

# A SCHLICHTNESS THEOREM FOR ENVELOPES OF HOLOMORPHY

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ABSTRACT. Let  $\Omega$  be a domain in  $\mathbb{C}^2$ . We prove the following theorem. If the envelope of holomorphy of  $\Omega$  is schlicht over  $\Omega$ , then the envelope is in fact schlicht. We provide examples showing that the conclusion of the theorem does not hold in  $\mathbb{C}^n$ ,  $n > 2$ . Additionally, we show that the theorem cannot be generalized to provide information about domains in  $\mathbb{C}^2$  whose envelopes are multiply sheeted.

## 1. INTRODUCTION

A distinct difference between complex analysis in one variable and in several variables is the existence in higher dimension of domains which are not Stein. We are led to consider the envelope of holomorphy of such a domain: the smallest Stein Riemann domain over  $\mathbb{C}^n$  to which all functions holomorphic on our domain extend holomorphically. (See, for example [Gun90, Chapter H].)

In the case of domains in  $\mathbb{C}^n$  with relatively simple structure, the envelope of holomorphy can, at least in principle, be calculated explicitly. Examples of such domains are Reinhardt, Hartogs, tubular and contractible domains. These classes of domains enjoy the distinct advantage that their envelopes are schlicht; that is, their envelopes are domains in  $\mathbb{C}^n$ .

In general, however, the envelope of a domain need not be schlicht. It may be a finitely or infinitely sheeted Riemann domain spread over  $\mathbb{C}^n$ . It is not obvious, given a description of a domain, whether its envelope is schlicht or multiply sheeted. Nor is it entirely clear how the geometry of a given domain relates to the geometry of its envelope. In fact, we have the following open question.

**Problem 1.1.** Does there exist a bounded domain,  $\Omega$ , in  $\mathbb{C}^n$ ,  $n \geq 2$ , such that  $\Omega$  has smooth boundary, and the envelope of  $\Omega$  is infinitely sheeted?

There are classical examples which show that there exist bounded domains whose envelopes are infinitely sheeted. Unfortunately, these domains have boundaries which are highly nonsmooth. (Such an example can be constructed, e.g., by nesting infinitely many Hartogs figures one inside the next, and carefully connecting them.)

An obvious problem suggested by the above discussion is that of determining why and when the envelope is schlicht or multiply sheeted. Can we, just by looking at a domain, determine whether its envelope is schlicht? Is there a way of controlling or understanding the number of sheets in the envelope of a domain? While we are unable to fully answer these questions, we present a theorem which indicates that

in certain situations there are criteria which allow us to better understand whether envelopes are schlicht or not.

Specifically, we prove the following.

**Theorem 1.2.** Let  $\Omega$  be a domain in  $\mathbb{C}^2$ , and let  $(\tilde{\Omega}, \pi)$  be its envelope of holomorphy. If  $\pi$  is injective on  $\pi^{-1}(\Omega)$ , then  $\pi$  is injective on  $\tilde{\Omega}$ . In other words, if  $\tilde{\Omega}$  is schlicht over  $\Omega$ , then  $\tilde{\Omega}$  is schlicht.

The theorem indicates that for domains in  $\mathbb{C}^2$  there may be a relationship between the number of sheets in the envelope lying above the domain, and the number of sheets in the envelope. As we shall see in Section 3, there appears to be a relationship only in the special case of domains whose envelope is schlicht over the domain. We shall also see in Section 3 that there is no such relationship in higher dimensions. Further examples in this vein can be found in [Jup03].

We note that Chirka and Stout proved a weaker result in [CS95]. Specifically, they proved Lemma 2.6. Our methods are different, and better suited to proving and understanding the results in which we are interested.

## 2. A SCHLICHTNESS THEOREM

We prove Theorem 1.2. Our proof proceeds as follows. Using the natural embedding of  $\Omega$  into  $\tilde{\Omega}$ , we are considering  $\Omega$  as a subset of  $\tilde{\Omega}$  rather than as a domain in  $\mathbb{C}^2$ . Let  $\Omega_1$  be the set of points in  $\tilde{\Omega}$  which can be reached from  $\Omega$  by pushing discs. Similarly, let  $\Omega_n$  be the set of points in  $\tilde{\Omega}$  which can be reached from  $\Omega_{n-1}$  by pushing discs. We recall that if two one-dimensional analytic varieties in  $\mathbb{C}^2$  intersect nontrivially, then so do slight perturbations of these varieties. We proceed inductively, showing that  $\pi$  is injective on  $\Omega_n$  for each  $n$ . In other words, we show that for each  $n$ ,  $\Omega_n$  can be identified with a domain in  $\mathbb{C}^2$ . To do so we push discs from  $\Omega_{n-1}$ , keeping track of their intersections. Using the above fact about analytic varieties, we show that if  $\Omega_n$  is not schlicht, then it is not schlicht over  $\Omega_k$ ,  $k \leq n - 1$ . In particular  $\Omega_n$  is not schlicht over  $\Omega$ , contradicting the hypothesis that  $\tilde{\Omega}$  is schlicht over  $\Omega$ .

We make several definitions which we shall need in the proof of the theorem.

**Definition 2.1** (Pushing Discs). We say that a point,  $p \in \tilde{\Omega}$ , can be reached from  $\Omega$  by pushing discs if there is a neighbourhood  $U$  of  $p$  in  $\tilde{\Omega}$  such that  $\pi|_U$  is a biholomorphism, and such that the following holds: there is a biholomorphism,  $F$ , of  $\Delta^2(0, 1)$  into  $U$  such that

- (1)  $p \in F(\Delta^2(0, 1))$ , and
- (2)  $F(H) \subset\subset U \cap \Omega$ ,

where  $H$  is the Hartogs figure,

$$H = \left( \Delta(0, 1) \times \left\{ \frac{1}{2} < |w| < 1 \right\} \right) \cup \left( \Delta\left(0, \frac{1}{2}\right) \times \Delta(0, 1) \right).$$

**Remark 2.2.** Several observations and remarks about disc pushing should be made.

- (1) We insist that our neighbourhoods,  $U$ , are biholomorphic to balls in  $\mathbb{C}^2$ , in particular  $\pi(U)$  is a ball.
- (2) The extension of a holomorphic function from  $F(H)$  to  $F(\Delta^2(0, 1))$  is single valued.
- (3) Given a point,  $p \in \tilde{\Omega}$ , which can be reached from  $\Omega$  by pushing discs as above, we can assume (by rotation and scaling in the first coordinate, if necessary) that  $p$  is in  $F(C)$ , where

$$C = [0, 1] \times \Delta(0, 1).$$

In other words, we have a continuous family of holomorphic discs in  $\tilde{\Omega}$  such that the “bottom” disc and the boundaries of the discs lie in  $\Omega$ , while the “top” disc is not contained in  $\Omega$ .

**Definition 2.3** ( $\Omega_n$ ). We inductively define the sets  $\Omega_n$ . Let

$$\Omega_0 = \Omega.$$

Let

$$\Omega_{n+1} = \{ \text{all points in } \tilde{\Omega} \text{ which can be reached from } \Omega_n \text{ by pushing discs } \}.$$

**Remark 2.4.** We make several observations about  $\Omega_n$ .

- (1) It is clear from the construction of  $\Omega_n$  that these sets are open subsets of  $\tilde{\Omega}$ .
- (2) We have defined  $\Omega_n$  as a subset of  $\tilde{\Omega}$ , and when building  $\Omega_n$  we push discs within  $\tilde{\Omega}$ . However, if  $\Omega_{n-1}$  is in fact a domain in  $\mathbb{C}^2$ , we can also view disc pushing as follows. We have

$$\pi(F(\Delta^2(0, 1))) \subset \mathbb{C}^2, \text{ where } \pi(F(H)) \subset \pi(\Omega_{n-1}) = \Omega_{n-1}.$$

In other words we push discs from  $\Omega_{n-1}$ , considering  $\Omega_{n-1}$  as a subset of  $\mathbb{C}^2$ . We then lift these discs to  $\Omega_n \subset \tilde{\Omega}$ . By the identity principle for liftings ([JP00, Proposition 1.1.5]) the liftings are unique.

We next observe that we can use  $\Omega_n$  to recover  $\tilde{\Omega}$ .

**Proposition 2.5.** Let  $\Omega' = \cup_{n=0}^{\infty} \Omega_n$ . Then  $\Omega' = \tilde{\Omega}$ .

*Proof.* Assume not, and let  $p \in \tilde{\Omega}$  be a point in  $\partial\Omega'$ . We claim that there is a neighbourhood  $U$  of  $p$  in  $\tilde{\Omega}$  such that  $U \cap \Omega'$  is pseudoconvex.

In fact, let  $U$  be a neighbourhood of  $p$  such that  $\pi|_U$  is a biholomorphism, and further choose  $U$  so that  $\pi(U)$  is a ball. If  $U \cap \Omega'$  is not pseudoconvex, then we can find a biholomorphism,  $F$ , of  $\Delta^2(0, 1)$  into  $U$  such that

- (1)  $F(H) \subset\subset U \cap \Omega'$ , and
- (2)  $F(\Delta^2(0, 1))$  contains points which are not in  $\Omega'$ .

This follows from the fact that  $\pi|_U$  is a biholomorphism, so that we can consider  $U \cap \Omega'$  as a domain in  $\mathbb{C}^2$ , and from the notion of II-pseudoconvexity as described in [Pf75, Chapter II.2].

Now  $F(H) \subset\subset \Omega'$ , so in fact  $F(H) \subset\subset \Omega_n$ , for some  $n$ . This implies that  $\Omega_{n+1}$  contains points which are not in  $\Omega'$ . This is a contradiction.

We conclude that every point in  $\partial\Omega'$  has a neighbourhood,  $U$ , such that  $U \cap \Omega'$  is pseudoconvex. By the equivalence of local and global pseudoconvexity of

unbranched Riemann domains over  $\mathbb{C}^n$  (see e.g. [JP00, Corollary 2.2.16]) we see that  $\Omega'$  is pseudoconvex. However, as  $\Omega'$  is a subset of  $\tilde{\Omega}$ , every holomorphic function on  $\Omega$  extends in a single valued fashion to  $\Omega'$ . We conclude that  $\Omega' = \tilde{\Omega}$ .  $\square$   $\square$

To prove Theorem 1.2 we require two lemmas.

**Lemma 2.6.** Assume that  $\Omega_n$  is schlicht. Assume on the other hand that  $\Omega_{n+1}$  is not schlicht. Then  $\Omega_{n+1}$  is not schlicht over  $\Omega_n$ .

Precisely, assume that there are points,  $p_1 \neq p_2$  in  $\Omega_{n+1}$  such that  $\pi(p_1) = \pi(p_2)$ . Then in fact there are points  $q_1 \neq q_2$  in  $\Omega_{n+1}$  such that  $q = \pi(q_1) = \pi(q_2) \in \Omega_n$ .

This lemma says, in particular, that if pushing discs from  $\Omega$  once does not create any sheets over  $\Omega$ , then in fact it does not create any sheets over  $\Omega_1$ . As noted in the Introduction, Chirka and Stout have also proved this lemma [CS95].

**Lemma 2.7.** Assume that  $\Omega_n$  is schlicht. If  $\Omega_{n+1}$  is not schlicht over  $\Omega_k$ ,  $k \leq n$ , then  $\Omega_{n+1}$  is not schlicht over  $\Omega_{k-1}$ .

Once we have these lemmas we prove the theorem as follows.

*Proof of Theorem 1.2.* We inductively show that  $\Omega_n$  is schlicht. By assumption we have that  $\Omega_0$  is schlicht.

If  $\Omega_1$  were not schlicht, then Lemma 2.6 would imply that it was not schlicht over  $\Omega_0$ . This in turn means that  $\tilde{\Omega}$  is not schlicht over  $\Omega_0$ . This is a contradiction, and we conclude that  $\Omega_1$  is schlicht.

Applying Lemma 2.6 to  $\Omega_2$  reveals that if  $\Omega_2$  were not schlicht, then it would not be schlicht over  $\Omega_1$ . Lemma 2.7 now implies that  $\Omega_2$  is not schlicht over  $\Omega_0$ . This again gives a contradiction, and we conclude that  $\Omega_2$  is schlicht.

We proceed similarly for  $\Omega_n$ ,  $n \geq 3$ .  $\square$   $\square$

We now prove Lemmas 2.6 and 2.7.

*Proof of Lemma 2.6.* We begin by choosing two points,  $p_1 \neq p_2$  in  $\Omega_{n+1}$  with  $p = \pi(p_1) = \pi(p_2)$ . By definition and our comments after Definition 2.1, for each point  $p_j$  there is a neighbourhood,  $U_j$ , of  $p_j$  in  $\tilde{\Omega}$ , and a biholomorphism  $F_j$  of  $\Delta^2(0, 1)$  into  $\tilde{\Omega}$  satisfying

- (1)  $p_j \in F_j(C)$ , and
- (2)  $F_j(H) \subset\subset U_j \cap \Omega_n$ ,

recalling that  $C$  is defined as

$$C = [0, 1] \times \Delta(0, 1).$$

We now define a subset  $c$  of  $C$  as

$$c = (\{0\} \times \Delta(0, 1)) \cup ([0, 1] \times \partial\Delta(0, 1)).$$

We have, by assumption, that  $\pi(F_1(C)) \cap \pi(F_2(C))$  is nonempty; in particular it contains  $p$ . We let  $M$  be the connected component of  $\pi(F_1(C)) \cap \pi(F_2(C))$  which contains  $p$ . We note that the  $\pi(F_j(C))$  can be assumed to be in general position, so that their intersection is a two dimensional manifold. (The intersection is a closed set, so strictly speaking the intersection is a manifold with ‘‘ends’’.) By the stability of intersection of one-dimensional analytic manifolds in  $\mathbb{C}^2$ , we conclude that  $M$  contains a point,  $q$ , which lies in  $\pi(F_1(c))$  or in  $\pi(F_2(c))$ . We then have

that  $q = \pi(F_1(z_1))$  for some  $z_1$  in  $c$ , or  $q = \pi(F_2(z_2))$  for some  $z_2$  in  $c$ . In the first case  $q_1 = F_1(z_1)$  is in  $\Omega_n$ , in the second case  $q_2 = F_2(z_2)$  is in  $\Omega_n$ .

We eliminate the possibility that both  $q_1 = F_1(z_1)$  and  $q_2 = F_2(z_2)$  are points in  $\Omega_n$ . Otherwise, since  $\Omega_n$  is schlicht and these two points have the same projection, we must have that  $q_1 = q_2$ . Let  $\gamma$  be a path in  $M$  connecting  $q$  to  $p$ . By the identity principle for liftings, we conclude that  $p_1 = p_2$ . This is a contradiction.

We examine the case where  $z_1 \in c$ , hence  $F_1(z_1)$  is in  $\Omega_n$ . Since  $q_2 = F_2(z_2)$  is not in  $\Omega_n$ ,  $q_1 \neq q_2$ . But  $\pi(q_1) = \pi(q_2)$ . We now conclude that  $\Omega_{n+1}$  is not schlicht over  $\Omega_n$ .

The case where  $q$  is in  $\pi(F_2(c))$  but not in  $\pi(F_1(c))$  proceeds analogously.  $\square$   $\square$

The proof of Lemma 2.7 is similar.

*Proof of Lemma 2.7.* We begin by choosing two points,  $p_1 \neq p_2$ , with  $p_1$  in  $\Omega_{n+1}$  and  $p_2$  in  $\Omega_k$ , and with  $p = \pi(p_1) = \pi(p_2)$ . By definition, for each point  $p_j$  there is a neighbourhood,  $U_j$ , of  $p_j$  in  $\tilde{\Omega}$ , and a biholomorphism  $F_j$  of  $\Delta^2(0, 1)$  into  $\tilde{\Omega}$  satisfying

- (1)  $p_1 \in F_1(C)$ ,  $F_1(H) \subset\subset U_1 \cap \Omega_n$ , and
- (2)  $p_2 \in F_2(C)$ ,  $F_2(H) \subset\subset U_2 \cap \Omega_{k-1}$ .

As in the proof of Lemma 2.6, we have that  $\pi(F_1(C)) \cap \pi(F_2(C))$  is nonempty; in particular it contains  $p$ . We let  $M$  be the connected component of  $\pi(F_1(C)) \cap \pi(F_2(C))$  which contains  $p$ . By the stability of intersection of one-dimensional analytic manifolds in  $\mathbb{C}^2$ , we conclude that  $M$  contains a point,  $q$ , which lies in  $\pi(F_1(c))$  or in  $\pi(F_2(c))$ .

It is not possible that  $F_1(z_1)$  is in  $\Omega_n$ . Indeed, since  $q_2 = F_2(z_2)$  is in  $\Omega_k \subset \Omega_n$ , this would imply that we have  $F_1(z_1) = F_2(z_2)$  by the fact that  $\Omega_n$  is schlicht. As above, the path  $\gamma \subset M$  connecting  $q$  and  $p$  yields the contradiction that  $p_1 = p_2$ .

We thus must have that  $q_1 = F_1(z_1)$  is in  $\Omega_{n+1}$  but not in  $\Omega_n$ , and that  $q_2 = F_2(z_2) \in F_2(c)$  and hence it is in  $\Omega_{k-1}$ . This means that  $\Omega_{n+1}$  is not schlicht over  $\Omega_{k-1}$ . With this, the proof of the lemma is complete.  $\square$   $\square$

**2.1. Remarks.** Theorem 1.2 also holds for two dimensional Riemann domains over  $\mathbb{C}^2$ . The proof proceeds exactly as above.

### 3. COUNTEREXAMPLES

**3.1. Counterexamples in  $\mathbb{C}^2$ .** We construct a domain,  $\Omega$ , in  $\mathbb{C}^2$  such that the envelope of  $\Omega$  has two sheets over  $\Omega$ , but three sheets over  $\mathbb{C}^2$ . This shows that Theorem 1.2 cannot be generalized to give information about domains in  $\mathbb{C}^2$  whose envelopes are multiply sheeted.

The example is obtained as follows. We construct three families of analytic discs whose intersection is a small set. For each family we find a pseudoconvex domain close to the family, making sure that certain boundary points of the new domain are in fact strictly pseudoconvex. We call this new domain a fattening of the family. We then build a frame for the family: a small neighbourhood of the union of the bottom disc of the family and the boundaries of the discs in the family. Pushing discs in this frame gives us the fattened family with which we started. We join the frames with well chosen paths and find pseudoconvex neighbourhoods of the paths.

The domain,  $\Omega$ , is the union of the frames and the paths. The correct choice of paths ensures that our domain has the desired properties.

We begin by constructing three families of discs,

$$\begin{aligned}\Sigma_s &: \Delta(0, 2) \rightarrow \mathbb{C}^2, & s \in (-1, \epsilon_\Sigma), \\ \Delta_t &: \Delta(0, 1/2) \rightarrow \mathbb{C}^2, & t \in (-\epsilon_\Delta, 1), \\ \Gamma_l &: \Delta(0, r_\Gamma) \rightarrow \mathbb{C}^2, & l \in (-\epsilon_\Gamma, \epsilon_\Gamma)\end{aligned}$$

where  $\epsilon_\Sigma$ ,  $\epsilon_\Delta$ ,  $\epsilon_\Gamma$  and  $r_\Gamma$  are small positive real numbers to be carefully chosen, and the maps are defined as

$$\begin{aligned}\Sigma_s(w) &= (s, w), \\ \Delta_t(z) &= (z, t), \\ \Gamma_l(\xi) &= (\xi + i\eta\xi^2, \xi - il),\end{aligned}$$

with  $\eta$  a small positive real number to be carefully chosen.

We shall refer to the families as  $\Sigma$ ,  $\Delta$  and  $\Gamma$ . By the bottom disc of the family  $\Sigma$ ,  $\Delta$  or  $\Gamma$  we mean, respectively, the discs  $\Sigma_{-1}$ ,  $\Delta_1$  or  $\Gamma_{\epsilon_\Gamma}$ . For ease of notation we denote these discs by  $\Sigma_B$ ,  $\Delta_B$  and  $\Gamma_B$  respectively. By the boundary of the family  $\Sigma$ ,  $\Delta$  or  $\Gamma$  we mean, respectively, the family of circles

$$\begin{aligned}\Sigma_s|_{\partial\Delta(0,2)}, & \quad s \in (-1, \epsilon_\Sigma), \\ \Delta_t|_{\partial\Delta(0,1/2)}, & \quad t \in (-\epsilon_\Delta, 1), \\ \Gamma_l|_{\partial\Delta(0,r_\Gamma)}, & \quad l \in (-\epsilon_\Gamma, \epsilon_\Gamma).\end{aligned}$$

We denote these boundaries by  $\partial\Sigma$ ,  $\partial\Delta$ , and  $\partial\Gamma$ , respectively. Finally,  $\Sigma' = \Sigma \cup \partial\Sigma$ ,  $\Delta' = \Delta \cup \partial\Delta$ , and  $\Gamma' = \Gamma \cup \partial\Gamma$ . Let  $\pi_j$  be the projection onto the  $j$ th coordinate of  $\mathbb{C}^2$ .

For each of the three following claims we must make suitable choices of  $\epsilon_\Sigma$ ,  $\epsilon_\Delta$ ,  $\epsilon_\Gamma$ ,  $r_\Gamma$  and  $\eta$ .

We would first like to show that the intersection of the three families,  $\Sigma \cap \Delta \cap \Gamma$ , is a small set. We let

$$P = \left\{ (z_1, z_2) \in \mathbb{C}^2; \operatorname{Im} z_1 = \operatorname{Im} z_2 = 0, \right. \\ \left. \frac{-1}{2} \leq \operatorname{Re} z_1 \leq \epsilon_\Sigma, -\epsilon_\Delta \leq \operatorname{Re} z_2 \leq 1 \right\}.$$

We see that

$$\Sigma' \cap \Delta' = P.$$

Let  $G$  be the union of the graphs of the functions

$$f(y) = \pm \left( y^2 - \frac{y}{\eta} \right)^{1/2}, \quad y \leq 0.$$

In other words

$$G = \{(x, y) \in \mathbb{R}^2; (f(y), y), y \leq 0\}.$$

A point in  $\Gamma_l$  looks like  $(\xi + i\eta\xi^2, \xi - il)$ . Let  $\xi = x + iy$ .

Choosing  $r_\Gamma < 1/\eta$  we see that for a point,  $\Gamma_l(\xi) = (\xi + i\eta\xi^2, \xi - il)$ , in  $\Gamma_l$  to be in  $P$  we must have that

- (1)  $y \leq 0$ ,
- (2)  $y = l$ , and
- (3)  $x = \pm \left( y^2 - \frac{y}{\eta} \right)^{1/2}$ .

Thus  $\Gamma_l(\xi) = \Gamma_y(\xi)$ . If  $l > 0$  and  $\xi \in \Delta(0, r_\Gamma)$  then  $\Gamma_l(\xi)$  is not in  $P$ . If  $l = 0$  and  $\Gamma_l(\xi)$  is in  $P$ , then  $\xi = 0$ . Finally, if  $l < 0$  and  $\Gamma_l(\xi)$  is in  $P$ , then  $\xi = \pm(l^2 - l/\eta)^{1/2} + il$ .

We have shown:

**Claim 3.1.** The intersection of the three families,  $\Sigma \cap \Delta \cap \Gamma$ , is a small set. Specifically,

$$\Sigma \cap \Delta \cap \Gamma = P \cap \{ \Gamma_y(\xi); \xi = x + iy \in G \cap \Delta(0, r_\Gamma), -\epsilon_\Gamma \leq y \leq 0 \}.$$

The following claim will allow us to choose the frames of our families to be disjoint.

**Claim 3.2.** The boundaries of the families are pairwise disjoint. Similarly, the bottom discs of the families are pairwise disjoint. The boundary of each family is disjoint from the bottom disc of the other families.

Finally, we note that the boundaries and bottoms of our families of discs are disjoint from the intersection of the three families.

**Claim 3.3.** The sets  $\Sigma_B, \Delta_B, \Gamma_B, \partial\Sigma, \partial\Delta$  and  $\partial\Gamma$  are all disjoint from  $\Sigma \cap \Delta \cap \Gamma$ .

Given a family of discs such as  $\Delta$ , and any  $\delta > 0$ , we can find a pseudoconvex domain,  $\Delta^\square$ , contained in a  $\delta$  neighbourhood of  $\Delta$ . We can then construct a subdomain,  $\Delta^\cup$ , of  $\Delta^\square$  such that

- (1)  $\Delta^\cup$  is contained within a  $\delta$  neighbourhood of  $\Delta_B \cup \partial\Delta$ ,
- (2)  $\tilde{\Delta}^\cup = \Delta^\square$ , and
- (3) the boundary of  $\Delta^\cup$  contains smooth points which consist of strictly pseudoconvex points.

We call  $\Delta^\square$  a fattening of  $\Delta$ , and we call  $\Delta^\cup$  the frame of  $\Delta$ .

An analogous construction can be carried out for  $\Gamma$  and  $\Sigma$ . In fact, by Claim 3.2 we can choose these frames to be pairwise disjoint. By Claim 3.3 we can also choose them so that they do not intersect  $\Sigma^\square \cap \Delta^\square \cap \Gamma^\square$ .

Let  $\gamma_1$  and  $\gamma_2$  be two disjoint smooth paths,  $\gamma_1 : [0, 1] \rightarrow \mathbb{C}^2$  and  $\gamma_2 : [0, 1] \rightarrow \mathbb{C}^2$ , satisfying:

- (1) The path  $\gamma_1$  connects a strictly pseudoconvex boundary point of  $\Delta^\cup$  and a strictly pseudoconvex boundary point of  $\Sigma^\cup$ . Similarly,  $\gamma_2$  connects a strictly pseudoconvex boundary point of  $\Sigma^\cup$  and a strictly pseudoconvex boundary point of  $\Gamma^\cup$ .
- (2) The intersection of  $\Sigma^\square \cup \Delta^\square \cup \Gamma^\square$  and  $\gamma_1 \cup \gamma_2$  consists of the four endpoints of the paths.
- (3) The curve  $\gamma_1$  is transversal to the boundaries of  $\Delta^\cup$  and  $\Sigma^\cup$  at its endpoints. The analogous statement holds for  $\gamma_2$ .
- (4) Let  $p$  be a point in the complement of  $\pi_1(\Sigma^\square \cup \Delta^\square \cup \Gamma^\square)$ , and define the set  $A$  as

$$A = \mathbb{C}^2 \setminus (\{z_1 = p\} \times \mathbb{C}).$$

The fundamental group of  $A$  is nontrivial. We choose  $\gamma_1$  in such a way that

- (a)  $\Delta^\sqcup \cup \gamma_1 \cup \Sigma^\sqcup$  contains no nontrivial element of the fundamental group, and
- (b)  $\Delta^\square \cup \gamma_1 \cup \Sigma^\square$  does contain a nontrivial element of the fundamental group.

We choose  $\gamma_2$  so that analogous statements hold for  $\gamma_2$ ,  $\Sigma$  and  $\Gamma$ .

Item 4 implies that  $f(z_1, z_2) = (z_1 - p)^{1/3}$  is holomorphic on a small neighbourhood of

$$\Sigma^\sqcup \cup \Delta^\sqcup \cup \Gamma^\sqcup \cup \gamma_1 \cup \gamma_2,$$

but triple valued on a small neighbourhood of

$$\Sigma^\square \cup \Delta^\square \cup \Gamma^\square \cup \gamma_1 \cup \gamma_2.$$

Indeed, as we follow  $\gamma_1$  from  $\Delta^\square$  to  $\Sigma^\square$  we change branches of the cube root function. Similarly, as we follow  $\gamma_2$  from  $\Sigma^\square$  to  $\Gamma^\square$  we change branches again.

By a theorem of Fornæss and Stout [FS77] we can find a neighbourhood,  $\Gamma_1$ , of  $\gamma_1$  such that  $\Gamma_1$  is pseudoconvex,  $\Gamma_1 \cup \Delta^\sqcup$  is locally pseudoconvex near the intersection of  $\gamma_1$  and  $\Delta^\sqcup$ , and  $\Gamma_1 \cup \Sigma^\sqcup$  is locally pseudoconvex near the intersection of  $\gamma_1$  and  $\Sigma^\sqcup$ . Similarly, we find a neighbourhood,  $\Gamma_2$ , of  $\gamma_2$  with analogous properties. These neighbourhoods can be made as small as we like.

We define  $\Omega$  as

$$\Omega = \Sigma^\sqcup \cup \Delta^\sqcup \cup \Gamma^\sqcup \cup \Gamma_1 \cup \Gamma_2.$$

As noted above,  $f(z_1, z_2) = (z_1 - p)^{1/3}$  is holomorphic on  $\Omega$ , but triple valued on  $\Sigma^\square \cup \Delta^\square \cup \Gamma^\square \cup \Gamma_1 \cup \Gamma_2$ . We conclude that the envelope of  $\Omega$  is triple sheeted. Over  $\Omega$ , however, the envelope is only double sheeted, specifically over those points in the pairwise intersections of our three families of discs. In fact, the envelope of  $\Omega$  can be identified with the Riemann domain given as the disjoint union of  $\Sigma^\square$ ,  $\Delta^\square$  and  $\Gamma^\square$ , connected by the sets  $\Gamma_1$  and  $\Gamma_2$ , equipped with the natural projection. Certainly every holomorphic function on  $\Omega$  extends to this Riemann domain. By construction it is an unbranched Riemann domain which is locally pseudoconvex. By the equivalence of local and global pseudoconvexity of unbranched Riemann domains ([JP00, Corollary 2.2.16]), this Riemann domain is Stein, and thus is the envelope of  $\Omega$ .

**3.2. Counterexamples in  $\mathbb{C}^3$ .** We construct a domain,  $\Omega$ , in  $\mathbb{C}^3$  such that the envelope of  $\Omega$  has one sheet over  $\Omega$ , but two sheets over  $\mathbb{C}^3$ . This shows that Theorem 1.2 cannot be generalized to give information about domains in  $\mathbb{C}^n$ ,  $n > 2$ .

The key point in the example is that the intersection of one dimensional varieties in  $\mathbb{C}^3$  is not generically preserved under perturbation.

We define  $\Omega$  as follows. We first build two domains,  $V_1$  and  $V_2$ , with strictly pseudoconvex boundary points. We join these domains with a well chosen path  $\gamma$ . We let  $\pi_j$  be the projection onto the  $j$ th coordinate of  $\mathbb{C}^3$ .

Begin by defining two domains,  $U_1$  and  $U_2$ . Let  $U_1$  be

$$U_1 = \left[ \{|z| < 8\} \times \left\{ \frac{1}{2} < |w| < 1 \right\} \times \left\{ |\zeta| < \frac{1}{2} \right\} \right] \\ \cup \left[ \left\{ \frac{1}{2} < |z| < 8 \right\} \times \left\{ \frac{1}{2} < |w| < 1 \right\} \times \{|\zeta| < 1\} \right].$$



We see that  $\tilde{U}_1$  is

$$\tilde{U}_1 = \{|z| < 8\} \times \left\{ \frac{1}{2} < |w| < 1 \right\} \times \{|\zeta| < 1\}.$$

Let  $U_2$  be

$$U_2 = \left[ \left\{ |z| < \frac{1}{4} \right\} \times \{|w| < 8\} \times \left\{ \frac{3}{2} < |\zeta| < 2 \right\} \right] \cup \left[ \left\{ |z| < \frac{1}{4} \right\} \times \left\{ \frac{3}{2} < |w| < 8 \right\} \times \left\{ \frac{3}{4} < |\zeta| < 2 \right\} \right].$$

We see that  $\tilde{U}_2$  is

$$\tilde{U}_2 = \left\{ |z| < \frac{1}{4} \right\} \times \{|w| < 8\} \times \left\{ \frac{3}{4} < |\zeta| < 2 \right\}.$$

Notice that

$$U_1 \cap U_2 = \emptyset, \quad \tilde{U}_1 \cap U_2 = \emptyset, \quad \tilde{U}_2 \cap U_1 = \emptyset,$$

but that  $\tilde{U}_1 \cap \tilde{U}_2$  is non empty.

We define

$$V_1 = U_1 \cap B^3(0, 6)$$

and

$$V_2 = U_2 \cap B^3(0, 6).$$

Just as with  $U_1$  and  $U_2$  we have that

$$V_1 \cap V_2 = \emptyset, \quad \tilde{V}_1 \cap V_2 = \emptyset, \quad \tilde{V}_2 \cap V_1 = \emptyset,$$

but that  $\tilde{V}_1 \cap \tilde{V}_2$  is non empty.

Unlike  $U_j$ , however, each  $V_j$  has strictly pseudoconvex boundary points: points in  $\partial V_j \cap \partial B^3(0, 6)$ . Let  $p_1$  and  $p_2$  be such boundary points in  $V_1$  and  $V_2$ , respectively. Let  $\gamma : [0, 1] \rightarrow \mathbb{C}^3$  be a smooth curve satisfying:

- (1)  $\gamma$  runs from  $p_1$  to  $p_2$ .
- (2)  $\gamma$  is transversal to  $\partial V_j$  at  $p_j$ .
- (3)  $\gamma$  intersects  $\overline{B^3(0, 6)}$  only at  $p_1$  and  $p_2$ .
- (4) Let  $p$  be a point in the complement of  $\pi_1(B^3(0, 6))$ , and define the set

$$A = \mathbb{C}^3 \setminus (\{z = p\} \times \mathbb{C}^2).$$

The fundamental group of  $A$  is nontrivial. We choose  $\gamma$  in such a way that

- (a)  $V_1 \cup V_2 \cup \gamma$  contains no nontrivial element of the fundamental group,
- and
- (b)  $\tilde{V}_1 \cup \tilde{V}_2 \cup \gamma$  does contain a nontrivial element of the fundamental group.

As in the previous example, we find a neighbourhood,  $\Gamma$ , of  $\gamma$  such that  $\Gamma$  is pseudoconvex,  $\Gamma \cup V_1$  is locally pseudoconvex near  $p_1$ , and  $\Gamma \cup V_2$  is locally pseudoconvex near  $p_2$ .

We define  $\Omega$  as  $\Omega = V_1 \cup V_2 \cup \Gamma$ .

Our choice of path ensures that  $f(z, w, \zeta) = (z - p)^{1/2}$  is a holomorphic function on  $\Omega$ . However,  $f$  is not holomorphic on  $\tilde{V}_1 \cup \tilde{V}_2 \cup \Gamma$ : as we follow  $\gamma$  from  $\tilde{V}_1$  to  $\tilde{V}_2$  we change branches of the square root function. We conclude that  $f$  is double valued on  $\tilde{V}_1 \cup \tilde{V}_2 \cup \Gamma$ . Thus the envelope of  $\Omega$  is double sheeted. The two sheets lie over  $\tilde{V}_1 \cap \tilde{V}_2$ . Since this intersection contains no points in  $\Omega$ , we see that the

envelope is single sheeted over  $\Omega$ . As in the previous example, we are viewing the envelope as a particular unbranched Riemann domain: the disjoint union of  $\tilde{V}_1$  and  $\tilde{V}_1$ , connected with the set  $\Gamma$ , and equipped with the natural projection.

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