

Schrödinger equation with critical Sobolev exponent

Alessio Pomponio*
 SISSA – via Beirut 2/4 – I-34014 Trieste
 pomponio@sissa.it

1 Introduction

In this paper we study the existence of solutions and their concentration phenomena of a singularly perturbed semilinear Schrödinger equation with the presence of the critical Sobolev exponent, that is:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = K(x)u^p + Q(x)u^\sigma & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (1)$$

where $N \geq 3$, $1 < p < \sigma = \frac{N+2}{N-2}$, V, K and Q are C^2 function from \mathbb{R}^N to \mathbb{R} . We will show that there exist solutions of (1) concentrating near the maximum and minimum points of an auxiliary functional which depends only on V, K and Q .

On the potentials, we will make the following assumptions:

(V) $V \in C^2(\mathbb{R}^N, \mathbb{R})$, V and D^2V are bounded; moreover,

$$V(x) \geq C > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

(K) $K \in C^2(\mathbb{R}^N, \mathbb{R})$, K and D^2K are bounded; moreover,

$$K(x) \geq C > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

(Q) $Q \in C^2(\mathbb{R}^N, \mathbb{R})$, Q and D^2Q are bounded; moreover, $Q(0) = 0$.

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We point out that while V and K must be strictly positive, Q can change sign and must vanish in 0.

Let us introduce an auxiliary function which will play a crucial rôle in the study of (1). Let $\Gamma: \mathbb{R}^N \rightarrow \mathbb{R}$ be a function so defined:

$$\Gamma(\xi) = \bar{C}_1 \Gamma_1(\xi) - \bar{C}_2 \Gamma_2(\xi), \quad (2)$$

where

$$\begin{aligned} \Gamma_1(\xi) &\equiv V(\xi)^{\frac{p+1}{p-1} - \frac{N}{2}} K(\xi)^{-\frac{2}{p-1}}, \\ \Gamma_2(\xi) &\equiv Q(\xi) V(\xi)^{\frac{\sigma+1}{p-1} - \frac{N}{2}} K(\xi)^{-\frac{\sigma+1}{p-1}}, \\ \bar{C}_1 &\equiv \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} U^{p+1}, \\ \bar{C}_2 &\equiv \frac{1}{\sigma+1} \int_{\mathbb{R}^N} U^{\sigma+1}, \end{aligned}$$

and U is the unique solution of

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N, \\ U(0) = \max_{\mathbb{R}^N} U. \end{cases} \quad (3)$$

Let us observe that by **(V)** and **(K)**, Γ is well defined.

Our main result is:

Theorem 1.1. *Let $\xi_0 \in \mathbb{R}^N$. Suppose **(V)**, **(K)** and **(Q)**. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then (1) possesses a solution u_ε which concentrates on ξ_ε with $\xi_\varepsilon \rightarrow \xi_0$, as $\varepsilon \rightarrow 0$, provided that one of the two following conditions holds:*

- (a) ξ_0 is a non-degenerate critical point of Γ ;
- (b) ξ_0 is an isolated local strict minimum or maximum of Γ .

In the case $V \equiv K \equiv 1$, by Theorem 1.1 and by the expression of Γ , see (2), we easily get:

Corollary 1.2. *Let $\xi_0 \in \mathbb{R}^N$. Let $V \equiv K \equiv 1$ and suppose **(Q)**. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then (1) possesses a solution u_ε which concentrates on ξ_ε with $\xi_\varepsilon \rightarrow \xi_0$, as $\varepsilon \rightarrow 0$, provided that one of the two following conditions holds:*

- (a) ξ_0 is a non-degenerate critical point of Q ;

(b) ξ_0 is an isolated local strict minimum or maximum of Q .

The existence of solutions of nonlinear Schrödinger equation like (1) with subcritical growth (i.e. $\sigma < \frac{N+2}{N-2}$) and their concentrations, as $\varepsilon \rightarrow 0$, have been extensively studied. In particular, we recall the paper [6, 8], where is proved the existence of solutions concentrating on the minima of the same Γ as in (2), under suitable conditions at infinity on the potentials.

The case $\sigma = \frac{N+2}{N-2}$ has been studied by Alves, João Marcos do Ó and Souto in [1], proving the existence of solutions of

$$-\varepsilon^2 \Delta u + V(x)u = f(u) + u^\sigma \quad \text{in } \mathbb{R}^N \quad (4)$$

concentrating on minima of V . In (4), $f(u)$ is a nonlinearity with subcritical growth.

On the other hand, when $K \equiv 0$ and $Q \equiv 1$, nonexistence results of single blow-up solutions have been proved in a recent work by Cingolani and Pistoia, see [7].

The new feature of the present paper is that the coefficient Q of $u^{\frac{N+2}{N-2}}$ vanishes at $x = 0$. After the rescaling $x \mapsto \varepsilon x$, equation (1) becomes

$$-\Delta u + V(\varepsilon x)u = K(\varepsilon x)u^p + Q(\varepsilon x)u^\sigma.$$

Then, assumption $Q(0) = 0$ implies that, roughly, the unperturbed problem, with $\varepsilon = 0$ is unaffected by the critical nonlinearity.

Theorem 1.1 will be proved as a particular case of two multiplicity results in Section 5. The proof of the theorem relies on a finite dimensional reduction, precisely on the perturbation technique developed in [4], where (1) with $Q \equiv 0$ is studied. For the sake of brevity, we will refer to [4] for some details. In Section 2 we present the variational framework. In Section 3 we perform the Liapunov-Schmidt reduction and in Section 4 we make the asymptotic expansion of the finite dimensional functional.

Notation

- If not written otherwise, all the integrals are calculated in dx .
- With $o_\varepsilon(1)$ we denote a function which tends to 0 as $\varepsilon \rightarrow 0$.
- We set $2^* = \frac{2N}{N-2}$, the critical Sobolev exponent.

2 The variational framework

Performing the change of variable $x \mapsto \varepsilon x$, equation (1) becomes

$$\begin{cases} -\Delta u + V(\varepsilon x)u = K(\varepsilon x)u^p + Q(\varepsilon x)u^\sigma & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (5)$$

Of course, if u is a solution of (5), then $u(\cdot/\varepsilon)$ is solution of (1). Solutions of (5) are critical points $u \in H^1(\mathbb{R}^N)$ of

$$\begin{aligned} f_\varepsilon(u) = & \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx \\ & - \frac{1}{p+1} \int_{\mathbb{R}^N} K(\varepsilon x)u^{p+1} dx - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} Q(\varepsilon x)u^{\sigma+1} dx. \end{aligned}$$

The solutions of (5) will be found near the solutions of

$$-\Delta u + V(\varepsilon \xi)u = K(\varepsilon \xi)u^p \quad \text{in } \mathbb{R}^N, \quad (6)$$

for an appropriate choice of $\xi \in \mathbb{R}^N$.

The solutions of (6) are critical points of the functional

$$F^{\varepsilon \xi}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon \xi)u^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} K(\varepsilon \xi)u^{p+1} dx \quad (7)$$

and can be found explicitly.

Let U denote the unique, positive, radial solution of (3), then a straight calculation shows that $\alpha U(\beta x)$ solves (6) whenever

$$\alpha = \alpha(\varepsilon \xi) = \left[\frac{V(\varepsilon \xi)}{K(\varepsilon \xi)} \right]^{1/(p-1)} \quad \text{and} \quad \beta = \beta(\varepsilon \xi) = [V(\varepsilon \xi)]^{1/2}.$$

We set

$$z^{\varepsilon \xi}(x) = \alpha(\varepsilon \xi)U(\beta(\varepsilon \xi)x) \quad (8)$$

and

$$Z^\varepsilon = \{z^{\varepsilon \xi}(x - \xi) : \xi \in \mathbb{R}^N\}.$$

When there is no possible misunderstanding, we will write z , resp. Z , instead of $z^{\varepsilon \xi}$, resp. Z^ε . We will also use the notation z_ξ to denote the function $z_\xi(x) \equiv z^{\varepsilon \xi}(x - \xi)$. Obviously all the functions in $z_\xi \in Z$ are solutions of (6) or, equivalently, critical points of $F^{\varepsilon \xi}$.

The next lemma shows that z_ξ is an ‘‘almost solution’’ of (5).

Lemma 2.1. *Given $\bar{\xi}$, for all $|\xi| \leq \bar{\xi}$ and for all ε sufficiently small, we have*

$$\|\nabla f_\varepsilon(z_\xi)\| = O(\varepsilon). \quad (9)$$

Proof Let $v \in H^1(\mathbb{R}^N)$, recalling that z_ξ is solution of (6), we have:

$$\begin{aligned} (\nabla f_\varepsilon(z_\xi) | v) &= \int_{\mathbb{R}^N} \nabla z_\xi \cdot \nabla v + \int_{\mathbb{R}^N} V(\varepsilon x) z_\xi v - \int_{\mathbb{R}^N} K(\varepsilon x) z_\xi^p v - \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v \\ &= \int_{\mathbb{R}^N} [\nabla z_\xi \cdot \nabla v + V(\varepsilon \xi) z_\xi v - K(\varepsilon \xi) z_\xi^p v] \\ &\quad + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi v - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^p v - \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v \\ &= \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi v - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^p v - \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v. \end{aligned} \quad (10)$$

Following [4], we infer that

$$\int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi v - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^p v = O(\varepsilon) \|v\|.$$

Let us study the last term in (10). We get

$$\left| \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v \right| \leq \left(\int_{\mathbb{R}^N} Q(\varepsilon x)^{\frac{2^*}{\sigma}} z_\xi^{2^*} \right)^{\frac{\sigma}{2^*}} \|v\|.$$

By assumption **(Q)**, we know that

$$|Q(\varepsilon x)| \leq \varepsilon |\nabla Q(0)| |x| + C \varepsilon^2 |x|^2,$$

therefore

$$\int_{\mathbb{R}^N} Q(\varepsilon x)^{\frac{2^*}{\sigma}} z_\xi^{2^*} \leq C_1 \varepsilon^{\frac{2^*}{\sigma}} \int_{\mathbb{R}^N} |x|^{\frac{2^*}{\sigma}} z^{2^*}(x - \xi) dx + C_2 \varepsilon^{2 \frac{2^*}{\sigma}} \int_{\mathbb{R}^N} |x|^{2 \frac{2^*}{\sigma}} z^{2^*}(x - \xi) dx.$$

By the exponential decay of z , it is easy to see that, if $|\xi| \leq \bar{\xi}$, then

$$\left(\int_{\mathbb{R}^N} Q(\varepsilon x)^{\frac{2^*}{\sigma}} z_\xi^{2^*} \right)^{\frac{\sigma}{2^*}} \|v\| = O(\varepsilon) \|v\|$$

and so the lemma is proved. \square

3 The finite dimensional reduction

In the next lemma we will show that $D^2 f_\varepsilon$ is invertible on $(T_{z_\xi} Z^\varepsilon)^\perp$, where $T_{z_\xi} Z^\varepsilon$ denotes the tangent space to Z^ε at z_ξ .

Let $L_{\varepsilon, \xi} : (T_{z_\xi} Z^\varepsilon)^\perp \rightarrow (T_{z_\xi} Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon, \xi} v \mid w) = D^2 f_\varepsilon(z_\xi)[v, w]$.

Lemma 3.1. *Given $\bar{\xi} > 0$, there exists $C > 0$ such that for ε small enough one has that*

$$\|L_{\varepsilon, \xi} v\| \geq C \|v\|, \quad \forall |\xi| \leq \bar{\xi}, \quad \forall v \in (T_{z_\xi} Z^\varepsilon)^\perp. \quad (11)$$

Proof We recall that $T_{z_\xi} Z^\varepsilon = \text{span}\{\partial_{\xi_1} z_\xi, \dots, \partial_{\xi_N} z_\xi\}$ and, moreover, by straightforward calculations, (see [4]), we get:

$$\partial_{\xi_i} z^{\varepsilon \xi}(x - \xi) = -\partial_{x_i} z^{\varepsilon \xi}(x - \xi) + O(\varepsilon). \quad (12)$$

Therefore, let $\mathcal{V} = \text{span}\{z_\xi, \partial_{x_1} z_\xi, \dots, \partial_{x_N} z_\xi\}$, by (12) it suffices to prove (11) for all $v \in \text{span}\{z_\xi, \phi\}$, where ϕ is orthogonal to \mathcal{V} . Precisely we shall prove that there exist $C_1, C_2 > 0$ such that, for all $\varepsilon > 0$ small and all $|\xi| \leq \bar{\xi}$, one has:

$$(L_{\varepsilon, \xi} z_\xi \mid z_\xi) \leq -C_1 < 0, \quad (13)$$

$$(L_{\varepsilon, \xi} \phi \mid \phi) \geq C_2 \|\phi\|^2, \quad \text{for all } \phi \perp \mathcal{V}. \quad (14)$$

The proof of (13) follows easily from the fact that z_ξ is a Mountain Pass critical point of $F^{\varepsilon \xi}$ and so from the fact that, given $\bar{\xi}$, there exists $c_0 > 0$ such that for all $\varepsilon > 0$ small and all $|\xi| \leq \bar{\xi}$ one finds:

$$D^2 F^{\varepsilon \xi}(z_\xi)[z_\xi, z_\xi] < -c_0 < 0.$$

Indeed, arguing as in the proof of Lemma 2.1, we have

$$\begin{aligned} (L_{\varepsilon, \xi} z_\xi \mid z_\xi) &= D^2 F^{\varepsilon \xi}(z_\xi)[z_\xi, z_\xi] + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi^2 \\ &\quad - p \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^{p+1} - \sigma \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^{\sigma+1} < -c_0 + O(\varepsilon) < -C_1. \end{aligned}$$

Let us prove (14). As before, the fact that z_ξ is a Mountain Pass critical point of $F^{\varepsilon \xi}$ implies that

$$D^2 F^{\varepsilon \xi}(z_\xi)[\phi, \phi] > c_1 \|\phi\|^2 \quad \text{for all } \phi \perp \mathcal{V}. \quad (15)$$

Consider a radial smooth function $\chi_1 : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \leq \varepsilon^{-1/2}; \quad \chi_1(x) = 0, \quad \text{for } |x| \geq 2\varepsilon^{-1/2};$$

$$|\nabla \chi_1(x)| \leq 2\varepsilon^{1/2}, \quad \text{for } \varepsilon^{-1/2} \leq |x| \leq 2\varepsilon^{-1/2}.$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x - \xi)\phi(x), \quad i = 1, 2.$$

As observed in [4], we have

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2 \underbrace{\int_{\mathbb{R}^N} \chi_1 \chi_2 (\phi^2 + |\nabla \phi|^2)}_{I_\phi} + o_\varepsilon(1) \|\phi\|^2.$$

We need to evaluate the three terms in the equation below:

$$(L_{\varepsilon, \xi} \phi \mid \phi) = (L_{\varepsilon, \xi} \phi_1 \mid \phi_1) + (L_{\varepsilon, \xi} \phi_2 \mid \phi_2) + 2(L_{\varepsilon, \xi} \phi_1 \mid \phi_2).$$

We have:

$$\begin{aligned} (L_{\varepsilon, \xi} \phi_1 \mid \phi_1) &= D^2 F^{\varepsilon \xi}(z_\xi)[\phi_1, \phi_1] + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) \phi_1^2 \\ &\quad - p \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^{p-1} \phi_1^2 - \sigma \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^{\sigma-1} \phi_1^2. \end{aligned}$$

Following [4], using (15) and the definition of χ_i , it is easy to see that

$$D^2 F^{\varepsilon \xi}(z_\xi)[\phi_1, \phi_1] \geq c_1 \|\phi_1\|^2 + o_\varepsilon(1) \|\phi\|^2$$

and

$$\left| \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) \phi_1^2 - p \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^{p-1} \phi_1^2 - \sigma \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^{\sigma-1} \phi_1^2 \right| \leq \varepsilon^{1/2} c_2 \|\phi\|^2,$$

hence

$$(L_{\varepsilon, \xi} \phi_1 \mid \phi_1) \geq c_1 \|\phi_1\|^2 - \varepsilon^{1/2} c_2 \|\phi\|^2 + o_\varepsilon(1) \|\phi\|^2.$$

Analogously

$$\begin{aligned} (L_{\varepsilon, \xi} \phi_2 \mid \phi_2) &\geq c_3 \|\phi_2\|^2 + o_\varepsilon(1) \|\phi\|^2, \\ (L_{\varepsilon, \xi} \phi_1 \mid \phi_2) &\geq c_4 I_\phi + o_\varepsilon(1) \|\phi\|^2. \end{aligned}$$

Therefore, we get

$$(L_{\varepsilon, \xi} \phi \mid \phi) \geq c_5 \|\phi\|^2 - c_6 \varepsilon^{1/2} \|\phi\|^2 + o(\varepsilon) \|\phi\|^2.$$

This proves (14) and completes the proof of the lemma. \square

We will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite dimensional functional. First of all we will make a Liapunov-Schmidt reduction, and successively we will study the behavior of an auxiliary finite dimensional functional.

Lemma 3.2. *For $\varepsilon > 0$ small and $|\xi| \leq \bar{\xi}$ there exists a unique $w = w(\varepsilon, \xi) \in (T_{z_\xi} Z)^\perp$ such that $\nabla f_\varepsilon(z_\xi + w) \in T_{z_\xi} Z$. Such a $w(\varepsilon, \xi)$ is of class C^2 , resp. $C^{1, p-1}$, with respect to ξ , provided that $p \geq 2$, resp. $1 < p < 2$. Moreover, the functional $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$ has the same regularity of w and satisfies:*

$$\nabla \Phi_\varepsilon(\xi_0) = 0 \iff \nabla f_\varepsilon(z_{\xi_0} + w(\varepsilon, \xi_0)) = 0.$$

Proof Let $P \equiv P_{\varepsilon, \xi}$ denote the projection onto $(T_{z_\xi} Z)^\perp$. We want to find a solution $w \in (T_{z_\xi} Z)^\perp$ of the equation $P \nabla f_\varepsilon(z_\xi + w) = 0$. One has that $\nabla f_\varepsilon(z_\xi + w) = \nabla f_\varepsilon(z_\xi) + D^2 f_\varepsilon(z_\xi)[w] + R(z_\xi, w)$ with $\|R(z, w)\| = o(\|w\|)$, uniformly with respect to z_ξ , for $|\xi| \leq \bar{\xi}$. Therefore, our equation is:

$$L_{\varepsilon, \xi} w + P \nabla f_\varepsilon(z_\xi) + P R(z_\xi, w) = 0. \quad (16)$$

According to Lemma 3.1, this is equivalent to

$$w = N_{\varepsilon, \xi}(w), \quad \text{where} \quad N_{\varepsilon, \xi}(w) = -(L_{\varepsilon, \xi})^{-1} (P \nabla f_\varepsilon(z_\xi) + P R(z_\xi, w)).$$

By (9) it follows that

$$\|N_{\varepsilon, \xi}(w)\| = O(\varepsilon) + o(\|w\|). \quad (17)$$

Then one readily checks that $N_{\varepsilon, \xi}$ is a contraction on some ball in $(T_{z_\xi} Z)^\perp$ provided that $\varepsilon > 0$ is small enough and $|\xi| \leq \bar{\xi}$. Then there exists a unique w such that $w = N_{\varepsilon, \xi}(w)$. Given $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(\xi, w) \mapsto P \nabla f_\varepsilon(z_\xi + w)$. Then, in particular, the function $w(\varepsilon, \xi)$ turns out to be of class C^1 with respect to ξ . Finally, it is a standard argument, see [2, 3], to check that the critical points of $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w)$ give rise to critical points of f_ε . \square

Now we will give two estimates on w and $\partial_{\xi_i} w$ which will be useful to study the finite dimensional functional $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$.

Remark 3.3. From (17) it immediately follows that:

$$\|w\| = O(\varepsilon). \quad (18)$$

Moreover, repeating the arguments of [4], if $\gamma = \min\{1, p - 1\}$ and $i = 1, \dots, N$, we infer that

$$\|\partial_{\xi_i} w\| = O(\varepsilon^\gamma). \quad (19)$$

4 The finite dimensional functional

Now we will use the estimates on w and $\partial_{\xi_i} w$ established in the previous section to find the expansion of $\nabla \Phi_\varepsilon(\xi)$, where $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$.

Lemma 4.1. Let $|\xi| \leq \bar{\xi}$. Suppose **(V)**, **(K)** and **(Q)**. Then, for ε sufficiently small, we get:

$$\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi)) = \Gamma(\varepsilon\xi) + O(\varepsilon), \quad (20)$$

where Γ is the auxiliary function introduced in (2).

Moreover, for all $i = 1, \dots, N$, we get:

$$\partial_{\xi_i} \Phi_\varepsilon(\xi) = \varepsilon \partial_{\xi_i} \Gamma(\varepsilon\xi) + o(\varepsilon). \quad (21)$$

Proof In the sequel, to be short, we will often write w instead of $w(\varepsilon, \xi)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

Since z_ξ is a solution of (6), we have:

$$\begin{aligned} \Phi_\varepsilon(\xi) &= f_\varepsilon(z_\xi + w(\varepsilon, Q)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla(z_\xi + w)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x)(z_\xi + w)^2 \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} K(\varepsilon x)(z_\xi + w)^{p+1} - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} Q(\varepsilon x)(z_\xi + w)^{\sigma+1} \\ &= \Sigma_\varepsilon(\xi) + \mathbf{L}_\varepsilon(\xi) + \Theta_\varepsilon(\xi), \end{aligned} \quad (22)$$

where

$$\Sigma_\varepsilon(\xi) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla z_\xi|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon\xi) z_\xi^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} K(\varepsilon\xi) z_\xi^{p+1}, \quad (23)$$

$$\Theta_\varepsilon(\xi) = -\frac{1}{\sigma+1} \int_{\mathbb{R}^N} Q(\varepsilon\xi) z_\xi^{\sigma+1} \quad (24)$$

and

$$\begin{aligned}
\mathbb{L}_\varepsilon(\xi) &= \frac{1}{2} \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi^2 + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi w \\
&\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^{p+1} - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon \xi)) z_\xi^p w \\
&\quad - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} (Q(\varepsilon x) - Q(\varepsilon \xi)) z_\xi^{\sigma+1} - \int_{\mathbb{R}^N} (Q(\varepsilon x) - Q(\varepsilon \xi)) z_\xi^\sigma w \\
&\quad \quad \quad + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) w^2 \\
&\quad \quad \quad - \frac{1}{p+1} \int_{\mathbb{R}^N} K(\varepsilon x) [(z_\xi + w)^{p+1} - z_\xi^{p+1} - (p+1) z_\xi^p w] \\
&\quad - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} Q(\varepsilon x) [(z_\xi + w)^{\sigma+1} - z_\xi^{\sigma+1} - (\sigma+1) z_\xi^\sigma w] - \int_{\mathbb{R}^N} Q(\varepsilon \xi) z_\xi^\sigma w.
\end{aligned}$$

Let us observe that, since, $Q(0) = 0$, arguing as in the proof of Lemma 2.1 and recalling (18), we get

$$\left| \int_{\mathbb{R}^N} Q(\varepsilon \xi) z_\xi^\sigma w \right| \leq \left(\int_{\mathbb{R}^N} Q(\varepsilon \xi)^{\frac{2^*}{\sigma}} z_\xi^{2^*} \right)^{\frac{\sigma}{2^*}} \|w\| = o(\varepsilon).$$

By this and with easy calculations, see also [4], we infer

$$\mathbb{L}_\varepsilon(\xi) = O(\varepsilon). \tag{25}$$

Moreover, since z_ξ is solution of (6), we get

$$\Sigma_\varepsilon(\xi) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} K(\varepsilon \xi) z_\xi^{p+1}.$$

By (8), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} K(\varepsilon \xi) z_\xi^{p+1} &= V(\varepsilon \xi)^{\frac{p+1}{p-1} - \frac{N}{2}} K(\varepsilon \xi)^{-\frac{2}{p-1}} \int_{\mathbb{R}^N} U^{p+1}. \\
\int_{\mathbb{R}^N} Q(\varepsilon \xi) z_\xi^{\sigma+1} &= Q(\varepsilon \xi) V(\varepsilon \xi)^{\frac{\sigma+1}{p-1} - \frac{N}{2}} K(\varepsilon \xi)^{-\frac{\sigma+1}{p-1}} \int_{\mathbb{R}^N} U^{\sigma+1}.
\end{aligned}$$

By these two equations and by (22), (23), (24) and (25) we prove the first part of the lemma.

Let us prove now the estimate on the derivatives of Φ_ε .

It is easy to see that

$$\nabla \Theta_\varepsilon(\xi) = o(\varepsilon).$$

With calculations similar to those of [4], we infer that

$$\nabla L_\varepsilon(\xi) = o(\varepsilon),$$

and so (21) follows immediately. \square

5 Proof of Theorem 1.1

In this section we will give two multiplicity results. Theorem 1.1 will follow from those as a particular case.

Theorem 5.1. *Let (V), (K) and (Q) hold. Suppose Γ has a nondegenerate smooth manifold of critical points M . Then for $\varepsilon > 0$ small, (1) has at least $l(M)$ solutions that concentrate near points of M . Here $l(M)$ denotes the cup long of M (for a precise definition see, for example, [4]).*

Proof First of all, we fix $\bar{\xi}$ in such a way that $|x| < \bar{\xi}$ for all $x \in M$. We will apply the finite dimensional procedure with such $\bar{\xi}$ fixed.

Fix a δ -neighborhood M_δ of M such that $M_\delta \subset \{|x| < \bar{\xi}\}$ and the only critical points of Γ in M_δ are those in M . We will take $U = M_\delta$.

By (20) and (21), $\Phi_\varepsilon(\cdot/\varepsilon)$ converges to $\Gamma(\cdot)$ in $C^1(\bar{U})$ and so, by Theorem 6.4 in Chapter II of [5], we have at least $l(M)$ critical points of l provided ε sufficiently small.

Let ξ be one of these critical points of Ψ_ε , then $u_\varepsilon^\xi = z_\xi + w(\varepsilon, \xi)$ is solution of (5) and so

$$u_\varepsilon^\xi(x/\varepsilon) \simeq z_\xi(x/\varepsilon) = z^{\varepsilon\xi} \left(\frac{x - \xi}{\varepsilon} \right)$$

is solution of (1) and concentrates on ξ . \square

Moreover, when we deal with local minima (resp. maxima) of Γ , the preceding results can be improved because the number of positive solutions of (1) can be estimated by means of the category and M does not need to be a manifold.

Theorem 5.2. *Let (V), (K) and (Q) hold and suppose Γ has a compact set X where Γ achieves a strict local minimum (resp. maximum), in the sense that there exists $\delta > 0$ and a δ -neighborhood X_δ of X such that*

$$b \equiv \inf\{\Gamma(x) : x \in \partial X_\delta\} > a \equiv \Gamma|_X, \quad (\text{resp. } \sup\{\Gamma(x) : x \in \partial X_\delta\} < a).$$

Then there exists $\varepsilon_\delta > 0$ such that (1) has at least $\text{cat}(X, X_\delta)$ solutions that concentrate near points of X_δ , provided $\varepsilon \in (0, \varepsilon_\delta)$.

Proof We will treat only the case of minima, being the other one similar. Fix again $\bar{\xi}$ in such a way that X_δ is contained in $\{x \in \mathbb{R}^N : |x| < \bar{\xi}\}$. We set $X^\varepsilon = \{\xi : \varepsilon\xi \in X\}$, $X_\delta^\varepsilon = \{\xi : \varepsilon\xi \in X_\delta\}$ and $Y^\varepsilon = \{\xi \in X_\delta^\varepsilon : \Phi_\varepsilon(\xi) \leq (a+b)/2\}$. By (20) it follows that there exists $\varepsilon_\delta > 0$ such that

$$X^\varepsilon \subset Y^\varepsilon \subset X_\delta^\varepsilon, \quad (26)$$

provided $\varepsilon \in (0, \varepsilon_\delta)$. Moreover, if $\xi \in \partial X_\delta^\varepsilon$ then $\Gamma(\varepsilon\xi) \geq b$ and hence

$$\Phi_\varepsilon(\xi) \geq \Gamma(\varepsilon\xi) + O(\varepsilon) \geq b + o_\varepsilon(1).$$

On the other side, if $\xi \in Y^\varepsilon$ then $\Phi_\varepsilon(\xi) \leq (a+b)/2$. Hence, for ε small, Y^ε cannot meet $\partial X_\delta^\varepsilon$ and this readily implies that Y^ε is compact. Then Φ_ε possesses at least $\text{cat}(Y^\varepsilon, X_\delta^\varepsilon)$ critical points in X_δ . Using (26) and the properties of the category one gets

$$\text{cat}(Y^\varepsilon, Y^\varepsilon) \geq \text{cat}(X^\varepsilon, X_\delta^\varepsilon) = \text{cat}(X, X_\delta).$$

The concentration statement follows as before. □

Remark 5.3. *Let us observe that the (a) of Theorem 1.1 is a particular case of Theorem 5.1, while the (b) is a particular case of Theorem 5.2.*

References

- [1] C. O. Alves, João Marcos do Ó, M. A. S. Souto, *Local mountain-pass for a class of elliptic problems in \mathbb{R}^N involving critical growth*, *Nonlinear Anal.*, **46**, (2001), 495–510.
- [2] A. Ambrosetti, M. Badiale, *Variational perturbative methods and bifurcation of bound states from the essential spectrum*, *Proc. Royal Soc. Edinburgh*, **128 A**, (1998), 1131–1161.
- [3] A. Ambrosetti, M. Badiale, S. Cingolani, *Semiclassical States of Nonlinear Schrödinger Equations*, *Arch. Rational Mech. Anal.*, **140**, (1997), 285–300.
- [4] A. Ambrosetti, A. Malchiodi, S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, *Arch. Rational Mech. Anal.*, **159**, (2001), 253–271.
- [5] K. C. Chang, *Infinite dimensional Morse theory and multiple solutions problems*, Birkhäuser, 1993.

- [6] S. Cingolani, M. Lazzo, *Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions*, Journal of Diff. Eq., **160**, (2000), 118–138.
- [7] S. Cingolani, A. Pistoia, *Nonexistence of single blow-up solutions for a nonlinear Schrödinger equation involving critical Sobolev exponent*, Zeit. Angew. Mathematik und Physik, **55**, (2004), 1–15.
- [8] X. F. Wang, B. Zeng, *On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions*, SIAM J. Math. Anal., **28**, (1997), 633–655.