

# Isometry groups of Lobachevskian spaces, similarity transformation groups of Euclidean spaces and Lorentzian holonomy groups

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## Abstract

Weakly-irreducible not irreducible subalgebras of  $\mathfrak{so}(1, n + 1)$  were classified by L. Berard Bergery and A. Ikemakhen. In the present paper a geometrical proof of this result is given. Transitivity acting isometry groups of Lobachevskian spaces and transitively acting similarity transformation groups of Euclidean spaces are classified.

## Introduction

In 1952 A. Borel and A. Lichnerowicz showed that *the holonomy group of a Riemannian manifold is a product of irreducible holonomy groups of Riemannian manifolds*, see [9]. The main reason is the following. If a subgroup  $G \subset SO(n)$  preserves a proper vector subspace, then  $G$  preserves also its orthogonal complement  $U^\perp$  and we have  $\mathbb{R}^n = U \oplus U^\perp$ , i.e. the group  $G$  is totally reducible. In 1955 M. Berger classified possible connected irreducible holonomy groups of Riemannian manifolds, see [8].

The Borel and Lichnerowicz theorem does not work in the pseudo-Riemannian case. Suppose a subgroup  $G \subset SO(r, s)$  preserves a proper vector subspace  $U \subset \mathbb{R}^{r,s}$  such that the restriction of the inner product to  $U$  is degenerate, then  $U \cap U^\perp \neq \{0\}$  and we have no orthogonal decomposition of  $\mathbb{R}^{r,s}$  into  $G$ -irreducible subspaces. A subgroup  $G \subset SO(r, s)$  is called *weakly-irreducible* if it preserves no nondegenerate proper subspace of  $\mathbb{R}^{r,s}$ . There is Wu's theorem that states that *the holonomy group of a pseudo-Riemannian manifold is a product of weakly-irreducible holonomy groups of pseudo-Riemannian manifolds*, see [19]. If a holonomy group is irreducible, then it is weakly-irreducible. In [8] M. Berger gave a classification of irreducible holonomy groups for pseudo-Riemannian manifolds. In particular, *the only connected irreducible*

*holonomy group of Lorentzian manifolds is  $SO^0(1, n + 1)$ , see [11] and [10] for direct proofs of this fact.*

There is still no classification of weakly-irreducible not irreducible holonomy groups of pseudo-Riemannian manifolds. The first step towards a classification of weakly-irreducible not irreducible holonomy groups of Lorentzian manifolds was made by L. Berard Bergery and A. Ikemakhen, who classified weakly-irreducible not irreducible subalgebras of  $\mathfrak{so}(1, n + 1)$ , see [6]. More precisely, they divided weakly-irreducible not irreducible subalgebras of  $\mathfrak{so}(1, n + 1)$  into 4 types. The proof of this result was purely algebraical.

We introduce a geometrical proof of the result of L. Berard Bergery and A. Ikemakhen. We consider an  $n + 2$ -dimensional Minkowski vector space  $(V, \eta)$  and fix an isotropic vector  $p \in V$ . We denote by  $SO(V)_{\mathbb{R}p}$  the Lie subgroup of  $SO(V)$  that preserves the isotropic line  $\mathbb{R}p$ . We denote by  $E$  a vector subspace  $E \subset V$  such that  $(\mathbb{R}p)^{\perp \eta} = \mathbb{R}p \oplus E$ , and by  $q$  an isotropic vector  $q \in V$  such that  $\eta(q, E) = 0$  and  $\eta(p, q) = 1$ . The vector space  $E$  is an Euclidean space. We consider the vector model of the  $n + 1$ -dimensional Lobachevskian space  $L^{n+1}$  and its boundary  $\partial L^{n+1}$ , which is diffeomorphic to the  $n$ -dimensional unit sphere. We have the natural isomorphisms

$$SO(V) \simeq \text{Isom } L^{n+1} \simeq \text{Conf } \partial L^{n+1} \text{ and } SO(V)_{\mathbb{R}p} \simeq \text{Sim } E,$$

where  $\text{Isom } L^{n+1}$  is the group of all isometries of  $L^{n+1}$ ,  $\text{Conf } \partial L^{n+1}$  is the group of all conformal transformations of  $\partial L^{n+1}$  and  $\text{Sim } E$  is the group of all similarity transformations of  $E$ . We identify the set  $\partial L^{n+1} \setminus \{\mathbb{R}p\}$  with the Euclidean space  $E$ . Then any subgroup  $G \subset SO(V)_{\mathbb{R}p}$  acts on  $E$ , moreover we have  $G \subset \text{Sim } E$ . We prove that *a connected subgroup  $G \subset SO(V)_{\mathbb{R}p}$  is weakly-irreducible iff the corresponding subgroup  $G \subset \text{Sim } E$  under the isomorphism  $SO(V)_{\mathbb{R}p} \simeq \text{Sim } E$  acts transitively on  $E$ .* This gives us a *one-to-one correspondence between connected weakly-irreducibly acting subgroups of  $SO(V)_{\mathbb{R}p}$  and connected transitively acting subgroups of  $\text{Sim } E$ .* Using a description for connected transitive subgroups of  $\text{Sim } E$  (see [2], [3]), we divide such subgroups into 4 types. We show that the corresponding weakly-irreducible subgroups of  $SO(V)_{\mathbb{R}p}$  have the same type introduced by L. Berard Bergery and A. Ikemakhen.

We also classify transitively acting isometry groups of the Lobachevskian space  $L^{n+1}$ . We show that these groups are exhausted by  $SO^0(V)$  and by the weakly-irreducible not irreducible subgroups of  $SO(V)_{\mathbb{R}p}$  of type 1 and type 3.

**Remark** In another paper we will use a similar ideas for complex Lobachevskian space in order to classify connected weakly-irreducible not irreducible subgroups of  $SU(1, n + 1) \subset SO(2, 2n + 2)$ .

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# 1 Results of L. Berard Bergery and A. Ikemakhen

Let  $(V, \eta)$  be a Minkowski space of dimension  $n + 2$ , where  $\eta$  is a metric on  $V$  of signature  $(1, n + 1)$ . We fix a basis  $p, e_1, \dots, e_n, q$  of  $V$  with respect to which the Gram matrix of  $\eta$  has

the form  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$ , where  $E_n$  is the  $n$ -dimensional identity matrix.

Let  $E \subset V$  be the vector subspace spanned by  $e_1, \dots, e_n$ . The vector space  $E$  is an Euclidean space with respect to the inner product  $\eta|_E$ .

Denote by  $\mathfrak{so}(V)$  the Lie algebra of all  $\eta$ -skew symmetric endomorphisms of  $V$  and by  $\mathfrak{so}(V)_{\mathbb{R}p}$  the subalgebra of  $\mathfrak{so}(V)$  that preserves the line  $\mathbb{R}p$ .

The Lie algebra  $\mathfrak{so}(V)_{\mathbb{R}p}$  can be identified with the following algebra of matrices:

$$\mathfrak{so}(V)_{\mathbb{R}p} = \left\{ \begin{pmatrix} a & -X^t & 0 \\ 0 & A & X \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, X \in E, A \in \mathfrak{so}(E) \right\}.$$

The above matrix can be identified with the triple  $(a, A, X)$ . Define the following subalgebras of  $\mathfrak{so}(V)_{\mathbb{R}p}$ ,  $\mathcal{A} = \{(a, 0, 0) : a \in \mathbb{R}\}$ ,  $\mathcal{K} = \{(0, A, 0) : A \in \mathfrak{so}(E)\}$  and  $\mathcal{N} = \{(0, 0, X) : X \in E\}$ . We see that  $\mathcal{A}$  commutes with  $\mathcal{K}$ , and  $\mathcal{N}$  is an ideal. We have the decomposition

$$\mathfrak{so}(V)_{\mathbb{R}p} = (\mathcal{A} \oplus \mathcal{K}) \ltimes \mathcal{N}.$$

A subalgebra  $\mathfrak{g} \subset \mathfrak{so}(V)$  is called *irreducible* if it preserves no proper subspace of  $V$ ;  $\mathfrak{g}$  is called *weakly-irreducible* if it preserves no nondegenerate proper subspace of  $V$ .

Obviously, if  $\mathfrak{g} \subset \mathfrak{so}(V)$  is irreducible, then it is weakly-irreducible. If  $\mathfrak{g} \subset \mathfrak{so}(V)$  preserves a degenerate proper subspace  $U \subset V$ , then it preserves the isotropic line  $U \cap U^\perp$ ; any such algebra is conjugated to a subalgebra of  $\mathfrak{so}(V)_{\mathbb{R}p}$ .

Let  $\mathcal{B} \subset \mathfrak{so}(E)$  be a subalgebra. Recall that  $\mathcal{B}$  is a compact Lie algebra and we have the decomposition  $\mathcal{B} = \mathcal{B}' \oplus \mathfrak{z}(\mathcal{B})$ , where  $\mathcal{B}'$  is the commutant of  $\mathcal{B}$  and  $\mathfrak{z}(\mathcal{B})$  is the center of  $\mathcal{B}$ .

The following result is due to L. Berard Bergery and A. Ikemakhen.

**Theorem** *Suppose  $\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}p}$  is a weakly-irreducible subalgebra. Then  $\mathfrak{g}$  belongs to one of the following types*

**type 1.**  $\mathfrak{g} = (\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{N}$ , where  $\mathcal{B} \subset \mathfrak{so}(E)$  is a subalgebra;

**type 2.**  $\mathfrak{g} = \mathcal{B} \ltimes \mathcal{N}$ ;

**type 3.**  $\mathfrak{g} = (\mathcal{B}' \oplus \{\varphi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes \mathcal{N}$ , where  $\varphi : \mathfrak{z}(\mathcal{B}) \rightarrow \mathcal{A}$  is a non-zero linear map;

**type 4.**  $\mathfrak{g} = (\mathcal{B}' \oplus \{\psi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes \mathcal{N}_W$ , where we have a non-trivial orthogonal decomposition  $E = U \oplus W$  such that  $\mathcal{B} \subset \mathfrak{so}(W)$ ;  $\mathcal{N}_W = \{(0, 0, X) : X \in W\}$ ;  $\mathcal{N}_U = \{(0, 0, X) : X \in U\}$  and  $\psi : \mathfrak{z}(\mathcal{B}) \rightarrow \mathcal{N}_U$  is a surjective linear map.

Denote by  $SO(V)$  the Lie group of all automorphisms of  $V$  that preserve the form  $\eta$ , and with  $\det f = 1$ , and by  $SO(V)_{\mathbb{R}p}$  the Lie subgroup of  $SO(V)$  that preserves the isotropic line  $\mathbb{R}p$ . Obviously,  $\mathfrak{so}(V)$  and  $\mathfrak{so}(V)_{\mathbb{R}p}$  are the Lie algebras of  $SO(V)$  and  $SO(V)_{\mathbb{R}p}$  respectively. By definition, the *type* of a connected weakly-irreducible Lie subgroup  $G \subset SO(V)_{\mathbb{R}p}$  is the type of its Lie algebra  $\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}p}$ .

## 2 Transitive similarity transformation groups of Euclidean spaces

In this section we recall a description for connected transitively acting groups of similarity transformations and isometries of Euclidean spaces, see [2] or [3].

Let  $(E, \eta)$  be an Euclidean space. A map  $f : E \rightarrow E$  is called a *similarity transformation* of  $E$  if there exists a  $\lambda > 0$  such that  $\|f(x_1) - f(x_2)\| = \lambda\|x_1 - x_2\|$  for all  $x_1, x_2 \in E$ , where  $\|x\|^2 = \eta(x, x)$ . If  $\lambda = 1$ , then  $f$  is called an *isometry*. Denote by  $\text{Sim } E$  and  $\text{Isom } E$  the groups of all similarity transformations and isometries of  $E$  respectively. A subgroup  $G \subset \text{Sim } E$  such that  $G \not\subset \text{Isom } E$  is called *essential*. A subgroup  $G \subset \text{Sim } E$  is called *irreducible* if it preserves no proper affine subspace of  $E$ .

We need the following theorem from [2] and [3].

**Theorem 1** (1) *Let  $G \subset \text{Isom } E$  be a connected subgroup that acts transitively on  $E$ . Then there exists a decomposition  $G = H \ltimes F$ , where  $H$  is the stabilizer of a point  $x \in E$  and  $F$  is a normal subgroup of  $G$  that acts simply transitively on  $E$ .*

(2) *Let  $F \subset \text{Isom } E$  be a connected subgroup that acts simply transitively on  $E$ . Then there exists an orthogonal decomposition  $E = U \oplus W$  and a Lie groups homomorphism  $\Psi : U \rightarrow SO(W)$  such that  $F = U^\Psi \ltimes W$ , where*

$$U^\Psi = \{\Psi(u) \cdot u : u \in U\} \subset SO(W) \times U$$

*is a group of screw isometries.*

(3) *Let  $G \subset \text{Sim } E$  be an essential connected subgroup that acts transitively on  $E$ . Then  $G = (A_1 \times H) \ltimes F$ , where  $A_1 \subset \text{Sim } E$  is a 1-parameterized essential subgroup that preserves a point  $x$ ,  $H \subset \text{Isom } E$  commutes with  $A_1$  and preserves the point  $x$ , and  $F$  is a normal subgroup of  $G$  that acts simply transitively on  $E$ .*

(4) *A connected subgroup  $G \subset \text{Isom } E$  acts irreducibly on  $E$  if and only if it acts transitively on  $E$ .*

From parts (3) and (4) of the theorem it follows that *a connected subgroup  $G \subset \text{Sim } E$  acts irreducibly on  $E$  if and only if it acts transitively on  $E$ .*

### 3 Isometries of Lobachevskian spaces

Let  $p, e_1, \dots, e_n, q$  be a basis of the vector space  $V$  as above. Consider the basis  $e_0, e_1, \dots, e_n, e_{n+1}$  of  $V$ , where  $e_0 = \frac{\sqrt{2}}{2}(p - q)$  and  $e_{n+1} = \frac{\sqrt{2}}{2}(p + q)$ . With respect to this basis the Gram matrix of  $\eta$  has the form  $\begin{pmatrix} -1 & 0 \\ 0 & E_{n+1} \end{pmatrix}$ , where  $E_{n+1}$  is the  $n + 1$ -dimensional identity matrix.

The vector model of the  $n + 1$ -dimensional *Lobachevskian space* is defined in the following way

$$L^{n+1} = \{x \in V : \eta(x, x) = -1, x_0 > 0\}.$$

Recall that  $L^{n+1}$  is an  $n + 1$ -dimensional Riemannian submanifold of  $V$ . The tangent space at a point  $x \in L^{n+1}$  is identified with the vector subspace  $(x)^{\perp_n} \subset V$  and the restriction of the form  $\eta$  to this subspace is positively definite.

Any element  $f \in SO(V)$  preserves the space  $L^{n+1}$ . Moreover, for any  $f \in SO(V)$ , the restriction  $f|_{L^{n+1}}$  is an isometry of  $L^{n+1}$  and any isometry of  $L^{n+1}$  can be obtained in this way. Hence we have the isomorphism

$$SO(V) \simeq \text{Isom } L^{n+1},$$

where  $\text{Isom } L^{n+1}$  is the group of all isometries of  $L^{n+1}$ .

Consider the *light-cone* of  $V$ ,

$$C = \{x \in V : \eta(x, x) = 0\}.$$

The subset of the  $n + 1$ -dimensional projective space  $\mathbb{P}V$  that consists of all *isotropic lines*  $l \subset C$  is called the *boundary of the Lobachevskian space*  $L^{n+1}$  and is denoted by  $\partial L^{n+1}$ .

We identify  $\partial L^{n+1}$  with the  $n$ -dimensional unit sphere  $S^n$  in the following way. Consider the vector subspace  $E_1 = E \oplus \mathbb{R}e_{n+1}$ . Each isotropic line intersects the hyperplane  $e_0 + E_1$  at a unique point. The intersection  $(e_0 + E_1) \cap C$  is the set

$$\{x \in V : x_0 = 1, x_1^2 + \dots + x_{n+1}^2 = 1\},$$

which is the  $n$ -dimensional sphere  $S^n$ . This gives us the identification  $\partial L^{n+1} \simeq S^n$ .

Denote by  $\text{Conf } S^n$  the group of all conformal transformations of  $S^n$ . Any transformation  $f \in SO(V)$  takes isotropic lines to isotropic lines. Moreover, under the above identification, we have  $f|_{\partial L^{n+1}} \in \text{Conf } \partial L^{n+1}$  and any transformation from  $\text{Conf } \partial L^{n+1}$  can be obtained in this way. Hence we have the isomorphism

$$SO(V) \simeq \text{Conf } \partial L^{n+1}.$$

Suppose  $f \in SO(V)_{\mathbb{R}p}$ . The corresponding element  $f \in \text{Conf } S^n$  (we denote it by the same letter) preserves the point  $p_0 = \mathbb{R}p \cap (e_0 + E_1)$ . Clearly,  $p_0 = \sqrt{2}p$ . Denote by  $s_0$  the stereographic projection  $s_0 : S^n \setminus \{p_0\} \rightarrow e_0 + E$ . Since  $f \in \text{Conf } S^n$ , we see that  $s_0 \circ f \circ s_0^{-1} : E \rightarrow E$  (here

we identify  $e_0 + E$  with  $E$ ) is a similarity transformation of the Euclidean space  $E$ . Conversely, any similarity transformation of  $E$  can be obtained in this way. Thus we have the isomorphism

$$SO(V)_{\mathbb{R}p} \simeq \text{Sim } E.$$

A *plane* in the Lobachevskian space  $L^{n+1}$  is the nonempty intersection of  $L^{n+1}$  and of a vector subspace  $U \subset V$ . The intersection  $L^{n+1} \cap U$  is not empty if and only if the restriction of the form  $\eta$  to  $U$  has signature  $(1, \dim U - 1)$ . A subgroup  $G \subset \text{Isom } L^{n+1}$  is called *irreducible* if it preserves no proper plane in  $L^{n+1}$ .

The following theorem is due to F.I. Karpelevich, see [3] or [15].

**Theorem 2** *Let  $G$  be a proper connected closed subgroup of  $\text{Isom } L^{n+1}$ . Then  $G$  acts irreducibly on  $L^{n+1}$  if and only if it preserves an isotropic line  $l \in \partial L^{n+1}$  and acts transitively on the Euclidean space  $E_l = \partial L^{n+1} \setminus \{l\}$ .*

Since the holonomy group of a Lorentzian manifold can be not closed, we need an analog of this theorem for not closed groups. In [11] was proved the following theorem.

**Theorem 3** *Let  $G$  be a connected (non nec. closed) subgroup of  $SO(V)$  that acts weakly-irreducibly. Then either  $G$  acts transitively on  $L^{n+1}$  or  $G$  acts transitively on the Euclidean space  $E_l = \partial L^{n+1} \setminus \{l\}$ .*

We will prove the following theorem.

**Theorem 4** *Let  $G$  be a proper connected subgroup of  $SO(V)_{\mathbb{R}p}$ . Then  $G$  acts weakly-irreducibly on  $V$  if and only if it acts transitively on the Euclidean space  $E = \partial L^{n+1} \setminus \{\mathbb{R}p\}$ .*

**Proof.** We claim that the subgroup  $G \subset SO(V)_{\mathbb{R}p}$  acts weakly-irreducibly on  $V$  if and only if the corresponding subgroup  $G \subset \text{Sim } E$  acts irreducibly on  $E$ . If  $G \subset SO(V)_{\mathbb{R}p}$  is not weakly-irreducible, then it preserves a not degenerate proper subspace  $U \subset V$ . Since the orthogonal complement  $U^\perp \subset V$  is also preserved and either  $U \cap C \neq \{0\}$  or  $U^\perp \cap C \neq \{0\}$ , we can assume that  $U \cap C \neq \{0\}$ . The subgroup  $G \subset \text{Sim } E$  preserves the affine subspace  $s_0((e_0 + E) \cap C \cap U) \subset E$ , which is not empty. Conversely, if the subgroup  $G \subset \text{Sim } E$  preserves a proper affine subspace  $W \subset E$ , then  $G \subset SO(V)_{\mathbb{R}p}$  preserves the vector subspace of  $V$  spanned by  $s_0^{-1}(W) \subset e_0 + E$ , which is not degenerate. Now the proof of the theorem follows from parts (3) and (4) of theorem 1.  $\square$

## 4 Application to holonomy groups of Lorentzian manifolds

Now we consider connected weakly-irreducible not irreducible subgroups of  $SO(V)$ . Any such group  $G$  preserves an isotropic line and is conjugated to a subgroup of  $SO(V)_{\mathbb{R}p}$ .

In section 2 we have constructed the isomorphism  $SO(V)_{\mathbb{R}p} \simeq \text{Sim } E$ . This isomorphism and theorem 4 gives us a *one-to-one correspondence between connected weakly-irreducible subgroups  $G \subset SO(V)_{\mathbb{R}p}$  and connected transitively acting subgroups  $G \subset \text{Sim } E$ .*

**Theorem 5** *Let  $G \subset \text{Sim } E$  be a transitively acting connected subgroup. Then  $G$  belongs to one of the following types*

**type 1.**  $G = (A \times H) \triangleleft E$ , where  $A = \mathbb{R}^+$  is the unite component for the group of all dilations of  $E$  about the origin 0,  $H \subset SO(E)$  is a Lie subgroup, and  $E$  is the group of all translations in  $E$ ;

**type 2.**  $G = H \triangleleft E$ ;

**type 3.**  $G = (A^\Phi \times H) \triangleleft E$ , where  $\Phi : A \rightarrow SO(E)$  is a homomorphism and

$$A^\Phi = \{\Phi(a) \cdot a : a \in A\} \subset SO(E) \times A$$

is a group of screw dilations of  $E$ ;

**type 4.**  $G = (H \times U^\Psi) \triangleleft W$ , where  $E = U \oplus W$  is an orthogonal decomposition,  $\Psi : U \rightarrow SO(W)$  is a homomorphism, and

$$U^\Psi = \{\Psi(u) \cdot u : u \in U\} \subset SO(E) \times U$$

is a group of screw isometries of  $E$ .

The corresponding subgroups of  $SO(V)_{\mathbb{R}p}$  under the isomorphism  $SO(V)_{\mathbb{R}p} \simeq \text{Sim } E$  are the groups of the same type introduced by L. Berard Bergery and A. Ikemakhen.

**Proof.** Denote by  $A$ ,  $K$  and  $N$  the connected Lie subgroups of  $SO(V)_{\mathbb{R}p}$  corresponding to the subalgebras  $\mathcal{A}$ ,  $\mathcal{K}$  and  $\mathcal{N} \subset \mathfrak{so}(V)_{\mathbb{R}p}$ . With respect to the basis  $p, e_1, \dots, e_n, q$  these groups have

$$\text{the following forms } A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R}, a > 0 \right\},$$

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in SO(E) \right\} \text{ and } N = \left\{ \begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^t X \\ 0 & \text{id} & X \\ 0 & 0 & 1 \end{pmatrix} : X \in E \right\}.$$

We have the decomposition  $SO^0(V)_{\mathbb{R}p} = (A \times K) \triangleleft N$ .

The computation shows that under the isomorphism  $SO(V)_{\mathbb{R}p} \simeq \text{Sim } E$

$$\text{the element } \begin{pmatrix} a & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \in A \text{ corresponds to the dilation } X \mapsto aX,$$

the element  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K$  corresponds to  $f \in SO(E)$ , and

the element  $\begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^t X \\ 0 & \text{id} & X \\ 0 & 0 & 1 \end{pmatrix} \in N$  corresponds to the translation  $Y \mapsto Y + X$ .

Let a subgroup  $G \subset \text{Sim } E$  act transitively. Denote by the same letter  $G$  the corresponding weakly-irreducible subgroup of  $SO(V)_{\mathbb{R}p}$ . Since we are interested in the groups up to conjugacy, in the theorem 1 we choose  $x = 0$ , then  $H \subset SO(E)$ .

For the subgroup  $G \subset SO(V)_{\mathbb{R}p}$  we have two cases:

**case 1.**  $G$  preserves the vector  $p$ ;

**case 2.**  $G$  preserves the isotropic line  $\mathbb{R}p$  but does not preserve the vector  $p$ .

Consider these cases.

**Case 1.** We have  $G \subset K \ltimes N$ . Hence the corresponding subgroup  $G \subset \text{Sim } E$  consists of isometries, i.e.  $G \subset \text{Isom } E$ . From the transitivity of  $G$  it follows that  $G = H \ltimes F$ , where  $H \subset SO(E)$  and  $F$  is a normal subgroup of  $G$  that acts simply transitively on  $E$ . Hence there exists an orthogonal decomposition  $E = U \oplus W$  and a homomorphism  $\Psi : U \rightarrow SO(W)$  such that  $F = U^\Psi \ltimes W$ .

There are two subcases

**Subcase 1.1.** The homomorphism  $\Psi$  is trivial. Hence  $F = E$  and  $G = H \ltimes E$ . From the classification of L. Berard Bergery and A. Ikemakhen we have  $G \subset SO(V)_{\mathbb{R}p}$  is a group of type 2.

**Subcase 1.2.** The homomorphism  $\Psi$  is not trivial. We can assume that the homomorphism  $d\Psi : U \rightarrow \mathfrak{so}(W)$  is injective. Indeed, if  $\ker d\Psi \neq \{0\}$ , then we choose the decomposition  $E = U_1 \oplus W_1$ , where  $W_1 = W \oplus \ker d\Psi$  and  $U_1 \subset U$  is the orthogonal complement of  $\ker d\Psi$  in  $U$ , and we consider  $\Psi_1 = \Psi|_{U_1}$ .

We claim that  $H$  commutes with  $\Psi(U) \subset SO(W)$ , moreover  $H$  acts trivially on  $U$  and  $H \subset SO(W)$ . Let  $f \in H$ ,  $u \in U$ . Since  $F$  is a normal subgroup of  $G$ , we have  $f \circ \Psi(u) \circ u \circ f^{-1} = w \circ \Psi(u_1) \circ u_1$  for some  $w \in W$  and  $u_1 \in U$ . Hence for all  $v \in E$  we have  $f(u) + f \circ \Psi(u) \circ f^{-1}(v) = w + u_1 + \Psi(u_1)v$ . Since this holds for all  $v \in E$ , we have  $f \circ \Psi(u) \circ f^{-1} = \Psi(u_1)$ . We will prove that  $\Psi(u) = \Psi(u_1)$ . Let  $\mathfrak{l}(\Psi(U))$  and  $\mathfrak{h} = \mathfrak{l}(H)$  be the Lie algebras of the Lie groups  $\Psi(U)$  and  $H$  respectively. We have  $(\mathfrak{h} + \mathfrak{l}(\Psi(U)))' = \mathfrak{h}' + [\mathfrak{h}, \Psi(U)]$ . Since  $[\mathfrak{h}, \Psi(U)] \subset \Psi(U)$  and the Lie algebra  $\mathfrak{l}(\Psi(U))$  is commutative, we have  $(\mathfrak{h} + \mathfrak{l}(\Psi(U)))'' = \mathfrak{h}'$ . If  $\Psi(u) \neq \Psi(u_1)$ , then  $[\mathfrak{h}, \Psi(U)] \neq \{0\}$  and  $(\mathfrak{h} + \mathfrak{l}(\Psi(U)))' \neq (\mathfrak{h} + \mathfrak{l}(\Psi(U)))''$ . Since the subalgebra  $\mathfrak{h} + \mathfrak{l}(\Psi(U)) \subset \mathfrak{so}(E)$  is compact, we have a contradiction. Thus,  $\Psi(u) = \Psi(u_1)$  and  $H$  commutes with  $\Psi(U)$ . Consider now the Lie algebra  $\mathfrak{l}(G)$  of the Lie group  $G$ . We have  $\mathfrak{l}(G) = (\mathfrak{h} \oplus \mathfrak{l}(U^\Psi)) \ltimes W$ . Since  $U^\Psi = \{\Psi(u) \circ u : u \in U\}$ , we see that  $\mathfrak{l}(U^\Psi) = \{d\Psi(u) + u : u \in U\}$ . For  $\xi \in \mathfrak{h}$  and



$d\Psi(u) + u \in l(U^\Psi)$  we have  $[\xi, d\Psi(u) + u] = \xi u \subset U$ . Since  $U \cap l(G) = \{\emptyset\}$ , we see that  $\xi u = 0$ . Hence  $H$  acts trivially on  $U$ . Since  $H \subset SO(E)$  and  $W$  is orthogonal to  $U$ , we see that  $H(W) \subset W$  and  $H \subset SO(W)$ .

We see now that  $d\Psi(U) \subset \mathfrak{so}(W)$  is a commutative subalgebra that commutes with  $\mathfrak{h}$ . Put  $\mathcal{B} = \mathfrak{h} \oplus d\Psi(U)$ . We have  $\mathfrak{z}(\mathcal{B}) = \mathfrak{z}(\mathfrak{h}) \oplus d\Psi(U)$ . Put  $\psi = d\Psi^{-1} : d\Psi(U) \rightarrow U$  and extend  $\psi$  to the linear map  $\psi : \mathfrak{z}(\mathcal{B}) \rightarrow U$  by putting  $\psi|_{\mathfrak{z}(\mathfrak{h})} = 0$ . Thus we have

$$l(G) = (\mathcal{B}' \oplus \{\psi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes W.$$

We see that  $l(G)$  is an algebra of type 4 and  $G$  is a group of type 4.

**Case 2.** In this case we have  $G \subset \text{Sim } E$ , hence  $G = (A_1 \times H) \ltimes F$ , where  $A_1$  is a 1-parameterized subgroup of  $G$  that preserves the point 0,  $H \subset SO(E)$  commutes with  $A_1$ , and  $F$  is a normal subgroup that acts simply transitively on  $E$ .

There are two subcases

**Subcase 2.1.** We have  $A_1 = A$  is the unity component of the group of all dilations of  $E$  about the origin  $0 \in E$ .

We claim that  $F = E$ . Indeed, suppose that  $F = U^\Psi \ltimes W$  and the homomorphism  $\Psi$  is not trivial. Let  $u \in U$ ,  $w \in W$  and  $1 \neq \lambda \in A = \mathbb{R}^+$ . Since the subgroup  $F \subset G$  is normal, we see that  $\lambda \circ \Psi(u) \circ u \circ w \circ \lambda^{-1} \in U^\Psi \ltimes W$ . Let  $v \in E$ . We have  $(\lambda \circ \Psi(u) \circ u \circ w \circ \lambda^{-1})v = \Psi(u)(\lambda \circ u \circ w \circ \lambda^{-1})v = \Psi(u)(\lambda \circ u \circ w(\lambda^{-1}v)) = \Psi(u)(\lambda(u + w + \lambda^{-1}v)) = \Psi(u)(\lambda u + \lambda w + v)$ . Hence,  $\lambda \circ \Psi(u) \circ u \circ w \circ \lambda^{-1} = \Psi(u) \circ (\lambda u) \circ (\lambda w) \in U^\Psi \ltimes W$ . This implies  $u = \lambda u$  for all  $u \in U$ , hence,  $\lambda = 1$ . This gives us a contradiction. Thus,  $F = E$ .

Now we see that  $G = (A_1 \times H) \ltimes F$  is a group of type 1.

**Subcase 2.2.** In this case  $A_1 \neq A$ , then  $A_1 \subset A \times SO(E)$ . By analogy with subcase 2.1., we can prove that  $F = E$ .

Let  $\xi : \mathbb{R} \rightarrow A_1$  be a parameterization of the group  $A_1$ . Define the homomorphisms  $\xi_1 : \mathbb{R} \rightarrow A$  and  $\xi_2 : \mathbb{R} \rightarrow SO(E)$  by condition  $\xi(t) = \xi_1(t) \cdot \xi_2(t)$  for all  $t \in \mathbb{R}$ . Since  $A_1 \not\subset SO(E)$ , we see that the homomorphism  $\xi_1$  is an isomorphism. Put  $\Phi = \xi_2 \circ \xi_1^{-1} : A \rightarrow SO(E)$ . We have

$$A_1 = \{\Phi(a) \cdot a : a \in A\} \subset SO(n) \times \mathbb{R}.$$

We see that  $l(G) = (l(A_1) \oplus \mathfrak{h}) \ltimes E$  and

$$l(A_1) = \{d\Phi(a) + a : a \in l(A)\}.$$

Note that the subalgebra  $l(d\Phi(l(A))) \subset \mathfrak{so}(E)$  is commutative and commutes with  $\mathfrak{h}$ . Put  $\mathcal{B} = \mathfrak{h} \oplus l(d\Phi(l(A)))$ . We see that  $\mathfrak{z}(\mathcal{B}) = \mathfrak{z}(\mathfrak{h}) \oplus l(d\Phi(l(A)))$ . Put  $\varphi = (d\Phi)^{-1} : d\Phi(l(A)) \rightarrow l(A)$  and extend  $\varphi$  to the linear map  $\varphi : \mathfrak{z}(\mathcal{B}) \rightarrow l(A)$  by putting  $\varphi|_{\mathfrak{z}(\mathfrak{h})} = 0$ . Thus,

$$l(G) = (\mathcal{B}' \oplus \{\varphi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes E.$$

We see that  $G$  is a group of type 3. This completes the proof of the theorem.  $\square$ .

## 5 Transitive isometry groups of the Lobachevskian space

$L^{n+1}$

Recall that we consider a Minkowski space  $(V, \eta)$  of dimension  $n + 2$  and a basis  $p, e_1, \dots, e_n, q$  of  $V$  with respect to which the Gram matrix of  $\eta$  has the form  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$ , where  $E_n$  is the  $n$ -dimensional identity matrix. We consider the vector subspace  $E \subset V$  spanned by  $e_1, \dots, e_n$  as an Euclidean space with respect to the inner product  $\eta|_E$ . We denote by  $SO(V)_{\mathbb{R}p}$  the subgroup of  $SO(V)$  that preserves the line  $\mathbb{R}p$ . For the Lie group  $SO^0(V)_{\mathbb{R}p}$  we have the decomposition  $SO^0(V)_{\mathbb{R}p} = (A \times K) \ltimes N$ , where with respect to the basis  $p, e_1, \dots, e_n, q$  the groups  $A, K$  and

$$N \text{ have the following matrix forms } A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R}, a > 0 \right\},$$

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in SO(E) \right\} \text{ and } N = \left\{ \begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^tX \\ 0 & \text{id} & X \\ 0 & 0 & 1 \end{pmatrix} : X \in E \right\}.$$

**Theorem 6** *Let  $G \subset SO(V)$  be a connected subgroup that acts transitively on the Lobachevskian space  $L^{n+1}$ . Then either  $G = SO^0(V)$  or  $G$  preserves an isotropic line  $l \subset V$  and there exists a basis  $p, e_1, \dots, e_n, q$  of  $V$  as above such that  $l = \mathbb{R}p$  and  $G$  is one of the following groups*

- (1)  $(A \times H) \ltimes N$ , where  $H \subset K$  is a subgroup;
- (2)  $(A^\Phi \times H) \ltimes N$ , where  $\Phi : A \rightarrow K$  is a not trivial homomorphism and

$$A^\Phi = \{\Phi(a) \cdot a : a \in A\} \subset K \times A.$$

Moreover the groups of the form  $A \ltimes N$  and  $A^\Phi \ltimes N$  exhaust all connected subgroups of  $SO(V)$  that act simply transitively on  $L^{n+1}$ .

Note that  $A$  is the group of translations in  $L^{n+1}$  along the line  $h = (\mathbb{R}p \oplus \mathbb{R}q) \cap L^{n+1}$ ,  $K$  is the group of rotations about  $h$ ,  $N$  is the group of parabolic translations along 2-dimension planes in  $L^{n+1}$  and  $A^\Phi$  is a group of screw translations along the line  $h$ .

**Proof.** Suppose a subgroup  $G \subset SO(V)$  acts transitively on  $L^{n+1}$ . Then it preserves no plane in  $L^{n+1}$ , hence  $G$  acts weakly-irreducibly on  $SO(V)$ . If  $G$  acts irreducibly on  $V$ , then  $G = SO^0(V)$ , see [11] or [10].

If  $G$  acts weakly-irreducibly not irreducibly on  $V$ , then  $G$  preserves an isotropic line  $l \subset V$ , we assume that  $l = \mathbb{R}p$ . Then  $G$  is the group of type 1,2,3 or 4.

We claim that the subgroup  $K \ltimes N \subset SO(V)$  does not act transitively on  $L^{n+1}$ . Indeed, any element of  $K \ltimes N$  takes the vector  $\frac{1}{2}p - q \in L^{n+1}$  to some vector  $u - q$ , where  $u \in$

span $\{p, e_1, \dots, e_n\}$ , hence there is no element of  $K \ltimes N$  that takes  $\frac{1}{2}p - q \in L^{n+1}$  to  $p - \frac{1}{2}q \in L^{n+1}$ . Hence the groups of type 2 and 4 does not act transitively on  $L^{n+1}$ .

We must prove that groups of type 1 and 3, i.e. groups of the form  $A \times H \ltimes N$  and  $A^\Phi \times H \ltimes N$  act transitively on  $L^{n+1}$ . Let  $v = xp + \alpha + yq \in L^{n+1}$  and  $w = xp + \beta + yq \in L^{n+1}$ , where  $\alpha, \beta \in E$ . Then we have  $2xy + \eta(\alpha, \alpha) = -1$  and  $2xy + \eta(\beta, \beta) = -1$ . Let  $X = \frac{\alpha - \beta}{y}$ . The

element  $\begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^tX \\ 0 & \text{id} & X \\ 0 & 0 & 1 \end{pmatrix} \in N$  takes  $u$  to  $w$ .

Let  $v = x_1p + \beta + y_1q \in L^{n+1}$ , i.e.  $2x_1y_1 + \eta(\beta, \beta) = -1$ .

The element  $\begin{pmatrix} \frac{x_1}{x} & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \frac{x}{x_1} \end{pmatrix} \in A$  takes  $w$  to  $v$ . The element  $\begin{pmatrix} \frac{x_1}{x} & 0 & 0 \\ 0 & \Phi(\frac{x_1}{x}) & 0 \\ 0 & 0 & \frac{x}{x_1} \end{pmatrix} \in A^\Phi$  takes  $w$  to  $xp + \Phi(\frac{x_1}{x})(\beta) + yq \in L^{n+1}$ . Thus there exist elements in  $(A \times H) \ltimes N$  and  $(A^\Phi \times H) \ltimes N$  that take  $u$  to  $v$ , i.e. the groups  $(A \times H) \ltimes N$  and  $(A^\Phi \times H) \ltimes N$  act transitively on  $L^{n+1}$ .

Note that the elements of the subgroup  $H \subset G$  preserve the point  $p - \frac{1}{2}q \in L^{n+1}$ . Since  $\dim L^{n+1} = \dim(A \ltimes N) = \dim(A^\Phi \ltimes N)$  and  $L^{n+1}$  is simply connected, we see that the groups of the form  $A \ltimes N$  and  $A^\Phi \ltimes N$  are the only connected subgroups of  $SO(V)$  that act simply transitively on  $L^{n+1}$ .  $\square$

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