

An asymptotic expansion for a ratio of products of gamma functions

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Abstract

An asymptotic expansion of a ratio of products of gamma functions is derived. It generalizes a formula which was stated by Dingle, first proved by Paris, and recently reconsidered by Olver.

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1 Introduction

Our starting point is the Gaussian hypergeometric function $F(a, b; c; z)$ and its series representation

$$\frac{1}{\Gamma(c)} F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\Gamma(c+n) n!} z^n, \quad |z| < 1,$$

which here is written in terms of Pochhammer symbols

$$(x)_n = x(x+1) \dots (x+n-1) = \Gamma(x+n)/\Gamma(x).$$

The hypergeometric series appears as one solution of the Gaussian (or hypergeometric) differential equation, which is characterized by its three regular

singular points at $z = 0, 1, \infty$. The local series solutions at 0 and 1 of this differential equation are connected by the continuation formula [1]

$$\begin{aligned} \frac{1}{\Gamma(c)} F(a, b; c; z) &= \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; 1+a+b-c; 1-z) \\ &+ \frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a, c-b; 1+c-a-b; 1-z), \quad (1) \end{aligned}$$

$(|\arg(1-z)| < \pi).$

Here we want to show that Eq. (1) implies an interesting asymptotic expansion for a ratio of products of gamma functions, of which only a special case was known before.

By applying the method of Darboux [4, 8] to (1), we derive in Sec. 2 the formula in question. The behaviour of this and a related formula is discussed in Sec. 3 and illustrated by a few numerical examples.

2 Derivation of an asymptotic expansion for a ratio of products of gamma functions

It is well-known that the late coefficients of a Taylor series expansion contain information about the nearest singular point of the expanded function [3]. In this respect we want to analyze the continuation formula (1), in which then only the second, at $z = 1$ singular term R is relevant, which may be written as

$$R = \frac{\Gamma(a+b-c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{\infty} \frac{(c-a)_m (c-b)_m}{\Gamma(1+c-a-b+m)m!} (1-z)^{c-a-b+m}.$$

By means of the binomial theorem in its hypergeometric-series-form, we may expand the power factor

$$(1-z)^{c-a-b+m} = \sum_{n=0}^{\infty} \frac{\Gamma(a+b-c-m+n)}{\Gamma(a+b-c-m)n!} z^n.$$

Interchanging then the order of the summations and simplifying by means of the reflection formula of the gamma function, we arrive at

$$R = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(c-a)_m (c-b)_m}{m!} \frac{\Gamma(a+b-c-m+n)}{n!} z^n.$$

This is to be compared with the left-hand side L of (1), which is

$$L = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n.$$

Comparison of the coefficients of these two power series, which according to Darboux [4] and Schäfke and Schmidt [8] should agree asymptotically as $n \rightarrow \infty$, then yields

$$\begin{aligned} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} &= \sum_{m=0}^M (-1)^m \frac{(c-a)_m (c-b)_m}{m!} \Gamma(a+b-c-m+n) \quad (2) \\ &+ O(\Gamma(a+b-c-M-1+n)). \end{aligned}$$

By means of

$$O(\Gamma(a+b-c-M-1+n)) = \Gamma(a+b-c+n) O(n^{-M-1})$$

and the reflection formula of the gamma function, the relevant formula (2) may also be written as

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} = 1 + \sum_{m=1}^M \frac{(c-a)_m (c-b)_m}{m!(1+c-a-b-n)_m} + O(n^{-M-1}). \quad (3)$$

The asymptotic expansion for a ratio of products of gamma functions in this form (3) or the other (2) seems to be new. It is only the special case when $c = 1$ which is known. This special case was stated by Dingle[2], first proved by Paris[7], and reconsidered recently by Olver[5], who has found a simple direct proof. His proof, as well as the proof of Paris, can be adapted easily to the more general case when c is different from 1. Still another proof is available [6] which includes an integral representation of the remainder term. Our derivation of Eq. (2) or (3) is significantly different from all the earlier proofs of the case when $c = 1$.

3 Discussion and numerical examples

We now want to discuss our result in the form (3). First we observe that the substitution $c \rightarrow a+b-c$ leads to the related formula

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} = 1 + \sum_{m=1}^M \frac{(a-c)_m (b-c)_m}{m!(1-c-n)_m} + O(n^{-M-1}). \quad (4)$$

Which of (3) or (4) is more advantageous numerically depends on the values of the parameters, and in this respect the two formulas complement each other. Table 1 shows an example with a set of parameters for which (3) gives more accurate values than (4), while Table 2 contains an example for which (4) is superior to (3).

For finite n and $M \rightarrow \infty$ the series on the right-hand side of (3) converges if $\text{Re}(1 - c - n) > 0$. The same is true for (4) if $\text{Re}(1 + c - a - b - n) > 0$. Then, in both cases, the Gaussian summation formula yields

$$\frac{\Gamma(1 - c - n)\Gamma(1 + c - a - b - n)}{\Gamma(1 - a - n)\Gamma(1 - b - n)},$$

which, by means of the reflection formula of the gamma function, is seen to be equal to

$$\frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)\Gamma(a + b - c + n)} \frac{\sin(\pi[a + n])\sin(\pi[b + n])}{\sin(\pi[c + n])\sin(\pi[a + b - c + n])}. \quad (5)$$

Otherwise (2) – (4) are divergent asymptotic expansions as $n \rightarrow \infty$.

Although in our derivation n is a sufficiently large positive integer, the asymptotic expansions (2) – (4) are expected to be valid in a certain sector of the complex n -plane, and in fact, the proofs of Paris [7] and of Olver [6] apply to complex values of n .

If the series in (3) or (4) converge, their sums are equal to (5), which generally (if neither $c - a$ nor $c - b$ is equal to an integer) is different from the left-hand side of (3) or (4). Therefore (3) and (4) can be valid only in the half-planes in which the series do not converge. This means that (3) is an asymptotic expansion as $n \rightarrow \infty$ in the half-plane $\text{Re}(c - 1 + n) \geq 0$, and (4) is an asymptotic expansion as $n \rightarrow \infty$ in the half-plane $\text{Re}(a + b - c - 1 + n) \geq 0$. Otherwise the series on the right-hand sides represent a different function, namely (5).

A few numerical examples may serve for demonstration of these facts. In Table 3, the series converge to (5) for $n = 10$, and therefore (3) and (4) are not valid. For $n = 20$, on the other hand, the series diverge and so (3) and (4) hold. The transition between the two regions is at the line $\text{Re}(n) = 12.4$ in case of (3) or $\text{Re}(n) = 12.5$ in case of (4). In Table 4, we see convergence for $n = -15$ and divergence for $n = -5$, the transition between the two regions being at the line $\text{Re}(n) = -10.4$ in case of (3) or $\text{Re}(n) = -10.5$ in case of (4).

References

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	M	right-hand side of (3)	right-hand side of(4)
$n = 10$	1	0.9771429	0.9744681
	2	0.9773113	0.9780243
	3	0.9772978	0.9769927
	4	0.9773005	0.9774980 ←
	5	0.9772995 ←	0.9771117 ←
	6	0.9773001 ←	0.9775615
	7	0.9772995	0.9767519
	8	0.9773003	0.9791530
	9	0.9772983	0.9652341
	10	0.9773079	1.2823765
		exact value of (3) or (4): 0.97729983	

Table 1: Values of the right-hand sides of (3) and (4) for the parameters $a = 0.7$, $b = 1.2$, $c = 0.4$.

	M	right-hand side of (3)	right-hand side of(4)
$n = 10$	1	0.968000	0.972093
	2	0.973760	0.972350
	3	0.971512 ←	0.972324
	4	0.973078 ←	0.972331
	5	0.971231	0.972327 ←
	6	0.975016	0.972330 ←
	7	0.959571	0.972325
	8	1.179434	0.972342
	9	4.748048	0.972163
	10	26.430946	0.968966
		exact value of (3) or (4): 0.97232850	

Table 2: Values of the right-hand sides of (3) and (4) for the parameters $a = -0.7$, $b = -1.2$, $c = -0.4$.

	M	right-hand side of (3)	right-hand side of(4)
$n = 10$	1	0.976000	0.975000
	2	0.972434	0.971912
	3	0.971341	0.971037
	4	0.970882	0.970687
	5	0.970651	0.970517
	6	0.970520	0.970423
	7	0.970440	0.970367
	8	0.970388	0.970331
	9	0.970352	0.970307
	10	0.970326	0.970290
		exact value of(3) or (4): 1.94045281	
		exact value of (5): 0.97022640 ←	
$n = 20$	1	1.008000	1.007895
	2	1.007360	1.007392
	3	1.007521	1.007504
	4	1.007438 ←	1.007452 ←
	5	1.007515 ←	1.007497 ←
	6	1.007385	1.007426
	7	1.007839	1.007650
	8	1.002201	1.005398
	9	0.921096	0.965891
	10	0.478588	0.740024
		exact value of (3) or (4): 1.00747290 ←	
		exact value of (5): 0.50373645	

Table 3: Values of the right-hand sides of (3) and (4) for the parameters $a = -11.7$, $b = -11.2$, $c = -11.4$.

	M	right-hand side of (3)	right-hand side of(4)
$n = -15$	1	0.986667	0.986957
	2	0.985648	0.985745
	3	0.985453	0.985492
	4	0.985397	0.985415
	5	0.985376	0.985386
	6	0.985368	0.985373
	7	0.985363	0.985367
	8	0.985361	0.985363
	9	0.985360	0.985361
	10	0.985359	0.985360
		exact value of (3) or (4): 1.97071532	
		exact value of (5): 0.98535766 ←	
$n = -5$	1	1.010909	1.011111
	2	1.009891	1.009798
	3	1.010254 ←	1.010331 ←
	4	1.009940 ←	1.009818 ←
	5	1.010589	1.011015
	6	1.005300	0.998322
	7	0.951894	0.887892
	8	0.737202	0.459630
	9	0.134729	-0.725230
	10	-1.243041	-3.418810
		exact value of (3) or (4): 1.01011438 ←	
		exact value of (5): 0.50505719	

Table 4: Values of the right-hand sides of (3) or (4) for the parameters $a = 11.7$, $b = 11.2$, $c = 11.4$.