > 0} \sum_{k=0}^\infty x^k \langle u, \mathcal{Q}_\lambda^k(\mathcal{P}_\lambda(\Phi_\epsilon))u \rangle \\
&= \sup_{\epsilon > 0} \langle u, \mathcal{R}_{\lambda, x}^{(\min)}(\Phi_\epsilon)u \rangle.
\end{aligned}$$

Let  $\tilde{\Phi} = \delta\Phi$ . For  $\epsilon > 0$ , it follows from Proposition 2.2.13 in [3] that the bounded positive operators  $\tilde{\Phi}_\epsilon$  and  $C_\epsilon$  satisfy the inequality  $\tilde{\Phi}_\epsilon \leq C_\epsilon$ . Applying Proposition 3.2 we obtain

the estimate

$$\begin{aligned}
\int_0^1 \sum_{k=0}^{\infty} x^{k+1} \langle u, \mathcal{Q}_\lambda^{k+1}(I)u \rangle dx &= \delta^{-1} \int_0^1 x \sup_{\epsilon > 0} \langle u, \mathcal{R}_{\lambda, x}^{(min)}(\tilde{\Phi}_\epsilon)u \rangle dx \\
&\leq \delta^{-1} \int_0^1 x \sup_{\epsilon > 0} \langle u, \mathcal{R}_{\lambda, x}^{(min)}(C_\epsilon)u \rangle dx \\
&\leq \delta^{-1} \int_0^1 x(\lambda - b)^{-1} \left( \|C^{1/2}u\|^2 + 2a(1-x)^{-p}\|u\|^2 \right) dx \\
&< \infty.
\end{aligned}$$

By (3.4) and Lemma 3.1 we have  $s - \lim_{n \rightarrow \infty} \mathcal{Q}_\lambda^n(I) = 0$ , which implies that the minimal q.d.s. is conservative.  $\square$ .

## 4 Applications

In this section we obtain some relative bounds to apply our sufficient conservativity condition of Theorem 3.1 to a concrete example.

Let  $\mathfrak{h} = L^2(\mathbb{R}^n, dx)$  and  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  be the real valued function. We are looking for the condition that there exist constants  $a > 0$  and  $p < 1$  such that

$$\|W\varphi\|^2 \leq \varepsilon \| -\Delta\varphi \|^2 + a\varepsilon^{-p} \|\varphi\|^2, \quad \varphi \in C_0^2(\mathbb{R}^n)$$

holds for any  $\varepsilon > 0$ , where  $\Delta$  is a Laplacian operator and  $C_0^2(\mathbb{R}^n)$  is the set of twice continuously differentiable functions with compact support on  $\mathbb{R}^n$ . We prove first the following :

**Lemma 4.1** *For a given  $n \in \mathbb{N}$ , let  $\alpha$  be a nonnegative real number satisfying  $n/(1+\alpha) < 2$ . If  $W \in L^{2+2\alpha}(\mathbb{R}^n)$ , there exist  $a > 0$  and  $p < 1$  such that the bound*

$$\|W\varphi\|^2 \leq \varepsilon \| -\Delta\varphi \|^2 + a\varepsilon^{-p} \|\varphi\|^2$$

*holds for any  $\varepsilon > 0$  and  $\varphi \in D(-\Delta)$ .*

*Proof:* Since  $C_0^\infty(\mathbb{R}^n)$ , the space of infinitely differentiable functions with compact support, is a core for  $-\Delta$ , it is sufficient to show the bound for any  $\varphi \in C_0^\infty(\mathbb{R}^n)$ .

We use the method employed in the proof of Theorem IX 28 in [25]. Assume  $W^{1+\alpha} \in L^2(\mathbb{R}^n)$ . Denote by  $\hat{f}$  the Fourier transform of  $f \in \mathfrak{h}$ . For  $\varphi \in C_0^\infty(\mathbb{R}^n)$ , we have

$$\begin{aligned} \|W^{1+\alpha}\varphi\|_2^2 &\leq \|W^{1+\alpha}\|_2^2 \|\varphi\|_\infty^2, \\ \|\varphi\|_\infty &\leq (2\pi)^{-n/2} \|\hat{\varphi}\|_1 \end{aligned} \quad (4.1)$$

and

$$\|\hat{\varphi}\|_1^2 \leq C \|(\lambda^4 + 1)^{(1+\alpha)/2} \hat{\varphi}\|_2^2, \quad (4.2)$$

where  $C = \|(\lambda^4 + 1)^{-(1+\alpha)/2}\|_2^2 < \infty$  since  $\alpha > \frac{n}{2} - 1$ .

For any  $r > 0$ , let  $\hat{\varphi}_r(\lambda) = r^n \hat{\varphi}(r\lambda)$ . Then

$$\begin{aligned} \|\hat{\varphi}_r\|_1 &= \|\hat{\varphi}\|_1, \\ \|(\lambda^4 + 1)^{(1+\alpha)/2} \hat{\varphi}_r\|_2^2 &= \int_{\mathbb{R}^n} (\lambda^4 + 1)^{1+\alpha} r^{2n} |\hat{\varphi}(r\lambda)|^2 d^n \lambda \\ &= r^n \|(r^{-4}\lambda^4 + 1)^{(1+\alpha)/2} \hat{\varphi}\|_2^2. \end{aligned}$$

Thus using (4.2) for  $\hat{\varphi}_r$ , and these equalities, we obtain

$$\|\hat{\varphi}\|_1^2 \leq C r^n \|(r^{-4}\lambda^4 + 1)^{(1+\alpha)/2} \hat{\varphi}\|_2^2. \quad (4.3)$$

Substituting (4.3) into (4.1), by Plancherel's Theorem, there is a constant  $C_1 > 0$  such that

$$\|W^{1+\alpha}\varphi\|_2^2 \leq C_1 r^n \|(r^{-4}\Delta^2 + 1)^{(1+\alpha)/2} \varphi\|_2^2,$$

which implies

$$W^{2+2\alpha} \leq C_1 r^n (r^{-4}\Delta^2 + 1)^{1+\alpha}. \quad (4.4)$$

Suppose that  $A$  and  $B$  are self-adjoint operators such that

$$0 \leq B \leq A.$$

Then the above implies that

$$0 \leq B^t \leq A^t$$

for any  $t \in [0, 1]$  (see Problem 51 of Chapter VIII of [25] and also the Heinz-Kato theorem in §2.3.3. of [26]). Thus we have

$$W^2 \leq C_2 r^{n/(1+\alpha)} (r^{-4} \Delta^2 + 1),$$

which yields

$$\|W\varphi\|^2 \leq C_2 r^{-(4-n/(1+\alpha))} \|\Delta\varphi\|^2 + C_2 r^{n/(1+\alpha)} \|\varphi\|^2.$$

Choose  $\varepsilon = C_2 r^{-(4-n/(1+\alpha))}$ . Then we obtain

$$\|W\varphi\|^2 \leq \varepsilon \|\Delta\varphi\|^2 + a\varepsilon^{-p} \|\varphi\|^2$$

where  $p = n(4(1+\alpha) - n)^{-1}$ . Since  $n/(1+\alpha) < 2$ ,  $p < 1$ . If we choose  $r$  large enough, the bound follows.  $\square$

**Remark 4.1** (a) In Lemma 4.1, one can choose  $\alpha = 0$  for  $n = 1$ . Notice that  $\alpha > 0$  for  $n = 2$  and  $\alpha > 1/2$  for  $n = 3$ , etc.

(b) Let the dimension  $n = 1, 2, 3$ . If  $W \in L^4(\mathbb{R}^n, dx)$ , then  $W^2 \in L^2(\mathbb{R}^n, dx)$  and so  $W^2$  is relatively bounded by  $-\Delta$  (see Theorem X.15 of [25]). Thus  $W^2$  is relatively form bounded by  $-\Delta$ , i.e.,

$$\|W\varphi\|^2 \leq b \langle \varphi, (-\Delta + 1)\varphi \rangle, \quad \varphi \in C_0^\infty(\mathbb{R}^n),$$

where  $b$  is a constant. See also Theorem X.18 (b) of [25].

In the rest of this section, we apply Theorem 3.1 and Lemma 4.1 to a model of heavy ion collision proposed by Alicki [27].

**Example 4.1** (Q.d.s. in a model for heavy ion collision)

Let  $\mathfrak{h} = L^2(\mathbb{R}^3)$ . We denote by  $\partial_k = \frac{\partial}{\partial x_k}$  ( $k = 1, 2, 3$ ) differential operators with respect to the  $k$  th coordinate and  $\partial_{lk} = \frac{\partial^2}{\partial x_k \partial x_l}$  ( $l, k = 1, 2, 3$ ). For any measurable function  $T$ , we denote the (distributional) derivative  $\frac{\partial T}{\partial x_l}$  by  $(T)_l$ ,  $l = 1, 2, 3$ . Consider the operators  $L_l$ , for  $l = 1, 2, 3$

$$L_l u = w(x_l + \alpha \partial_l) u, \quad (4.5)$$

$$D(L_l) = \{u \in L^2(\mathbb{R}^3) : \text{the distribution } L_l u \in L^2(\mathbb{R}^3)\}$$

where  $w, \alpha \in \mathbb{R}$  are non-zero real constants, and  $L_l = 0$  for  $l \geq 4$ . Let  $V$  be a real measurable function. Consider the operators  $H$  and  $G$  given by

$$Hu = \left(-\frac{1}{2}\Delta + V\right) u, \quad (4.6)$$

$$Gu = -iHu - \frac{1}{2} \sum_{l=1}^{\infty} L_l^* L_l u$$

for  $u \in C_0^\infty(\mathbb{R}^3)$ . Let us assume that the following properties hold:

- (1)  $w^2 \alpha^2 \geq 2$
- (2)  $|V(x)| \leq \frac{1}{4} w^2 (x^2 + b_1)$  for some constant  $b_1 > 0$ , where  $x^2 = x_1^2 + x_2^2 + x_3^2$ .
- (3) There exist real measurable functions  $U_1$  and  $U_2$  and positive constants  $b_2, b_3$  such that  $U_1 \in L^\beta(\mathbb{R}^3)$  for some  $\beta > 3$ ,  $U_2(x) \leq b_2(|x| + b_3)$  and the bounds

$$|(V)_l| \leq U_1 + U_2 \quad (4.7)$$

hold for  $l = 1, 2, 3$ .

For an instance the function  $V(x) = \frac{1}{4} w^2 |x|^\nu$ ,  $0 < \nu \leq 2$ , satisfies the conditions (2) and (3) in the above. Let us mention that in the example proposed by Alicki [27], the constant  $w$  in (4.5) is a function  $W(x)$  proportional to  $\sqrt{\gamma(x)}$  where  $\gamma(x)$  represents a friction force. The conservativity of this q.d.s. has been already investigated in [1]

under appropriate (boundedness) assumptions on  $V, W$  and their derivatives. In this paper we only consider the case that  $W(x)$  is a constant to avoid unnecessary notational complications involved.

We apply Theorem 3.1 and Lemma 4.1 to show that the minimal q.d.s. constructed from above operators  $L_l$  and  $G$  given in (4.5) and (4.6) respectively is conservative. We will check that the main inequalities (3.1) and (3.2) hold for  $u \in C_0^\infty(\mathbb{R}^3)$ . The most difficult problem is to extend the inequalities to every  $u \in D(G^2)$ . In order to overcome this problem, we need technical estimates.

**Lemma 4.2** *For all  $u \in C_0^\infty(\mathbb{R}^3)$ , the bounds*

$$\langle u, (\alpha^4 \Delta^2 + x^4)u \rangle \leq \langle u, (-\alpha^2 \Delta + x^2 + 3|\alpha|)^2 u \rangle \quad (4.8)$$

and

$$\left\| \frac{1}{2} w^2 (-\alpha^2 \Delta + x^2 - 3\alpha) u \right\|^2 \leq b_4 \|Gu\|^2 + b_5 \|u\|^2 \quad (4.9)$$

for some  $b_4 > 1$  and  $b_5 > 0$  hold.

*Proof:* A direct computation shows that

$$\begin{aligned} (-\alpha^2 \Delta + x^2)^2 &= \alpha^4 \Delta^2 + x^4 - \alpha^2 (\Delta x^2 + x^2 \Delta) \\ &= \alpha^4 \Delta^2 + x^4 - \alpha^2 \left( 2 \sum_{k=1}^3 \partial_k x^2 \partial_k + 6 \right) \\ &\geq \alpha^4 \Delta^2 + x^4 - 6\alpha^2, \end{aligned}$$

as a bilinear form on the domain  $C_0^\infty(\mathbb{R}^3)$ . This proves the bound (4.8).

Next we prove the bound (4.9). Put

$$\begin{aligned} G_0 &= -\frac{1}{2} \sum_{l=1}^3 L_l^* L_l \\ &= -\frac{1}{2} w^2 (-\alpha^2 \Delta + x^2 - 3\alpha). \end{aligned} \quad (4.10)$$

We have that as bilinear forms on  $C_0^\infty(\mathbb{R}^3)$

$$\begin{aligned}
G^*G &= (iH + G_0)(-iH + G_0) \\
&= H^2 + G_0^2 + i[H, G_0] \\
&\geq G_0^2 + i[H, G_0],
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
i[H, G_0] &= \frac{iw^2}{4}[\Delta, x^2] + \frac{iw^2\alpha^2}{2}[V, \Delta] \\
&= \frac{iw^2}{2} \sum_{l=1}^3 (\partial_l x_l + x_l \partial_l) - \frac{iw^2\alpha^2}{2} \sum_{l=1}^3 (\partial_l(V)_l + (V)_l \partial_l) \\
&\geq -\frac{w^2}{2}(-\Delta + x^2) - \frac{w^2\alpha^2}{2}(-\Delta + \sum_{l=1}^3 |(V)_l|^2).
\end{aligned}$$

It follows from (4.7) that

$$i[H, G_0] \geq -\frac{w^2}{2}(1 + \alpha^2)(-\Delta + x^2) - 3w^2\alpha^2(U_1^2 + U_2^2).$$

The bound (4.8) implies that  $(-\Delta + x^2)^{1/2}$  is infinitesimally small with respect to  $G_0$ . By the condition (3) and Lemma 4.1 (Remark 4.1 (a)),  $U_1$  and  $U_2$  are also infinitesimally small with respect to  $G_0$ . Thus there exist constants  $0 < a < 1$  and  $b > 0$  such that

$$i[H, G_0] \geq -aG_0^2 - b$$

as a bilinear form on  $C_0^\infty(\mathbb{R}^3)$ . The bound (4.9) follows from (4.11) and the above bound.

□

Recall that

$$G = -i\left(-\frac{1}{2}\Delta + V\right) + G_0$$

where  $G_0$  is given as (4.10). Notice that  $G_0$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^3)$ . Since  $w^2\alpha^2 \geq 2$  by the condition (1), the bound (4.8) implies that  $-\frac{1}{2}\Delta$  is  $G_0$ -bounded with

relative bound smaller than equal to  $1/2$ . The condition (2) and the bound (4.8) imply that  $V$  is  $G_0$ -bounded with relative bound smaller than  $1/2$ . Thus  $-iH$  is relatively bounded perturbation of  $G_0$  with relative bound smaller than 1. Thus Assumption **A** holds.

We show that the minimal q.d.s. is conservative applying Theorem 3.1. Let us choose the operator  $C$ ,

$$C = w^2(-\alpha^2\Delta + x^2 + 3|\alpha|) = \sum_{l=1}^3 L_l^* L_l + b_6 = -2G_0 + b_6, \quad (4.12)$$

$$D(C) = \{u \in L^2(\mathbb{R}^3) \mid \text{the distribution } Cu \in L^2(\mathbb{R}^3)\}$$

where  $b_6 = 3w^2(|\alpha| - \alpha)$ . Using the relation (4.9) and the fact that  $-iH$  is relatively bounded perturbation of  $G_0$ , we obtain that  $G$  and  $C$  are relatively bounded with respect to each other and so  $D(G) = D(C)$ .

We will check that the operator  $C$  satisfies the assumption **C**. Hypothesis (a) and (b) are trivially fulfilled. Now we will check (c). First, we have that as bilinear forms on  $C_0^\infty(\mathbb{R}^3)$ ,

$$\begin{aligned} [C, -\frac{1}{2}\Delta + V] &= -\alpha^2 w^2 [\Delta, V] - \frac{1}{2} w^2 [x^2, \Delta] \\ &= -\alpha^2 w^2 \sum_{l=1}^3 (\partial_l(V)_l + (V)_l \partial_l) + w^2 \sum_{l=1}^3 (\partial_l x_l + x_l \partial_l) \\ &\leq \alpha^2 w^2 (-\Delta + \sum_{l=1}^3 (V)_l^2) + w^2 (-\Delta + x^2), \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} [C, L_l] &= w^3 (-\alpha^2 [\Delta, x_l] + \alpha [x^2, \partial_l]) \\ &= -2w^3 \alpha (\alpha \partial_l + x_l) = -2w^2 \alpha L_l, \end{aligned}$$

and so

$$\sum_{l=1}^3 L_l^* [C, L_l] = -2w^2 \alpha \sum_{l=1}^3 L_l^* L_l = -2w^2 \alpha C + b_6. \quad (4.14)$$



By direct computation, we have

$$CG + G^*C + C^2 = -i[C, -\frac{1}{2}\Delta + V] + b_6C,$$

and

$$\begin{aligned} CG + G^*C + \sum_{l=1}^3 L_l^* C L_l \\ &= -i[C, -\frac{1}{2}\Delta + V] + \frac{1}{2} \sum_{l=1}^3 (L_l^*[C, L_l] + (L_l^*[C, L_l])^*) \\ &= -i[C, -\frac{1}{2}\Delta + V] - 2w^2\alpha C + b_6, \end{aligned}$$

as bilinear forms on  $C_0^\infty(\mathbb{R}^3)$ . Substituting (4.13) and (4.14) into the above equations, and using the fact that  $-\Delta, -\Delta + x^2$  are relatively form bounded with respect to  $C$ , we have that for  $u \in C_0^\infty(\mathbb{R}^3)$ ,

$$2\operatorname{Re}\langle Cu, Gu \rangle + \|Cu\|^2 \leq b_7\langle u, Cu \rangle + \alpha^2 w^2 \sum_{l=1}^3 \|(V)_l u\|^2, \quad (4.15)$$

and

$$\begin{aligned} 2\operatorname{Re}\langle Cu, Gu \rangle + \sum_{l=1}^3 \langle L_l u, C L_l u \rangle \\ \leq b_8\langle u, Cu \rangle + \alpha^2 w^2 \sum_{l=1}^3 \|(V)_l u\|^2, \end{aligned} \quad (4.16)$$

where  $b_7, b_8 > 0$ .

Note  $|(V)_l| \leq U_1 + U_2$  for  $l = 1, 2, 3$  with  $U_1 \in L^\infty(\mathbb{R}^3)$  where  $\beta > 3$ , and  $U_2(x) \leq b_2(|x| + b_3)$ . Applying Lemma 4.1 to (4.15) and (4.16), we obtain (3.1) and (3.2) for  $u \in C_0^\infty(\mathbb{R}^3)$ .

We want to extend the inequality (3.1) and (3.2) to the domain  $D(G)$ . For  $u \in D(G)$ , there exists a sequence  $\{u_n\}$  of elements of  $C_0^\infty(\mathbb{R}^3)$  such that

$$\lim_{n \rightarrow \infty} u_n = u, \quad \lim_{n \rightarrow \infty} C u_n = Cu, \quad \lim_{n \rightarrow \infty} G u_n = Gu,$$

by the relation (4.9). Then the relation (3.1) holds for  $u \in D(G)$ . Also the relation (3.2) implies that  $\{C^{1/2}L_l u_n\}_{n \geq 1}$  is a Cauchy sequence. Therefore it is convergent and it is easy to deduce that (3.2) holds for  $u \in D(G)$ .

Recall that  $\Phi = \sum_{l=1}^3 L_l^* L_l$  and  $C = \sum_{l=1}^3 L_l^* L_l + b_6$ . Hence the conditions of Theorem 3.1 also hold and the minimal q.d.s. is conservative.

**Remark 4.2** *Let us remind the condition of derivatives of  $V$ ,  $|(V)_l| \leq U_1 + U_2$  for  $l = 1, 2, 3$ . One can use the previous criterion in [1] to show the conservativity for  $U_1 \in L^4(\mathbb{R}^3)$  (see Remark 4.1 (b)). Applying our result, we extend the range of  $(V)_l$ , i.e.,  $U_1 \in L^\beta(\mathbb{R}^3)$  where  $\beta > 3$ .*

**Acknowledgement** : This work was supported by Korea Research Foundation Grant (KRF-2003-005-C00010).

## References

- [1] A. M. Chebotarev and F. Fagnola, Sufficient conditions for conservativity of minimal quantum dynamical semigroups, *J. Funct. Anal.* **153** (1998), 382-404.
- [2] R. Alicki and K. Lendi, Quantum dynamical semigroups and applications, *Lecture Notes Physics*, Vol 286 (1987), Springer.
- [3] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, second edition, Springer-Verlag, New York-Heidelberg-Berlin, vol I, 1987, vol. II, 1997.
- [4] E. B. Davies, *Quantum theory of open systems*, Academic Press, London-New York-San Francisco, 1976.
- [5] L. Accardi, A. Frigerio, and J. T. Lewis, Quantum stochastic processes, *Publ. Res. Inst. Math. Sci.* **18** (1982), 97-133 .

- [6] P. A. Meyer, *Quantum probability for probabilists*, in *Lect. Notes Math.*, Springer Verlag, Berlin, Heidelberg, New York (1993).
- [7] K. R. Parthasarathy *An Introduction to Quantum stochastic Calculus*, Monographs in Mathematics, Birkhäuser, Basel, 1992.
- [8] A. M. Chebotarev, Sufficient conditions for conservativeness of dynamical semigroups, *Theor. Math. Phys.* **80** (2) (1989).
- [9] A. M. Chebotarev, Sufficient conditions for conservativity of a minimal dynamical semigroup, *Math. Notes* **52** (1993), 1067-1077.
- [10] A. M. Chebotarev and F. Fagnola, Sufficient conditions for conservativity of quantum dynamical semigroups, *J. Funct. Anal.* **118** (1993), 131-153.
- [11] F. Fagnola, Chebotarev's sufficient conditions for conservativity of quantum dynamical semigroups, in : L. Accardi (Ed.), *Quantum Probab. Related Topics VIII* (1993), 123-142.
- [12] A. M. Chebotarev and S. Yu. Shustikov, Conditions sufficient for the conservativity of a minimal quantum dynamical semigroup, *Math. Notes*, **71** (2002), 692-710.
- [13] A. Arnold and S. Sparber, Quantum dynamical semigroups for diffusion models with Hartree interaction, *Commun. Math. Phys.* **251** (2004) 179-207.
- [14] F. Fagnola and R. Rebolledo, On the existence of stationary states for quantum dynamical semigroup, *J. Math. Phys.* **42** (2001), 1296-1308.
- [15] F. Fagnola and R. Rebolledo, Subharmonic projections for a quantum Markov semigroup, *J. Math. Phys.* **43**, **2** (2002), 1074-1082.
- [16] G. Lindblad, On the generator on dynamical semigroups, *Comm. Math. Phys.* **48** (1976), 119-130.

- [17] B. V. R. Bhat and K. B. Sinha, Examples of unbounded generators leading to non-conservative minimal semigroups, *Quantum Probab. Related Topics IX* (1994), 89-103.
- [18] F. Fagnola, Characterization of isometric and unitary weakly differentiable cocycles in Fock space, in : L. Accardi (Ed.), *Quantum Probab. Related Topics VIII* (1993), 143-164.
- [19] F. Fagnola, Diffusion processes in Fock space, *Quantum Probab. Related Topics IX* (1994), 189-214.
- [20] F. Fagnola and S.J. Wills, Solving quantum stochastic differential equations with unbounded coefficients, *J. Func. Anal.* **198** (2003), 279-310.
- [21] E. B. Davies, Quantum dynamical semigroups and the neutron diffusion equation, *Rep. Math. Phys.* **11** (1977), 169-188.
- [22] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.
- [23] J. Dereziński and V. Jaksic, Spectral theory of Paul-Fierz operators, *J. Funct. Anal.* **180** (2001), 243-327.
- [24] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
- [25] M. Reed and B. Simon, *Method of modern mathematical physics I, II*, Academic press, 1980.
- [26] H. Tanabe, *Equations in evolutions*, Pitman Press, London, 1979.
- [27] L. Alicki, Scattering theory for quantum dynamical semigroups in Quantum probability and applications to the quantum theory of irreversible processes, 20-31, *Lecture Notes in Mathematics*, Vol 1055 (1984).