

WEIL-ÉTALE COHOMOLOGY OVER FINITE FIELDS

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ABSTRACT. We calculate the derived functors $R\gamma_*$ for the base change γ from the Weil-étale site to the étale site for a variety over a finite field. For smooth and proper varieties, we apply this to express Tate's conjecture and Lichtenbaum's conjecture on special values of ζ -functions in terms of Weil-étale cohomology of the motivic complex $\mathbb{Z}(n)$.

1. INTRODUCTION

In [11], Lichtenbaum defined Weil-étale cohomology groups of varieties over finite fields in order to produce finitely generated cohomology groups which are related to special values of zeta functions. He gave several examples where these groups were indeed finitely generated. The purpose of this paper is to elucidate the precise relationship between Weil-étale cohomology groups and étale cohomology groups. This is applied to give necessary and sufficient conditions for the Weil-étale cohomology groups to be finitely generated, and to be related to special values of zeta functions.

Recall that an étale sheaf on a variety X over a finite field \mathbb{F}_q corresponds to a sheaf on $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$, together with a continuous action of the Galois group $\hat{G} = \text{Gal}(\bar{\mathbb{F}}_q : \mathbb{F}_q)$. In the Weil-étale topology, the role of the Galois group is replaced by the Weil group G , which is the subgroup of \hat{G} generated by the Frobenius operator φ : A Weil-étale sheaf is an étale sheaf on \bar{X} , together with an action of G . If we denote the category of Weil-étale sheaves by \mathcal{T}_G and the category of étale sheaves by $\mathcal{T}_{\hat{G}}$, then there is a morphism of topoi $\gamma : \mathcal{T}_G \rightarrow \mathcal{T}_{\hat{G}}$. The functor γ^* is the restriction functor, and for U étale over \bar{X} , $\gamma_*\mathcal{F}(U) = \text{colim}_H \mathcal{F}(U)^H$, where H runs through sufficiently small subgroups of G .

We give an explicit description of the total derived functor $R\gamma_*$ and derive formulas for $\gamma_*\mathcal{F}$, $R^1\gamma_*\mathcal{F}$; for $i > 1$, $R^i\gamma_*\mathcal{F} = 0$. If $\mathcal{F} = \gamma^*\mathcal{G}$ is the restriction of an étale sheaf, then the formula can be simplified to the following projection formula:

Theorem 1.1. *For every complex \mathcal{G} of étale sheaves, there is a quasi-isomorphism of complexes of étale sheaves*

$$R\gamma_*\mathbb{Z} \otimes^L \mathcal{G} \cong R\gamma_*(\gamma^*\mathcal{G}).$$

This raises the question of calculating $R\gamma_*\mathbb{Z}$. We show that $\gamma_*\mathbb{Z} \cong \mathbb{Z}$, $R^1\gamma_*\mathbb{Z} \cong \mathbb{Q}$, which gives the distinguished triangle

$$\mathcal{G} \rightarrow R\gamma_*\gamma^*\mathcal{G} \rightarrow \mathcal{G} \otimes \mathbb{Q}[-1] \xrightarrow{\delta} \mathcal{G}[1], \quad (1)$$

and implies that for a complex with torsion cohomology sheaves $\mathcal{G} \cong R\gamma_*\gamma^*\mathcal{G}$. We show that $R\gamma_*\mathbb{Z}$ is quasi-isomorphic to a complex considered by Kahn in [7], hence

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the map δ is induced by the composition

$$\mathbb{Q}[-1] \rightarrow \mathbb{Q}/\mathbb{Z}[-1] \xrightarrow{\cup \epsilon} \mathbb{Q}/\mathbb{Z}[0] \xrightarrow{\beta} \mathbb{Z}[1],$$

with β the Bockstein-homomorphism and $\epsilon \in \text{Ext}_{\hat{\mathcal{G}}}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ the class of the $\hat{\mathcal{G}}$ -module $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ with action $g \cdot (a, b) = (a + gb, b)$. In particular, $\delta = 0$ for a complex with \mathbb{Q} -vector spaces as cohomology sheaves, hence the sequence (1) splits and $R\gamma_*\gamma^*\mathcal{G} \cong \mathcal{G} \oplus \mathcal{G}[-1]$. We show that under the latter isomorphism, the cup-product with a generator $e \in H_W^1(\mathbb{F}_q, \mathbb{Z})$ is given by multiplication with the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

In the second half of the paper, we give applications of the above calculations to the Weil-étale hypercohomology groups $H_W^i(X, \mathbb{Z}(n))$ of the motivic complex $\gamma^*\mathbb{Z}(n)$. We assume that X is smooth over \mathbb{F}_q , because Weil-étale cohomology groups for singular schemes are not well-behaved. (We discuss in a forthcoming paper how to refine the Weil-étale topology to get reasonable cohomology groups for singular schemes). The general results above specialize to this situation, and we show that if X is of dimension d , then $H_W^i(X, \mathbb{Z}(n)) = 0$ for $i > \max\{2d+1, n+d+1\}$. If X is connected and proper, then there is an isomorphism $H_W^{2d+1}(X, \mathbb{Z}(d)) \xrightarrow{\text{deg}} \mathbb{Z}$, and the composition $H_W^{2d}(X, \mathbb{Z}(d)) \xrightarrow{\text{deg}(-\cup e)} \mathbb{Z}$ is surjective.

Lichtenbaum expected statement $L(X, n)$: If X is smooth and proper, then the cohomology groups $H_W^i(X, \mathbb{Z}(n))$ are finitely generated for all i . On the other hand, a conjecture of Kahn [7] can be reformulated with the above results into statement $K(X, n)$: If X is smooth and proper, then Weil-étale motivic cohomology is an integral model for l -adic cohomology, i.e. for every prime l (including p),

$$H_W^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \cong H_{\text{cont}}^i(X, \mathbb{Z}_l(n)).$$

Statements $K(X, n)$ and $L(X, n)$ are related to the conjunction $T(X, n)$ of Tate's conjecture on the surjectivity of the cycle map $CH^n(X) \otimes \mathbb{Q}_l \rightarrow H_{\text{cont}}^{2n}(\bar{X}, \mathbb{Q}_l(n))^{\hat{\mathcal{G}}}$, and semi-simplicity of $H_{\text{cont}}^{2n}(\bar{X}, \mathbb{Q}_l(n))$ at the eigenvalue 1, together with Beilinson's conjecture that rational and numerical equivalence on X agree up to torsion in codimension n (see also [9]):

Theorem 1.2. *Let X be a smooth projective variety over \mathbb{F}_q , and n an integer. Then*

$$K(X, n) + K(X, d-n) \Rightarrow L(X, n) \Rightarrow K(X, n) \Rightarrow T(X, n).$$

Conversely, if $T(X, n)$ holds for all smooth and projective varieties over \mathbb{F}_q and all n , then $K(X, n)$ and $L(X, n)$ hold for all X and n .

Finally we reinterpret a result of Milne [13] to show that Weil-étale motivic cohomology can be used to give formulas for special values of ζ -functions of varieties over finite fields, as anticipated by Lichtenbaum. For a complex with finitely many finite cohomology groups, define $\chi(C^\cdot) := \prod_i |H^i(C^\cdot)|^{(-1)^i}$ and let

$$\chi(X, \mathcal{O}_X, n) = \sum_{i \leq n, j \leq d} (-1)^{i+j} (n-i) \dim H^j(X, \Omega^i).$$

Since $e^2 = 0$, the groups $H_W^*(X, \mathbb{Z}(n))$ form a complex with differential e .

Theorem 1.3. *Let X be a smooth projective variety such that $K(X, n)$ holds. Then the order ρ_n of the pole of $\zeta(X, s)$ at $s = n$ is $\text{rank } H_W^{2n}(X, \mathbb{Z}(n))$, and*

$$\zeta(X, s) = \pm(1 - q^{n-s})^{-\rho_n} \cdot \chi(H_W^*(X, \mathbb{Z}(n)), e) \cdot q^{\chi(X, \mathcal{O}_X, n)} \quad \text{as } s \rightarrow n.$$

If furthermore $K(X, d - n)$ holds, then

$$\chi(H_W^*(X, \mathbb{Z}(n)), e) = \prod_i |H_W^i(X, \mathbb{Z}(n))_{\text{tor}}|^{(-1)^i} \cdot R^{-1},$$

where R is the determinant of the pairing

$$H_W^{2n}(X, \mathbb{Z}(n)) \times H_W^{2(d-n)}(X, \mathbb{Z}(d-n)) \rightarrow H_W^{2d}(X, \mathbb{Z}(d)) \rightarrow \mathbb{Z}.$$

To give explicit evidence, we show that $K(X, 0)$ holds, and that the surjectivity of the cycle map $\text{Pic } X \otimes \mathbb{Q}_l \rightarrow H_{\text{cont}}^2(X, \mathbb{Q}_l(1))$ implies $K(X, 1)$. In particular, $K(X, 1)$ holds for Hilbert modular surfaces, Picard modular surfaces, Siegel modular threefolds, and in characteristic at least 5 for supersingular and elliptic K3 surfaces. Using the method of Soulé [18], we also show that $K(X, n)$ holds for a smooth projective variety X of dimension d , which can be constructed out of products of smooth projective curves by union, base extension and blow-ups, and for $n \leq 1$ or $n \geq d - 1$. This applies to abelian varieties, unirational varieties of dimension at most 3, and to Fermat hypersurfaces. In [9], Kahn shows that conjecture $K(X, n)$ is true for arbitrary n if X is of abelian type and satisfies Tate's conjecture. This applies in particular to the product of elliptic curves.

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2. PROFINITE COMPLETION

We fix a finite field \mathbb{F}_q , let $\bar{\mathbb{F}}_q$ be the algebraic closure of \mathbb{F}_q , and φ the arithmetic Frobenius endomorphism $x \rightarrow x^q$ of $\bar{\mathbb{F}}_q$ over \mathbb{F}_q . The Galois group \hat{G} of $\bar{\mathbb{F}}_q/\mathbb{F}_q$ is isomorphic to the profinite completion $\lim_m \mathbb{Z}/m$ of \mathbb{Z} , and we let G be the subgroup of \hat{G} generated by φ . Of course, G is isomorphic to \mathbb{Z} , but we want to avoid confusing G -modules and abelian groups. The fixed field of mG and of $m\hat{G}$ is \mathbb{F}_{q^m} .

Let \mathcal{E} be a full subcategory of the category of separated schemes of finite type over \mathbb{F}_q , which contains with every scheme X also every scheme U which is étale and of finite type over X . Our main examples will be the category of separated schemes of finite type over \mathbb{F}_q , the category of smooth schemes of finite type over \mathbb{F}_q , and the small étale site of a scheme X separated and of finite type over \mathbb{F}_q . Let $\bar{\mathcal{E}}$ be the full subcategory of separated schemes of finite type over $\bar{\mathbb{F}}_q$ which are connected components of the base-change of a scheme in \mathcal{E} ; note that every scheme of finite type over $\bar{\mathbb{F}}_q$ is the base-change of a scheme over some \mathbb{F}_{q^r} . We equip \mathcal{E} and $\bar{\mathcal{E}}$ with the étale topology, although all arguments below hold for any Grothendieck topology τ which is at least as fine as the étale topology.

For $U \in \bar{\mathcal{E}}$, and $g \in \hat{G}$ we let $gU = U \times_{\bar{\mathbb{F}}_q, g^{-1}} \bar{\mathbb{F}}_q$, so that for every sheaf \mathcal{F} on $\bar{\mathcal{E}}$ we have $g^*\mathcal{F}(U) = \mathcal{F}(gU)$ and $g_*\mathcal{F}(U) = \mathcal{F}(g^{-1}U)$. We say that \hat{G} acts on \mathcal{F} , if for every $g \in \hat{G}$ there is an isomorphism $\sigma_g : \mathcal{F} \rightarrow g^*\mathcal{F}$ satisfying $\sigma_{gh} = h^*\sigma_g \circ \sigma_h$. For $f \in \mathcal{F}(U)$, we will abbreviate $\sigma_g(f) \in \mathcal{F}(gU)$ by gf .

Let $\hat{S}(U) \subseteq \hat{G}$ be the Galois group of the smallest field extension \mathbb{F}_{q^r} of \mathbb{F}_q over which U has a model U' , i.e. $U = U' \times_{\mathbb{F}_{q^r}} \bar{\mathbb{F}}_q$, and let $S(U) = \hat{S}(U) \cap G$. If \hat{G} acts on \mathcal{F} , then $\hat{S}(U)$ acts on $\mathcal{F}(U)$. In particular, \hat{G} acts on $\mathcal{F}(V \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q)$ for every $V \in \mathcal{E}$. We say that \hat{G} acts continuously on \mathcal{F} , if for each étale $U \in \bar{\mathcal{E}}$, $\hat{S}(U)$

acts continuously on $\mathcal{F}(U)$ equipped with the discrete topology, i.e. if the map $\hat{S}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ is continuous. Let $\mathcal{T}_{\hat{G}}$ be the topos of sheaves on $\bar{\mathcal{E}}$ equipped with a continuous action of \hat{G} .

Lemma 2.1. *a) If \mathcal{F} is a sheaf on $\bar{\mathcal{E}}$, then \hat{G} acts continuously on \mathcal{F} if and only if $\text{colim}_{H \subseteq \hat{S}(U)} \mathcal{F}(U)^H \xrightarrow{\sim} \mathcal{F}(U)$ for every $U \in \bar{\mathcal{E}}$. The maps in the direct system are the natural inclusion maps.*

b) There is an equivalence of categories between the category of sheaves on \mathcal{E} and the category $\mathcal{T}_{\hat{G}}$.

Proof. a) This is well-known.

b) This is Deligne [SGA 7 XIII, 1.1.3]. Explicitly, if $\pi : \text{Spec } \bar{\mathbb{F}}_q \rightarrow \text{Spec } \mathbb{F}_q$ is the structure map, then the sheaf \mathcal{G} on \mathcal{E} corresponds to the sheaf $\pi^* \mathcal{G}$ on $\bar{\mathcal{E}}$, sending $U \rightarrow \bar{X}$ with model U' over \mathbb{F}_{q^r} to $\text{colim}_m \mathcal{G}(U' \times_{\mathbb{F}_{q^r}} \mathbb{F}_{q^{rm}})$. The actions of $\hat{G}/m\hat{G}$ on $\mathcal{G}(U' \times_{\mathbb{F}_{q^r}} \mathbb{F}_{q^{rm}})$ are compatible and give an action of \hat{G} on the colimit. Conversely, a sheaf \mathcal{F} in $\mathcal{T}_{\hat{G}}$ corresponds to the sheaf $\pi_* \hat{\mathcal{G}} \mathcal{F}$ on \mathcal{E} , sending V to $\mathcal{F}(V \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q)^{\hat{G}}$. \square

In [11], Lichtenbaum defines the Weil-étale topology on the small étale site of a scheme X of finite type over \mathbb{F}_q . He shows that a Weil-étale sheaf is equivalent to an étale sheaf on \bar{X} together with a G -action, where $n \in G$ acts on \bar{X} via φ^n and on \mathcal{F} via $\sigma_\varphi^n : \mathcal{F}(U) \rightarrow \mathcal{F}(\varphi^n U)$. In accordance with Lichtenbaum's definition, we let \mathcal{T}_G be the topos of sheaves on $\bar{\mathcal{E}}$ equipped with an action of G , and call it the Weil-étale topos.

Lemma 2.2. *The forgetful functor from abelian groups of \mathcal{T}_G to sheaves of abelian groups on $\bar{\mathcal{E}}$ has an exact left adjoint and a right adjoint. In particular, it preserves injectives, and \mathcal{T}_G has enough injectives.*

Proof. The left adjoint is $\mathcal{F} \rightarrow \bigoplus_{g \in G} g^* \mathcal{F}$ and the right adjoint is $\mathcal{F} \rightarrow \prod_{g \in G} g^* \mathcal{F}$. In both cases, the action of G is the shift functor. A map $\alpha \in \text{Hom}_{\bar{\mathcal{E}}}(\mathcal{F}, \mathcal{G})$, corresponds to the G -invariant map $\text{Hom}_{\mathcal{T}_G}(\bigoplus_{g \in G} g^* \mathcal{F}, \mathcal{G})$ which on the summand indexed by g is the composition $g^* \mathcal{F} \xrightarrow{g^*(\alpha)} g^* \mathcal{G} \xrightarrow{\sigma_g^{-1}} \mathcal{G}$. The right adjoint case is analog. Since g^* and coproducts are exact, the left adjoint is exact and hence the forgetful functor preserves injectives. On the other hand, given a sheaf \mathcal{F} in \mathcal{T}_G , we can embed it into an injective étale sheaf \mathcal{I} on $\bar{\mathcal{E}}$. This gives rise to a G -invariant injection of \mathcal{F} into the sheaf $\prod_{g \in G} g^* \mathcal{I}$, which is injective in \mathcal{T}_G . \square

Recall that a morphism of topoi $\alpha : \mathcal{T} \rightarrow \mathcal{S}$ is a pair of adjoint functors $\alpha^* \dashv \alpha_*$ such that α^* commutes with finite limits.

Proposition 2.3. *There is a morphism of topoi $\gamma : \mathcal{T}_G \rightarrow \mathcal{T}_{\hat{G}}$. The functor γ^* is the forgetful functor, and*

$$\gamma_* \mathcal{F}(U) = \text{colim}_{H \subseteq S(U)} \mathcal{F}(U)^H,$$

where H runs through the subgroups of finite index in G which are contained in $S(U)$. In particular, γ_* is left exact and preserves injectives. The adjoint transformation $\text{id} \rightarrow \gamma_* \gamma^*$ is an isomorphism.

Proof. Since the invariant functor is left exact, γ_* of a sheaf is sheaf. The action of \hat{G} on $\gamma_*\mathcal{F}$ is given as follows. Given U and $H \subseteq S(U)$, $g \in \hat{G}$ acts as $\sigma_g = \sigma_\varphi^i : \mathcal{F}(U)^H \rightarrow \mathcal{F}(gU)^H$, if $g \equiv \varphi^i \pmod{H}$. It is easy to check that this is compatible with the inclusion $\mathcal{F}(U)^H \hookrightarrow \mathcal{F}(U)^{H'}$ for $H' \subseteq H \subseteq S(U)$, and hence induces an action of \hat{G} on the colimit.

Let \mathcal{F} be a sheaf with G -action and \mathcal{G} be a sheaf with continuous \hat{G} -action. Then $\mathcal{G} \cong \operatorname{colim}_H \mathcal{G}^H$, and the map

$$\begin{aligned} \operatorname{Hom}_G(\gamma^*\mathcal{G}, \mathcal{F}) &\rightarrow \operatorname{Hom}_{\hat{G}}(\mathcal{G}, \gamma_*\mathcal{F}) \\ \alpha &\rightarrow \operatorname{colim}_H \alpha|_{\mathcal{G}^H} \end{aligned}$$

is an isomorphism with inverse "composition with the adjoint inclusion $\gamma^*\gamma_*\mathcal{F} \rightarrow \mathcal{F}$ ". The fact $\mathcal{F} \xrightarrow{\sim} \gamma_*\gamma^*\mathcal{F}$ follows from the explicit description of γ_* and γ^* . \square

Since subgroups $H \subseteq \hat{S}(U)$ are cofinal in the set of all subgroups of finite index of \hat{G} , we will write by abuse of notation $\gamma_*\mathcal{F} = \operatorname{colim}_H \mathcal{F}^H$, remembering that even though not every \mathcal{F}^H is defined, the colimit is.

3. THE FUNCTOR $R\gamma_*$

Given two sheaves \mathcal{F} and \mathcal{G} in \mathcal{T}_G , the sheaf $\mathcal{H}\operatorname{om}(\mathcal{G}, \mathcal{F})$ is equipped with a G -action by $f^g = \sigma_g \circ f \circ \sigma_g^{-1}$. Then

$$\gamma_* \mathcal{H}\operatorname{om}(\mathcal{G}, \mathcal{F})(U) = \operatorname{colim}_{H \subseteq S(U)} \operatorname{Hom}(\mathcal{G}|_U, \mathcal{F}|_U)^H,$$

where the latter are the homomorphisms which are compatible with the action of H . If $\mathcal{G} = A$ is constant, then by adjointness of global section and constant sheaf functor we have $\operatorname{Hom}_{\mathcal{S}hv}(A, \mathcal{F}|_U)^H \cong \operatorname{Hom}_{\operatorname{Ab}}(A, \mathcal{F}(U))^H$, and the formula simplifies to $\gamma_* \mathcal{H}\operatorname{om}(A, \mathcal{F})(U) = \operatorname{colim}_{H \subseteq S(U)} \operatorname{Hom}(A, \mathcal{F}(U))^H$.

If $mG \subseteq S(U)$ and $i \in \mathbb{Z}/m$, then $\varphi^i U$ does not depend on the representative $\bar{i} \in \mathbb{Z}$ of i , and we simply write $\varphi^i U$. We denote the i th summand of $f \in \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U)$ by $f^{(i)}$.

Lemma 3.1. *Let \mathcal{F} in \mathcal{T}_G , $U \in \bar{\mathcal{E}}$, and $H = mG \subseteq S(U) \subseteq G$.*

a) *If H acts on $\operatorname{Hom}(\mathbb{Z}[G], \mathcal{F}(U)) \cong \mathcal{H}\operatorname{om}(\mathbb{Z}[G], \mathcal{F})(U)$ via $F \mapsto h \circ F \circ h^{-1}$, then there are isomorphisms*

$$\begin{aligned} \mathbb{Z}[G] \otimes_H \mathcal{F}(U) &\xrightarrow[\sim]{\alpha} \operatorname{Hom}(\mathbb{Z}[G], \mathcal{F}(U))^H, \\ \alpha(s \otimes f)(g) &= \begin{cases} gs f & \text{if } gs \in H; \\ 0 & \text{otherwise.} \end{cases} \\ \mathbb{Z}[G] \otimes_H \mathcal{F}(U) &\xrightarrow[\sim]{\beta} \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U), \\ \beta(\varphi^a \otimes f)^{(i)} &= \begin{cases} \varphi^a f & \text{if } i \equiv a \pmod{m}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

b) *The action of $\varphi \in G$ on $\operatorname{Hom}(\mathbb{Z}[G], \mathcal{F}(U))^H$ via multiplication on $\mathbb{Z}[G]$ corresponds under $\beta\alpha^{-1}$ to the automorphism $\zeta(f)^{(i)} = \varphi f^{(i-1)}$ of $\bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U)$.*

c) If $g \in \hat{G}/m\hat{G}$ satisfies $g \equiv \varphi^a \pmod{m\hat{G}}$, then the map $\text{Hom}(\mathbb{Z}[G], \mathcal{F}(U))^H \rightarrow \text{Hom}(\mathbb{Z}[G], \mathcal{F}(gU))^H$, $F \mapsto g \circ F \circ g^{-1}$ corresponds under $\beta\alpha^{-1}$ to the cyclic permutation

$$\tau_g : \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U) \rightarrow \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i gU), \quad \tau_g(f)^{(i)} = f^{(i+a)}.$$

d) Given a second subgroup $H' = mnG \subseteq H \subseteq S(U)$, the inclusion of fixed points $\text{Hom}(\mathbb{Z}[G], \mathcal{F}(U))^H \hookrightarrow \text{Hom}(\mathbb{Z}[G], \mathcal{F}(U))^{H'}$ corresponds under $\beta\alpha^{-1}$ to the map

$$\delta_m^n : \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U) \rightarrow \bigoplus_{j \in \mathbb{Z}/mn} \mathcal{F}(\varphi^j U), \quad \delta_m^n(f)^{(j)} = f^{(j \bmod m)}.$$

e) The map δ_m^n is compatible with the action of G and of \hat{G} given in b) and c), respectively.

Proof. a) This is an easy verification.

b) Consider the action of G on $\mathbb{Z}[G] \otimes_H \mathcal{F}(U)$ by left multiplication on $\mathbb{Z}[G]$. Then it is easy to verify that the three actions are compatible with α and β .

c) The conjugation map is well defined, because since F is H -invariant, we have for $h \in H$, $g \circ h \circ F \circ h^{-1} \circ g^{-1} = g \circ F \circ g^{-1}$. If $g \equiv \varphi^a \pmod{m\hat{G}}$, then $gU = \varphi^a U$, so that $\mathcal{F}(\varphi^i gU) = \mathcal{F}(\varphi^{i+a} U)$. It is easy to check that the conjugation map $F \mapsto g \circ F \circ g^{-1}$ corresponds under α to the map

$$\begin{aligned} \mathbb{Z}[G] \otimes_H \mathcal{F}(U) &\rightarrow \mathbb{Z}[G] \otimes_H \mathcal{F}(gU) \\ s \otimes f &\mapsto g^{-1} s \otimes g f \end{aligned}$$

and this corresponds to τ under β .

d) Let u be the inclusion of H -invariant maps into H' -invariant maps, and let $v : \mathbb{Z}[G] \otimes_H \mathcal{F}(U) \rightarrow \mathbb{Z}[G] \otimes_{H'} \mathcal{F}(U)$ be the map $s \otimes f \mapsto \sum_{j=0}^{n-1} s \varphi^{jm} \otimes \varphi^{-jm} f$. It is easy to check that $u \circ \alpha = \alpha \circ v$ and $\delta_m^n \circ \beta = \beta \circ v$.

e) This is clear because the actions of G and \hat{G} are compatible with the inclusion map, and with α and β . Explicitly,

$$\delta_m^n \zeta(f)^{(j)} = \zeta(f)^{(j \bmod m)} = \varphi f^{(j-1 \bmod m)} = \varphi \delta_m^n(f)^{(j-1)} = \zeta \delta_m^n(f)^{(j)},$$

and for $g \in \hat{G}$ with $g \equiv \varphi^a \pmod{mn\hat{G}}$,

$$\delta_m^n \tau_g(f)^{(j)} = \tau_g(f)^{(j \bmod m)} = f^{(j+a \bmod m)} = \delta_m^n(f)^{(j+a)} = \tau_g \delta_m^n(f)^{(j)}.$$

□

Consider the presheaf

$$\Xi(\mathcal{F}) : U \mapsto \text{colim}_{mG \subseteq S(U)} \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U),$$

where the index set is ordered by divisibility, and the maps in the direct system are the maps δ_m^n . The presheaf $\Xi(\mathcal{F})$ is a sheaf, because filtered direct limits and direct sums are left exact. Moreover, the action of $g \in \hat{G}$ is compatible with δ_m^n , so that we get an action of \hat{G} on $\Xi(\mathcal{F})$.

Lemma 3.2. *The functor $\mathcal{F} \mapsto \gamma_* \mathcal{H}\text{om}(\mathbb{Z}[G], -)$ from \mathcal{T}_G to $\mathcal{T}_{\hat{G}}$ is exact. In particular, the derived functors $R^i \text{colim}_H \mathcal{H}\text{om}(\mathbb{Z}[G], -)^H$ are zero for $i > 0$.*

Proof. By Lemma 3.1, $\text{colim}_m \mathcal{H}\text{om}(\mathbb{Z}[G], \mathcal{F})^{mG} \cong \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} (\varphi^i)^* \mathcal{F}$. Now the functor $\mathcal{F} \mapsto (\varphi^i)^* \mathcal{F}$ and filtered colimits of sheaves are exact. \square

Theorem 3.3. *Let \mathcal{F} be a sheaf in \mathcal{T}_G . Then the complex $R\gamma_* \mathcal{F}$ is quasi-isomorphic to the complex of sheaves of continuous \hat{G} -modules sending $U \in \hat{\mathcal{E}}$ to*

$$\Xi(\mathcal{F})(U) \xrightarrow{t-1} \Xi(\mathcal{F})(U). \quad (2)$$

Here $(tf)^{(i)} = \varphi f^{(i-1)}$ and $g \in \hat{G}$ with $\varphi^a \equiv g \pmod{m\hat{G}}$ acts as $(gf)^{(i)} = f^{(i+a)}$ on $\bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U)$.

Proof. Let P be the free resolution $0 \rightarrow \mathbb{Z}[G] \xrightarrow{t-1} \mathbb{Z}[G] \rightarrow 0$ of the constant sheaf \mathbb{Z} , and let $\mathcal{F} \rightarrow I$ be an injective resolution. Then $R\gamma_* \mathcal{F}$ is quasi-isomorphic to $\text{colim}_m (I)^{mG} \cong \text{colim}_m \mathcal{H}\text{om}(\mathbb{Z}, I)^{mG} \cong \text{colim}_m \mathcal{H}\text{om}(P, I)^{mG}$. If we take vertical cohomology in the latter double complex, we get complexes

$$R^b \left(\text{colim}_m \mathcal{H}\text{om}(\mathbb{Z}[G], -)^{mG} \right) (\mathcal{F}) \xrightarrow{t-1} R^b \left(\text{colim}_m \mathcal{H}\text{om}(\mathbb{Z}[G], -)^{mG} \right) (\mathcal{F})$$

concentrated in degree $a = 0, 1$ for each b . But by Lemma 3.2 the derived functors vanish for $b > 0$, and the double complex is quasi-isomorphic to

$$\text{colim}_m \mathcal{H}\text{om}(\mathbb{Z}[G], \mathcal{F})^{mG} \xrightarrow{t-1} \text{colim}_m \mathcal{H}\text{om}(\mathbb{Z}[G], \mathcal{F})^{mG},$$

where the map t is given in Lemma 3.1 b) and the \hat{G} -action in Lemma 3.1 c). By Lemma 3.1 a), this complex is isomorphic to the complex of the theorem. \square

4. $\gamma_* \mathcal{F}$, $R^1 \gamma_* \mathcal{F}$, AND $-\cup e$

To calculate the cohomology sheaves of $R\gamma_* \mathcal{F}$ explicitly, let $N_m^n : \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ be the map $f \mapsto \sum_{l=0}^{n-1} \varphi^{-lm} f$ for $mG \subseteq S(U)$. This descends to a map $N_m^n : \mathcal{F}(U)_{m\mathbb{Z}} \rightarrow \mathcal{F}(U)_{mn\mathbb{Z}}$, because

$$N_m^n(\varphi^m f) = \sum_{l=0}^{n-1} \varphi^{-(l-1)m} f = N_m^n(f) + \varphi^m f - \varphi^{-(n-1)m} f = N_m^n(f) + (1 - \varphi^{-nm}) \varphi^m f.$$

Proposition 4.1. *a) Let \mathcal{F} be a sheaf in \mathcal{T}_G . Then there is an exact sequence*

$$0 \rightarrow \text{colim}_m \mathcal{F}^{mG} \xrightarrow{\Delta} \Xi(\mathcal{F}) \xrightarrow{t-1} \Xi(\mathcal{F}) \xrightarrow{S} \text{colim}_{m, N_m^n} \mathcal{F}^{mG} \rightarrow 0.$$

b) If \mathcal{F} is a sheaf of \mathbb{Q} -vector spaces, then Δ is split, hence

$$R\gamma_* \mathcal{F} \cong \gamma_* \mathcal{F} \oplus R^1 \gamma_* \mathcal{F}[-1].$$

Proof. a) We have to calculate $\gamma_* \mathcal{F} = \ker t - 1$ and $R^1 \gamma_* \mathcal{F} = \text{coker } t - 1$. To construct the isomorphism $\Delta : \text{colim}_m \mathcal{F}^{mG} \rightarrow \ker t - 1$, let \mathcal{F} in \mathcal{T}_G , fix $U \in \hat{\mathcal{E}}$, and let $mG \subseteq S(U)$. Define

$$\Delta_m : \mathcal{F}(U)^{mG} \rightarrow \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U), \quad \Delta_m(c)^{(i)} = \varphi^{\bar{i}} c,$$

where $\bar{i} \in \mathbb{Z}$ is the representative of $i \in \mathbb{Z}/m$ with $0 \leq \bar{i} \leq m-1$. The image of Δ_m is contained in the kernel of $t - 1$, because

$$((t-1)\Delta_m(c))^{(i)} = \varphi \Delta_m(c)^{(i-1)} - \Delta_m(c)^{(i)} = \varphi^{\bar{i}-1-\bar{i}+1} f = 0.$$

Clearly Δ_m is injective, and if $f \in \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U)$ satisfies $(t-1)f = 0$, then $f^{(i)} = \varphi^{\bar{i}} f^{(0)}$, so that $\Delta_m(f^{(0)}) = f$, and Δ_m is an isomorphism to the kernel of $t-1$. It is easy to check that $\delta_m^n \circ \Delta_m = \Delta_{mn}$, hence we get a map

$$\Delta : \operatorname{colim}_m \mathcal{F}(U)^{mG} \rightarrow \Xi(\mathcal{F})(U) \quad (3)$$

which is an isomorphism to the kernel of $t-1$.

To construct the isomorphism $S : \operatorname{coker} t-1 \rightarrow \operatorname{colim}_{m, N_m^n} \mathcal{F}_{mG}$, consider the map

$$S_m : \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U) \rightarrow \mathcal{F}(U), \quad f \mapsto \sum_{i \in \mathbb{Z}/m} \varphi^{-\bar{i}} f^{(i)}.$$

We have

$$\begin{aligned} S_{mn}(\delta_m^n(f)) &= \sum_{j \in \mathbb{Z}/mn} \varphi^{-\bar{j}} \delta_m^n(f)^{(j)} = \sum_{j \in \mathbb{Z}/mn} \varphi^{-\bar{j}} f^{(j \bmod m)} \\ &= \sum_{l=0}^{n-1} \sum_{i \in \mathbb{Z}/m} \varphi^{-\bar{i}-lm} f^{(i)} = N_m^n \left(\sum_{i \in \mathbb{Z}/m} \varphi^{-\bar{i}} f^{(i)} \right) = N_m^n(S_m(f)). \end{aligned}$$

hence a surjective map

$$\tilde{S} : \Xi(\mathcal{F})(U) \rightarrow \operatorname{colim}_{m, N_m^n} \mathcal{F}(U).$$

Since

$$\begin{aligned} S_m((t-1)f) &= \sum_{i \in \mathbb{Z}/m} \varphi^{\bar{i}} (\varphi f^{(i-1)} - f^{(i)}) \\ &= \varphi f^{(m-1)} - \varphi^{-m+1} f^{(m-1)} = (1 - \varphi^{-m})(\varphi f^{(m-1)}), \end{aligned}$$

this map induces a map

$$S : \left(\operatorname{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U) \right) / t-1 \rightarrow \operatorname{colim}_{m, N_m^n} \mathcal{F}(U)_{mG}, \quad (4)$$

and we claim that S is an isomorphism. Indeed, for $c \in \mathcal{F}(\varphi^j U)$ define $R_j(c) \in \bigoplus_{i \in \mathbb{Z}/m} \mathcal{F}(\varphi^i U)$ by

$$R_i(c)^{(l)} = \begin{cases} \varphi^{l-\bar{j}} c & l < \bar{j}; \\ 0 & l \geq \bar{j}. \end{cases}$$

Then

$$((t-1)R_i(c))^{(i)} = \begin{cases} c & i = j; \\ -\varphi^{-\bar{j}} c & i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$(f - (t-1) \sum_{i \in \mathbb{Z}/m} R_i(f^{(i)}))^{(l)} = \begin{cases} \sum_{i \in \mathbb{Z}/m} \varphi^{-\bar{i}} f^{(i)} & l = 0 \\ 0 & l \neq 0. \end{cases}$$

We conclude that $S_m(f) = \sum_i \varphi^{-\bar{i}} f^{(i)} = 0$ implies that f is in the image of $t-1$.

b) This follows because the map $S' = \text{colim}_m \frac{1}{m} S_m : \Xi(\mathcal{F})(U) \rightarrow \gamma_* \mathcal{F}(U)$ satisfies $S' \circ \Delta = \text{id}$, hence we get a quasi-isomorphism

$$\begin{array}{ccc} \Xi(\mathcal{F})(U) & \xrightarrow{t^{-1}} & \Xi(\mathcal{F})(U) \\ s' \downarrow & & s \downarrow \\ \gamma_* \mathcal{F}(U) & \xrightarrow{0} & R^1 \gamma_* \mathcal{F}(U). \end{array}$$

□

Corollary 4.2. *We have $\gamma_* \mathbb{Z} \cong \mathbb{Z}$, $R^1 \gamma_* \mathbb{Z} \cong \mathbb{Q}$, and $R\gamma_* \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q}[-1]$.*

Proof. The map which is multiplication by $\frac{1}{m}$ on the copy of \mathbb{Z} indexed by m induces an isomorphism $\text{colim}_{m, N_m^n} \mathbb{Z} \xrightarrow{\sim} \mathbb{Q}$. □

Examples. 1) If a generator of G acts on the constant sheaf $\mathcal{F} = \mathbb{Q}$ as multiplication by $r \neq \pm 1$, then $R\gamma_* \mathcal{F} = 0$. More generally, for any constant sheaf \mathcal{F} of rational vector spaces, $\gamma_* \mathcal{F}$ is the largest subspace on which φ acts as multiplication by some root of unity, and $R^1 \gamma_* \mathcal{F}$ is the largest quotient space on which φ acts as multiplication by some root of unity.

2) Let \mathcal{F} be the sheaf $\oplus_{i \in G} A$ for an abelian group A , where G acts by shifting the factors. Then $\gamma_* \mathcal{F} = 0$, whereas $R^1 \gamma_* \mathcal{F} \subseteq \prod_i A$ consists of elements which are periodic for some period. Indeed, for an element of \mathcal{F}^{mG} , the entries in the sum are m -periodic, hence they must be zero. On the other hand, $\mathcal{F}_{mG} \cong \oplus_{i \in \mathbb{Z}/m} A$ and under this isomorphism the map $\mathcal{F}_{mG} \rightarrow \mathcal{F}_{mnG}$ is the n -fold concatenation map.

Consider the extension $e \in \text{Ext}_G^1(\mathbb{Z}, \mathbb{Z})$ of G -modules, given by $N = \mathbb{Z} \oplus \mathbb{Z}$ as an abelian group, and $g \cdot (r, s) = (r + as, s)$ for $g \in G$.

Lemma 4.3. *The extension e is a generator of $\text{Ext}_G^1(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. First note that $\text{Ext}_G^1(\mathbb{Z}, \mathbb{Z}) \cong H^1(G, \mathbb{Z}) \cong \mathbb{Z}$. Any extension $E \in \text{Ext}_G^1(\mathbb{Z}, \mathbb{Z})$, is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ as an abelian group. The action of a generator of G is given by a matrix of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for n an integer. The formula for addition of extensions classes then shows that $E = n \cdot e \in \text{Ext}_G^1(\mathbb{Z}, \mathbb{Z})$. □

For every complex of Weil-étale sheaves \mathcal{F}^\cdot , the connecting homomorphism of the distinguished triangle

$$\mathcal{F}^\cdot \rightarrow \mathcal{F}^\cdot \otimes N \rightarrow \mathcal{F}^\cdot \xrightarrow{\beta} \mathcal{F}^\cdot[1]$$

induces cup product with e on cohomology (up to sign). Indeed, e is the image of id under the induced map $\text{Hom}_G(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\beta} \text{Ext}_G^1(\mathbb{Z}, \mathbb{Z})$, and cup product is compatible with the Bockstein homomorphism, so that $e \cup x = \beta(\text{id}) \cup x = \beta(\text{id} \cup x) = \beta(x)$.

Proposition 4.4. *Let \mathcal{F} be a Weil-étale sheaf. Then under the identification of (2), the cup product with e on $R\gamma_* \mathcal{F}$ is induced by the following map $R\gamma_* \mathcal{F} \rightarrow R\gamma_* \mathcal{F}[1]$ in the derived category of étale sheaves*

$$\begin{array}{ccc} R\gamma_* \mathcal{F} & & \Xi(\mathcal{F}) \xrightarrow{t^{-1}} \Xi(\mathcal{F}) \\ e \downarrow & & \text{id} \downarrow \\ R\gamma_* \mathcal{F}[1] & & \Xi(\mathcal{F}) \xrightarrow{t^{-1}} \Xi(\mathcal{F}) \end{array}$$

Proof. Cup product with e is induced by the following vertical map of double complexes

$$\begin{array}{ccc} R\gamma_*\mathcal{F} & \longrightarrow & R\gamma_*(\mathcal{F} \otimes N) \\ \text{id} \downarrow & & \\ R\gamma_*\mathcal{F} & & \end{array}$$

Let $\alpha : R\gamma_*\mathcal{F} \rightarrow R\gamma_*\mathcal{F}[1]$ be the map in the statement of the proposition, except that we replace the vertical identity map $\Xi(\mathcal{F}) \rightarrow \Xi(\mathcal{F})$ by the cyclic permutation map $\Xi(\mathcal{F}) \xrightarrow{t} \Xi(\mathcal{F})$. In the derived category, α is quasi-isomorphic to the map

$$\begin{array}{ccc} R\gamma_*\mathcal{F} & \longrightarrow & \text{cone}(\alpha) \\ \text{id} \downarrow & & \\ R\gamma_*\mathcal{F} & & \end{array}$$

Cup product with e and α agree in the derived category, because in both diagrams, the upper row is the following double complex

$$\begin{array}{ccc} \Xi(\mathcal{F}) & \xrightarrow{i_1} & \Xi(\mathcal{F}) \oplus \Xi(\mathcal{F}) \\ t-1 \downarrow & & \begin{pmatrix} t-1 & t \\ 0 & t-1 \end{pmatrix} \downarrow \\ \Xi(\mathcal{F}) & \xrightarrow{i_1} & \Xi(\mathcal{F}) \oplus \Xi(\mathcal{F}). \end{array}$$

Finally, the map α and the map of the proposition are homotopic via the chain homotopy $h : R\gamma_*\mathcal{F} \rightarrow R\gamma_*\mathcal{F}$, which is the identity map in degree 0 and the zero map in degree 1. \square

As a consequence, we see that under the identification of Proposition 4.1 a), the cup product with e induces the colimit of the canonical maps $\mathcal{F}^{mG} \rightarrow \mathcal{F}_{mG}$, $f \mapsto mf$ on cohomology sheaves. Indeed, cup product with e is the composition

$$\gamma_*\mathcal{F} \cong \text{colim}_m \mathcal{F}^{mG} \xrightarrow{\Delta} \Xi(\mathcal{F}) \xrightarrow{S} \text{colim}_{m, N_m^n} \mathcal{F}_{mG} \cong R^1\gamma_*\mathcal{F},$$

and $S_m \circ \Delta_m(f) = mf$.

5. THE FUNCTOR $R\gamma_*\gamma^*$

Theorem 5.1. *Let \mathcal{G} be a complex of sheaves in $\mathcal{T}_{\hat{G}}$. Then there is a quasi-isomorphism*

$$R\gamma_*\mathbb{Z} \otimes^L \mathcal{G} \cong R\gamma_*(\gamma^*\mathcal{G}).$$

Proof. The complex $R\gamma_*\mathbb{Z}$ of Theorem 3.3 consists of *flat* sheaves. Hence we get a quasi-isomorphism of complexes

$$R\gamma_*\mathbb{Z} \otimes^L \mathcal{G}(U) \cong \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}(U) \xrightarrow{t-1} \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}(U).$$

Here t is the cyclic permutation of the summands. On the other hand, we have a quasi-isomorphism

$$R\gamma_*(\gamma^*\mathcal{G})(U) \cong \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}(\varphi^i U) \xrightarrow{t-1} \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}(\varphi^i U).$$

Here t acts as in Theorem 3.3. We claim that the following maps induce a quasi-isomorphism of the two complexes (we only write one of the two identical terms of the complexes):

$$\operatorname{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}^\cdot(U) \supseteq \operatorname{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}^\cdot(U)^{mG} \hookrightarrow \operatorname{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathcal{G}^\cdot(\varphi^i U). \quad (5)$$

The restriction of δ_m^n to the middle subgroup is defined because $\mathcal{G}^j(U)^{mG} \subseteq \mathcal{G}^j(U)^{mnG}$. The right map ν is given by $\nu(f)^{(i)} = \varphi^{\bar{i}} f^{(i)}$. It is easy to verify that ν is compatible with the maps δ_m^n , and the action of $g \in \hat{G}$. It also commutes with the action of t , because $(t\nu(f))^{(0)} = \varphi\nu(f)^{(m-1)} = \varphi^m f^{(m-1)} = f^{(m-1)} = (tf)^{(0)} = \nu(tf)^{(0)}$ (here we see why we need to restrict ν to the the intermediate group).

The two inclusions are in fact bijections. Indeed, because $\mathcal{G}^j(U)$ is a continuous \hat{G} -module, every element $x \in \mathcal{G}^j(U)$ in the m th term of the colimit on the left is contained in $\mathcal{G}^j(U)^{nG}$ for some n . Then $\delta_m^n(x)$ will be in the image of the inclusion map in the mn th term of the colimit. The same argument works for ν . \square

Corollary 5.2. *If \mathcal{G}^\cdot is a complex of sheaves in $\mathcal{T}_{\hat{G}}$, then there is a distinguished triangle*

$$\mathcal{G}^\cdot \rightarrow R\gamma_* \gamma^* \mathcal{G}^\cdot \rightarrow \mathcal{G}^\cdot \otimes \mathbb{Q}[-1] \xrightarrow{\delta} \mathcal{G}^\cdot[1]. \quad (6)$$

In particular, if \mathcal{G}^\cdot is a complex with torsion cohomology sheaves, then

$$\mathcal{G}^\cdot \cong R\gamma_* \gamma^* \mathcal{G}^\cdot.$$

If \mathcal{G}^\cdot is a complex with \mathbb{Q} -vector spaces as cohomology sheaves, then

$$R\gamma_* \gamma^* \mathcal{G}^\cdot \cong \mathcal{G}^\cdot \oplus \mathcal{G}^\cdot[-1],$$

and under this isomorphism, cup product with e is given by multiplication by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Proof. Everything is clear from the previous section except the last statement. It is easy to check that the quasi-isomorphism (5) is compatible with the description of the cup product with e in Proposition 4.4, hence it suffices to calculate the action of cup product with e on $R\gamma_* \mathbb{Z}$. By the remark after Proposition 4.4, this is induced by the composition

$$\mathbb{Z} \cong \operatorname{colim}_m \mathbb{Z} \xrightarrow{\Delta} \Xi(\mathbb{Z}) \xrightarrow{S} \operatorname{colim}_{m, N_m^n} \mathbb{Z} \cong \mathbb{Q},$$

which is the inclusion map. \square

We now calculate the map δ . Consider the extension $\epsilon \in \operatorname{Ext}_{\hat{G}}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, which is $\bar{N} = \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ as an abelian group, and $g \in \hat{G}$ acts via $g \cdot (x, y) = (x + gy, y)$. The image of ϵ under the composition of the canonical projection and the Bockstein homomorphism

$$\operatorname{Ext}_{\hat{G}}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \rightarrow \operatorname{Ext}_{\hat{G}}^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \operatorname{Ext}_{\hat{G}}^2(\mathbb{Q}, \mathbb{Z})$$

is calculated by the following pull-back commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \xrightarrow{\xi} & M & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \parallel & & \parallel & & \\
& & 0 & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & M & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\
& & & & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & \tilde{N} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Note that the cup product with ϵ is the boundary map induced by the lower sequence. These extensions have been studied by Kahn [7, Def. 4.1]. He denotes the complex $\mathbb{Q} \xrightarrow{\xi} M$ by \mathbb{Z}^c , where \mathbb{Q} is in degree 0, M is in degree 1.

Theorem 5.3. *There is a quasi-isomorphism of complexes of \hat{G} -modules $R\gamma_*\mathbb{Z} \cong \mathbb{Z}^c$. In particular, the boundary map δ in (6) is the composition*

$$\mathbb{Q}[-1] \rightarrow \mathbb{Q}/\mathbb{Z}[-1] \xrightarrow{\epsilon} \mathbb{Q}/\mathbb{Z}[0] \xrightarrow{\beta} \mathbb{Z}[1]. \quad (7)$$

Proof. Consider the map

$$\begin{aligned}
\mu_m : \bigoplus_{i \in \mathbb{Z}/m} \mathbb{Z} &\rightarrow M = \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}, \\
x &\mapsto \sum_{i \in \mathbb{Z}/m} \left(\frac{1}{2}x^{(i)} + \frac{i}{m}x^{(i)} + \mathbb{Z}, \frac{-1}{m}x^{(i)} \right).
\end{aligned}$$

This is a map of \hat{G} -modules, because for $a \in \hat{G}/m\hat{G}$, we have

$$\begin{aligned}
\mu_m(ax) &= \sum_{i \in \mathbb{Z}/m} \left(\frac{1}{2}x^{(i+a)} + \frac{i}{m}x^{(i+a)} + \mathbb{Z}, \frac{-1}{m}x^{(i+a)} \right) \\
&= \sum_{i \in \mathbb{Z}/m} \left(\frac{1}{2}x^{(i)} + \frac{i-a}{m}x^{(i)} + \mathbb{Z}, \frac{-1}{m}x^{(i)} \right) = a\mu_m(x).
\end{aligned}$$

The maps μ_m are compatible with δ_m^n (here we need the correcting summand $\frac{x^{(i)}}{2}$, and use the identity $\sum_{l=0}^{n-1} \frac{i+lm}{nm} = \frac{i}{m} + \frac{n-1}{2}$):

$$\begin{aligned}
\mu_{mn}(\delta_m^n(x)) &= \sum_{j \in \mathbb{Z}/mn} \left(\frac{1}{2}x^{(j \bmod m)} + \frac{j}{mn}x^{(j \bmod m)} + \mathbb{Z}, \frac{-1}{mn}x^{(j \bmod m)} \right) \\
&= \sum_{i \in \mathbb{Z}/m} \left(\frac{n}{2}x^{(i)} + \sum_{l=0}^{n-1} \frac{i+lm}{mn}x^{(i)} + \mathbb{Z}, \frac{-n}{mn}x^{(i)} \right) = \mu_m(x).
\end{aligned}$$

Hence we get a map $\mu : \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathbb{Z} \rightarrow M$, which is compatible with the action of \hat{G} . Consider the following diagram of maps of \hat{G} -modules with exact rows, where $S' = \text{colim}_m \frac{1}{m}S_m$.

$$\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{\Delta} & \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathbb{Z} & \xrightarrow{t-1} & \text{colim}_m \bigoplus_{i \in \mathbb{Z}/m} \mathbb{Z} & \xrightarrow{S'} & \mathbb{Q} \\
\parallel & & \downarrow S' & & \downarrow \mu & & \parallel \\
\mathbb{Z} & \longrightarrow & \mathbb{Q} & \xrightarrow{\xi} & M & \xrightarrow{p} & \mathbb{Q}
\end{array}$$

It is easy to see that the outer squares commute. On the other hand, if $x \in \bigoplus_{i \in \mathbb{Z}/m} \mathbb{Z}$, then

$$\begin{aligned}
\mu_m(t-1)x &= \sum_{i \in \mathbb{Z}/m} \left(\frac{1}{2}(x^{(i-1)} - x^{(i)}) + \frac{i}{m}(x^{(i-1)} - x^{(i)}) + \mathbb{Z}, \frac{-1}{m}(x^{(i-1)} - x^{(i)}) \right) \\
&= \sum_{i \in \mathbb{Z}/m} \left(\frac{1}{m}x^{(i)} + \mathbb{Z}, 0 \right) = \xi S'(x).
\end{aligned}$$

□

6. WEIL-ÉTALE COHOMOLOGY

Let Ab be the category of abelian groups, and Mod_G and $\text{Mod}_{\hat{G}}$ are the categories of G -modules and continuous \hat{G} -modules (for the discrete topology), respectively. Consider the following commutative diagram of functors, where $\Gamma_{\bar{X}}(\mathcal{F}) = \mathcal{F}(\bar{X})$,

$$\begin{array}{ccc} \mathcal{T}_G & \xrightarrow{\gamma_*} & \mathcal{T}_{\hat{G}} \\ \Gamma_{\bar{X}} \downarrow & & \Gamma_{\bar{X}} \downarrow \\ \text{Mod}_G & \xrightarrow{\gamma_*} & \text{Mod}_{\hat{G}} \\ \Gamma_G \downarrow & & \Gamma_{\hat{G}} \downarrow \\ \text{Ab} & \xlongequal{\quad} & \text{Ab} \end{array}$$

For a complex of sheaves \mathcal{F} in \mathcal{T}_G , we define the derived functors

$$\begin{aligned} H_W^i(X, \mathcal{F}) &= R^i(\Gamma_G \circ \Gamma_{\bar{X}})(\mathcal{F}) \in \text{Ab} \\ H_{\hat{G}}^i(\bar{X}, \mathcal{F}) &= R^i(\gamma_* \circ \Gamma_{\bar{X}})(\mathcal{F}) \in \text{Mod}_{\hat{G}} \end{aligned}$$

Following Lichtenbaum, we call the first groups Weil-étale cohomology groups.

Lemma 6.1. *For a complex of Weil-étale sheaves \mathcal{F} , the underlying abelian group of the derived functors $R^i\Gamma_{\bar{X}}(\mathcal{F}) \in \text{Mod}_G$ agree with the usual étale cohomology groups $H_{\text{ét}}^i(\bar{X}, \mathcal{F})$.*

Proof. The horizontal forgetful functors in the diagram

$$\begin{array}{ccc} \mathcal{T}_G & \xrightarrow{F} & \bar{\mathcal{E}} \\ \Gamma_{\bar{X}} \downarrow & & \Gamma_{\bar{X}} \downarrow \\ \text{Mod}_G & \xrightarrow{F} & \text{Ab} \end{array}$$

are exact, and all functors involved have an exact left adjoint by Lemma 2.2, hence preserve injectives. Thus $F \circ R\Gamma_{\bar{X}} = R\Gamma_{\bar{X}} \circ F$. □

We get the following spectral sequences for composition of functors

$$\begin{array}{ccc} E_2^{s,t} = H_{\text{cont}}^s(\hat{G}, H_{\hat{G}}^t(\bar{X}, \mathcal{F})) & \Rightarrow & H_W^{s+t}(X, \mathcal{F}) \\ \downarrow & & \parallel \\ E_2^{s,t} = H^s(G, H_{\text{ét}}^t(\bar{X}, \mathcal{F})) & \Rightarrow & H_W^{s+t}(X, \mathcal{F}). \end{array} \quad (8)$$

Since G has cohomological dimension 1, the latter spectral sequence breaks up into short exact sequence

$$0 \rightarrow H_{\text{ét}}^{t-1}(\bar{X}, \mathcal{F})_G \rightarrow H_W^t(X, \mathcal{F}) \rightarrow H_{\text{ét}}^t(\bar{X}, \mathcal{F})^G \rightarrow 0. \quad (9)$$

The identity $\Gamma_{\hat{G}} \circ \Gamma_{\bar{X}} \circ \gamma_* \cong \Gamma_G \circ \Gamma_{\bar{X}}$ gives another spectral sequence

$$H_{\text{ét}}^s(X, R^t\gamma_*\mathcal{F}) \Rightarrow H_W^{s+t}(X, \mathcal{F}).$$

Lemma 6.2. a) For a G -module A , the cup product with $e \in H^1(G, \mathbb{Z})$ induces the canonical map $H^0(G, A) \cong A^G \rightarrow A_G \cong H^1(G, A)$ on cohomology.

b) Cup product with e agrees with the composition

$$H_W^t(X, \mathcal{F}) \rightarrow H_{\text{ét}}^t(\bar{X}, \mathcal{F})^G \xrightarrow{\text{can}} H_{\text{ét}}^t(\bar{X}, \mathcal{F})_G \rightarrow H_W^{t+1}(X, \mathcal{F}).$$

Proof. a) This is proved as Proposition 4.4, see [16, Lemma 1.2].

b) Because the Hochschild-Serre spectral sequence is multiplicative, and $H^2(G, M) = 0$ for every G -module M , the cup product with $e \in H_W^1(\mathbb{F}_q, \mathbb{Z}) \cong H^1(G, \mathbb{Z}) = E_2^{1,0}(\mathbb{F}_q)$ on the abutment of (8) is induced by the cup product $E_2^{1,0}(\mathbb{F}_q) \times E_2^{0,t}(X) \rightarrow E_2^{1,t}(X)$. Hence the result follows from a). \square

6.1. Continuous Weil-étale cohomology. For pro-étale sheaves, Jannsen [6] defined continuous cohomology as the derived functors of $\lim \circ \Gamma_X$. We are introducing the analog for the Weil-étale topology.

Lemma 6.3. Limits in the categories \mathcal{T}_G and $\mathcal{T}_{\hat{G}}$ exist. More precisely, \lim_G agrees with the limit in the category of étale sheaves on \bar{X} , and $\lim_{\hat{G}} \cong \gamma_* \lim_G \gamma^*$. In particular, $R \lim_{\hat{G}} \cong R \gamma_* R \lim_G \gamma^*$.

Proof. The first statement follows from Lemma 2.2. Since γ_* has a left adjoint, it commutes with limits, and we get $\gamma_* \lim_G \gamma^* \cong \lim_{\hat{G}} \gamma_* \gamma^* \cong \lim_{\hat{G}}$. \square

Let $\mathcal{T}_G^{\mathbb{N}}$ and $\mathcal{T}_{\hat{G}}^{\mathbb{N}}$ be the category of inverse systems, indexed by the natural numbers, of sheaves in \mathcal{T}_G and $\mathcal{T}_{\hat{G}}$, respectively. There is a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{T}_G^{\mathbb{N}} & \xrightarrow{\gamma_*} & \mathcal{T}_{\hat{G}}^{\mathbb{N}} \\ \lim_G \downarrow & & \lim_{\hat{G}} \downarrow \\ \mathcal{T}_G & \xrightarrow{\gamma_*} & \mathcal{T}_{\hat{G}}. \end{array}$$

By Lemma 2.1 b), the functor $\lim_{\hat{G}}$ on $\mathcal{T}_{\hat{G}}^{\mathbb{N}}$ corresponds to the functor \lim on \mathcal{E} under the identification of Lemma 2.1. For $\mathcal{F} \in \mathcal{T}_G^{\mathbb{N}}$ a pro-system of sheaves of G -modules on \bar{X} , we define derived functors

$$H_W^i(X, (\mathcal{F})) = R^i(\Gamma_G \circ \lim_G \circ \Gamma_{\bar{X}})(\mathcal{F}) \in \text{Ab}$$

$$H_{\hat{G}}^i(\bar{X}, (\mathcal{F})) = R^i(\gamma_* \circ \lim_G \circ \Gamma_{\bar{X}})(\mathcal{F}) \in \text{Mod}_{\hat{G}}.$$

The second groups are continuous \hat{G} -modules for the *discrete* topology. It follows as in Lemma 6.1 that the underlying abelian group of the derived functors $R^i(\lim_G \circ \Gamma_{\bar{X}})(\mathcal{F}) \in \text{Mod}_G$ agrees with the usual continuous cohomology groups of Jannsen of \bar{X} . There is a map of spectral sequences

$$\begin{array}{ccc} E_2^{s,t} = H_{\text{cont}}^s(\hat{G}, H_{\hat{G}}^t(\bar{X}, (\mathcal{F}))) & \Rightarrow & H_W^{s+t}(X, (\mathcal{F})) \\ \downarrow & & \parallel \\ E_2^{s,t} = H^s(G, H^t(\bar{X}, (\mathcal{F}))) & \Rightarrow & H_W^{s+t}(X, (\mathcal{F})). \end{array} \quad (10)$$

Lemma 6.4. For an inverse system \mathcal{G} of torsion sheaves in $\mathcal{T}_{\hat{G}}^{\mathbb{N}}$, the cohomology groups $H_W^i(X, (\gamma^* \mathcal{G}))$ agree with the continuous cohomology groups $H^i(X, (\mathcal{G}))$ of

Jannsen on X , and the groups $H_{\hat{G}}^i(\bar{X}, (\gamma^*\mathcal{G}))$ agree with the continuous cohomology groups $H^i(\bar{X}, (\mathcal{G}))$ of Jannsen on \bar{X} . In particular,

$$H_{cont}^i(X, \mathbb{Z}/l(n)) := H^i(X, (\mathbb{Z}/l(n))) \cong H_W^i(X, (\mathbb{Z}/l(n))).$$

Proof. Note first that $\Gamma_G \circ \lim_G \circ \Gamma_{\bar{X}} = \lim_{\hat{G}} \circ \Gamma_{\hat{G}} \circ \Gamma_{\bar{X}} \circ \gamma_*$. Now $R\gamma_*\gamma^*\mathcal{G} \cong \mathcal{G}$ for torsion sheaves, and the derived functors of $\lim_{\hat{G}} \circ \Gamma_{\hat{G}} \circ \Gamma_{\bar{X}}$ are the continuous cohomology groups in the sense of Jannsen. The second statement follows similarly. \square

The upper spectral sequence (10) differs from Jannsen's spectral sequence [6, Cor. 3.4], for finitely generated cohomology groups

$$H_{cont}^s(\hat{G}, H^t(\bar{X}, (\mathcal{G}))) \Rightarrow H^{s+t}(X, (\mathcal{G})),$$

even though for étale torsion sheaves the E_2 -terms and the abutment agree. This is because Jannsen considers the coefficients with the limit topology, whereas we work with the discrete topology.

Lemma 6.5. *a) If (\mathcal{F}) is an inverse system of sheaves in \mathcal{T}_G , then the sheaf $R^s \lim \mathcal{F}$ is the sheaf associated to the presheaf which sends $U \in \mathcal{E}$ to $H_{\hat{G}}^s(U, (\mathcal{F}))$.*

b) If (\mathcal{G}) is an inverse system of sheaves in $\mathcal{T}_{\hat{G}}$, then the sheaf $R^s \lim_{\hat{G}} \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto H_{\hat{G}}^s(U, (\gamma^\mathcal{G}))$.*

Proof. a) follows with the same proof as b) by erasing all γ_* and γ^* .

b) Let $\mathcal{G} \rightarrow I^*$ be an injective resolution of inverse systems. Then $R^s \lim_{\hat{G}}(\mathcal{G})$ is by definition $\mathcal{H}^s(\gamma_* \lim \gamma^* I^*) = a\mathcal{H}^s(i\gamma_* \lim \gamma^* I^*)$, where a is the sheafification functor and i the inclusion of sheaves into presheaves. On the other hand, for every pro-system of sheaves $(\lim \mathcal{G})(U) = \lim(\mathcal{G}(U))$, for every sheaf $(\gamma_*\mathcal{G})(U) = \gamma_*(\mathcal{G}(U))$ and $\gamma^*\mathcal{G}(U) = \mathcal{G}(U)$, hence

$$\mathcal{H}^s(i\gamma_* \lim \gamma^* I^*)(U) = H^s(\gamma_* \lim I^*(U)) =: H_{\hat{G}}^s(U, (\gamma^*\mathcal{G})).$$

\square

7. MOTIVIC COHOMOLOGY

From now on we will assume that \mathcal{E} is the category of smooth schemes of finite type over \mathbb{F}_q , or the small étale site of a smooth scheme over \mathbb{F}_q (Weil-étale cohomology does not have good properties for non-smooth schemes). For $n \geq 0$, let $\mathbb{Z}(n)$ be the (étale) motivic complex of Voevodsky [20]; for example $\mathbb{Z}(0) \cong \mathbb{Z}$ and $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$. For an abelian group A , we define $A(n)$ to be $\mathbb{Z}(n) \otimes A$. In order to make our formulas work in general, we also define

$$\mathbb{Z}(n) = \mathbb{Q}/\mathbb{Z}'(n)[-1] \quad \text{for } n < 0,$$

where $\mathbb{Q}/\mathbb{Z}'(n) = \text{colim}_{p \nmid m} \mu_m^{\otimes n}$ is the prime to p -part of $\mathbb{Q}/\mathbb{Z}(n)$. Then for any n , there is are quasi-isomorphisms of complexes of étale sheaves [20, Prop. 6.7] and [4],

$$\mathbb{Z}/m(n) \cong \begin{cases} \mu_m^{\otimes n}[0] & p \nmid m \\ \nu_r^n[-n] & m = p^r, \end{cases}$$

where $\nu_r^n = W_r\Omega_{X, \log}^n$ is the logarithmic de Rham-Witt sheaf. We let $H_{\mathcal{M}}^i(X, A(n))$ be the Nisnevich and $H_{\hat{G}}^i(X, A(n))$ be the étale hypercohomology of $A(n)$, and abbreviate the Weil-étale hypercohomology $H_W^i(X, \gamma^*A(n))$ by $H_W^i(X, A(n))$. Then

$H_{\text{ét}}^i(X, \mathbb{Q}(n)) \cong H_{\mathcal{M}}^i(X, \mathbb{Q}(n))$. Theorems 5.1, 5.3, Corollary 5.2, and Proposition 4.4 specialized to this situation give:

Theorem 7.1. *Let X be a smooth variety over \mathbb{F}_q .*

a) *There is a long exact sequence*

$$\cdots \rightarrow H_{\text{ét}}^i(X, \mathbb{Z}(n)) \rightarrow H_W^i(X, \mathbb{Z}(n)) \rightarrow H_{\text{ét}}^{i-1}(X, \mathbb{Q}(n)) \xrightarrow{\delta} H_{\text{ét}}^{i+1}(X, \mathbb{Z}(n)) \rightarrow \cdots, \quad (11)$$

where the map δ is the composition

$$H_{\text{ét}}^{i-1}(X, \mathbb{Q}(n)) \rightarrow H_{\text{ét}}^{i-1}(X, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{-\epsilon} H_{\text{ét}}^i(X, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\beta} H_{\text{ét}}^{i+1}(X, \mathbb{Z}(n)).$$

b) *For torsion coefficients, we have*

$$H_{\text{ét}}^i(X, \mathbb{Z}/m(n)) \cong H_W^i(X, \mathbb{Z}/m(n)).$$

c) *With rational coefficients,*

$$H_W^i(X, \mathbb{Q}(n)) \cong H_{\mathcal{M}}^i(X, \mathbb{Q}(n)) \oplus H_{\mathcal{M}}^{i-1}(X, \mathbb{Q}(n)).$$

The cup product with $e \in \text{Ext}_G^1(\mathbb{Z}, \mathbb{Z})$ is multiplication by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

□

Lemma 7.2. *Let \bar{X} be a smooth variety of dimension d over an algebraically closed field of characteristic p , and let $l \neq p$.*

a) *If $n \geq d$, then $H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n)) \cong H_{\mathcal{M}}^i(\bar{X}, \mathbb{Z}(n))$, and the latter group is zero for $i > n + d$.*

b) *If $n < d$, then $H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n))$ is zero for $i > 2d + 1$, torsion for $i > 2n$, p -torsion free for $i > n + d + 1$, p -divisible for $i > n + d$ and l -divisible for $i > 2d$.*

Proof. a) We consider the statement rationally, with prime to p -coefficients and p -power coefficients separately. Let $\epsilon : \bar{X}_{\text{ét}} \rightarrow \bar{X}_{\text{zar}}$ be the canonical morphism of sites. Rationally, $R\epsilon_*\mathbb{Q}(n)_{\text{ét}} \cong \mathbb{Q}(n)_{\text{zar}}$ for any n . For $p \nmid m$, it follows from Suslin [19] that $R\epsilon_*\mathbb{Z}/m(n)_{\text{ét}} \cong \mathbb{Z}/m(n)_{\text{zar}}$ for $n \geq d$. Finally, $R\epsilon_*\mathbb{Z}/p^r(n)_{\text{ét}} \cong \mathbb{Z}/p^r(n)_{\text{zar}}$, because both sides are zero for $n > d$ by [4], and $R^j\epsilon_*\nu_r^d = 0$ for $j > 0$ by Gros-Suwa [5, III Lemme 3.16]. Hence we have $H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n)) \cong H_{\mathcal{M}}^i(\bar{X}, \mathbb{Z}(n))$ for any i and $n \geq d$. The latter group is zero for $i > d + n$ by definition.

b) Note first that $H_{\text{ét}}^i(\bar{X}, \mathbb{Q}(n)) = H_{\mathcal{M}}^i(\bar{X}, \mathbb{Q}(n)) = 0$ for $i > 2n$. For mod p coefficients, $\mathbb{Z}/p(n)_{\text{ét}} \cong \nu_1^n[-n]$ implies that $H_{\text{ét}}^i(\bar{X}, \mathbb{Z}/p(n)) = H_{\text{ét}}^{i-n}(\bar{X}, \nu_1^n) = 0$ for $i > d + n$, because $\text{cd}_p \bar{X} = d$. For mod l coefficients, $\mathbb{Z}/l(n)_{\text{ét}} \cong \mu_l^{\otimes n}[0]$ implies that $H_{\text{ét}}^i(\bar{X}, \mathbb{Z}/l(n)) = 0$ for $i > 2d$ because $\text{cd}_l \bar{X} = 2d$. The statement now follows using the short exact sequence

$$0 \rightarrow H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n))/l \rightarrow H_{\text{ét}}^i(\bar{X}, \mathbb{Z}/l(n)) \rightarrow {}_l H_{\text{ét}}^{i+1}(\bar{X}, \mathbb{Z}(n)) \rightarrow 0.$$

□

Theorem 7.3. *Let X be a smooth variety over a finite field of dimension d . Then $H_W^i(X, \mathbb{Z}(n)) = 0$ for $i > \max\{2d + 1, n + d + 1\}$.*

Proof. In view of (9) and the previous lemma, the result is clear for $n \geq d$. Similarly, for $n < d$ and $i > 2d + 2$, the group in question vanishes, because $H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n)) = 0$ for $i > 2d + 1$. It remains to show $H_W^{2d+2}(X, \mathbb{Z}(n)) \cong H_{\text{ét}}^{2d+1}(\bar{X}, \mathbb{Z}(n))_G = 0$. By the

lemma, $H_{\text{ét}}^{2d+1}(\bar{X}, \mathbb{Z}(n))$ is a divisible torsion group, and this property is inherited by its quotient group $H_W^{2d+2}(X, \mathbb{Z}(n))$. On the other hand, by Theorem 7.1 b), the limit of the surjections $H_W^{2d+1}(X, \mathbb{Z}/l^r(n)) \rightarrow {}_{l^r}H_W^{2d+2}(X, \mathbb{Z}(n))$ gives a surjection $H_{\text{cont}}^{2d+1}(X, \mathbb{Z}_l(n)) \rightarrow T_l H_W^{2d+2}(X, \mathbb{Z}(n))$ for every prime number l . By Deligne's proof of the Weil conjectures, $H_{\text{cont}}^i(X, \mathbb{Z}_l(n))$ is torsion for $i > n + d + 1$, hence $T_l H_W^{2d+2}(X, \mathbb{Z}(n)) = 0$. But a divisible l -torsion group A with $T_l A = 0$ is trivial. Indeed, any non-zero element $a \in A$ gives by divisibility rise to a non-zero element $(a_i)_i$ in the Tate-module. \square

We can give more explicit formulas for the case $n = 0, 1$:

Proposition 7.4. *Let X be smooth and connected.*

a) *We have*

$$H_W^i(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 1; \\ H_{\text{ét}}^{i-1}(X, \mathbb{Q}/\mathbb{Z}) & i > 2. \end{cases}$$

b) *If X is proper, then $H_W^i(X, \mathbb{Z})$ is finite for $i > 1$.*

c) *We have*

$$H_W^i(X, \mathbb{Z}(1)) = \begin{cases} 0 & i = 0; \\ \mathcal{O}(X)^\times & i = 1; \\ H_{\text{ét}}^{i-1}(X, \mathbb{Q}/\mathbb{Z}(1)) & i \geq 5, \end{cases}$$

and there is an exact sequence

$$0 \rightarrow \text{Pic } X \rightarrow H_W^2(X, \mathbb{Z}(1)) \rightarrow \mathcal{O}(X)^\times \otimes \mathbb{Q} \rightarrow \text{Br } X \rightarrow H_W^3(X, \mathbb{Z}(1)) \rightarrow \text{NS } X \otimes \mathbb{Q} \rightarrow H_{\text{ét}}^3(X, \mathbb{Q}/\mathbb{Z}(1)) \rightarrow H_W^4(X, \mathbb{Z}(1)) \rightarrow 0.$$

d) *If X is proper, then $H_W^2(X, \mathbb{Z}(1)) \cong \text{Pic } X$, the groups $H_W^i(X, \mathbb{Z}(1))$ are finite for $i \geq 5$, and there is an exact sequence*

$$0 \rightarrow \text{Br } X \rightarrow H_W^3(X, \mathbb{Z}(1)) \rightarrow \text{NS } X \otimes \mathbb{Q} \rightarrow H_{\text{ét}}^3(X, \mathbb{Q}/\mathbb{Z}(1)) \rightarrow H_W^4(X, \mathbb{Z}(1)) \rightarrow 0.$$

If $H_W^3(X, \mathbb{Z}(1))$ is finitely generated, then $\text{Br}(X)$ is finite, we have a short exact sequence

$$0 \rightarrow \text{Br } X \rightarrow H_W^3(X, \mathbb{Z}(1)) \rightarrow \text{NS } X \rightarrow 0,$$

and $H_W^4(X, \mathbb{Z}(1)) \cong H_{\text{ét}}^3(X, \mathbb{Q}/\mathbb{Z}(1))_{\text{cotor}}$ is finite.

Proof. a) Since $H_{\text{ét}}^1(\bar{X}, \mathbb{Z}) = 0$, we have $H_W^1(X, \mathbb{Z}) = H_{\text{ét}}^0(\bar{X}, \mathbb{Z})_G = \mathbb{Z}$. From (11) and $H_{\text{ét}}^i(X, \mathbb{Q}) = 0$ for $i > 0$ we get $H_{\text{ét}}^{i-1}(X, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H_{\text{ét}}^i(X, \mathbb{Z}) \xrightarrow{\sim} H_W^i(X, \mathbb{Z})$.

b) This follows from Deligne's proof of the Weil conjectures, and Gabber's finiteness result [1].

c) Since $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ and $H_{\text{ét}}^i(X, \mathbb{Q}(1)) = 0$ for $i \geq 3$, this follows from sequence (11) together with $H_{\text{ét}}^2(X, \mathbb{Z}(1)) = \text{Pic } X$ and $H_{\text{ét}}^3(X, \mathbb{Z}(1)) = \text{Br } X$.

d) The exact sequence follows because $\mathcal{O}(X)^\times$ is torsion, and the finiteness again follows from Deligne's and Gabber's results. If $H_W^3(X, \mathbb{Z}(1))$ is finitely generated, then its image in $\text{NS } X \otimes \mathbb{Q}$ is a lattice isomorphic to the torsion free finitely generated group $\text{NS } X$ in order for the quotient to be torsion. Finally,

$$\begin{aligned} \text{corank } H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(1)) &= \dim H_{\text{cont}}^3(X, \mathbb{Q}_l(1)) = \dim H_{\text{cont}}^2(\bar{X}, \mathbb{Q}_l(1))_{\hat{G}} \\ &= \dim H_{\text{cont}}^2(\bar{X}, \mathbb{Q}_l(1))^{\hat{G}} = \dim H_{\text{cont}}^2(X, \mathbb{Q}_l(1)) = \text{rank NS } X. \end{aligned}$$

□

Example. By Artin-Schreier theory, $H_{\text{ét}}^1(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Z}/p) \subseteq H_{\text{ét}}^1(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Q}_p/\mathbb{Z}_p) \cong H_{\text{ét}}^2(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Z})$ contains an infinite number of copies of \mathbb{Z}/p . From this and the exact sequence

$$\mathbb{Q} \cong H_{\text{ét}}^0(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Q}) \rightarrow H_{\text{ét}}^2(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Z}) \rightarrow H_W^2(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Z}) \rightarrow 0$$

we see that $H_W^2(\mathbb{A}_{\mathbb{F}_q}^1, \mathbb{Z})$ is not finitely generated.

Theorem 7.5. *Let X be a connected smooth projective variety of dimension d over \mathbb{F}_q with Albanese variety A . Then there is a commutative diagram*

$$\begin{array}{ccccc} H_W^{2d}(X, \mathbb{Z}(d)) & \longrightarrow & H_{\text{ét}}^{2d}(\bar{X}, \mathbb{Z}(d))^G & \xrightarrow[\sim]{(deg', Alb)} & \mathbb{Z} \oplus A(\mathbb{F}_q) \\ e \downarrow & & \downarrow & & \downarrow \\ H_W^{2d+1}(X, \mathbb{Z}(d)) & \xleftarrow[\sim]{} & H_{\text{ét}}^{2d}(\bar{X}, \mathbb{Z}(d))_G & \xrightarrow[\sim]{deg} & \mathbb{Z} \end{array}$$

Proof. By Lemma 7.2, $H_{\text{ét}}^{2d}(\bar{X}, \mathbb{Z}(d)) \cong \text{CH}^d(\bar{X})$. The kernel of the degree map $\text{CH}^d(\bar{X}) \xrightarrow{deg} \mathbb{Z}$ is divisible, and by Rojzman's theorem [17, 12] it agrees with the $\bar{\mathbb{F}}_q$ -rational torsion points $A(\bar{\mathbb{F}}_q)$. Since $A(\bar{\mathbb{F}}_q)^G = A(\mathbb{F}_q)$ is finite, we get $A(\bar{\mathbb{F}}_q)_G = 0$. The statement now follows from the short exact sequence (9) and Lemma 6.2 b). □

8. COMPARISON TO l -ADIC COHOMOLOGY

Fix a smooth projective variety X over \mathbb{F}_q and an integer n . There are two fundamental conjectures on Weil motivic cohomology. The first one is due to Lichtenbaum:

Conjecture 8.1. L(X, n) *For every i , the group $H_W^i(X, \mathbb{Z}(n))$ is finitely generated.*

Note that $H_W^i(X, \mathbb{Z}(n))$ may be not finitely generated if X is not smooth or not projective. The homomorphisms $\mathbb{Z}(n) \otimes \mathbb{Z}_l \rightarrow \mathbb{Z}/l^r(n)$ in the derived category of sheaves of \mathcal{T}_G are compatible, and induce a morphism $\mathbb{Z}(n) \otimes \mathbb{Z}_l \rightarrow R\lim \mathbb{Z}/l^r(n)$, hence in view of Lemma 6.3 upon applying $R\gamma_*$ a map $c : R\gamma_* \mathbb{Z}(n) \otimes \mathbb{Z}_l \rightarrow R\lim \mathbb{Z}/l^r(n)$ in the derived category of sheaves of $\mathcal{T}_{\hat{G}}$. In view of [8, Lemma 3.8] and Theorem 5.3, the following conjecture is equivalent to Conjecture 3.2 of Kahn [8].

Conjecture 8.2. K(X, n) *For every prime l , and any i , the map c induces an isomorphism*

$$H_W^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\sim} H_{\text{cont}}^i(X, \mathbb{Z}_l(n)).$$

This implies in particular that Weil motivic cohomology is an integral model for l -adic and p -adic cohomology. Finally, there is the classical conjecture, due to Tate (part 1,2 for $l \neq p$) and Beilinson (part 3):

Conjecture 8.3. T(X, n) *For every prime l ,*

- (1) *The cycle map $CH^n(X) \otimes \mathbb{Q}_l \rightarrow H_{\text{cont}}^{2n}(\bar{X}, \mathbb{Q}_l(n))^{\hat{G}}$ is surjective.*
- (2) *The \hat{G} -module $H_{\text{cont}}^{2n}(\bar{X}, \mathbb{Q}_l(n))$ is semi-simple at the eigenvalue 1.*
- (3) *Rational and numerical equivalence agree with rational coefficients.*

Theorem 8.4. *Let X be a smooth projective variety over \mathbb{F}_q , and n an integer. Then*

$$K(X, n) + K(X, d - n) \Rightarrow L(X, n) \Rightarrow K(X, n) \Rightarrow T(X, n).$$

Conversely, if $T(X, n)$ holds for all smooth and projective varieties X over \mathbb{F}_q and all n , then $K(X, n)$ holds for all X and n .

Proof. $K(X, n) + K(X, d - n) \Rightarrow L(X, n)$: This has been proved in by Kahn [9].
 $L(X, n) \Rightarrow K(X, n)$: By finite generation and Theorem 7.1 b), we have

$$\begin{aligned} H_W^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l &\cong \lim H_W^i(X, \mathbb{Z}(n))/l^r \cong \lim H_W^i(X, \mathbb{Z}/l^r(n)) \\ &\cong \lim H_{\text{ét}}^i(X, \mathbb{Z}/l^r(n)) \cong H_{\text{cont}}^i(X, \mathbb{Z}_l(n)). \end{aligned}$$

$K(X, n) \Rightarrow T(X, n)$: This is proved exactly as in [8, Prop. 3.9].

$T(-, -) \Rightarrow K(X, n)$: The hypothesis implies that $H_{\mathcal{M}}^i(X, \mathbb{Q}(n)) \cong H_{\text{ét}}^i(X, \mathbb{Q}(n)) = 0$ for $i \neq 2n$, [2]. One can now use the argument of [8, Prop. 3.9]. \square

Remark. 1) Conjecture $K(X, n)$ for all smooth and proper varieties implies the same statement for all smooth varieties as long as $l \neq p$. This follows by using localization sequences for l -adic cohomology and Weil-cohomology, and de Jong's theorem on alterations, see [3, Lemma 4.1] or [7, Section 5] for details.

2) For $l \neq p$, an equivalent formulation of Conjecture 8.2 is that the cycle map induces a quasi-isomorphism

$$R\gamma_*\mathbb{Z}(n) \otimes \mathbb{Z}_l \xrightarrow{\sim} R\lim_{\hat{G}} \mathbb{Z}/l^r(n)$$

in the derived category of étale sheaves on smooth schemes over \mathbb{F}_q . The right hand term is quasi-isomorphic to $R\gamma_*R\lim \mathbb{Z}/l^r(n)$ by Lemma 6.3, but the above quasi-isomorphism is not induced by a quasi-isomorphism between $\mathbb{Z}(n) \otimes \mathbb{Z}_l$ and $R\lim \mathbb{Z}/l^r(n)$ in the derived category of Weil-étale sheaves on smooth schemes over \mathbb{F}_q . For example, $\mathcal{H}^2(\mathbb{Z}(1) \otimes \mathbb{Z}_l) = 0$ by Hilbert's Theorem 90, but one can show that $R^2\lim \mathbb{Z}/l^r(1) \neq 0$.

3) In view of the spectral sequences (8) and (10), Conjecture $K(X, n)$ would follow if for all i and l , the map $H_{\hat{G}}^i(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \rightarrow H_{\hat{G}}^i(\bar{X}, (\mathbb{Z}/l^r(n)))$ is an isomorphism of \hat{G} -modules. These maps fit into a commutative diagram of short exact sequences of \hat{G} -modules

$$\begin{array}{ccccc} R^1\gamma_*H_{\text{ét}}^{i-1}(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l & \longrightarrow & H_{\hat{G}}^i(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l & \longrightarrow & \gamma_*H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \\ \downarrow & & \downarrow & & \downarrow \\ R^1\gamma_*H_{\text{cont}}^{i-1}(\bar{X}, \mathbb{Z}_l(n)) & \longrightarrow & H_{\hat{G}}^i(\bar{X}, (\mathbb{Z}/l^r(n))) & \longrightarrow & \gamma_*H_{\text{cont}}^i(\bar{X}, \mathbb{Z}_l(n)). \end{array}$$

Thus $K(X, n)$ follows from isomorphisms of \hat{G} -modules

$$\begin{aligned} H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l &\cong \gamma_*H_{\text{ét}}^i(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\sim} \gamma_*H_{\text{cont}}^i(\bar{X}, \mathbb{Z}_l(n)) \\ H_{\text{ét}}^{i-1}(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Q}_l &\cong R^1\gamma_*H_{\text{ét}}^{i-1}(\bar{X}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\sim} R^1\gamma_*H_{\text{cont}}^i(\bar{X}, \mathbb{Z}_l(n)). \end{aligned}$$

The former isomorphism for $i = 2n$ was Tate's original formulation of his conjecture [23].

9. VALUES OF ZETA-FUNCTIONS, EXAMPLES

We can reformulate results of Milne [13] to find expressions for values of zeta functions as conjectured by Lichtenbaum [10]. Since $e \in \text{Ext}_G^1(\mathbb{Z}, \mathbb{Z})$ satisfies $e^2 = 0$, the Weil étale cohomology groups $H_W^*(X, \mathbb{Z}(n))$ form a complex under cup product with e . By Theorems 7.1 c) and 7.3, the cohomology groups of this complex are torsion, and only finitely many are non-zero. For a complex C^\cdot of abelian groups with finitely many finite cohomology groups, one defines

$$\chi(C^\cdot) := \prod_i |H^i(C^\cdot)|^{(-1)^i}.$$

Let X be a smooth projective scheme over \mathbb{F}_q , and $\zeta(X, s) = Z(X, q^{-s})$ be its zeta function. Following Milne [13], we let

$$\chi(X, \mathcal{O}_X, n) = \sum_{i \leq n, j \leq d} (-1)^{i+j} (n-i) \dim H^j(X, \Omega^i).$$

The conclusion of the following theorem has been proved by Lichtenbaum for $n = 0$ in [11].

Theorem 9.1. *Let X be a smooth projective variety such that $K(X, n)$ holds. Then the order ρ_n of the pole of $\zeta(X, s)$ at $s = n$ is $\text{rank } H_W^{2n}(X, \mathbb{Z}(n))$, and*

$$\zeta(X, s) = \pm(1 - q^{n-s})^{-\rho_n} \cdot \chi(H_W^*(X, \mathbb{Z}(n)), e) \cdot q^{\chi(X, \mathcal{O}_X, n)} \quad \text{as } s \rightarrow n. \quad (12)$$

If $K(X, d-n)$ holds also, then

$$\chi(H_W^*(X, \mathbb{Z}(n)), e) = \prod_i |H_W^i(X, \mathbb{Z}(n))_{\text{tor}}|^{(-1)^i} \cdot R^{-1},$$

where R is the determinant of the pairing

$$H_W^{2n}(X, \mathbb{Z}(n)) \times H_W^{2(d-n)}(X, \mathbb{Z}(d-n)) \rightarrow H_W^{2d}(X, \mathbb{Z}(d)) \xrightarrow{\text{deg}(-\cup e)} \mathbb{Z}.$$

Proof. Since $K(X, n)$ implies semi-simplicity of l -adic cohomology, the first formula follows by comparing to the formulas for l -adic cohomology in [13, Thm. 0.1].

Recall that for a short exact sequence $0 \rightarrow A^\cdot \rightarrow B^\cdot \rightarrow C^\cdot \rightarrow 0$ we have $\chi(A^\cdot) \cdot \chi(C^\cdot) = \chi(B^\cdot)$. Thus it suffices to show that $R^{-1} = \chi(H_W^*(X, \mathbb{Z}(n))/\text{tor}, e)$. By hypothesis, the latter complex consists only of the upper map in the following commutative diagram

$$\begin{array}{ccc} H_W^{2n}(X, \mathbb{Z}(n))/\text{tor} & \xrightarrow{e} & H_W^{2n+1}(X, \mathbb{Z}(n))/\text{tor} \\ d' \downarrow & & d \downarrow \\ \text{Hom}(H_W^{2(d-n)}(X, \mathbb{Z}(d-n)), \mathbb{Z}) & \xlongequal{\quad} & \text{Hom}(H_W^{2(d-n)}(X, \mathbb{Z}(d-n)), \mathbb{Z}). \end{array}$$

The maps d' and d are given by $d'(x)(y) = \text{deg}(x \cdot y \cdot e)$ and $d(x)(y) = \text{deg}(x \cdot y)$, where deg is the map of Theorem 7.5. Comparing with l -adic cohomology and using [14, Lemma 5.3], one sees that d is an isomorphism. Hence

$$\chi(H_W^*(X, \mathbb{Z}(n))/\text{tor}, e) = \frac{|\ker e|}{|\text{coker } e|} = \frac{|\ker d'|}{|\text{coker } d'|} = \frac{1}{R}.$$

□

We give some explicit examples for varieties satisfying the hypothesis of the previous theorem.

Proposition 9.2. *Conjecture $L(X, n)$ holds for $n \leq 0$. In particular, (12) holds for all smooth projective X and $n \leq 0$.*

Proof. For $n = 0$, this is Proposition 7.4 b). For $n < 0$, the proposition follows because $H_{cont}^i(X, \mathbb{Q}_l(n)) = 0$ by Deligne's proof of the Weil conjectures, hence

$$H_W^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l := H_{\text{ét}}^{i-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H_{cont}^i(X, \mathbb{Z}_l(n))$$

is finitely generated and torsion. \square

Theorem 9.3. *Assume that X is smooth and projective and that the cycle map $\text{Pic } X \otimes \mathbb{Q}_l \rightarrow H_{cont}^2(X, \mathbb{Q}_l(1))$ is surjective for some l . Then $K(X, 1)$ holds. In particular, (12) holds for X and $n = 1$.*

Proof. In view of Theorem 7.1 b) we can verify $K(X, 1)$ after tensoring with \mathbb{Q} . We have $H_{\text{ét}}^i(X, \mathbb{Q}(1)) = 0$ for $i \neq 2$, $H_{\text{ét}}^2(X, \mathbb{Q}(1)) = \text{Pic } X \otimes \mathbb{Q}$ and $H_{cont}^i(X, \mathbb{Q}_l(1)) = 0$ for $i \neq 2, 3$. Hence we get the following diagram from Theorem 7.1 c)

$$\begin{array}{ccccc} \text{Pic } X \otimes \mathbb{Q}_l & \xrightarrow{\sim} & H_W^2(X, \mathbb{Z}(1)) \otimes \mathbb{Q}_l & \xrightarrow{\text{surj}} & H_{cont}^2(X, \mathbb{Q}_l(1)) \\ \parallel & & e \downarrow & & e \downarrow \\ \text{Pic } X \otimes \mathbb{Q}_l & \xleftarrow{\sim} & H_W^3(X, \mathbb{Z}(1)) \otimes \mathbb{Q}_l & \longrightarrow & H_{cont}^3(X, \mathbb{Q}_l(1)). \end{array}$$

In codimension 1, rational and homological equivalence agree rationally, hence the upper composition is an isomorphism. On the other hand, by Milne [13, Prop. 0.3], the surjectivity of the cycle map for one l implies semi-simplicity of $H_{cont}^2(\bar{X}, \mathbb{Q}_l(1))$, for all l including $l = p$, hence the right vertical map is an isomorphism. \square

In particular, the conclusion holds for Hilbert modular surfaces, Picard modular surfaces, Siegel modular threefolds, and in characteristic at least 5 for supersingular and elliptic K3 surfaces [22]. We use Soulé's method to produce more examples.

Proposition 9.4. *Let $X = X_1 \times \dots \times X_d$ be a product of smooth projective curves over \mathbb{F}_q , and let $n \leq 1$ or $n \geq d - 1$. Then $K(X, n)$ holds for X .*

Proof. By [8, Prop. 3.9], it suffices to show that $H_{\mathcal{M}}^i(X, \mathbb{Z}(n)) \otimes \mathbb{Q} = 0$ for $i < 2n$, that $H_{\mathcal{M}}^{2n}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_l \cong H_{cont}^{2n}(X, \mathbb{Q}_l(n))$, and that the Frobenius acts semi-simply at 1 on $H_{cont}^{2n}(\bar{X}, \mathbb{Q}_l(n))$. We essentially repeat the proof of Soulé [18, Thm. 3], adapted to our situation.

Write $X_i \cong \mathbf{1} \oplus X_i^+ \oplus \mathcal{L}$ in the category of Chow motives, where $\mathbf{1} = \text{Spec } \mathbb{F}_q$ and $\mathbb{P}^1 \cong \mathbf{1} \oplus \mathcal{L}$. Then X is a direct sum of motives of the form $M = \otimes_{s=1}^j X_{n_s}^+ \otimes \mathcal{L}^k$, with $0 \leq j + k \leq d$. Such a motive M has a Frobenius endomorphism F_M , and F_M has a minimal polynomial $P_M(u)$ such that all roots of $P_M(u)$ have absolute value equal to $q^{\frac{j+2k}{2}}$ [18, Prop. 3.1.2]. Since the Frobenius F_M acts on $H_{\mathcal{M}}^i(M, \mathbb{Z}(n)) \otimes \mathbb{Q}$ as multiplication by q^n [18, Prop. 1.5.2], we get $0 = P_M(F_M) = P_M(q^n)$ on $H_{\mathcal{M}}^i(M, \mathbb{Z}(n)) \otimes \mathbb{Q}$. For $j + 2k \neq 2n$, $P_M(q^n)$ is non-zero, hence $H_{\mathcal{M}}^i(M, \mathbb{Z}(n)) \otimes \mathbb{Q} = 0$. By our choice of n , $j + 2k = 2n$ can only happen for $j = 0$ or $j = 2$.

If $j = 0$, then $M = \mathcal{L}^n$, and by the projective bundle formula for motivic cohomology, $H_{\mathcal{M}}^i(\mathcal{L}^n, \mathbb{Z}(n)) \otimes \mathbb{Q}_l = H_{\mathcal{M}}^{i-2n}(\mathbb{F}_q, \mathbb{Z}(0)) \otimes \mathbb{Q}_l$. The latter group is zero

except for $i = 2n$, in which case it is isomorphic to $H_{cont}^{2n}(\bar{\mathcal{L}}^n, \mathbb{Q}_l(n)) = \mathbb{Q}_l$. The Galois group acts trivially, in particular semi-simply, on the latter group.

If $j = 2$, then $M = X^+ \otimes Y^+ \otimes \mathcal{L}^{n-1}$, and

$$H_{\mathcal{M}}^i(M, \mathbb{Z}(n)) \otimes \mathbb{Q}_l = H_{\mathcal{M}}^{i-2n+2}(X^+ \otimes Y^+, \mathbb{Z}(1)) \otimes \mathbb{Q}_l = H_{\mathcal{M}}^{i-2n+1}(X^+ \otimes Y^+, \mathbb{G}_m) \otimes \mathbb{Q}_l.$$

The latter group is zero for $i < 2n$, because the group of global sections of a projective variety over a finite field is finite. On the other hand,

$$\begin{aligned} H_{\mathcal{M}}^{2n}(M, \mathbb{Z}(n)) \otimes \mathbb{Q}_l &\cong \mathrm{CH}^1(X^+ \otimes Y^+) \otimes \mathbb{Q}_l \\ &\cong H_{cont}^2(X^+ \otimes Y^+, \mathbb{Q}_l(1)) \cong H_{cont}^{2n}(M, \mathbb{Q}_l(n)) \end{aligned}$$

by Tate's theorem [21]. Tate's theorem also implies that the Galois group acts semi-simply at 1 on the module $H_{cont}^2(\bar{X}^+ \otimes \bar{Y}^+, \mathbb{Q}_l(1))$. For $l = p$, the same statement follows by [13, Prop. 0.3]. \square

As in Soulé, let $A(k)$ be the subcategory of smooth projective varieties generated by products of curves and the following operations:

- (1) If X and Y are in $A(k)$, then $X \amalg Y$ is in $A(k)$.
- (2) If Y is in $A(k)$, and there are morphisms $c : X \rightarrow Y$ and $c' : Y \rightarrow X$ in the category of Chow motives, such that $c' \circ c : X \rightarrow X$ is multiplication by a constant, then X is in $A(k)$.
- (3) If k' is a finite extension of k , and $X \times_k k'$ is in $A(k')$, then X is in $A(k)$.
- (4) If Y is a closed subscheme of X and Y and X are in $A(k)$, then the blow-up X' of X along Y is in $A(k)$.

Theorem 9.5. *Let X be a variety of dimension d in $A(\mathbb{F}_q)$. Then $K(X, n)$ and $L(X, n)$ hold for $n \leq 1$ or $n \geq d - 1$. In particular, (12) and (13) hold for X and $n \leq 1$ or $n \geq d - 1$.*

Proof. The statement holds for products of curves, and it is clear that if X and Y satisfy $K(X, n)$ then $X \amalg Y$ also does. In 2) and 3), the map $H_W^i(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_l \rightarrow H_{cont}^i(X, \mathbb{Q}_l(n))$ is a direct summand for the corresponding map for Y and $X \times_k k'$, respectively. Finally, if X' is the blow-up of X along Y , and Y has codimension c in X , then one has $X' = X \oplus (\oplus_{j=1}^{c-1} Y \otimes \mathcal{L}^j)$. \square

In particular, the conclusion of theorem holds for abelian varieties, unirational varieties of dimension at most 3, or Fermat hypersurfaces.

In [9], Kahn shows that $K(X, n)$ holds, if the Chow motive of X is in the subcategory of Chow motives generated by abelian varieties and Artin motives, and if Tate's conjecture holds for X . This applies in particular to products of elliptic curves.

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