A Lax Pair for the 2D Euler Equation

Yanguang (Charles) Li *

Department of Mathematics,

University of Missouri

Columbia, MO 65211

E-mail: cli@math.missouri.edu

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Abstract

A Lax pair for the 2D Euler equation is found.

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1. A Lax Pair for the 2D Euler Equation

This is to report that a Lax pair for the 2D Euler equation is found. We write the 2D Euler equation in the vorticity form,

$$\frac{\partial\Omega}{\partial t} + \{\Psi, \Omega\} = 0 , \qquad (1.1)$$

where Ω is the vorticity, Ψ is the stream function, and the bracket $\{ \}$ is defined as

$$\{f,g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)$$

Let us denote the x-directional and the y-directional velocities by u and v respectively. Then

$$u = -\frac{\partial \Psi}{\partial y}$$
, $v = \frac{\partial \Psi}{\partial x}$, $\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, $\Delta \Psi = \Omega$.

The Lax pair is given as

$$\begin{cases} L\varphi = \lambda\varphi ,\\ \partial_t \varphi + A\varphi = 0 , \end{cases}$$
(1.2)

where

$$L\varphi = \{\Omega,\varphi\} \ , \quad A\varphi = \{\Psi,\varphi\} \ ,$$

and λ is a complex constant, and φ is a complex-valued function. The compatibility condition of the Lax pair (1.2) gives the 2D Euler equation (1.1), i.e.

$$\partial_t L = [L, A]$$

where [L, A] = LA - AL, gives the Lax representation of the 2D Euler equation (1.1).

Remark 1.1 With the recent development on chaos in partial differential equations [1] [2] [3], I am interested in building a dynamical system theory for 2D Euler equation under periodic boundary condition [4] [5]. In particular, I am investigating the existence v.s. nonexistence of homoclinic structure. For such studies, it will be fundamentally important to find a Lax pair (if it exists) for the 2D Euler equation. Then I started with Vladimir Zakharov's paper [6]. Zakharov proposed the Lax pair

$$\begin{cases} \lambda D_1 \varphi + \{\Omega, \varphi\} = 0 ,\\ \partial_t \varphi + \lambda D_2 \varphi + \{S, \varphi\} = 0 , \end{cases}$$

where

$$D_1 = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} , \quad D_2 = \gamma \frac{\partial}{\partial x} + \delta \frac{\partial}{\partial y} ,$$

 α, β, γ , and δ are real constants, λ is a complex constant, S is a real-valued function, and φ is a complex-valued function. The compatibility condition of this Lax pair gives the following equation instead of the 2D Euler equation,

$$\begin{cases} \frac{\partial\Omega}{\partial t} + \{S,\Omega\} = 0 \\ D_1 S = D_2 \Omega \end{cases}.$$

(Notice the misprints in the English translation of the article [6].)

Remark 1.2 The author is also aware of the Lax pair in the inverse Cauchy-Green tensor variable of the Lagrangian formulations of both 2D and 3D Euler equations found by Susan Friedlander and Misha Vishik [7] [8].

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