SOME PROPERTIES OF SUBSETS OF HYPERBOLIC GROUPS

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Abstract

We present some results about quasiconvex subgroups of infinite index and their products. After that we extend the standard notion of a subgroup commensurator to an arbitrary subset of a group, and generalize some of the previously known results.

1 Introduction

Assume that G is a δ-hyperbolic group for some $\delta \geq 0$ and $\Gamma(G, \mathcal{A})$ is its Cayley graph corresponding to some finite symmetrized (i.e. $\mathcal{A} = \mathcal{A}^{-1}$) generating set $\mathcal{A}$. $\Gamma(G, \mathcal{A})$ is a proper geodesic metric space; a subset $Q \subseteq G$ is said to be *ε-quasiconvex*, if any geodesic connecting two elements from Q belongs to a closed ε-neighborhood $\mathcal{O}_\varepsilon(Q)$ of Q in $\Gamma(G, \mathcal{A})$ for some $\varepsilon \geq 0$. Q will be called *quasiconvex* if there exists $\varepsilon > 0$ for which it is ε-quasiconvex.

In [6] Gromov shows that the notion of quasiconvexity in a hyperbolic group does not depend on the choice of a finite generating set .

If $A, B \subset G$, $x \in G$ we define $A^B = \{bab^{-1} \mid a \in A, b \in B\}$, $A^x = xAx^{-1}$. For any $x \in G$ the set $x^G = \{x\}^G$ is the conjugacy class of the element x.

If $x \in G$, $o(x)$ will denote the order of the element x in G, $|x|_G$ − the length of a shortest representation of x in terms of the generators from $\mathcal{A}$. $\langle x \rangle$ will be the cyclic subgroup of G generated by x (sometimes, if $o(x) = n$, we will write $\langle x \rangle_n$). 1_G will denote the identity element of G. For a subgroup $H \leq G$, $|G : H|$ will be the index of H in G.

Theorem 1. *Let $H_1, H_2, \ldots, H_s$ be quasiconvex subgroups of a hyperbolic group G. Let K be an arbitrary subgroup of G. Then the following two conditions are equivalent:*

(a) $|K : (K \cap H_j^g)| = \infty$ for every $j \in \{1, 2, \ldots, s\}$ and every $g \in G$;

(b) there exists an element of infinite order $x \in K$ such that the intersection $\langle x \rangle_\infty \cap (H_1^G \cup H_2^G \cup \ldots \cup H_s^G)$ is trivial.

Let $G_1, G_2, \ldots, G_n$ be quasiconvex subgroups of G, $f_0, f_1, \ldots, f_n \in G$, $n \in \mathbb{N} \cup \{0\}$. Using the same terminology as in [9], the set

$$P = f_0 G_1 f_1 G_2 \cdot \ldots \cdot f_{n-1} G_n f_n = \{f_0 g_1 f_1 \cdot \ldots \cdot g_n f_n \in G \mid g_i \in G_i,\ i = 1, \ldots, n\}$$

will be called a *quasiconvex product*.
The quasiconvex subgroups G_i, $i = 1, 2, \ldots, n$, are *members* of the product P.

Let $U = \bigcup_{k=1}^{q} P_k$ be a finite union of quasiconvex products P_k, $k = 1, \ldots, q$. A subgroup $H \leq G$ will be called a *member* of U, by definition, if H is a member of P_k for some $1 \leq k \leq q$. For any such set U we fix its representation as a finite union of quasiconvex products and fix its members.

Theorem 2. *Assume that U is a finite union of quasiconvex products in a hyperbolic group G and the subgroups $H_1, H_2, \ldots, H_s$ are all the members of U. If K is a subgroup of G and $K \subseteq U$ then for some $g \in G$ and $j \in \{1, 2, \ldots, s\}$ one has $|K : (K \cap H_j^g)| < \infty$.*

We will say that a finite union of quasiconvex products has *infinite index* in G if each of its members has infinite index in G. As an immediate consequence of Theorem2 applied to the case when $K = G$ we achieve

Corollary 1. *Let G be a hyperbolic group and U be a finite union of quasiconvex products of infinite index in G. Then U is a proper subset of G, i.e. $G \neq U$.*

In section 2 we generalize the definition of a subgroup commensurator. More precisely, to any subset $A \subset G$ we correspond a subgroup $Comm_G(A) \leq G$. In sections 7 and 8 we list several known results about quasiconvex subgroups and their commensurators and then extend them to more general settings.

A group is said to be non-elementary if it is not virtually cyclic. In section 9 we investigate some properties of infinite conjugacy classes and prove

Theorem 3. *Let G be a non-elementary hyperbolic group and A be a finite union of conjugacy classes in G. If the subset A is infinite then it is not quasiconvex.*

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2 Some concepts and definitions

Suppose G is an arbitrary group and 2^G is the set of all its subsets. Below we establish some auxiliary relations on 2^G. Assume $A, B \subseteq G$.

Definition. We will write $B \preceq A$ if there exist elements $x_1, \ldots, x_n \in G$ such that

$$B \subset Ax_1 \cup Ax_2 \cup \ldots \cup Ax_n \ .$$

If $G \preceq A$, the subset A will be called *quasidense*.

Obviously, the relation "$\preceq$" is transitive and reflexive.

Any subgroup H of finite index in G is quasidense; the complement $H^{(c)} = G \backslash H$ in this case is also quasidense (if $H \neq G$) since it contains a left coset modulo H, and a shift (left or right) of a quasidense subset is quasidense.

On the other hand, if $H \leq G$ and $|G : H| = \infty$, the set of elements of H is not quasidense in G. There is $y \in G \backslash H$, hence for any $x \in G$ either $x \in G \backslash H$ or $xy \in G \backslash H$, thus $G = H^{(c)} \cup H^{(c)} y^{-1}$, i.e. $H^{(c)}$ is still quasidense.

Definition. A and B will be called *equivalent* if $A \preceq B$ and $B \preceq A$. In this case we will use the notation $A \approx B$.

It is easy to check that "$\approx$" is an equivalence relation on 2^G. Now, let $[A]$ denote the equivalence class of a subset $A \subseteq G$ and let $\mathcal{M}$ be the set of all such equivalence classes. Evidently, the relation "$\preceq$" induces a partial order on $\mathcal{M}$: $[A], [B] \in \mathcal{M}$, $[A] \leq [B]$ if and only if $A \preceq B$.

The group G acts on $\mathcal{M}$ as follows: $g \in G$, $[A] \in \mathcal{M}$, then $g \circ [A] = [gA]$. Indeed, the verification of the group action axioms is straightforward:

1. If $g, h \in G$, $[A] \in \mathcal{M}$ then $(gh) \circ [A] = g \circ (h \circ [A])$;
2. If $1_G \in G$ is the identity element and $[A] \in \mathcal{M}$ then $1_G \circ [A] = [A]$.

This action is well defined because if $A \approx B$, $g \in G$, then $gA \approx gB$.

If $A \subseteq G$, the stabilizer of $[A]$ under this action is the subgroup

$$St_G([A]) = \{g \in G \mid g \circ [A] = [A]\}.$$

Definition. For a given subset A of the group G the subgroup $St_G([A])$ will be called *commensurator* of A in G and denoted $Comm_G(A)$. In other words,

$$Comm_G(A) = \{g \in G \mid gA \approx A\} .$$

Thus, to an arbitrary subset A of the group we corresponded a subgroup in G. Now, let's list some

Properties of $Comm_G(A)$:

1) If $card(A) < \infty$ or A is quasidense then $Comm_G(A) = G$ (because any two finite non-empty subsets are equivalent and a left shift of a quasidense subset is quasidense);

2) If $A, B \subseteq G$ and $A \approx B$ then $Comm_G(A) = Comm_G(B)$;

3) The commensurator of $A \subset G$ contains (as its subgroups) the normalizer of A $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$ and the stabilizer under the action of the group G on itself by left multiplication $St_G(A) = \{g \in G \mid gA = A\}$.

4) For any $h \in G$ $Comm_G(hA) = hComm_G(A)h^{-1}$.

Lemma 2.1. *Let A, B be subgroups of G. Then $A \preceq B$ if and only if the index $|A : (A \cap B)|$ is finite.*

Proof. The sufficiency is trivial. To prove the necessity, suppose there exist $y_j \in G$, $j = 1, 2, \ldots, m$, such that $A \subset By_1 \cup \ldots \cup By_m$. Without loss of generality we can assume that $A \cap By_j \neq \emptyset$ for every $j = 1, 2, \ldots, m$. Then for each $j = 1, 2, \ldots, m$, there are $a_j \in A, b_j \in B$ such that $y_j = b_j a_j$. Hence $By_j = Ba_j$ for all j, and therefore

$$A = \bigcup_{j=1}^{m} By_j \cap A = \bigcup_{j=1}^{m} (Ba_j \cap A) = \bigcup_{j=1}^{m} (B \cap A)a_j,$$

i.e. $|A : (B \cap A)| < \infty$. $\square$

For a subgroup $H \leq G$ the standard notion of the commensurator (virtual normalizer) subgroup of H is given by

$$VN_G(H) = \{g \in G \mid |H : (H \cap gHg^{-1})| < \infty, |gHg^{-1} : (H \cap gHg^{-1})| < \infty\} \ .$$

Now we are going to show that our new definition is just a generalization of it:

Remark 1. If H is a subgroup of the group G then $Comm_G(H) = VN_G(H)$.

Indeed, let $g \in VN_G(H)$. Then, by definition,

$$H \preceq (H \cap gHg^{-1}) \preceq gHg^{-1} \preceq gH \ \text{ and } \ gH \preceq gHg^{-1} \preceq (H \cap gHg^{-1}) \preceq H \ ,$$

thus $H \approx gH$ and $g \in Comm_G(H)$. So, $VN_G(H) \subseteq Comm_G(H)$.

Now, suppose $g \in Comm_G(H)$, implying $H \approx gH$ but $gH \approx gHg^{-1}$, hence $H \preceq gHg^{-1}$ and $gHg^{-1} \preceq H$. By Lemma 2.1, $g \in VN_G(H)$. Therefore $VN_G(H) = Comm_G(H)$.

If the group G is finitely generated, one can fix a finite symmetrized generating set $\mathcal{A}$ and define the word metric $d(\cdot, \cdot)$ corresponding to $\mathcal{A}$ in the standard way: first for every $g \in G$ we define $l_{\mathcal{A}}(g)$ to be the length of a shortest word in $\mathcal{A}$ representing g; second, for any $x, y \in G$ we set $d(x, y) = l_{\mathcal{A}}(x^{-1}y)$. Now, for arbitrary two subsets $A, B \subseteq G$ one can establish

$$h(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{O}_\varepsilon(B), B \subset \mathcal{O}_\varepsilon(A)\} -$$

the Hausdorff distance between A and B ($\mathcal{O}_\varepsilon(B)$ is the closed ε-neighborhood of B in G). Where an infinum over the empty set is defined to be positive infinity. In this case, for any $A, B \subseteq G$ we observe that $B \preceq A$ if and only if there exists $c > 0$ such that $B \subset \mathcal{O}_c(A)$, and, therefore, $A \approx B$ if and only if $h(A, B) < \infty$.

Now we investigate the special case, when G is δ-hyperbolic (for the definition of a hyperbolic group see section 3) and, therefore, finitely generated. We will need the following

Remark 2. ([9, Remark 4, lemma 2.1]) Let $Q, A, B \subseteq G$ be quasiconvex subsets, $g \in G$. Then (a) the left shift $gQ = \{gx \mid x \in Q\}$ is quasiconvex ; (b) the right shift $Qg = \{xg \mid x \in Q\}$ is quasiconvex; (c)$A \cup B$ is quasiconvex.

Therefore, a left coset modulo a quasiconvex subgroup and a conjugate subgroup to it are quasiconvex in G.

Remark 3. Let the group G be hyperbolic.

1) Suppose a subset $A \subset G$ is quasiconvex and $A \approx B$ for some $B \subset G$. Then B is also quasiconvex.

Indeed, as we saw above, there exist $c_1, c_2 \geq 0$ such that $B \subset \mathcal{O}_{c_1}(A)$ and $A \subset \mathcal{O}_{c_2}(B)$. Consider arbitrary $x, y \in B$ and a geodesic segment $[x, y]$ connecting them. Then

$$x, y \in \mathcal{O}_{c_1}(A) = \bigcup_{g \in G, |g|_G \leq c_1} Ag \ ,$$

which is ε-quasiconvex by Remark 2 for some $\varepsilon \geq 0$. Therefore,

$$[x,y] \subset \mathcal{O}_{c_1+\varepsilon}(A) \subset \mathcal{O}_{c_1+\varepsilon+c_2}(B)$$

implying that B is $(c_1+c_2+\varepsilon)$-quasiconvex.

2) A subset Q of the group G (or of the Cayley graph $\Gamma(G,\mathcal{A})$) is quasidense if and only if there exists $\alpha \geq 0$ such that for every $x \in G$ (or $\Gamma(G,\mathcal{A})$) the distance $d(x,Q) = \inf\{d(x,y) \mid y \in Q\}$ is at most α, i.e. $G \subseteq \O_\alpha(Q)$.

Indeed, if $G = Qg_1 \cup Qg_2 \cup \ldots \cup Qg_n$, where $g_i \in G$, $i = 1,2,\ldots,n$, Denote $\alpha = max\{|g_i|_G \ : \ 1 \leq i \leq n\}$. Then for any $x \in G$, there are $i \in \{1,\ldots,n\}$ and $y \in Q$ with $x = yg_i$, hence $d(y,x) = |g_i|_G \leq \alpha$.

For demonstrating the sufficiency, let $\{g_1, g_2, \ldots, g_n\}$ be the set of all elements in G of length at most α. Then for every $x \in G$ there exists $y \in Q$ with $d(y,x) = |y^{-1}x|_G \leq \alpha$; hence, $y^{-1}x = g_i$ for some $i \in \{1,2,\ldots,n\}$. Thus, $x = yg_i \in Qg_i$.

3) A quasidense subset $Q \subseteq G$ is quasiconvex.

This is an immediate consequence of the part 2).

3 Preliminaries

Assume $\big(X, d(\cdot,\cdot)\big)$ is a proper geodesic metric space. If $Q \subset X$, $N \geq 0$, the closed N-neighborhood of Q will be denoted by

$$\mathcal{O}_N(Q) \overset{def}{=} \{x \in X \mid d(x,Q) \leq N\} \ .$$

If $x,y,w \in X$, then the number

$$(x|y)_w \overset{def}{=} \frac{1}{2}\Big(d(x,w) + d(y,w) - d(x,y)\Big)$$

is called the *Gromov product* of x and y with respect to w.

Let abc be a geodesic triangle in the space X and $[a,b]$, $[b,c]$, $[a,c]$ be its sides between the corresponding vertices. There exist "special" points $O_a \in [b,c]$, $O_b \in [a,c]$, $O_c \in [a,b]$ with the properties: $d(a,O_b) = d(a,O_c) = \alpha$, $d(b,O_a) = = d(b,O_c) = \beta$, $d(c,O_a) = d(c,O_b) = \gamma$. From a corresponding system of linear equations one can find that $\alpha = (b|c)_a$, $\beta = (a|c)_b$, $\gamma = (a|b)_c$. Two points $O \in [a,b]$ and $O' \in [a,c]$ are called *a-equidistant* if $d(a,O) = d(a,O') \leq \alpha$. The triangle abc is said to be δ*-thin* if for any two points O, O' lying on its sides and equidistant from one of its vertices, $d(O,O') \leq \delta$ holds (Figure 0).

A geodesic n-gon in the space X is said to be δ*-slim* if each of its sides belongs to a closed δ-neighborhood of the union of the others.

We assume the following equivalent definitions of hyperbolicity of the space X to be known to the reader (see [4],[14]):

1°. There exists $\delta \geq 0$ such that for any four points $x,y,z,w \in X$ their Gromov products satisfy

$$(x|y)_w \geq min\{(x|z)_w, (y|z)_w\} - \delta \ ;$$

2°. All triangles in X are δ-thin for some $\delta \geq 0$;

3°. All triangles in X are δ-slim for some $\delta \geq 0$.

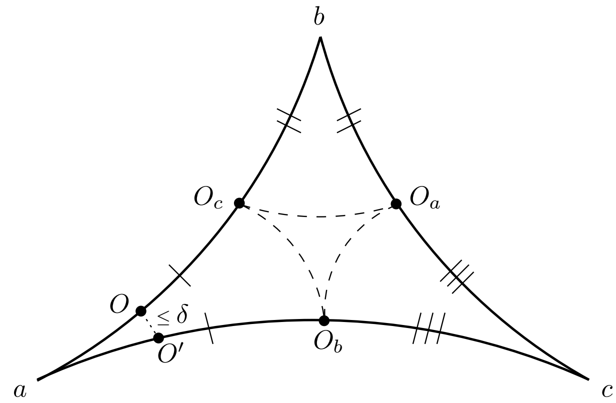

Figure 0

Now, suppose G is finitely generated group with a fixed finite symmetrized generating set $\mathcal{A}$. One can define $d(\cdot,\cdot)$ to be the usual left-invariant metric on the Cayley graph of the group G corresponding to $\mathcal{A}$. Then the Cayley graph $\Gamma(G,\mathcal{A})$ becomes a proper geodesic metric space. G is called hyperbolic if $\Gamma(G,\mathcal{A})$ is a hyperbolic metric space. It is easy to show that this definition does not depend on the choice of the finite generating set $\mathcal{A}$ in G, thus hyperbolicity is a group-theoretical property. It is well known that free groups of finite rank are hyperbolic.

Further on we will assume that $\Gamma(G,\mathcal{A})$ meets $1^\circ, 2^\circ$ and 3° for a fixed (sufficiently large) $\delta \geq 0$.

For any two points $x, y \in \Gamma(G,\mathcal{A})$ we fix a geodesic path between them and denote it by $[x,y]$.

Let p be a path in the Cayley graph of G. Further on by p_-, p_+ we will denote the startpoint and the endpoint of p, by $||p||$ – its length; $lab(p)$, as usual, will mean the word in the alphabet $\mathcal{A}$ written on p. $elem(p) \in G$ will denote the element of the group G represented by the word $lab(p)$.

A path q is called (λ, c)- *quasigeodesic* if there exist $0 < \lambda \leq 1$, $c \geq 0$, such that for any subpath p of q the inequality $\lambda ||p|| - c \leq d(p_-, p_+)$ holds.
In a hyperbolic space quasigeodesics and geodesics with same ends are mutually close :

Lemma 3.1. ([4, 5.6,5.11],[14, 3.3]) *There is a constant $\nu = \nu(\delta, \lambda, c)$ such that for any (λ, c)-quasigeodesic path p in $\Gamma(G,\mathcal{A})$ and a geodesic q with $p_- = q_-$, $p_+ = q_+$, one has $p \subset \mathcal{O}_\nu(q)$ and $q \subset \mathcal{O}_\nu(p)$.*

An important property of cyclic subgroups in a hyperbolic group states

Lemma 3.2. ([4, 8.21],[14, 3.2]) *For any word w representing an element $g \in G$ of infinite order there exist constants $\lambda > 0$, $c \geq 0$, such that any path with a label w^m in the Cayley graph of G is (λ, c)-quasigeodesic for arbitrary integer m.*

In particular, it follows from lemmas 3.1 and 3.2 that any cyclic subgroup of a hyperbolic group is quasiconvex.

Recall that a group H is called *elementary* if it has a cyclic subgroup $\langle h \rangle$ of finite index. It is known that every element $g \in G$ of infinite order is contained in a unique maximal elementary subgroup $E(g)$ of G ([6],[12]), and

$$E(g) = \{x \in G \mid \exists\, n \in \mathbb{N} \text{ such that } xg^n x^{-1} = g^{\pm n}\}.$$

Let $W_1, W_2, \ldots, W_l$ be words in $\mathcal{A}$ representing elements $g_1, g_2, \ldots, g_l$ of infinite order, where $E(g_i) \neq E(g_j)$ for $i \neq j$. The following lemma will be useful:

Lemma 3.3. ([12, Lemma 2.3]) *There exist $\lambda = \lambda(W_1, W_2, \ldots, W_l) > 0$, $c = c(W_1, W_2, \ldots, W_l) \geq 0$ and $N = N(W_1, W_2, \ldots, W_l) > 0$ such that any path p in the Cayley graph $\Gamma(G, \mathcal{A})$ with label $W_{i_1}^{m_1} W_{i_2}^{m_2} \ldots W_{i_s}^{m_s}$ is (λ, c)-quasigeodesic if $i_k \neq i_{k+1}$ for $k = 1, 2, \ldots, s-1$, and $|m_k| > N$ for $k = 2, 3, \ldots, s-1$ (each i_k belongs to $\{1, \ldots, l\}$).*

If $X_1, X_2, \ldots, X_n$ are points in $\Gamma(G, \mathcal{A})$, the notation $X_1 X_2 \ldots X_n$ will be used for the geodesic n-gon with vertices X_i, $i = 1, \ldots, n$, and sides $[X_i, X_{i+1}]$, $i = 1, 2, \ldots, n-1$, $[X_n, X_0]$. $[X_1, X_2, \ldots, X_n]$ will denote the broken line with these vertices in the corresponding order.

Lemma 3.4. ([11, Lemma 21]) *Let $p = [X_0, X_1, \ldots, X_n]$ be a broken line in $\Gamma(G, \mathcal{A})$ such that $||[X_{i-1}, X_i]|| > C_1$ $\forall\ i = 1, \ldots, n$, and $(X_{i-1}|X_{i+1})_{X_i} \leq C_0$ $\forall\ i = 1, \ldots, n-1$, where $C_0 \geq 14\delta$, $C_1 > 12(C_0 + \delta)$. Then p is contained in the closed $2C_0$-neighborhood $\mathcal{O}_{2C_0}([X_0, X_n])$ of the geodesic segment $[X_0, X_n]$.*

Lemma 3.5. *In the conditions of Lemma 3.4, $\|[X_0, X_n]\| \geq \|p\|/2$.*

Proof. Induction on n. If $n = 1$ the statement is trivial. So, assume $n > 1$. By the induction hypothesis $\|[X_0, X_{n-1}]\| \geq \|q\|/2$ where q is the broken line $[X_0, X_1, \ldots, X_{n-1}]$. It is shown in the proof of [11, Lemma 21] that our conditions imply $(X_0|X_n)_{X_{n-1}} \leq C_0 + \delta$, hence

$$\|[X_0, X_n]\| = \|[X_0, X_{n-1}]\| + \|[X_{n-1}, X_n]\| - 2(X_0|X_n)_{X_{n-1}} \geq$$

$$\geq \|q\|/2 + \|[X_{n-1}, X_n]\|/2 + C_1/2 - 2(C_0 + \delta) \geq \|p\|/2 \ .$$

Q.e.d. $\square$

Lemma 3.6. ([13, Prop. 3]) *Let G be a group generated by a finite set $\mathcal{A}$. Let A, B be subgroups of G quasiconvex with respect to $\mathcal{A}$. Then $A \cap B$ is quasiconvex with respect to $\mathcal{A}$.*

Lemma 3.7. ([1, Prop. 1]) *Let G be a hyperbolic group and H a quasiconvex subgroup of G of infinite index. Then the number of double cosets of G modulo H is infinite.*

Lemma 3.8. ([1, Lemma 10],[5, Lemma 1.3]) *For any integer $m \geq 1$ and numbers $\delta, \varepsilon, C \geq 0$, there exists $A = A(m, \delta, \varepsilon, C) \geq 0$ with the following property.*

Let G be a δ-hyperbolic group with a generating set containing at most m elements and H a ε-quasiconvex subgroup of G. Let $g_1, \ldots, g_n, s$ be elements of G such that

(i) *cosets Hg_i and Hg_j are different for $i \neq j$;*

(ii) *g_n is a shortest representative of the coset Hg_n;*

(iii) *$|g_i|_G \leq |g_n|_G$ for $1 \leq i < n$;*

(iv) *for $i \neq n$, all the products $g_i g_n^{-1}$ belong to the same double coset HsH with $|s|_G \leq C$.*

Then $n \leq A = A(m, \delta, \varepsilon, C)$.

Lemma 3.9. ([4, 8.3.36]) *Any infinite subgroup of a hyperbolic group contains an element of infinite order.*

Let H be a subgroup of G. Inheriting the terminology from [5] we will say that the elements $\{g_i \mid 1 \leq i \leq n\}$ of G are *essentially distinct* (relatively to H) if $Hg_i \neq Hg_j$ for $i \neq j$. Conjugates $g_i^{-1}Hg_i$ of H in this case are called *essentially distinct conjugates*.

Definition. ([5, Def. 0.3]) The *width* of an infinite subgroup H in G is n if there exists a collection of essentially distinct conjugates of H such that the intersection of any two elements of the collection is infinite and n is maximal possible. The width of a finite subgroup is defined to be 0.

Lemma 3.10. ([5, Main Thm.]) *A quasiconvex subgroup of a hyperbolic group has a finite width.*

Lemma 3.11. ([9, Cor. 1,Lemma 2.1]) *In a hyperbolic group a finite union of quasiconvex products is a quasiconvex set.*

Lemma 3.12. ([9, Thm. 1]) *Suppose $G_1, \ldots, G_n$, $H_1, \ldots, H_m$ are quasiconvex subgroups of the group G, $n, m \in \mathbb{N}$; $f, e \in G$. Then there exist numbers $r, t_1, \ldots, t_r \in \mathbb{N} \cup \{0\}$ and $f_l, \alpha_{lk}, \beta_{lk} \in G$, $k = 1, 2, \ldots, t_l$ (for every fixed l), $l = 1, 2, \ldots, r$, such that*

$$fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_m = \bigcup_{l=1}^{r} f_l S_l$$

where for each l, $t = t_l$, there are indices $1 \leq i_1 \leq i_2 \leq \ldots \leq i_t \leq n$, $1 \leq j_1 \leq \leq j_2 \leq \ldots \leq j_t \leq m$:

$$S_l = (G_{i_1}^{\alpha_{l1}} \cap H_{j_1}^{\beta_{l1}}) \cdot \ldots \cdot (G_{i_t}^{\alpha_{lt}} \cap H_{j_t}^{\beta_{lt}}).$$

Remark 4. Observe that arbitrary quasiconvex product $f_0G_1f_1G_2 \cdot \ldots \cdot G_nf_n$ is equal to a "transformed" product $fG_1'G_2' \cdot \ldots \cdot G_n'$ (which appears in the formulation of Lemma 3.12) where $G_i' = (f_i \cdot \ldots \cdot f_n)^{-1}G_i(f_i \cdot \ldots \cdot f_n)$, $i = 1, \ldots, n$, are quasiconvex subgroups of G by Remark 2 and $f = f_0f_1 \cdot \ldots \cdot f_n \in G$.

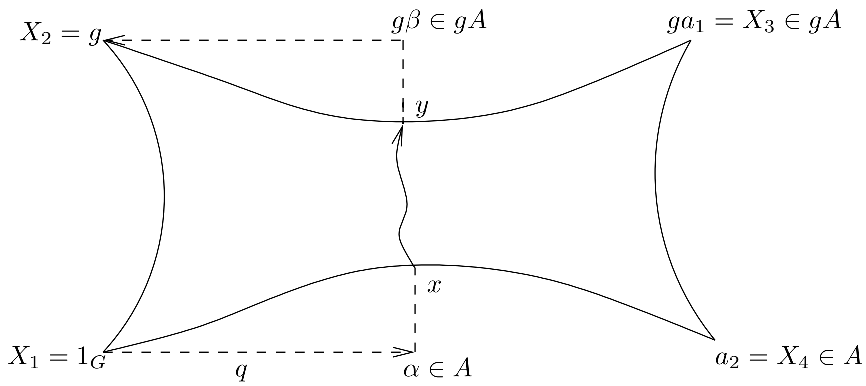

Figure 1

4 Properties of Quasiconvex Subgroups of Infinite Index

Lemma 4.1. *Consider a geodesic quadrangle $X_1X_2X_3X_4$ in $\Gamma(G, \mathcal{A})$ with $d(X_2, X_3) > d(X_1, X_2) + d(X_3, X_4)$. Then there are points $U, V \in [X_2, X_3]$ such that $d(X_2, U) \le d(X_1, X_2)$, $d(V, X_3) \le d(X_3, X_4)$ and the geodesic subsegment $[U, V]$ of $[X_2, X_3]$ lies 2δ-close to the side $[X_1, X_4]$.*

Proof. Since $(X_1|X_3)_{X_2} \le d(X_1, X_2)$ and $(X_1|X_4)_{X_3} \le d(X_3, X_4)$, one can choose $U, V \in [X_2, X_3]$ satisfying $d(X_2, U) = (X_1|X_3)_{X_2}$, $d(X_3, V) = (X_1|X_4)_{X_3}$. The triangle $X_1X_3X_2$ is δ-thin, therefore, after taking $V' \in [X_1, X_3]$ at distance $d(X_3, V)$ from X_3, one obtains $[U, V] \subset \O_\delta([X_1, V'])$. Finally, since V' is the special point of triangle $X_1X_3X_4$ by construction, $[X_1, V']$ is in the closed δ-neighborhood of the side $[X_1, X_4]$, and thus, $[U, V] \subset \O_{2\delta}([X_1, X_4])$. $\square$

Lemma 4.2. *Let A be an infinite ε-quasiconvex set in G and $g \in G$. Then if the intersection $A \cap gAg^{-1}$ is infinite, there exists an element $r \in G$ with $|r|_G \le 4\delta + 2\varepsilon + 2\varkappa$ such that $g \in ArA^{-1}$, where $\varkappa$ is the length of a shortest element from A.*

Proof. Note, at first, that for every $a \in A$ the geodesic segment $[1_G, a]$ belongs to a closed $(\delta + \varepsilon + \varkappa)$-neighborhood of A in $\Gamma(G, \mathcal{A})$. Indeed, pick $b \in A$ with $d(1_G, b) = |b|_G = \varkappa$ and consider the geodesic triangle 1_Gab. Using δ-hyperbolicity of the Cayley graph one achieves

$$[1_G, a] \subset \O_\delta([a, b] \cup [1_G, b]) \subset \O_{\delta+\varkappa}([a, b]) \subset \O_{\delta+\varkappa+\varepsilon}(A) \ .$$

By the conditions of the lemma there is an element $a_1 \in A$ such that $ga_1g^{-1} = a_2 \in A$ and $|a_1|_G > 2|g|_G$. Set $X_1 = 1_G$, $X_2 = g$, $X_3 = ga_1$,$X_4 = a_2$ (Fig. 1). Then $d(X_2, X_3) = |a_1|_G$, $d(X_1, X_2) = |g|_G = |a_2^{-1}ga_1|_G = d(X_3, X_4)$ and in the geodesic quadrangle $X_1X_2X_3X_4$ one has $d(X_2, X_3) > d(X_1, X_2) + d(X_3, X_4)$ and, so, by lemma 1 there exist $x \in [X_1, X_4]$, $y \in [X_2, X_3]$ with $d(x, y) \le 2\delta$. As we showed above, $[1_G, a_i] \subset \O_{\delta+\varkappa+\varepsilon}(A)$ for $i = 1, 2$, hence there is $\alpha \in A$ such that $d(\alpha, x) \leq \delta + \varepsilon + \varkappa$. A left shift is an isometry of $\Gamma(G, \mathcal{A})$, thus, $[X_2, X_3] = [g, ga_1] \subset \O_{\delta+\varkappa+\varepsilon}(gA)$ and we can obtain an element $\beta \in A$ such that $d(y, g\beta) \leq \delta + \varepsilon + \varkappa$.

Consider the broken line $q = [X_1, \alpha, g\beta, g]$ in $\Gamma(G, \mathcal{A})$; then $elem(q) = g$ in G. $d(\alpha, g\beta) \leq 4\delta + 2\varepsilon + 2\varkappa$ by construction, hence we achieved $g = elem(q) = \alpha \cdot r \cdot \beta^{-1}$ where $r = elem([\alpha, g\beta])$, $|r|_G \leq d(\alpha, g\beta) \leq 4\delta + 2\varepsilon + 2\varkappa$. □

For the case when A is a quasiconvex subgroup of a hyperbolic group G, Lemma 4.2 was proved in [5, Lemma 1.2].

Corollary 2. *Suppose H is a quasiconvex subgroup of infinite index in a hyperbolic group G. Then H contains no infinite normal subgroups of G.*

Proof. Indeed, assume $N \trianglelefteq G$ and $N \subset H$. By Lemma 3.7, there is a double coset HrH, $r \in G$, with the length of a shortest representative greater than $(4\delta + 2\varepsilon)$ (ε is the quasiconvexity constant of H). Thus, according to the Lemma 4.2, $N \subset H \cap rHr^{-1}$ is finite. □

Lemma 4.3. *Let G be a δ-hyperbolic group, H and K – its subgroups with H quasiconvex. If* $K \subset \bigcup_{j=1}^{N} Hs_jH$ *for some $s_1, \ldots, s_N \in G$ then $K \preceq H$, i.e. $|K : (K \cap H)| < \infty$.*

Proof. By contradiction, assume $K = \bigsqcup_{i=1}^{\infty} (K \cap H)x_i$ – disjoint union of right cosets with $x_i \in K$ for all $i \in \mathbb{N}$. For every i choose a shortest representative g_i of the coset Hx_i in G. Then for arbitrary $i \neq k$, $Hg_i = Hx_i \neq Hx_k = Hg_k$ and $x_i x_k^{-1} \in Hs_jH$ for some $j \in \{1, 2, \ldots, N\}$, $j = j(i, k)$, hence $g_i g_k^{-1} \in Hs_jH$.

Let A_j be the constants corresponding to Hs_jH, $j = 1, \ldots, N$, from Lemma 3.8. Pick a natural number $n > \sum_{j=1}^{N} A_j$ and consider $g_1, g_2, \ldots, g_n$. Without loss of generality, assume $|g_n|_G \geq |g_i|_G$ for $1 \leq i < n$.

By the choice of n, there exits $l \in \{1, \ldots, N\}$ such that

$$card\left\{i \in \{1, 2, \ldots, n-1\} \mid g_i g_n^{-1} \in Hs_lH\right\} \geq A_l \ .$$

This leads to a contradiction to Lemma 3.8. Q.e.d. □

Lemma 4.4. *Suppose H is a quasiconvex subgroup of a hyperbolic group G and K is a subgroup of G with $|K : (K \cap H)| = \infty$. Then there is an element $x \in K$ of infinite order such that the intersection $\langle x \rangle_\infty \cap H$ is trivial.*

Proof. Observe that our conditions imply that K is infinite. If $K \cap H$ is finite, the statement follows from Lemma 3.9 applied to K.

So, suppose $K \cap H$ is infinite; hence there is an element $y \in K \cap H$ of infinite order. $|K : (K \cap H)| = \infty$, therefore, applying lemmas 4.3 and 4.2 one obtains $g \in K$ such that $H \cap gHg^{-1}$ is a finite subgroup of H. Thus $x = gyg^{-1} \in gHg^{-1}$ satisfies the needed property (x is an element of K because $g, y \in K$). □

Lemma 4.5. *Let G be a δ-hyperbolic group with respect to some finite generating set $\mathcal{A}$, and let H_i be ε_i-quasiconvex subgroups of G, $i = 1, 2$. If one has $\sup\{(h_1|h_2)_{1_G} \ : \ h_1 \in H_1, h_2 \in H_2\} = \infty$ then $card(H_1 \cap H_2) = \infty$.*

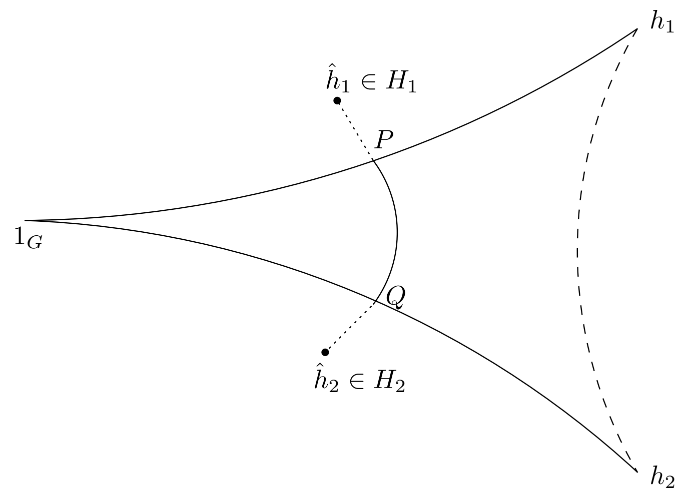

Figure 2

<u>Proof.</u> Define a finite subset Δ of G by $\Delta = \{g \in H_1H_2 \ : \ |g|_G \le \delta+\varepsilon_1+\varepsilon_2\}$. For each $g \in \Delta$ pick a pair $(x, y) \in H_1 \times H_2$ such that $x^{-1}y = g$, and let $\Omega \subset H_1 \times H_2$ denote the (finite) set of the chosen pairs. Define Ω_1 to be the projection of Ω on H_1, i.e. $\Omega_1 = \{x \in H_1 \mid \exists\ y \in H_2 : (x, y) \in \Omega\}$.

By construction, $card(\Omega_1) < \infty$, and thus, $D \stackrel{def}{=} max\{|h|_G \ : \ h \in \Omega_1\} < \infty$.

Assume $h_1 \in H_1$ and $h_2 \in H_2$ and consider the geodesic triangle $1_Gh_1h_2$ in $\Gamma(G, \mathcal{A})$ (Figure 2). Let P, Q be its "special" points on the sides $[1_G, h_1]$ and $[1_G, h_2]$ correspondingly. Since the triangles in $\Gamma(G, \mathcal{A})$ are δ-thin and H_i are ε_i-quasiconvex, $i = 1, 2$, we have $d(P, Q) \le \delta$, and there exist $\hat{h}_i \in H_i$, $i = 1, 2$, with $d(\hat{h}_1, P) \le \varepsilon_1$, $d(\hat{h}_2, Q) \le \varepsilon_2$. By the triangle inequality $d(\hat{h}_1, \hat{h}_2) = |\hat{h}_1^{-1}\hat{h}_2|_G \le \delta + \varepsilon_1 + \varepsilon_2$, therefore, there is a pair $(x, y) \in \Omega$ such that $\hat{h}_1^{-1}\hat{h}_2 = x^{-1}y$, and so

$$\hat{h}_1x^{-1} = \hat{h}_2y^{-1} \in H_1 \cap H_2 \ . \tag{1}$$

From our construction it also follows that

$$|\hat{h}_1x^{-1}|_G \ge |\hat{h}_1|_G - |x|_G \ge d(1_G, P) - \varepsilon_1 - D = (h_1|h_2)_{1_G} - \varepsilon_1 - D \ . \tag{2}$$

Hence, if $\sup\{(h_1|h_2)_{1_G} \ : \ h_1 \in H_1, h_2 \in H_2\} = \infty$ then, as we see from (1) and (2), $H_1 \cap H_2$ has elements of arbitrary large lengths, and thus, it is infinite. $\square$

Suppose $H_1, H_2, \ldots, H_s$ are quasiconvex subgroups of the group G, and K is a subgroup of G satisfying $|K : (K \cap H_j)| = \infty$, $j = 1, 2, \ldots, s$.

Lemma 4.6. *There exists an element of infinite order $x \in K$ with the property $\langle x\rangle_\infty \cap (H_1 \cup H_2 \cup \ldots \cup H_s) = \{1_G\}$.*

Proof. By Lemma 4.4 for every $i = 1, 2, \ldots, s$ there is an element $\tilde{x}_i \in K$ of infinite order such that $\tilde{x}_i^n \notin H_i$ for every $n \in \mathbb{Z}\backslash\{0\}$. If for some $i, j \in \{1, \ldots, s\}$, $i \neq j$, one has $E(\tilde{x}_i) = E(\tilde{x}_j)$, then $\langle \tilde{x}_i \rangle \cap \langle \tilde{x}_j \rangle$ is non-trivial, therefore one can remove $\tilde{x}_j$ from the collection $\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_s\}$ because in this case one has $\langle \tilde{x}_i \rangle_\infty \cap H_j = \{1_G\}$ (since any two non-trivial subgroups of $\mathbb{Z}$ intersect non-trivially). Thus, after performing this procedure a finite number of times, we will obtain a collection of elements of infinite order $\{x_1, x_2, \ldots, x_r\} \subset G$, $r \leq s$, satisfying the properties: $E(x_i) \neq E(x_j)$ for $i \neq j$, and for every $k = 1, 2, \ldots, s$ there is $i_k \in \{1, \ldots, r\}$ such that $\langle x_{i_k} \rangle_\infty \cap H_k = \{1_G\}$.

Now, for each $i = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, s$, define

$$\alpha_{ik} = \begin{cases} min\{m \in \mathbb{N} \mid x_i^m \in H_k\}, & \text{if } \langle x_i \rangle \cap H_k \neq \{1_G\} \\ 0, & \text{otherwise} \end{cases}.$$

Denote

$$\alpha_i = \begin{cases} 1, & \text{if } \alpha_{ik} = 0 \text{ for every } k = 1, \ldots, s \\ l.c.m.\{\alpha_{ik} \mid 1 \leq k \leq s, \alpha_{ik} > 0\}, & \text{otherwise} \end{cases}.$$

$\alpha_i \in \mathbb{N}$, so one can set $y_i = x_i^{\alpha_i}$, $i = 1, 2, \ldots, r$. Then for any distinct $i, j \in \{1, \ldots, r\}$, and any $k = 1, \ldots, s$, $E(x_i) = E(y_i) \neq E(y_j) = E(x_j)$, and

(i) either $y_i \in H_k$ or $\langle y_i \rangle_\infty \cap H_k = \{1_G\}$ (by construction, the latter holds for $i = i_k$).

At last, for every natural n we define $z_n = y_1^n y_2^n \cdot \ldots \cdot y_r^n \in K$.

Assume, by contradiction, that for each $n \in N$ there exits $l = l(n) \in \mathbb{N}$ such that $z_n^l \in H_1 \cup \ldots \cup H_s$ (this obviously holds if z_n has a finite order in G). Then there is an index $k_0 \in \{1, 2, \ldots, s\}$ such that $(z_n)^{l(n)} \in H_{k_0}$ for infinitely many $n \in \mathbb{N}$. Without loss of generality, assume $k_0 = 1$.

Let $w_1, w_2, \ldots, w_r$ be words over the alphabet $\mathcal{A}$ representing the elements $y_1, y_2, \ldots, y_r$ correspondingly. We apply Lemma 3.3 to obtain $\lambda > 0$, $c \geq 0$ and $N > 0$ (depending on $w_1, w_2, \ldots, w_r$) such that any path p in the Cayley graph $\Gamma(G, \mathcal{A})$ with label $(w_1^n w_2^n \ldots w_r^n)^{l(n)}$ is (λ, c)-quasigeodesic if $n > N$. Let $\nu = \nu(\delta, \lambda, c)$ be the constant from Lemma 3.1, and ε be the quasiconvexity constant of H_1.

Let p_n be the path in $\Gamma(G, \mathcal{A})$ starting at 1_G labelled by $(w_1^n w_2^n \ldots w_r^n)^{l(n)}$, $n \in \mathbb{N}$, $n > N$. p_n ends at the element $z_n^{l(n)}$ which belongs to H_1 for infinitely many n. Then $p_n \subset \O_{\nu+\varepsilon}(H_1)$ for infinitely many n.

Denote $t = min\{i \mid 1 \leq i \leq r, y_i \notin H_1\}$ (such t exists by construction of y_i). By definition, $y_1^n y_2^n \cdot \ldots \cdot y_t^n$ lies on p, therefore, $d(y_1^n y_2^n \cdot \ldots \cdot y_t^n, H_1) \leq \nu + \varepsilon$ for infinitely many n, i.e. for those n one can find elements $a_n \in G$ such that $y_1^n y_2^n \cdot \ldots \cdot y_t^n a_n \in H_1$ and $|a_n|_G \leq \nu + \varepsilon$. Remark that $y_1, \ldots, y_{t-1} \in H_1$ by definition, hence $y_t^n a_n = h_n \in H_1$ for infinitely many $n \in \mathbb{N}$.

Finally, since one has

$$(y_t^n | h_n)_{1_G} = 1/2(|y_t^n|_G + |h_n|_G - |a_n|_G) \geq 1/2(|y_t^n|_G - \nu - \varepsilon)$$

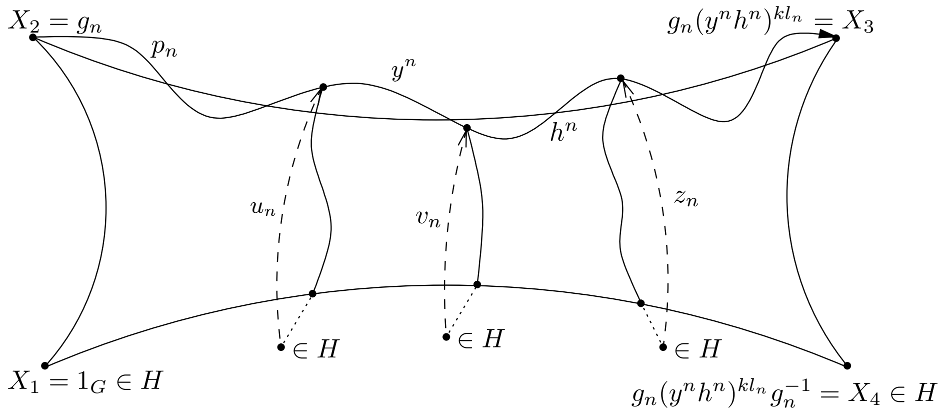

Figure 3

for infinitely many n, we apply Lemma 3.2 to achieve $\sup\{(y_t^n|h_n)_{1_G}\} = \infty$. Hence, by Lemma 4.5, $card(\langle y_t\rangle_\infty \cap H_1) = \infty$, thus, from (i) it follows that $y_t \in H_1$ – a contradiction.

Therefore, there exist $n \in \mathbb{N}$ such that $z_n^m \notin H_1 \cup \ldots \cup H_s$ for every $m \in N$, consequently, $x = z_n \in K$ has an infinite order and $\langle x\rangle_\infty \cap (H_1 \cup \ldots \cup H_s) = \{1_G\}$. The proof of the lemma is finished. □

Proposition 1. *Suppose H is a quasiconvex subgroup of a hyperbolic group G and K is any subgroup of G that satisfies $|K : (K \cap H^g)| = \infty$ for all $g \in G$. Then there exists an element $x \in K$ having infinite order, such that $\langle x\rangle_\infty \cap H^G = \{1_G\}$.*

Proof of Proposition 1. Observe that the subgroup K is infinite because $|K : (K \cap H)| = \infty$, thus applying Lemma 3.9 we obtain an element $h \in K$ of infinite order.

Assume, by contradiction, that for every $x \in K$ there exist $l = l(x) \in \mathbb{N}$ and $g = g(x) \in G$ such that $gx^lg^{-1} \in H$. In particular, $h^{l_0} \in g_0^{-1}Hg_0$ for some $l_0 \in \mathbb{N}$, $g_0 \in G$.

Take an arbitrary $y \in K$ of infinite order. If $E(y) = E(h)$ then $y^m \in \langle h\rangle$ for some $m \in \mathbb{N}$, hence $y^{ml_0} \in g_0^{-1}Hg_0$.

If $E(y) \neq E(h)$, choose words w_1, w_2 over the alphabet $\mathcal{A}$ representing y and h. Then by Lemma 3.3 there exist $\lambda = \lambda(w_1, w_2) > 0$, $c = c(w_1, w_2) \geq 0$ and $N = N(w_1, w_2) > 0$ such that any path p in the Cayley graph $\Gamma(G, \mathcal{A})$ with label $(w_1^n w_2^n)^l$ is (λ, c)-quasigeodesic if $n > N$, for every $l \in \mathbb{N}$.

Let $\nu = \nu(\delta, \lambda, c)$ be the constant from Lemma 3.1 and let ε denote the quasiconvexity constant for H.

By our assumption for every $n \in \mathbb{N}$, $n > N$, there are $l_n \in \mathbb{N}$ and $g_n \in G$ satisfying $g_n(y^nh^n)^{l_n}g_n^{-1} \in H$, then $g_n(y^nh^n)^{kl_n}g_n^{-1} \in H \ \forall\ k \in \mathbb{N}$. Consider a path p_n in $\Gamma(G, \mathcal{A})$ with $(p_n)_- = g_n$ and $lab(p_n) \equiv (w_1^n w_2^n)^{kl_n}$ and a geodesic quadrangle $X_1X_2X_3X_4$ in $\Gamma(G,\mathcal{A})$ where $X_1 = 1_G$, $X_2 = g_n$, $X_3 = g_n(y^n h^n)^{kl_n}$, $X_4 = g_n(y^n h^n)^{kl_n} g_n^{-1} \in H$.

Now, if $n > N$, the path p_n is (λ, c)-quasigeodesic and ν-close to $[X_2, X_3]$, therefore, applying Lemma 4.1, we conclude that for sufficiently large k (compared to ν and $|g_n|_G$) there is a subpath q_n of p_n labelled by $w_1^n w_2^n$ which lies in the closed $C = C(\delta, \nu)$-neighborhood of $[X_1, X_4]$, consequently, $q_n \subset \text{Ø}_{C+\varepsilon}(H)$. Note that C depends only on δ, y, h but does not depend on n, k and g_n.

Hence, for each natural $n > N$ we obtained elements $u_n, v_n, z_n \in G$ with $|u_n|_G, |v_n|_G, |z_n|_G \leq C + \varepsilon$ and $u_n y^n v_n^{-1} \in H$, $v_n h^n z_n^{-1} \in H$ (see Figure 3). There are infinitely many such n and only finitely many possible u_n, v_n, z_n, hence for some indices $i, j \in \mathbb{N}$, $i < j$, one will have $u_i = u_j$, $v_i = v_j$, $z_i = z_j$. Thus, $u_i y^i v_i^{-1} \in H$, $u_i y^j v_i^{-1} \in H$, implying $v_i y^{j-i} v_i^{-1} \in H$. Similarly, $v_i h^{j-i} v_i^{-1} \in H$.

Thus for each $y \in K$ of infinite order we found an element $a = a(y) \in G$ and $l = l(y) \in \mathbb{N}$ satisfying $a y^l a^{-1} \in H$ and $a h^l a^{-1} \in H$ (in the case when $E(y) = E(h)$, $a = g_0$, $l = ml_0$).

This implies that for arbitrary $y_1, y_2 \in K$ with $o(y_i) = \infty$, $i = 1, 2$, we have $a_i = a(y_i) \in G$ and $t_i = t(y_i) \in \mathbb{N}$ such that $y_i^{t_i} \in a_i^{-1} H a_i$ and $h^{t_i} \in a_i^{-1} H a_i$, $i = 1, 2$. Therefore, $h^{t_1 t_2} \in a_1^{-1} H a_1 \cap a_2^{-1} H a_2$, and, thus, this intersection is infinite (because this subgroup contains an element of infinite order). But by Lemma 3.10 there can be only finitely many of such conjugates of H in G, therefore, there are $a_1, a_2, \ldots, a_s \in G$ such that for every element $y \in G$ of infinite order one has

$$y^l \in \bigcup_{i=1}^{s} a_i^{-1} H a_i$$

for some $l \in \mathbb{N}$, which contradicts to Lemma 4.6 because of Remark 2 and the conditions of the proposition.

Hence there is an element $x \in K$ such that for any $l \in \mathbb{N}$, $x^l \notin H^G$. It follows that x has infinite order and $\langle x \rangle_\infty \cap H^G = \{1_G\}$. $\square$

5 Proofs of Theorems 1,2

Proof of Theorem1. The direction (b) $\Rightarrow$ (a) is trivial, so let's focus on the direction (a) $\Rightarrow$ (b).

Using Proposition 1 for every $i = 1, 2, \ldots, s$ one finds $\tilde{x}_i \in K$ such that $o(\tilde{x}_i) = \infty$ and $\langle \tilde{x}_i \rangle_\infty \cap H_i^G = \{1_G\}$. Now as in the proof of the Lemma 4.6 we narrow down this collection to $\{x_1, x_2, \ldots, x_r\}$ satisfying the properties: $E(x_i) \neq E(x_j)$ for $i \neq j$, and for every $k = 1, 2, \ldots, s$ there is $i_k \in \{1, 2 \ldots, r\}$ such that $\langle x_{i_k} \rangle_\infty \cap H_k^G = \{1_G\}$.

Assuming the contrary of the statement, for every natural number n and $z_n = x_1^n x_2^n \cdot \ldots \cdot x_r^n \in K$ we obtain $l_n \in \mathbb{N}$ such that $z_n^{l_n} \in H_1^G \cup \ldots \cup H_s^G$. Thus, there is an index $k_0 \in \{1, 2, \ldots, s\}$ such that $z_n^{l_n} \in H_{k_0}^G$ for every $n \in \Delta$ where Δ is an infinite subset of $\mathbb{N}$. Without loss of generality, assume $k_0 = 1$.

Thus, for every $n \in \Delta$ there exists $g_n \in G$ such that $g_n z_n^{l_n} g_n^{-1} \in H_1$, hence $g_n z_n^{kl_n} g_n^{-1} \in H_1$ for any $k \in \mathbb{N}$. Now, as in the proof of the Proposition 1, we take words $w_1, w_2, \ldots, w_r$ representing $x_1, x_2, \ldots, x_r$ and apply Lemma 3.3 to the path p_n in $\Gamma(G,\mathcal{A})$ starting at g_n and labelled by $(w_1^n w_2^n \dots w_r^n)^{kl_n}$. Hence, for every sufficiently large $n \in \Delta$ and $k \in \mathbb{N}$ we find a subpath q_n of p_n labelled by $w_1^n \dots w_r^n$ which is $(C+\varepsilon)$-close to H_1 where $C = C(\delta, w_1, \dots, w_r)$ (but is independent of n,k and g_n) and ε is the quasiconvexity constant of H_1.

Again, similarly to the proof of Proposition 1, for each $i = 1, 2, \dots, r$ we obtain an element $v_i \in G$ and $m_i \in \mathbb{N}$ such that $v_i x_i^{m_i} v_i^{-1} \in H_1$. In particular that should hold for $i = i_1$. The contradiction achieved finishes the proof of the theorem. □

Proof of Theorem2. Assume the contrary. Then using Theorem1 we obtain an element $x \in K$, $o(x) = \infty$, such that $\langle x \rangle_\infty \cap (H_1^G \cup H_2^G \cup \dots \cup H_s^G)$ is trivial, hence $\langle x \rangle^G \cap H_j^G = \{1_G\}$ for every $j = 1, 2, \dots, s$.

By the conditions $U = \bigcup_{k=1}^q P_k$ where each P_k is a quasiconvex product, $k = 1, \dots, q$. Since any cyclic subgroup is quasiconvex in G (by lemmas 3.2,3.1), and in view of Remark 4, application of the Lemma 3.12 to the intersection

$$\langle x \rangle = \langle x \rangle \cap U = \bigcup_{k=1}^{q} \big(\langle x \rangle \cap P_k \big)$$

shows that it is finite (because each product S_l will be trivial). A contradiction with $o(x) = \infty$. Hence, the theorem is proved. □

Corollary 3. *Let U be a finite union of quasiconvex products having infinite index in a hyperbolic group G. Then U is not quasidense but its complement $U^{(c)} = G \backslash U$ is quasidense in G.*

Proof. Let $g_1, g_2, \dots, g_n \in G$. It is easy see that the sets $\bigcup_{i=1}^n U g_i$ and $U^{-1}U$ are finite unions of quasiconvex products of infinite index.

Thus, the fact that $U \subset G$ is not quasidense follows from the definition of a quasidense subset and Theorem2 applied to the case when $K = G$.

By Theorem2, there exists $y \in G$ such that $y \notin U^{-1}U$. Consider an arbitrary $x \in G$. If $x \in U$ then $xy \in U^{(c)}$ (because of the choice of y) hence

$$G \subseteq U^{(c)} \cup U^{(c)} y^{-1} \ . \ \square$$

6 Boundary and Limit Sets

Let X be a proper geodesic metric space with metric $d(\cdot,\cdot)$. Assume also that X is δ-hyperbolic for some $\delta \geq 0$. Further in this paper we will need the construction of Gromov *boundary* ∂X for the space X (for more detailed theory the reader is referred to the corresponding chapters in [4],[2]). The points of the boundary are equivalence classes of geodesic rays $r : [0, \infty) \to X$ where rays r_1, r_2 are equivalent if $\sup\{d(r_1(t), r_2(t))\} < \infty$ ($\Leftrightarrow$ $h(r_1, r_2) < \infty$ – Hausdorff distance between the images of these rays).

For another definition of the boundary, fix a basepoint $p \in X$. A sequence $(x_i)_{i \in \mathbb{N}} \subset X$ is called *converging to infinity* if

$$\lim_{i,j \to \infty} (x_i | x_j)_p = \infty \ .$$

Two sequences $(x_i)_{i\in\mathbb{N}}, (y_j)_{i\in\mathbb{N}}$ converging to infinity are said to be equivalent if

$$\lim_{i\to\infty}(x_i|y_i)_p = \infty .$$

The points of the boundary ∂X are identified with the equivalence classes of sequences converging to infinity. It is easy to see that this definition does not depend on the choice of a basepoint. If α is the equivalence class of $(x_i)_{i\in\mathbb{N}}$ we will write $\lim\limits_{i\to\infty} x_i = \alpha$.

It is known that the two objects defined above are homeomorphic through the map sending a geodesic ray $r : [0,\infty) \to X$ into the sequence $\big(r(i)\big)_{i\in\mathbb{N}}$.

For any two distinct points $\alpha, \beta \in \partial X$ there exists at least one bi-infinite geodesic $r : (-\infty, +\infty) \to X$ such that $\lim\limits_{i\to\infty} r(-i) = \alpha$ and $\lim\limits_{i\to\infty} r(i) = \beta$. We will say that this geodesic joins α and β; it will be denoted (α, β).

The spaces ∂X and $X \cup \partial X$ can be topologized so that they become compact and Hausdorff (see [14],[4]).

Every isometry ψ of the space X induces a homeomorphism of ∂X in a natural way: for every equivalence class of geodesic rays $[r] \in \partial X$ choose a representative $r : [0,\infty) \to X$ and set $\psi([r]) = [\psi \circ r]$.

For a subset $A \subseteq X$ the *limit set* $\Lambda(A)$ of A is the collection of the points $\alpha \in \partial X$ that are limits of the sequences from A.

Let Ω be a subset of ∂X containing at least two distinct points. We define the *convex hull* $CH(\Omega)$ of Ω to be the set of all points in X lying on bi-infinite geodesics that join elements from Ω.

Below we list some known properties of limit sets and convex hulls:

Lemma 6.1. ([8, Lemmas 3.2,3.6]) *Let Ω be an arbitrary subset of ∂X having at least two elements. Then*

(a) $CH(\Omega)$ is ε-quasiconvex where $\varepsilon \geq 0$ depends only on δ;

(b) If the subset Ω is closed then $\Lambda\left(CH(\Omega)\right) = \Omega$.

In this paper our main interest concerns hyperbolic groups, so further we will assume that the space X is the Cayley graph $\Gamma(G, \mathcal{A})$ of some δ-hyperbolic group G with a fixed finite generating set $\mathcal{A}$. Because of the natural embedding of G (as a metric subspace) into $\Gamma(G, \mathcal{A})$, we will identify subsets of G with subsets of its Cayley graph. The boundary ∂G, by definition, coincides with the boundary of $\Gamma(G, \mathcal{A})$.

Left multiplication by elements of the group induces the isometric action of G on $\Gamma(G, \mathcal{A})$. Hence, G acts homeomorphically on the boundary ∂G as described above.

If $g \in G$ is an element of infinite order in G then the sequences $(g^i)_{i\in\mathbb{N}}$ and $(g^{-i})_{i\in\mathbb{N}}$ converge to infinity and we will use the notation

$$\lim_{i\to\infty} g^i = g^{\infty},\ \ \lim_{i\to\infty} g^{-i} = g^{-\infty} .$$

Lemma 6.2. ([8],[15]) *Suppose A, B are arbitrary subsets of G, $g \in G$. Then*

(a) $\Lambda(A) = \emptyset$ if and only if A is finite;

(b) $\Lambda(A)$ *is a closed subset of the boundary* ∂G*;*
(c) $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$*;*
(d) $\Lambda(Ag) = \Lambda(A)$*,* $g \circ \Lambda(A) = \Lambda(gA)$*;*
(e) If $A \preceq B$ *then* $\Lambda(A) \subseteq \Lambda(B)$*. Hence,* $A \approx B$ *implies* $\Lambda(A) = \Lambda(B)$*.*

(b),(c) and (d) are easy consequences of the definition and (a) is obtained after a standard application of Ascoli theorem; (e) follows from (c) and (d).

If H is a subgroup of G, it is known that ΛH is either empty (if H is finite) or consists of two distinct points (if H is infinite elementary) or is uncountable (if H is non-elementary). In the second case, i.e. when there exists $g \in H$ such that $o(g) = \infty$ and $|H : \langle g \rangle| < \infty$, $\Lambda H = \{g^{\infty}, g^{-\infty}\}$.

Lemma 6.3. ([8, Lemma 3.3]) *If* H *is an infinite subgroup of* G *then* $\Lambda(H)$ *contains at least two distinct points and the sets* $\Lambda(H)$*,* $CH\big(\Lambda(H)\big)$ *are* H*-invariant, i.e. for every* $h \in H \quad h \circ \Lambda(H) = \Lambda(H)$*,* $h \cdot CH\big(\Lambda(H)\big) = CH\big(\Lambda(H)\big)$*.*

As the hyperbolic group G acts on its boundary, for every subset $\Omega \subset \partial G$ one can define the stabilizer subgroup by $St_G(\Omega) = \{g \in G \mid g \circ \Omega = \Omega\}$. For our convenience, we set $St_G(\emptyset) = G$.

It is proved in [4, thm. 8.3.30] that for any point $\alpha \in \partial G \quad St_G(\{\alpha\})$ is an elementary subgroup of the group G (in fact, if $\alpha = g^{\infty}$ for some element of infinite order $g \in G$ then

$$St_G(\{\alpha\}) = E^+(g) \stackrel{def}{=} \{x \in G \mid \exists\, n \in \mathbb{N} \text{ such that } xg^n x^{-1} = g^n\} \leq E(g);$$

otherwise the subgroup $St_G(\{\alpha\})$ is finite). In addition, if $g \in G$, $o(g) = \infty$, then $St_G(\{g^{\infty}, g^{-\infty}\}) = E(g)$.

Remark 5. For an arbitrary subset A of G $Comm_G(A) \subseteq St_G\big(\Lambda(A)\big)$.

Indeed, if $g \in Comm_G(A)$, then $gA \approx A$, hence after applying claims (d),(e) of Lemma 6.2, we obtain $g \circ \Lambda(A) = \Lambda(gA) = \Lambda(A)$, i.e. $g \in St_G\big(\Lambda(A)\big)$.

Remark 6. Suppose $\Omega \subseteq \partial G$ has at least two distinct points. Denote by $cl(\Omega) \subseteq \partial G$ the closure of Ω in the topology of the group boundary. Then $\Lambda\left(CH(\Omega)\right) = cl(\Omega)$.

Indeed, since $CH(\Omega) \subseteq CH\left(cl(\Omega)\right)$ we obtain

$$\Lambda\left(CH(\Omega)\right) \subseteq \Lambda\left(CH\big(cl(\Omega)\big)\right) = cl(\Omega)\ ,$$

where the last equality is achieved using Lemma 6.1. Finally, $\Lambda\left(CH(\Omega)\right)$ is a closed subset of ∂G containing Ω (by part (b) of Lemma 6.2), which implies statement of the Remark 6.

The following lemma (in a slightly different form) can be found in [15, Cor. to Lemma 13]:

Lemma 6.4. *Suppose* $\Omega \subset \partial G$ *is a subset having at least two distinct points. Then* $\Lambda\big(St_G(\Omega)\big) \subseteq cl(\Omega)$*.*

Proof. Since Ω has at least two points, it makes sense to consider the convex hull $CH(\Omega)$. Observe that for any $g \in St_G(\Omega)$, $gCH(\Omega) \subseteq CH(\Omega)$: the left translation by the element $g \in G$ is an isometry of $\Gamma(G, \mathcal{A})$, therefore a bi-infinite geodesic (α, β), $\alpha, \beta \in \Omega$ goes to a bi-infinite geodesic $(g \circ \alpha, g \circ \beta) \subset CH(\Omega)$ since Ω is $St_G(\Omega)$-invariant.

Fix any point $x \in CH(\Omega)$. By our observation above, $St_G(\Omega)x \subset CH(\Omega)$, hence $\Lambda\big(St_G(\Omega)x\big) \subset \Lambda\big(CH(\Omega)\big)$. The claim of the lemma now follows by applying Lemma 6.2.(d) and Remark 6. $\square$

7 Known Results and Examples

In this section we list some known results about the boundary and limit sets and give (counter)examples in order to motivate the rest of the paper where we extend these results to larger classes of subsets.

Result 1: if A and B are quasiconvex subgroups of a hyperbolic group G then $A \approx B$ if and only if $\Lambda(A) = \Lambda(B)$.

Indeed, the necessity follows by Lemma 6.2.(e). For proving the sufficiency we note that by [15, thm. 8] $\Lambda(A \cap B) = \Lambda(A) \cap \Lambda(B) = \Lambda(A) = \Lambda(B)$. But $A \cap B \leq A$ and $A \cap B \leq B$, so, by Lemma 3.6 and [15, thm. 4]

$$|A : (A \cap B)| < \infty,\ |B : (A \cap B)| < \infty,$$

i.e. the subgroups A and B are commensurable. Hence $A \approx B$.

However, if one removes at least one of the conditions on A and B, the claim of the result 1 fails:

Example 1. Let $G = F(x, y)$ – the free group with two free generators x, y. Define $A = \{x^n \mid n \geq 0\}$, $B = \{x^n y^m \mid 0 \leq m \leq n\}$. These are quasiconvex subsets (not subgroups) of G because any prefix of an element from one of these sets is still contained in the same set. Evidently, $\Lambda(A) = \{x^\infty\}$.
Suppose $(x^{n_i} y^{m_i})_{i \in \mathbb{N}}$, $0 \leq m_i \leq n_i$, $i \in \mathbb{N}$, is a sequence converging to infinity in B. If the sequence of integers $(n_i)_{i \in \mathbb{N}}$ is bounded then the sequence $(m_i)_{i \in \mathbb{N}}$ is also bounded, hence the set the group of elements in $(x^{n_i} y^{m_i})_{i \in \mathbb{N}}$ is finite which contradicts to the definition of a sequence that converges to infinity. Thus, $\sup_{i \in \mathbb{N}}\{n_i\} = \infty$ and, passing to a subsequence, we can assume $\lim_{i \to \infty} n_i = \infty$. Then

$$(x^{n_i} y^{m_i} | x^{n_i})_{1_G} = n_i \to \infty \text{ as } i \to \infty\ ,$$

thus, $\lim_{i \to \infty} (x^{n_i} y^{m_i}) = \lim_{i \to \infty} x^{n_i} = x^\infty$. Therefore, $\Lambda(B) = \{x^\infty\} = \Lambda(A)$ but $A \not\approx B$.

Example 2. If G is an arbitrary hyperbolic group and H is its infinite normal subgroup of infinite index, then H is not quasiconvex by Corollary 2 and $\Lambda(H) = \Lambda(G) = \partial G$ by [8, Lemma 3.8.(2)], thus the quasiconvexity of A, B in result 1 is important.

Result 2: If A is a quasiconvex subgroup of a hyperbolic group G then we have an equality in the Remark 5: $Comm_G(A) = St_G\big(\Lambda(A)\big)$.

By [15, thm. 17] or [8, cor. 3.10], $VN_G(A) = St_G\big(\Lambda(A)\big)$ and $VN_G(A) = Comm_G(A)$ by Remark 1.

Again, both of the conditions of A being a subgroup and A being quasiconvex are not redundant:

Example 3. Choose $G = F(x, y)$ as in example 1 and let

$$B = \{x^n y^m \mid 0 \leq m \leq n^2, n \geq 0\},\ C = \{x^{-n} \mid n \in \mathbb{N}\},\ A = B \cup C\ .$$

A is quasiconvex since B and C are, $\Lambda(A) = \Lambda(B) \cup \Lambda(C) = \{x^{\infty}, x^{-\infty}\}$ ($\Lambda B = \{x^{\infty}\}$ by a similar argument to the one presented in example 1). Then $St_G\big(\Lambda(A)\big) = \langle x \rangle$ – the infinite cyclic subgroup generated by x.
Let's show that $Comm_G(A) = \{1_G\}$. By Remark 5 and since $Comm_G(A)$ is a subgroup, it is enough to prove that $x^{-k} \notin Comm_G(A)$ for any integer $k > 0$. Indeed, for any $n > k \quad x^{n-k}y^{n^2} \in x^{-k}A$ and

$$d\left(x^{n-k}y^{n^2}, A\right) = d\left(x^{n-k}y^{n^2}, x^{n-k}y^{(n-k)^2}\right) = n^2 - (n-k)^2 = 2nk - k^2 \to \infty$$

when $n \to \infty$. Implying that $x^{-k}A \not\approx A$.

Example 4. Consider a finitely generated group M containing a normal subgroup $N \lhd M$ and an infinite subnormal subgroup $K \lhd N$ such that $|M : N| = \infty$, $|N : K| = \infty$ and for any $x \in M \backslash N \quad xKx^{-1} \cap K = \{1_M\}$ (for example, one can take $M = \mathbb{Z}\ wr\ \mathbb{Z}$).
Then M is isomorphic to a quotient of some free group G of finite rank by its normal subgroup H: $M \cong G/H$. Let $\phi : G \to G/H$ be the natural homomorphism and $A, B \leq G$ be the preimages of K and N under ϕ correspondingly. Then $H \lhd A \lhd B \lhd G$, $|G : A| = \infty$. $\Lambda(A) = \Lambda(B) = \Lambda(G) = \partial G$ by [8, lemma 3.8,(2)], hence $St_G\big(\Lambda(A)\big) = G$.
We claim that $Comm_G(A) = B$. As we know $Comm_G(A) = VN_G(A)$, therefore $B \subset Comm_G(A)$. Now, for an arbitrary $g \in G \backslash B$, by construction, one has $\phi(A \cap gAg^{-1}) = \{1_M\}$, hence $(A \cap gAg^{-1}) \subset H$. Since K is infinite, we get $|A : H| = \infty$, and thus, $|A : (A \cap A^g)| = \infty$, so, $g \notin Comm_G(A)$.

In this example the subgroup A of G is not quasiconvex and
$|St_G\big(\Lambda(A)\big) : Comm_G(A)| = \infty$.

Result 3: ([1, Thm. 2],[8, Lemma 3.9]) If A is an infinite quasiconvex subgroup of a hyperbolic group G then A has a finite index in its commensurator $Comm_G(A)$.

By Lemma 2.1, the condition $|Comm_G(A) : A| < \infty$ is equivalent to

$$Comm_G(A) \preceq A\ .$$

It is easy to construct an example of a quasiconvex subset (not subgroup) A with exactly one limit point demonstrating that the latter fails, more precisely, $Comm_G(H)$ can have two limit points.

However, in the next section this result will be extended to the class of all quasiconvex subsets A with $card(\Lambda(A)) \geq 2$.

Result 4: Let A be a quasiconvex subgroup of a hyperbolic group G. Then $Comm_G(A)$ is quasiconvex.

If the subgroup A is infinite, this is a consequence of the result 3 by Remark 2. On the other hand, if A is finite, then $Comm_G(A) = G$.

Below we give an example of an infinite quasiconvex set $A \subset G$ such that $Comm_G(A)$ is not quasiconvex.

Example 5. We use the counterexample 12 from [15]. Again, let $G = F(x, y)$ be the free group of rank 2. Let $K = \langle x^n y x^{-n} \mid n \geq 0 \rangle$. It is shown in [15] that $\Lambda(K)$ is not a limit set of a quasiconvex subgroup in G (because $\Lambda(K)$ is not "symmetric": $x^\infty \in \Lambda(K)$ but $x^{-\infty} \notin \Lambda(K)$).
As the subgroup K is infinite, we can consider the convex hull $A = CH\big(\Lambda(K)\big)$. By Lemma 6.1 A is quasiconvex and $\Lambda(A) = \Lambda(K)$ ($\Lambda(K) \subset \partial G$ is closed by the claim (b) of Lemma 6.2). $K \subset Comm_G(K) \subset St_G\big(\Lambda(K)\big)$, hence, as we saw in the proof of Lemma 6.4, A is K-invariant. Consequently, $K \subset Comm_G(A)$. Remark 5 and Lemma 6.4 imply

$$\Lambda(K) \subset \Lambda\big(Comm_G(A)\big) \subset \Lambda\left(St_G\big(\Lambda(A)\big)\right) \subset \Lambda(A) = \Lambda(K) \ .$$

Thus $\Lambda\big(Comm_G(A)\big) = \Lambda\left(St_G\big(\Lambda(A)\big)\right) = \Lambda(K)$, therefore the subgroups $Comm_G(A)$ and $St_G\big(\Lambda(A)\big)$ are not quasiconvex.

In the next section we are going to extend the results 1-3 to a broader class of quasiconvex subsets of the hyperbolic group G. In particular, we will substitute the requirement for A and B to be subgroups by a weaker condition.

8 Tame Subsets

Again, let G be a δ-hyperbolic group with fixed finite generating set $\mathcal{A}$.

Definition. A subset A of the group G will be called *tame* if A has at least two limit points on ∂G and $A \preceq CH\big(\Lambda(A)\big)$. I.e. there exists $\nu \geq 0$ such that $A \subset \mathcal{O}_\nu(C)$ where $C = CH\big(\Lambda(A)\big)$.

In particular, this definition implies that any tame subset is infinite.

Remark 7. If A and D are subsets of G such that $A \approx D$ and A is tame then D is also tame.

Indeed, By Lemma 6.2.(e) $\Lambda(A) = \Lambda(D)$, hence,

$$D \preceq A \preceq CH\big(\Lambda(A)\big) = CH\big(\Lambda(D)\big) \ .$$

Thus "tameness" of a subset is preserved under the equivalence relation "$\approx$".

Lemma 8.1. *Let A, B, C, D be non-empty subsets of the group G where A and B are tame, C is finite and D is arbitrary. Let $H \leq G$ be an infinite subgroup. Then the following sets are tame: 1) $A \cup B$; 2) $A \cup C$; 3)$A \cdot C$; 4) $D \cdot A$; 5) H.*

Proof. 1) Since $\Lambda(A), \Lambda(B) \subset \Lambda(A \cup B)$, we have

$$CH\big(\Lambda(A)\big) \cup CH\big(\Lambda(B)\big) \subseteq CH\big(\Lambda(A \cup B)\big) \ .$$

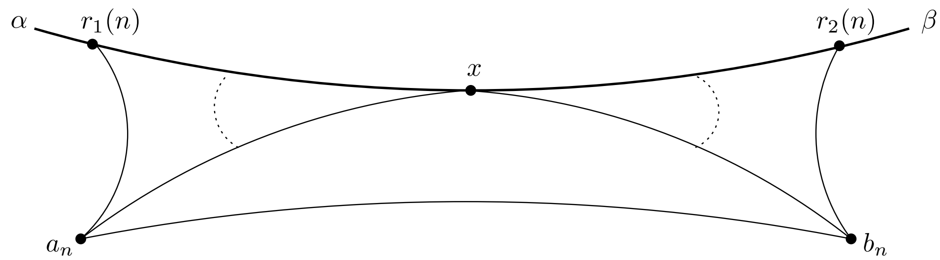

Figure 4

$A \preceq CH\big(\Lambda(A)\big)$ and $B \preceq CH\big(\Lambda(B)\big)$ by conditions of the lemma, hence

$$A \cup B \preceq CH\big(\Lambda(A)\big) \cup CH\big(\Lambda(B)\big) \preceq CH\big(\Lambda(A \cup B)\big)\ ,$$

which shows that $A \cup B$ is tame.

2) and 3) are immediate consequences of the fact that $A \cup C \approx A$, $A \cdot C \approx A$, and Remark 7.

4) Denote $K = CH\big(\Lambda(A)\big)$. By definition, $A \preceq K$, therefore, $DA \preceq DK$. Now, since for every $y \in D$ $yK = CH\big(\Lambda(yA)\big) \subset CH\big(\Lambda(DA)\big)$, we obtain $DK \subset CH\big(\Lambda(DA)\big)$. Hence $DA \preceq CH\big(\Lambda(DA)\big)$.

5) The set $CH\big(\Lambda(H)\big)$ is H-invariant by Lemma 6.3, therefore, for any $x \in CH\big(\Lambda(H)\big)$ we have $Hx \subset CH\big(\Lambda(H)\big)$. But $H \preceq Hx$, hence H is a tame subset. □

In particular, this lemma shows that any infinite set U that is a finite union of quasiconvex products in a hyperbolic group G is tame.

In example 3 from section 7 we constructed a quasiconvex subset A in the group $G = F(x, y)$ with exactly two limit points $x^{\infty}, x^{-\infty}$. Therefore, $CH\big(\Lambda(A)\big)$ consists of one bi-infinite geodesic and $CH\big(\Lambda(A)\big) \cap G = \{x^n \mid n \in \mathbb{Z}\}$. Now, for each $n \in \mathbb{N}$, $x^n y^{n^2} \in A$ and $d\left(x^n y^{n^2}, CH\big(\Lambda(A)\big)\right) = n^2 \to \infty$, as $n \to \infty$. Thus, the subset A from example 3 is not tame.

Lemma 8.2. *Suppose A is a tame subset of a hyperbolic group G and $B \subseteq G$ is a quasiconvex subset such that $\Lambda(A) \subseteq \Lambda(B)$. Then $A \preceq B$.*

Proof. By the conditions of the lemma, $A \preceq CH\big(\Lambda(A)\big) \preceq CH\big(\Lambda(B)\big)$. Therefore, it remains to show that $CH\big(\Lambda(B)\big) \preceq B$, i.e. there exists $\varkappa \geq 0$ such that $CH\big(\Lambda(B)\big) \subset \mathcal{O}_{\varkappa}(B)$.

Let ε be the quasiconvexity constant for B. Consider any $x \in CH\big(\Lambda(B)\big)$. By definition, there exist $\alpha, \beta \in \Lambda(B)$ such that $x \in (\alpha, \beta)$.
Let $r_1, r_2 : [0, \infty) \to \Gamma(G, \mathcal{A})$ be the geodesic half-lines obtained by bisecting (α, β) at the point x. Thus, $r_1(0) = r_2(0) = x$, $\lim\limits_{i \to \infty} r_1(i) = \alpha$, $\lim\limits_{i \to \infty} r_2(i) = \beta$ (see Figure 4).

There are sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ in B converging to infinity such that $\lim\limits_{i\to\infty} a_i = \alpha$, $\lim\limits_{i\to\infty} b_i = \beta$. Hence, $(r_1(i)|a_i)_x \to \infty$, $(r_2(i)|b_i)_x \to \infty$ as $i \to \infty$. Consequently, for some $n \in \mathbb{N}$ we have

$$(r_1(n)|a_n)_x > 2\delta, \quad (r_2(n)|b_n)_x > 2\delta \ . \tag{$*$}$$

Remark 8. Let PQR be a geodesic triangle in $\Gamma(G,\mathcal{A})$ and $(P|Q)_R > 2\delta$. Then $d(R,[P,Q]) > 2\delta$.

Indeed, assume, by contradiction, that there exists $S \in [P,Q]$ satisfying $d(R,S) \le 2\delta$. By definition of the Gromov product,

$$(P|Q)_R = \frac{1}{2}\big(d(P,R)+d(Q,R)-d(P,Q)\big) \le$$

$$\le \frac{1}{2}\big(d(P,S)+d(S,R)+d(Q,S)+d(S,R)-d(P,Q)\big) \ .$$

But $d(P,S)+d(Q,S) = d(P,Q)$ since $[P,Q]$ is a geodesic segment, therefore $(P|Q)_R \le d(S,R) \le 2\delta$. A contradiction.

Consider, now, the geodesic quadrangle in $\Gamma(G,\mathcal{A})$ with vertices a_n, $r_1(n)$, $r_2(n)$, b_n. $x \in [r_1(n), r_2(n)]$ (Fig. 4). Applying $(*)$ and the Remark 8 we obtain

$$d\big(x,[a_n,r_1(n)]\big) > 2\delta, \ \ d\big(x,[b_n,r_2(n)]\big) > 2\delta \ .$$

Since the Cayley graph $\Gamma(G,\mathcal{A})$ is δ-hyperbolic, all quadrangles are 2δ-slim, thus

$$[r_1(n),r_2(n)] \subset \mathcal{O}_{2\delta}\big([a_n,r_1(n)] \cup [b_n,r_2(n)] \cup [a_n,b_n]\big) \ .$$

Consequently, $d(x,[a_n,b_n]) \le 2\delta$. $a_n, b_n \in B$ and B is ε-quasiconvex, therefore $[a_n,b_n] \subset \mathcal{O}_\varepsilon(B)$.

So, $d(x,B) \le 2\delta+\varepsilon$ for every $x \in CH\big(\Lambda(B)\big)$. After denoting $\varkappa = 2\delta+\varepsilon$ we achieve $CH\big(\Lambda(B)\big) \subset \mathcal{O}_\varkappa(B)$. The Lemma 8.2 is proved. □

Corollary 4. *Let $A \preceq B$ be subsets of G where A has at least two limit points on ∂G and B is quasiconvex. Then $Comm_G(A) \preceq B$, $St_G\big(\Lambda(A)\big) \preceq B$.*

Proof. By Remark 5 it is enough to prove the second inequality. Using lemmas 6.4, 6.2 we get

$$\Lambda\Big(St_G\big(\Lambda(A)\big)\Big) \subseteq \Lambda(A) \subseteq \Lambda(B) \ .$$

Since any subgroup is a tame subset (Lemma 8.1), applying Lemma 8.2 we obtain $St_G\big(\Lambda(A)\big) \preceq B$. □

The statement of corollary 2 can be generalized as follows:

Corollary 5. *Let G be a hyperbolic group and let U be a finite union of quasiconvex products of infinite index in G. Suppose $A \subset U$ is an infinite subset. Then $|G : Comm_G(A)| = \infty$.*

Proof. First, notice that since A is infinite, U is also infinite, hence U has an infinite member $H \leq G$ of infinite index. Therefore, G is non-elementary. There are two possibilities: either $card\big(\Lambda(A)\big) \leq 1$ or $card\big(\Lambda(A)\big) \geq 2$.

In the first case $card\big(\Lambda(A)\big) = 1$ (by Lemma 6.2.(a)), so $\Lambda(A) = \{\alpha\} \in \partial G$. Hence $St_G\big(\Lambda(A)\big) = St_G(\{\alpha\})$ is an elementary subgroup, thus $Comm_G(A)$ is also elementary. Consequently, $|G : Comm_G(A)| = \infty$.

In the second case, when $card\big(\Lambda(A)\big) \geq 2$, we can use corollary 4 to obtain $Comm_G(A) \preceq U$. But if $Comm_G(A)$ were quasidense (or, equivalently, $|G : Comm_G(A)| < \infty$), we would have $G \preceq Comm_G(A) \preceq U$, so U would have to be also quasidense. The latter contradicts to the statement of corollary 3. □

Now we are going to extend the results 1,2 to all tame quasiconvex subsets:

Proposition 2. *Suppose A and B are tame quasiconvex subsets of a hyperbolic group G. Then $A \approx B$ if and only if $\Lambda(A) = \Lambda(B)$.*

Proof. The necessity is given by Lemma 6.2.(e); the sufficiency immediately follows from Lemma 8.2. □

Corollary 6. *Let A be a subset of a hyperbolic group G having at least two distinct limit points on ∂G. Then the following two conditions are equivalent:*

1) A is tame and quasiconvex;

2) $A \approx CH\big(\Lambda(A)\big)$.

Proof. Denote $C = CH\big(\Lambda(A)\big)$. By Lemma 6.2.(b) $\Lambda(A)$ is a closed subset of ∂G, hence from Lemma 6.1.(b) we get $\Lambda(A) = \Lambda(C)$. Therefore $CH\big(\Lambda(C)\big) = C$ implying that C is tame. C is quasiconvex by Lemma 6.1.(a).

Now, 2) follows from 1) by Proposition 2.

1) follows from 2) because the equivalence relation "$\approx$" preserves quasiconvexity and "tameness" of a subset. □

Proposition 3. *For any tame quasiconvex subset A of a hyperbolic group G $Comm_G(A) = St_G\big(\Lambda(A)\big)$.*

Proof. By Remark 5 it is enough to show that $St_G\big(\Lambda(A)\big) \subseteq Comm_G(A)$. Take an arbitrary $g \in St_G\big(\Lambda(A)\big)$. Then $\Lambda(gA) = g \circ \Lambda(A) = \Lambda(A)$. The subset gA is tame and quasiconvex since A is so, hence, by Proposition 2, $gA \approx A$. Thus, $g \in Comm_G(A)$. Q.e.d. □

Proposition 4. *Let G be a hyperbolic group and let $A \subset G$ be a quasiconvex subset that has at least two distinct limit points on the boundary ∂G. Then $St_G\big(\Lambda(A)\big) \preceq A$. Consequently, $Comm_G(A) \preceq A$.*

Proof. Denote $H = St_G\big(\Lambda(A)\big)$. Using lemmas 6.2.(b) and 6.4 we obtain $\Lambda(H) \subseteq \Lambda(A)$. $H \leq G$ is a subgroup, hence it is tame (Lemma 8.1); A is quasiconvex by the conditions. The statement of the proposition now follows from Lemma 8.2. □

The Proposition 4 generalizes the result 3 from section 7.

Now, it is easy to see that the set A from example 5 in section 7 is tame and quasiconvex, thus, $Comm_G(A) = St_G\big(\Lambda(A)\big)$. However $Comm_G(A)$ is not quasiconvex. Thus we can not extend the result 4 from section 7 in the same way we did the previous ones.

9 Proof of the Theorem 3

A subset B of the group G will be called *normal* if for every $g \in G \quad B^g = B$.

Lemma 9.1. *Suppose G is a non-elementary hyperbolic group and A is its subset containing an infinite normal subset B. Then A is quasiconvex if and only if it is quasidense in G.*

Proof. The sufficiency is trivial (see part 3) of Remark 3). To show the necessity, first we observe that $Comm_G(B) = G$, hence $St_G\big(\Lambda(B)\big) = G$ by Remark 5. Since B is infinite $\Lambda(B)$ is a non-empty subset of the boundary ∂G, thus there exists $\alpha \in \Lambda(B)$. Now, if $\Lambda(B) = \{\alpha\}$ then, as we know, $G = St_G\big(\Lambda(B)\big) = St_G(\{\alpha\})$ is an elementary subgroup of G, but G is non-elementary. Therefore $\Lambda(B)$ has at least two distinct points. Now we can apply Lemma 6.4 to obtain

$$\partial G = \Lambda(G) = \Lambda\left(St_G\big(\Lambda(B)\big)\right) \subseteq \Lambda(B) \subseteq \partial G$$

and conclude that $\Lambda(B) = \partial G$. $B \subset A$, thus $\Lambda(A) = \partial G$. Finally, since G is a tame subset of itself (by Lemma 8.1) and A is quasiconvex, we apply Lemma 8.2 and achieve $G \preceq A$. $\square$

Proposition 5. *Let G be a hyperbolic group, $K, H_1, \ldots, H_s$ be its subgroups, where K is non-elementary and H_j are ε_j-quasiconvex for all $j = 1, 2, \ldots, s$. If $K \preceq \bigcup_{j=1}^{s} H_j^G$ then for some $k \in \{1, \ldots, s\}$ and $g \in G$ one has $K \preceq H_k^g$, i.e. the index $|K : (K \cap H_k^g)|$ is finite.*

Proof. Denote $\varepsilon = max\{\varepsilon_1, \ldots, \varepsilon_s\}$ and assume the contrary to the statement we need to prove. Then by Theorem1 there exists an element $x \in K$ of infinite order satisfying $\langle x\rangle \cap \bigcup_{j=1}^{s} H_j^G = \{1_G\}$. Now, since K is non-elementary we can apply Proposition 1 to obtain an element $y \in K$ with $o(y) = \infty$ such that $\langle x\rangle^G \cap \langle y\rangle$ is trivial. Consequently, $\langle x\rangle \cap \langle y\rangle^G = \{1_G\}$.

By the conditions of the proposition

$$K \subset \bigcup_{i=1}^{r} \left(\bigcup_{j=1}^{s} H_j^G \right) b_i = \bigcup_{i=1}^{r} \left(\bigcup_{j=1}^{s} H_j^G b_i \right)$$

for some $b_1, b_2, \ldots, b_r \in G$. For every $n \in \mathbb{N} \quad x^n y^n \in K$, hence there exist $j_0 \in \{1, \ldots, s\}$ and $i_0 \in \{1, \ldots, r\}$ such that $x^n y^n \in (H_{j_0})^G \cdot b_{i_0}$ for infinitely many $n \in \mathbb{N}$. Without loss of generality, assume $j_0 = 1$, $i_0 = 1$.

Let w_1, w_2, w_3 be words in alphabet $\mathcal{A}$ representing elements x, y, b_1^{-1} correspondingly.

Observe that the word $(w_1^n w_2^n w_3)^l$ represents the element $(x^n y^n b_1^{-1})^l$. For infinitely many $n \in \mathbb{N}$, $n \geq N$, there exists $g_n \in G$ such that $x^n y^n b_1^{-1} \in g_n H_1 g_n^{-1}$, thus, $(x^n y^n b_1^{-1})^l \in g_n H_1 g_n^{-1}$ for all $l \in \mathbb{N}$.

For any such n and l, define the points Y_j in $\Gamma(G, \mathcal{A})$, $j = 0, 1, \ldots, 2l$, as follows: $Y_0 = g_n$, $Y_1 = g_n x^n$, $Y_2 = g_n x^n y^n b_1^{-1}$, $Y_3 = g_n x^n y^n b_1^{-1} x^n$, $Y_4 = g_n (x^n y^n b_1^{-1})^2$, $\ldots$, $Y_{2l-1} = g_n (x^n y^n b_1^{-1})^{l-1} x^n$, $Y_{2l} = g_n (x^n y^n b_1^{-1})^l$. Consider the path q starting at Y_0 and labelled by the word $(w_1^n w_2^n w_3)^l$. Thus, q ends at Y_{2l} and passes through each Y_j, $j = 1, \ldots, 2l-1$. By lemmas 3.2 and 3.1 applied to the segment of q between Y_j and Y_{j+1} and the geodesic $[Y_j, Y_{j+1}]$, for each $j = 0, \ldots, 2l-1$, there exists a constant $\nu = \nu(w_1, w_2, w_3) \geq 0$ such that $q \subset \mathcal{O}_\nu([Y_0, \ldots, Y_{2l}])$.

By the choice of y and Lemma 4.5 there exists $\varkappa = \varkappa(x, y, b_1) \geq 0$ such that $(x^{-n}|y^n)_{1_G} \leq \varkappa$ and $(b_1 y^{-n} b_1^{-1}|x^n)_{1_G} \leq \varkappa$ for all n.

Denote $C_0 = \varkappa + |b_1|_G$. Then, for any odd $j \in \{1, 2, \ldots, 2l-1\}$,

$$(Y_{j-1}|Y_{j+1})_{Y_j} = (x^{-n}|y^n b_1^{-1})_{1_G} \leq (x^{-n}|y^n)_{1_G} + |b_1|_G \leq C_0.$$

And for any even $j \in \{1, 2, \ldots, 2l-1\}$,

$$(Y_{j-1}|Y_{j+1})_{Y_j} = (b_1 y^{-n}|x^n)_{1_G} \leq (b_1 y^{-n} b_1^{-1}|x^n)_{1_G} + |b_1|_G \leq C_0.$$

Choose an arbitrary $C_1 > 12(C_0 + \delta)$. Lemma 3.2 implies that for any sufficiently large n, $\|[Y_j, Y_{j+1}]\| > C_1$, $j = 1, 2, \ldots, 2l-1$, hence, we can use lemmas 3.4 and 3.5 to obtain

$$[Y_0, \ldots, Y_{2l}] \subset \mathcal{O}_{2C_0}([Y_0, Y_{2l}]) \text{ and } \|[Y_0, Y_{2l}]\| \geq \|[Y_0, \ldots, Y_{2l}]\|/2 \geq \frac{2lC_0}{2} .$$

Let k be an even number from the set $\{l-1, l\}$. Then the subpath p of q between Y_k and Y_{k+1} is labelled by the word w_1^n and $p \subset \mathcal{O}_\nu([Y_k, Y_{k+1}])$. As we showed, there are points $U, V \in [Y_0, Y_{2l}]$ such that $d(Y_k, U) \leq 2C_0$ and $d(Y_{k+1}, V) \leq 2C_0$. Since the quadrangles in a δ-hyperbolic space are 2δ-slim, we obtain $[Y_k, Y_{k+1}] \subset \mathcal{O}_{2C_0+2\delta}([U, V])$. Applying lemmas lemmas 3.4 and 3.5 to $[Y_0, \ldots, Y_k]$ and $[Y_{k+1}, \ldots, Y_{2l}]$ we achieve

$$d(Y_0, U) \geq d(Y_0, Y_k) - 2C_0 \geq \frac{\|[Y_0, \ldots, Y_k]\|}{2} - 2C_0 \geq \frac{C_1(l-1)}{2} - 2C_0 \text{ and}$$

$$d(V, Y_{2l}) \geq d(Y_{k+1}, Y_{2l}) - 2C_0 \geq \frac{\|[Y_{k+1}, \ldots, Y_{2l}]\|}{2} - 2C_0 \geq \frac{C_1 l}{2} - 2C_0 .$$

Hence, if $l \in \mathbb{N}$ is sufficiently large, we will have $d(Y_0, Y_{2l}) > 2|g_n|_G$, $d(Y_0, U) > |g_n|_G$ and $d(V, Y_{2l}) > |g_n|_G$ (and, similarly, $d(U, Y_{2l}) > |g_n|_G$, $d(V, Y_0) > |g_n|_G$).

Let $X_1 = 1_G$, $X_2 = g_n (x^n y^n b_1^{-1})^l g_n^{-1} \in H_1$ for infinitely many $n \in \mathbb{N}$. By Lemma 4.1 applied to the geodesic quadrangle $X_0 Y_0 Y_{2l} X_1$, the subsegment $[U, V]$ of $[Y_0, Y_{2l}]$ lies in the 2δ-neighborhood of $[X_0, X_1]$ and $[X_0, X_1] \subset \mathcal{O}_\varepsilon(H_1)$

(for infinitely many n and sufficiently large l). Accumulating all of the above, we obtain

$$p \subset \mathcal{O}_{\nu+2C_0+2\delta+\varepsilon}(H_1)$$

(for infinitely many $n \in \mathbb{N}$ and sufficiently large $l = l(n) \in \mathbb{N}$). As in the proof of Proposition 1, the latter implies that for some $t \in \mathbb{N}$ we have $x^t \in H_1^G$ – a contradiction to the construction of x.

The proposition is proved. □

Proof of Theorem3. It is given that $A = a_1^G \cup a_2^G \ldots \cup a_s^G$ for some elements $a_1, \ldots, a_s \in G$.

Suppose that A is quasiconvex. A is infinite, therefore by Lemma 9.1 the subset A is quasidense in G, hence

$$G \preceq A \subset \bigcup_{j=1}^{s} \langle a_j \rangle^G .$$

But since G is non-elementary, $|G : \langle a_j \rangle^g| = \infty$ for each $j = 1, 2, \ldots, s$, and for any $g \in G$. Thus, we achieve a contradiction with Proposition 5 applied to $K = G$. Q.e.d. □

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