

A CONTROLLED APPROACH TO THE ISOMORPHISM CONJECTURE

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ABSTRACT. We use a *hocolim* approach to the Isomorphism Conjecture in K -Theory to analyze the case of groups of the form $G \rtimes \mathbb{Z}$ and $G_1 *_G G_2$. As an important corollary we prove that the isomorphism conjecture in K -Theory holds for a finitely generated free group.

1. INTRODUCTION

The Isomorphism Conjecture is, currently, one of the most important tools in calculating algebraic invariants that appear in classification problems in Topology. It should be considered as an *induction* technique that evaluates geometrically important obstruction groups of a space from the virtually cyclic subgroups of the fundamental group of the space. The algebraic invariants relevant to the rigidity problems in topology are the A and K -invariants, $L^{-\infty}$ -invariants and pseudoisotopy invariants. They all can be characterized as elements of homotopy groups of the corresponding spectrum.

More specifically, let \mathcal{G}_Γ be the class of virtually cyclic subgroups of a group Γ , $\mathcal{E}\mathcal{G}_\Gamma$ the classifying space for the class \mathcal{G}_Γ , and

$$p : E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma \rightarrow \mathcal{E}\mathcal{G}_\Gamma / \Gamma = \mathcal{B}\mathcal{G}_\Gamma$$

the projection map. Let \mathbb{S} be any of the \mathbb{A} , \mathbb{K} , $\mathbb{L}^{-\infty}$ or the pseudoisotopy spectra.

Conjecture (Isomorphism Conjecture ([10])). *The assembly map*

$$\mathbb{H}(\mathcal{B}\mathcal{G}_\Gamma, \mathbb{S}(p)) \rightarrow \mathbb{S}(B\Gamma)$$

induced by the commutative diagram:

$$\begin{array}{ccc} E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\ p \downarrow & & \downarrow \\ \mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & * \end{array}$$

is a homotopy equivalence.

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The homology spectrum that appears in the Isomorphism Conjecture can be computed, via a spectral sequence, from the \mathbb{S} -groups of the virtually cyclic subgroups of Γ .

Farrell–Jones have proved the isomorphism conjecture for a large class of geometrically significant groups and for the pseudoisotopy spectrum ([10], [11]). That implies the isomorphism conjecture for lower K -groups.

In this paper, we will study the Isomorphism Conjecture using homotopy colimits. The homology spectrum in the statement of the conjecture can be described as a homotopy colimit. In this context, the Isomorphism Conjecture reflects the extend of how much the factors mentioned above commute with certain homotopy colimits.

Homotopy colimits in the Isomorphism Conjecture have been used before ([1], [8], [17], [20]). In all the references the homotopy colimit machinery encodes, in a homological/homotopical way, controlled problems.

The Isomorphism Conjecture can also be considered as a statement about a “forget control” map. In this context, we will use Segal’s Pushdown Construction ([15]) in homotopy colimits to capture the geometric idea of change of control. This approach is applied to two cases, with $\mathbb{S} = \mathbb{K}$, the K -theory spectrum:

- Groups that admit an epimorphism to \mathbb{Z} and the kernel satisfying the IC.
- Amalgamated free products where each factor satisfies the IC.

In the first case (Section 4), we will describe the part of K -theory that is controlled over the circle and the forget control map in this case. More precisely, let $\Gamma = G \rtimes_{\alpha} \mathbb{Z}$ where G satisfies the \mathbb{S} -IC. The main result in this case (Theorem 4.7) states that the homology spectrum of $B\mathcal{G}_{\Gamma} \times S^1$ is homotopy equivalent to the “controlled” part, over S^1 , of the theory, in analogy with the results in [16] and [17] in K -theory. Let \mathbb{K}_R denotes the functor that maps a space X to $\mathbb{K}(R[\pi_1(X)])$.

Theorem (Semidirect Product with \mathbb{Z}). *Let $\Gamma = G \rtimes_{\alpha} \mathbb{Z}$ such that G satisfies the \mathbb{K}_R -IC. Let*

$$q_{\Gamma} : E\Gamma \times_{\Gamma} E\Gamma \rightarrow S^1 \times B\Gamma.$$

Then

- (1) *there is an exact sequence:*

$$\dots \rightarrow \pi_i(BG) \xrightarrow{1-\alpha_*} \pi_i(BG) \rightarrow H_i(S^1 \times B\Gamma, \mathbb{K}_R(q_{\Gamma})) \rightarrow \pi_{i-1}(BG) \xrightarrow{1-\alpha_*} \pi_{i-1}(BG) \dots$$

- (2) *If RG is regular coherent, then the assembly map*

$$H_i(B\Gamma, \mathbb{K}_R(p_{\Gamma})) \rightarrow K_i(R\Gamma)$$

is an epimorphism for all $i \in \mathbb{Z}$.

In the special case when Γ is a trivial extension, we derive information about the cokernel of the forget control map i.e. the Nil-part of the theory. That leads to a proof of the IC for the K -theory

reduced Nil-spectrum (Section 5), under the assumption that groups of the form $G_0 \times \mathbb{Z}$, with G_0 virtually infinite group, satisfy the \mathbb{K}_R -IC.

Theorem (Product with \mathbb{Z}). *Let G satisfy the Bundle \mathbb{K}_R -IC and, for every virtually infinite cyclic subgroup $G_0 < G$, $G_0 \times \mathbb{Z}$ satisfies the bundle \mathbb{K}_R -IC. Let $\Gamma = G \times \mathbb{Z}$. Then*

- (1) Γ satisfies the Bundle \mathbb{K}_R -IC.
- (2) G satisfies the Bundle Nil_R -IC

In the second case (Section 6), we prove that the part of the theory that is controlled over the interval satisfies a Mayer-Vietoris property in analogy to the result in [17].

Theorem (Amalgamated free products). *Let $\Gamma = G_1 *_{G_0} G_2$ such that G_i , $i = 0, 1, 2$, satisfy the \mathbb{K}_R -IC. Let*

$$q_\Gamma : E\Gamma \times_\Gamma \mathcal{E}\Gamma \rightarrow I \times \mathcal{B}\Gamma.$$

Then

- (1) there is an exact sequence:

$$\pi_i(BG_0) \rightarrow \pi_i(BG_1) \oplus \pi_i(BG_2) \rightarrow H_i(I \times \mathcal{B}\Gamma, \mathbb{K}_R(q_\Gamma)) \rightarrow \pi_i(BG_0) \rightarrow \pi_{i-1}(BG_1) \oplus \pi_{i-1}(BG_2)$$

- (2) If RG_0 is regular coherent, then the assembly map

$$H_i(\mathcal{B}\Gamma, \mathbb{K}_R(p_\Gamma)) \rightarrow K_i(R\Gamma)$$

is an epimorphism for all $i \in \mathbb{Z}$.

If in addition the *base groups* (G in the semidirect product case and G_0 in the amalgamated free product case) are torsion free and satisfy the integral Novikov conjecture, then the assembly map is a homotopy equivalence. Using these ideas we prove (Corollary 7.4):

Corollary. *Let F be a finitely generated free group and R a regular coherent ring. Then*

- (1) F satisfies the \mathbb{K}_R -IC.
- (2) $F \rtimes \mathbb{Z}$ satisfies the \mathbb{K}_R -IC

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2. NOTATION

Let Γ be a discrete group and \mathcal{C}_Γ a class of subgroups of Γ (i.e. a collection of subgroups of Γ closed under taking conjugates and subgroups). The *classifying space* of the class \mathcal{C}_Γ , $\mathcal{E}\mathcal{C}_\Gamma$ is the Γ -complex whose isotropy groups are in \mathcal{C}_Γ and its non-empty fixed point sets are contractible.

A model for this space, reminiscent of the bar construction ([10]), is given as follows: It is the realization of a semi-simplicial complex with n -simplex given by a sequence

$$\sigma = \gamma_0 \Gamma_0(\gamma_1 \Gamma_1, \gamma_2 \Gamma_2, \dots, \gamma_n \Gamma_n)$$

with $\Gamma_i \in \mathcal{C}$ such that $\gamma_i^{-1} \Gamma_{i-1} \gamma_i \subset \Gamma_i$ for $i = 1, \dots, n$. The face operator is given by

$$\partial_i \sigma = \begin{cases} \gamma_0 \gamma_1 \Gamma_1(\gamma_2 \Gamma_2, \dots, \gamma_n \Gamma_n), & i = 0, \\ \gamma_0 \Gamma_0(\gamma_1 \Gamma_1, \dots, \gamma_{i-1} \Gamma_{i-1}, \gamma_i \gamma_{i+1} \Gamma_{i+1}, \gamma_{i+1} \Gamma_{i+1}, \dots, \gamma_n \Gamma_n), & 0 < i < n, \\ \gamma_0 \Gamma_0(\gamma_1 \Gamma_1, \dots, \gamma_{n-1} \Gamma_{n-1}), & i = n. \end{cases}$$

The group Γ acts on EC_Γ by

$$\gamma \sigma = (\gamma \gamma_0) \Gamma_0(\gamma_1 \Gamma_1, \gamma_2 \Gamma_2, \dots, \gamma_n \Gamma_n), \text{ for } \gamma \in \Gamma.$$

We write \mathcal{BC}_Γ for the orbit space EC_Γ/Γ . To ensure that \mathcal{BC}_Γ is a simplicial complex, we subdivide EC_Γ twice, i.e. $\mathcal{BC}_\Gamma = (EC_\Gamma)''/\Gamma$.

The construction of the classifying space is functorial with respect to group homomorphisms. Let \mathcal{C}_Γ be a class of subgroups of Γ . For if $\rho : \Gamma \rightarrow G$ be a group homomorphism then ρ induces a ρ -equivariant map

$$\bar{\rho} : EC_\Gamma \rightarrow EC_G, \quad \gamma_0 \Gamma_0(\gamma_1 \Gamma_1, \gamma_2 \Gamma_2, \dots, \gamma_n \Gamma_n) \mapsto \rho(\gamma_0) \rho(\Gamma_0)(\rho(\gamma_1) \rho(\Gamma_1), \rho(\gamma_2) \rho(\Gamma_2), \dots, \rho(\gamma_n) \rho(\Gamma_n))$$

where \mathcal{C}_G is a class of subgroups of G that contains the images, under ρ , of the elements of \mathcal{C}_Γ . The map $\bar{\rho}$ induces a map ρ' to the quotient spaces.

For each simplicial complex K , we write $\text{cat}(K)$ for the category of simplices of K , viewed as a partially ordered set. Thus objects are the simplices of K and there is a single morphism from σ to τ whenever $\sigma \leq \tau$.

Definition 2.1 ([1]). Let $p : E \rightarrow B$ be a map with $B = |K|$, the geometric realization of a simplicial complex. The map is said to have a *homotopy colimit structure* if there is a functor

$$F : \text{cat}(K)^{op} \rightarrow \mathbf{Top}$$

such that:

- $E = \text{hocolim}_{\text{cat}(K)^{op}}(F)$.
- $p = \text{hocolim}_{\text{cat}(K)^{op}}(\nu)$, where ν is the natural transformation from F to the constant point functor.

Notation. Let $p : E \rightarrow B$ be a map, $B = |K|$ and $\text{cat}(K)^{op}$ the category of the simplicial complex K . We define the *barycentre functor*

$$\text{bar}(p) : \text{cat}(K)^{op} \rightarrow \mathbf{Top}, \quad \sigma \mapsto p^{-1}(\hat{\sigma})$$

where $\hat{\sigma}$ is the barycentre of σ .

Remark 2.2. We give basic examples of maps that admit a homotopy colimit structure. The proofs follow from direct calculations (also [20]).

- (1) Let $p : E \rightarrow B$, $B = |K|$, be a map that admits a homotopy colimit structure relative to the functor $\text{bar}(p)$. Let $p' : E' \rightarrow E$ be a fiber bundle. Then the composition

$$q : E' \xrightarrow{p'} E \xrightarrow{p} B$$

admits a homotopy colimit structure relative to the functor $\text{bar}(q)$.

- (2) Let Γ be a discrete group, $E\Gamma$ a free contractible Γ -complex and \mathcal{EC}_Γ the classifying complex for the family of subgroups of Γ . Let

$$p_\Gamma : E\Gamma \times_\Gamma \mathcal{EC}_\Gamma \rightarrow \mathcal{EC}_\Gamma / \Gamma = \mathcal{BC}_\Gamma$$

be the projection map to the second coordinate. Then p_Γ has a homotopy colimit structure with respect to the functor $\text{bar}(p_\Gamma)$. Notice that, in this case, $p_\Gamma^{-1}(\hat{\sigma})$ is a space of type $B\Gamma_\sigma$, where Γ_σ is the isotropy group of σ , an element in the class \mathcal{C}_Γ .

- (3) Let $\rho : \Gamma \rightarrow G$ be a group epimorphism. Then the map

$$q : E\Gamma \times_\Gamma \mathcal{EC}_\Gamma \xrightarrow{p} \mathcal{BC}_\Gamma \xrightarrow{\rho'} \mathcal{BC}_G$$

has a homotopy colimit structure with respect to the functor $\text{bar}(q)$, where $\mathcal{C}_G = \rho(\mathcal{C}_\Gamma)$, the class of subgroups of G consisting of the images of elements of \mathcal{C}_Γ .

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $X : \mathcal{C} \rightarrow \mathbf{Top}$ be two functors. Then *Segal's Pushdown Construction* (see for example [15]) defines a functor $F_*X : \mathcal{D} \rightarrow \mathbf{Top}$ such that

$$\text{hocolim}_{\mathcal{C}} X \simeq \text{hocolim}_{\mathcal{D}} F_*X.$$

We will explicitly describe the construction to the case of Part(3) in Remark 2.2. In this case, we start with a map:

$$q : E\Gamma \times_\Gamma \mathcal{EC}_\Gamma \xrightarrow{p} \mathcal{BC}_\Gamma \xrightarrow{\rho'} \mathcal{BC}_G$$

The map ρ' induces a functor

$$P : \text{cat}(\mathcal{BC}_\Gamma)^{op} \rightarrow \text{cat}(\mathcal{BC}_G)^{op}$$

We will describe the functor

$$P_*\text{bar}(p) : \text{cat}(\mathcal{BC}_G)^{op} \rightarrow \mathbf{Top}.$$

For each simplex σ of \mathcal{BC}_G , let $P\downarrow\sigma$ be the over category. In this case, the objects of $P\downarrow\sigma$ are simplices τ of \mathcal{BC}_Γ such that $\rho'(\tau)$ contains σ as a face. Set

$$p_\sigma = p| : p^{-1}(|P\downarrow\sigma|) \rightarrow |P\downarrow\sigma|.$$

Then $P_*\text{bar}(p)(\sigma) = \text{hocolim}_{P\downarrow\sigma} \text{bar}(p_\sigma)$. Summarizing:

Proposition 2.3. *There is a homotopy equivalence:*

$$\text{hocolim}_{\text{cat}(\mathcal{BC}_\Gamma)^{op}}(\text{bar}(p)) \simeq \text{hocolim}_{\text{cat}(\mathcal{BC}_G)^{op}}(P_* \text{bar}(p)).$$

The class of subgroups of Γ of interest in the Isomorphism Conjecture is the class of *virtually cyclic subgroups*, denoted \mathcal{G}_Γ . They naturally split into two categories:

- Finite subgroups of Γ .
- Virtually infinite cyclic subgroups of Γ i.e. subgroups which contain an infinite cyclic subgroup of finite index.

The subgroups of the second type are two-ended subgroups of Γ ([9]) and they split into two types:

- Groups H that admit an epimorphism to \mathbb{Z} with finite kernel i.e.

$$H \cong K \rtimes \mathbb{Z}.$$

- Groups H that admit an epimorphism to the infinite dihedral subgroup D_∞ with finite kernel, i.e.

$$H \cong A *_B C, \quad [B : 1] < \infty, [A : B] = [C : B] = 2.$$

3. A HOMOTOPY APPROACH TO THE ISOMORPHISM CONJECTURE

We will present a reformulation of the IC using homotopy colimits. This approach depends heavily on the interpretation of the homology spectrum as a homotopy colimit ([1], [20]).

Let Γ be a discrete group and \mathbb{S} a homotopy invariant functor from spaces to spectra. As before, let \mathcal{G}_Γ be the class of virtually cyclic (finite or infinite) subgroups of Γ . Let \mathcal{EG}_Γ be the classifying Γ -complex for the class \mathcal{G}_Γ and

$$p_\Gamma : E\Gamma \times_\Gamma \mathcal{EG}_\Gamma \rightarrow \mathcal{EG}_\Gamma / \Gamma = \mathcal{BG}_\Gamma$$

be the projection map. Let $\text{cat}(\mathcal{BG}_\Gamma)^{op}$ be the category corresponding to the partially ordered set of simplices of \mathcal{BG}_Γ . Let $\text{bar}(p_\Gamma)$ be the barycentre functor. Let $r : Y \rightarrow B\Gamma$ be a bundle. Form the pull-back:

$$\begin{array}{ccc} \bar{Y} & \longrightarrow & Y \\ \rho \downarrow & & \downarrow r \\ E\Gamma \times_\Gamma \mathcal{EG}_\Gamma & \longrightarrow & B\Gamma \\ p_\Gamma \downarrow & & \downarrow \\ \mathcal{BG}_\Gamma & \longrightarrow & * \end{array}$$

Notice that

$$\text{hocolim}_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \text{bar}(p_\Gamma \circ \rho) \cong \bar{Y} \simeq Y$$

Let $*$ denote the category with a single object and a single morphism. Let

$$F_\Gamma : * \rightarrow \mathbf{Top}, \quad F_\Gamma(*) = Y$$

The Bundle Isomorphism Conjecture. With the notation above, the functor from $\text{cat}(\mathcal{BG}_\Gamma)^{op}$ to $*$, induces a homotopy equivalence of spectra:

$$\text{hocolim}_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \mathbb{S} \circ \text{bar}(p_\Gamma \circ \rho) \rightarrow \text{hocolim}_{\text{cat}(*)} \mathbb{S} \circ \text{bar}(F_\Gamma) \cong \mathbb{S}(Y).$$

Remark 3.1.

- (1) In the interesting cases, the functor \mathbb{S} factors as

$$\mathbb{S} : \mathbf{Top} \xrightarrow{\mathcal{S}} \mathcal{C} \xrightarrow{B} \mathbf{Spectra}$$

where \mathcal{C} is the category of small categories and B is the classifying spectrum functor. Using the notation in [21], Theorem 3.19, the left hand side can also be re-written:

$$\text{hocolim}_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} B(\mathbb{S} \circ \text{bar}(p_\Gamma \circ \rho)) \simeq B \left(\int_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \mathbb{S} \circ \text{bar}(p_\Gamma \circ \rho) \right)$$

Thus the isomorphism conjecture can be reformulated as follows:

Categorical Reformulation of the Bundle IC. The natural functor $\text{cat}(\mathcal{BG}_\Gamma)^{op} \rightarrow *$ induces a homotopy equivalence of categories:

$$\int_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \mathbb{S} \circ \text{bar}(p_\Gamma \circ \rho) \rightarrow \int_* \mathbb{S} \circ F_\Gamma \cong \mathbb{S}(Y)$$

- (2) Since \mathbb{S} is homotopy invariant and $Y \simeq \text{hocolim}_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \text{bar}(p_\Gamma \circ \rho)$ we see that the Bundle IC states that \mathcal{S} commutes, up to homotopy, with the homotopy colimit above, i.e.:

IC Using Homotopy Colimits. With the above notation,

$$\int_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \mathbb{S} \circ \text{bar}(p_\Gamma \circ \rho) \simeq \mathbb{S}(\text{hocolim}_{\text{cat}(\mathcal{BG}_\Gamma)^{op}} \text{bar}(p_\Gamma \circ \rho)).$$

- (3) The above formulation is related to the classical statement of the conjecture in [10] as follows:
- (a) If r is the identity map, then the Bundle IC is the Isomorphism Conjecture in [10].
 - (b) In [10], the Fibered Isomorphism Conjecture is stated under the assumption that r is a fibration. Our assumption that r is a bundle is needed to preserve the homotopy colimit structure of the composition $p_\Gamma \circ \rho$.

As an illustration of the categorical approach to the Isomorphism Conjecture, we study the Isomorphism Conjecture under certain group extensions.

Let \mathbb{S} be a homotopy invariant functor from spaces to spectra and $\rho : \Gamma \rightarrow G$ a group epimorphism. Then the map ρ induces a commutative diagram (Section 2):

$$\begin{array}{ccc} E\Gamma \times_{\Gamma} \mathcal{E}\mathcal{G}_{\Gamma} & \longrightarrow & EG \times_G \mathcal{E}\mathcal{G}_G \\ p_{\Gamma} \downarrow & & \downarrow p_G \\ \mathcal{B}\mathcal{G}_{\Gamma} & \xrightarrow{\rho'} & \mathcal{B}\mathcal{G}_G \end{array}$$

The top horizontal map is induced by the map between classifying spaces and the map

$$\bar{\rho} : \mathcal{E}\mathcal{G}_{\Gamma} \rightarrow \mathcal{E}\mathcal{G}_G.$$

The map p_{Γ} induces a functor:

$$P_{\Gamma} : \text{cat}(\mathcal{B}\mathcal{G}_{\Gamma})^{op} \rightarrow \text{cat}(\mathcal{B}\mathcal{G}_G)^{op}.$$

Let $[\sigma]_G$ represent a simplex of $\mathcal{B}\mathcal{G}_G$. The space $\mathcal{E}\mathcal{G}_G$ with the G_{σ} action is a space of type $\mathcal{E}\mathcal{G}_{G_{\sigma}}$ and the quotient $\mathcal{E}\mathcal{G}_G/G_{\sigma}$ is a space of type $\mathcal{B}\mathcal{G}_{G_{\sigma}}$. Since G_{σ} is virtually cyclic, $\mathcal{E}\mathcal{G}_G$ is G_{σ} -contractible and $\mathcal{B}\mathcal{G}_{G_{\sigma}}$ is contractible. We write

$$p_{G_{\sigma}} : EG \times_{G_{\sigma}} \mathcal{E}\mathcal{G}_G \rightarrow \mathcal{B}\mathcal{G}_{G_{\sigma}}$$

for the projection map.

Let $\Delta_{\sigma} = \rho^{-1}(G_{\sigma})$, a subgroup of Γ . Then $\mathcal{E}\mathcal{G}_{\Gamma}$, with the Δ_{σ} action, is a space of type $\mathcal{E}\mathcal{G}_{\Delta_{\sigma}}$. Consider the projection map:

$$p_{\Delta_{\sigma}} : E\Gamma \times_{\Delta_{\sigma}} \mathcal{E}\mathcal{G}_{\Gamma} \rightarrow \mathcal{E}\mathcal{G}_{\Gamma}/\Delta_{\sigma} = \mathcal{B}\mathcal{G}_{\Delta_{\sigma}}$$

Then the functor $\text{bar}(p_{\Delta_{\sigma}})$ induces the homotopy colimit structure on $p_{\Delta_{\sigma}}$. The map ρ induces a commutative diagram:

$$\begin{array}{ccc} E\Gamma \times_{\Delta_{\sigma}} \mathcal{E}\mathcal{G}_{\Gamma} & \longrightarrow & EG \times_{G_{\sigma}} \mathcal{E}\mathcal{G}_G \\ p_{\Delta_{\sigma}} \downarrow & & \downarrow p_{G_{\sigma}} \\ \mathcal{B}\mathcal{G}_{\Delta_{\sigma}} & \xrightarrow{\rho_{\sigma}} & \mathcal{B}\mathcal{G}_{G_{\sigma}} \end{array}$$

With this set up, the map ρ_{σ} maps simplices to simplices and induces a functor

$$P_{\sigma} : \text{cat}(\mathcal{B}\mathcal{G}_{\Delta_{\sigma}})^{op} \rightarrow \text{cat}(\mathcal{B}\mathcal{G}_{G_{\sigma}})^{op}$$

Lemma 3.2. *For each simplex $[\sigma]$ of $\mathcal{B}\mathcal{G}_G$ there are homeomorphisms:*

$$\phi_{\sigma} : |P_{\sigma} \downarrow [\sigma]_G| \xrightarrow{\cong} |P_{\Gamma} \downarrow [\sigma]_G|, \quad \Phi_{\sigma} : p_{\Delta_{\sigma}}^{-1} |P_{\sigma} \downarrow [\sigma]_G| \xrightarrow{\cong} p_{\Gamma}^{-1} |P_{\Gamma} \downarrow [\sigma]_G|$$

natural in σ .

Proof. We will show that the first map is a homeomorphism. The inclusion map $\Delta_\sigma \rightarrow \Gamma$ induces a map

$$\phi : \mathcal{BG}_{\Delta_\sigma} = \mathcal{EG}_\Gamma / \Delta_\sigma \rightarrow \mathcal{EG}_\Gamma / \Gamma = \mathcal{BG}_\Gamma.$$

The restriction of ϕ induces a map

$$\phi_\sigma : |P_{\sigma\downarrow}[\sigma]_G| \rightarrow |P_\Gamma\downarrow[\sigma]_G|$$

We will define the inverse of ϕ_σ . For this, let $[\tau]_\Gamma$ represent a simplex in $|P_\Gamma\downarrow[\sigma]_G|$. Thus there is $g \in G$ such that $g\bar{\rho}(\tau) > \sigma$. Let $\gamma \in \Gamma$ be such that $\rho(\gamma) = g$. Define

$$\psi_\sigma : |P_\Gamma\downarrow[\sigma]_G| \rightarrow |P_{\sigma\downarrow}[\sigma]_G|, \quad \psi_\sigma([\tau]_\Gamma) = [\gamma\tau]_{\Delta_\sigma}$$

- ψ_σ is well defined:
 - $\psi_\sigma([\tau]_\Gamma) \in |P_{\sigma\downarrow}[\sigma]_G|$: That follows because

$$\bar{\rho}(\gamma\tau) = \rho(\gamma)\bar{\rho}(\tau) = g\bar{\rho}(\tau) > \sigma.$$

- The definition of ψ_σ does not depend on the choice of γ : Let $\gamma' \in \Gamma$ be such that $\rho(\gamma') = g$. Then

$$\gamma^{-1}\gamma' \in \text{Ker}(\rho) \subset \Delta_\sigma.$$

Thus $[\gamma\tau]_{\Delta_\sigma} = [\gamma'\tau]_{\Delta_\sigma}$.

- The definition of ψ_σ does not depend on the choice of the representative of the orbit of τ : Let $g' \in G$ such that $g'\bar{\rho}(\tau) > \sigma$. Then $\bar{\rho}(\tau)$ contains both $g^{-1}\sigma$ and $(g')^{-1}\sigma$. Since \mathcal{BG}_G is a simplicial complex, that implies

$$g^{-1}\sigma = (g')^{-1}\sigma \Rightarrow g'g^{-1} \in G_\sigma.$$

Since $\Delta_\sigma = \rho^{-1}(G_\sigma)$, ψ_σ is well defined.

- ψ_σ is the inverse of ϕ_σ : That follows directly from the definition of the functions.

For the construction of Φ_σ we start by defining a subcomplex of \mathcal{EG}_Γ :

$$Q_\sigma = \{\tau \in \mathcal{EG}_\Gamma, \bar{\rho}(\tau) > \sigma\}$$

Then

$$p_{\Delta_\sigma}^{-1}(|P_{\sigma\downarrow}[\sigma]|) = E\Gamma \times_{\Delta_\sigma} \Delta_\sigma Q_\sigma, \quad p_\Gamma^{-1}(|P_\Gamma\downarrow[\sigma]|) = E\Gamma \times_\Gamma \Gamma Q_\sigma.$$

Since each simplex in Q_σ has isotropy group a subgroup of Δ_σ ,

$$p_\Gamma^{-1}(|P_\Gamma\downarrow[\sigma]|) = E\Gamma \times_\Gamma \Gamma Q_\sigma \cong E\Gamma \times_{\Delta_\sigma} \Delta_\sigma Q_\sigma.$$

The naturality of the homeomorphisms is immediate from their construction. □

As in Proposition 2.3, there is a homotopy equivalence

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(P_{\Gamma*}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma))).$$

Using Lemma 3.2, we can reformulate the description of the functor $P_{\Gamma*}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma))$. For this, we define

$$h_\Gamma : \mathrm{cat}(\mathcal{BG}_G)^{op} \rightarrow \mathrm{Spectra}, \quad \sigma \mapsto \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_{\Delta_\sigma})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})).$$

Proposition 3.3. *With the above notation, there is a homotopy equivalence*

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(h_\Gamma).$$

Proof. We put everything together:

- Using the commutative diagram for G_σ and Segal's Pushdown Construction,

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_{\Delta_\sigma})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_{G_\sigma})^{op}}(P_{\sigma*}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})))$$

- Since \mathcal{BG}_{G_σ} contracts to $[\sigma]_G$, which is also contractible,

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_{\Delta_\sigma})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})) \simeq P_{\sigma*}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma}))([\sigma]_G) \simeq \mathrm{hocolim}_{P_\sigma\downarrow[\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})).$$

- By Lemma 3.2, the diagram

$$\begin{array}{ccc} p_{\Delta_\sigma}^{-1}(|P_\sigma\downarrow[\sigma]_G|) & \longrightarrow & p_\Gamma^{-1}(|P_\Gamma\downarrow[\sigma]_G|) \\ p_{\Delta_\sigma} \downarrow & & \downarrow p_\Gamma \\ |P_\sigma\downarrow[\sigma]_G| & \longrightarrow & |P_\Gamma\downarrow[\sigma]_G| \end{array}$$

commutes and the horizontal maps are homeomorphisms. Thus, by naturality,

$$\mathrm{hocolim}_{P_\sigma\downarrow[\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})) \cong \mathrm{hocolim}_{P_\Gamma\downarrow[\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma)).$$

- Thus

$$h_\Gamma(\sigma) = \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_{\Delta_\sigma})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{\Delta_\sigma})) \simeq \mathrm{hocolim}_{P_\Gamma\downarrow[\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma))$$

and the equivalence is natural in σ .

- Since $P_{\Gamma*}\mathrm{bar}(p_\Gamma)(\sigma) = \mathrm{hocolim}_{P_\Gamma\downarrow[\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma))$,

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(P_{\Gamma*}\mathrm{bar}(p_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(h_\Gamma)$$

That completes the proof of the proposition. □

The next Lemma is an application of Proposition 2.3 to the barycentre functors.

Lemma 3.4. *With the above notation, there is a homotopy equivalence*

$$B\Gamma \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(\beta)$$

where $\beta(\sigma) = \mathrm{hocolim}_{P_\Gamma\downarrow[\sigma]_G}\mathrm{bar}(p_\Gamma)$ is a space of type $B\Delta_\sigma$.

Proof. Notice that a model for $B\Gamma$ is given by $\text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}}(\text{bar}(p_\Gamma))$. The result follows from applying Proposition 2.3. \square

Theorem 3.5. *Let \mathbb{S} be a homotopy invariant functor from spaces to spectra and $\rho : \Gamma \rightarrow G$ a group epimorphism. Assume that*

- (1) *The Bundle \mathbb{S} -IC is true for G .*
- (2) *For each $F \in \mathcal{G}_G$, the Bundle \mathbb{S} -IC is true for $\rho^{-1}(F)$.*

Then the Bundle \mathbb{S} -IC holds for Γ .

Proof. We will give the proof when the bundle over $B\Gamma$ is the identity. The general case follows similarly. By Proposition 3.3,

$$\text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}}(\mathbb{S} \circ \text{bar}(p_\Gamma)) \simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_G)^{op}}(h_\Gamma)$$

where

$$h_\Gamma([\sigma]_G) = \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_{\Delta_\sigma})^{op}}(\mathbb{S} \circ \text{bar}(p_{\Delta_\sigma}))$$

with $\Delta_\sigma = \rho^{-1}(G_\sigma)$. By Assumption (2), the functor h_Γ is naturally homotopy equivalent to the functor h' , given by

$$h'([\sigma]_G) = \mathbb{S}(B\Delta_\sigma) \simeq \mathbb{S}(\beta([\sigma]_G))$$

The group Γ acts on EG via ρ . The action is not free because elements of the kernel fix EG . Consider the bundle map:

$$B\Gamma = E\Gamma \times_\Gamma EG \rightarrow EG/G = BG$$

induced by projection to the second coordinate. Form the pull-back diagram:

$$\begin{array}{ccc} Y & \longrightarrow & B\Gamma \\ r \downarrow & & \downarrow \\ EG \times_G \mathcal{E}\mathcal{G}_G & \longrightarrow & BG \\ p_G \downarrow & & \downarrow \\ \mathcal{B}\mathcal{G}_G & \longrightarrow & * \end{array}$$

For each $[\sigma]_G$ a simplex in $\mathcal{B}\mathcal{G}_G$, the inverse image of its barycentre under p_G is a space of type BG_σ . Thus the inverse image

$$(p_G \circ r)^{-1}([\hat{\sigma}]_G) \simeq B\Delta_\sigma.$$

Since the Bundle \mathbb{S} -IC holds for G

$$\begin{aligned} \mathbb{S}(B\Gamma) &\simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_G)^{op}}(\mathbb{S} \circ \text{bar}(p_G \circ r)) \\ &\simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_G)^{op}}(\mathbb{S}(B\Delta_\sigma)) \\ &\simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_G)^{op}}(h') \\ &\simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_G)^{op}}(h_\Gamma) \\ &\simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}}(\mathbb{S} \circ \text{bar}(p_\Gamma)) \end{aligned}$$

proving the \mathbb{S} -IC for Γ . □

Corollary 3.6. *The proof implies that if the Bundle \mathbb{S} -IC holds for G and the \mathbb{S} -IC holds for the inverse images of virtually cyclic subgroups of G , then the \mathbb{S} -IC holds for Γ .*

4. SPACES OVER THE CIRCLE

Let $\Gamma = G \rtimes_{\alpha} \mathbb{Z}$. Here α is the automorphism of G induced by the action of the generator of \mathbb{Z} . The automorphism α is well-defined up to inner automorphisms. By choosing a suitable right G -space for EG , there is an α -equivariant homeomorphism

$$\psi : EG \rightarrow EG$$

i.e. $\psi(xg) = \psi(x)\alpha(g)$. By taking quotients, we see that there is a homeomorphism

$$\phi : BG \rightarrow BG$$

that induces the map α in the fundamental group, again up to inner automorphisms. Choose as a model for $B\Gamma$ the mapping torus of ϕ :

$$B\Gamma = BG \times [0, 1] / \sim, \quad (\phi(x), 0) \sim (x, 1).$$

Then a model for $E\Gamma$ is the infinite mapping telescope of ψ :

$$E\Gamma = EG \times [0, 1] \times \mathbb{Z} / \sim, \quad (x, 1, n) \sim (\psi(x), 0, n + 1).$$

The right action of Γ on $E\Gamma$ is given by:

$$(x, t, n)(g, m) = (x\alpha^n(g), t, n + m).$$

The map

$$\Phi : BG \times [0, 1] \rightarrow B\Gamma, \quad \Phi(x, t) = [x, t]$$

has the property that:

$$\Phi(x, 0) = [x, 0], \quad \Phi(x, 1) = [x, 1] = [\phi(x), 0]$$

i.e., it defines a homotopy between the identity map and the map ϕ on BG inside $B\Gamma$. Also, the natural projection map to the second coordinate:

$$\rho : B\Gamma \rightarrow S^1$$

is a bundle. Consider the commutative diagram

$$\begin{array}{ccc} E\Gamma \times_{\Gamma} \mathcal{E}\mathcal{G}_{\Gamma} & \longrightarrow & E\Gamma \times_{\Gamma} \mathcal{E}\mathcal{G}_{\Gamma} \\ q_{\Gamma} \downarrow & & \downarrow p_{\Gamma} \\ S^1 \times \mathcal{B}\mathcal{G}_{\Gamma} & \longrightarrow & \mathcal{B}\mathcal{G}_{\Gamma} \end{array}$$

In the applications the spectrum \mathbb{S} has an infinite loop space structure. Let \mathcal{CF} be the homotopy cofiber:

$$\mathbb{S}(BG) \xrightarrow{1-\phi_*} \mathbb{S}(BG) \rightarrow \mathcal{CF}$$

Since id_{BG} and ϕ are homotopic in $B\Gamma$, there is an induced map:

$$f : \mathcal{CF} \rightarrow \mathbb{S}(B\Gamma).$$

This is the map that ‘forgets the control’ over S^1 . We will give a homological description of \mathcal{CF} .

With the above notation:

- Every simplex σ of \mathcal{EG}_Γ defines an equivalence relation on \mathbb{Z} ,

$$m_1 \sim_\sigma m_2 \iff \Gamma_\sigma \cap (1, m_1)G(1, -m_2) \neq \emptyset.$$

- For each $t \in S^1$, $m \in \mathbb{Z}$, a map

$$i_m : (EG \times \{t\} \times \{m\}) \times_{\Gamma_\sigma \cap G} \hat{\sigma} \rightarrow (EG \times \{t\} \times \mathbb{Z}) \times_{\Gamma} \hat{\sigma}, [(x, t, m), \hat{\sigma}] \mapsto [(x, t, m), \hat{\sigma}]$$

here $EG \times \{t\} \times \mathbb{Z} \subset E\Gamma$. Notice that the image of i_m depends on the choice of $t \in S^1$.

Lemma 4.1. *Fix $t \in S^1$. Let σ be a simplex of \mathcal{EG}_Γ and $t \in S^1$. Then:*

- (1) i_m is a monomorphism for each $m \in \mathbb{Z}$.
- (2) $\text{Im}(i_{m_1}) = \text{Im}(i_{m_2})$ if and only if $m_1 \sim_\sigma m_2$.
- (3) The images of i_{m_1} and i_{m_2} are either equal or disjoint.
- (4) The induced map:

$$i = \coprod_{[m_k]} i_{m_k} : \coprod_{[m_k]} \text{Im}(i_k) \rightarrow (EG \times \{t\} \times \mathbb{Z}) \times_{\Gamma} \hat{\sigma}$$

where m_k runs over a complete set of representatives of \sim_σ , is a homeomorphism.

Proof. For (1), assume that $i_m([(x, t, m), \hat{\sigma}]) = i_m([(y, t, m), \hat{\sigma}])$. Then, there is $(g, n) \in \Gamma$, such that

$$((x, t, m), \hat{\sigma}) = ((y, t, m)(g, n), (g, n)^{-1} \hat{\sigma}) = ((y\alpha^m(g), t, m+n), (g, n)^{-1} \hat{\sigma}).$$

That implies that $(g, n) \in \Gamma_\sigma$ and $n = 0$. Thus $(g, n) \in \Gamma_\sigma \cap G$ and $[(x, t, m), \hat{\sigma}] = [(y, t, m), \hat{\sigma}]$.

Similar calculations show that in $\text{Im}(i_{m_1}) = \text{Im}(i_{m_2})$ then $m_1 \sim_\sigma m_2$. If we assume that $m_1 \sim_\sigma m_2$, then there is $g \in G$ such that $(g, m_1 - m_2) \in \Gamma_\sigma$. Then, if $[(x, t, m_1), \hat{\sigma}] \in \text{Im}(i_{m_1})$, then

$$[(x, t, m_1), \hat{\sigma}] = [(x, t, m_1), (g, m_1 - m_2) \hat{\sigma}] = [(x\alpha^{m_1}(g), t, m_2), \hat{\sigma}] \in \text{Im}(i_{m_2}),$$

proving (2).

For (3), assume that m_1 and m_2 are not equivalent and let $[(x, m_1), \hat{\sigma}] \in \text{Im}(i_{m_1}) \cap \text{Im}(i_{m_2})$. But by (2), $m \sim_\sigma m_1$ and $m \sim_\sigma m_2$. Thus $m_1 \sim_\sigma m_2$, contrary to our assumption.

It is immediate that i is surjective. Parts (1)–(3) show that i is a bijection. The inverse of i is given by:

$$i^{-1} : (EG \times \{t\} \times \mathbb{Z}) \times_{\Gamma} \hat{\sigma} \rightarrow \coprod_{[m_k]} \text{Im}(i_k), [(x, t, m), (g, n) \hat{\sigma}] \mapsto [(x\alpha^m(g), t, m+n), \hat{\sigma}] \in \text{Im}(i_{m+n}),$$

and thus i is a homeomorphism. \square

The following result is immediate from the proof.

Lemma 4.2. *Fix $\sigma \in \mathcal{EG}_\Gamma$ and $t \in S^1$. Then the map*

$$\chi : EG \times_G G(1, m)\hat{\sigma} \rightarrow EG \times_\Gamma \mathcal{EG}_\Gamma, [x, (g, m)\hat{\sigma}] \mapsto [(xg, t, m), \hat{\sigma}]$$

induces a homeomorphism onto $Im(i_m)$.

We will study the homotopy colimit structure of the quotient map:

$$q_\Gamma : E\Gamma \times_\Gamma \mathcal{EG}_\Gamma \rightarrow \mathcal{BG}_\Gamma \times S^1.$$

Equip S^1 with the structure of a simplicial complex with three 0-simplices v_i , $i = 0, 1, 3$ and three 1-simplices $e_i = \{v_i, v_{i+1}\}$, where i is taken mod 3. The projection map induces a functor:

$$\rho : \text{cat}(S^1 \times \mathcal{BG}_\Gamma)^{op} \rightarrow \text{cat}(S^1)^{op}.$$

Using Segal's Pushdown Construction we get a homotopy equivalence:

$$\text{hocolim}_{\text{cat}(S^1 \times \mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\text{bar}(q_\Gamma)) \simeq \text{hocolim}_{\text{cat}(S^1)^{op}}(P_*\rho).$$

We will describe explicitly the functor $P_*\rho$ on $\text{cat}(S^1)^{op}$:

For each simplex t of S^1 , let

$$\mathbb{S}\circ\text{bar}(q_\Gamma)(t, -) : \text{cat}(\mathcal{BG}_\Gamma)^{op} \rightarrow \text{Spectra}, \sigma \mapsto \mathbb{S}\circ\text{bar}(q_\Gamma)(t, \sigma)$$

and thus

$$P_*\rho(\hat{t}) = \text{hocolim}_{\text{cat}(\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\text{bar}(q_\Gamma)(t, -)).$$

So if we fix t , the functor $\text{bar}(q_\Gamma)$ associates to σ , the space

$$q_\Gamma^{-1}(\hat{t} \times \hat{\sigma}) = (EG \times \{\hat{t}\} \times \mathbb{Z}) \times_\Gamma \Gamma \hat{\sigma} \cong (EG \times \{\hat{t}\} \times \mathbb{Z}) \times_{\Gamma_\sigma} \hat{\sigma},$$

after subdividing.

The inclusion map $G \rightarrow \Gamma$ induces a commutative diagram:

$$\begin{array}{ccc} EG \times_G \mathcal{EG}_\Gamma & \xrightarrow{f} & (EG \times \{\hat{t}\} \times \mathbb{Z}) \times_\Gamma \mathcal{EG}_\Gamma \\ p_G \downarrow & & \downarrow q_\Gamma \\ \mathcal{EG}_\Gamma / G = \mathcal{BG}_G & \xrightarrow{u} & \hat{t} \times \mathcal{BG}_\Gamma = \hat{t} \times \mathcal{EG}_\Gamma / \Gamma \end{array}$$

where u is just the quotient map. Notice that \mathcal{EG}_Γ is a model for the space of type \mathcal{EG}_G . Then u induces a functor:

$$U : \text{cat}(\mathcal{BG}_G)^{op} \rightarrow \text{cat}(\mathcal{BG}_\Gamma)^{op}$$

Let $[\sigma]_\Gamma$ denote the Γ -orbit of the simplex σ of \mathcal{EG}_Γ . Then the over category:

$$U\downarrow[\sigma]_\Gamma = \{[\gamma\tau]_G : \tau \geq \sigma, \gamma \in \Gamma\} = \{[(1, m)\tau]_G : \tau \geq \sigma, m \in \mathbb{Z}\},$$

where $[-]_G$ denotes the G -orbit of the simplex.

Lemma 4.3. *With the above notation,*

$$(1) |U\downarrow[\sigma]_\Gamma| = \coprod_{[m_k]} |Id\downarrow[(1, m_k)\sigma]_G|.$$

$$(2) p_G^{-1}(|U\downarrow[\sigma]_\Gamma|) = \coprod_{[m_k]} p_G^{-1}(|Id\downarrow[(1, m_k)\sigma]_G|) =$$

$$\coprod_{[m_k]} EG \times_G G(|Id\downarrow(1, m_k)\sigma|) \cong \coprod_{[m_k]} EG \times_{\Gamma_\sigma \cap G} |Id\downarrow(1, m_k)\sigma|.$$

where m_k runs over a complete set of representatives of \sim_σ .

Proof. Notice that:

$$[\gamma_1\sigma]_G = [\gamma_2\sigma]_G \iff \Gamma_\sigma \cap \gamma_1 G \gamma_2^{-1} \neq \emptyset$$

In this case, each orbit admits a representative of the form $(1, m)\tau$, with $m \in \mathbb{Z}$,

$$[(1, m_1)\sigma]_G = [(1, m_2)\sigma]_G \iff \Gamma_\sigma \cap (1, m_1)G(1, -m_2) \neq \emptyset \iff m_1 \sim_\sigma m_2,$$

which implies that $|U\downarrow[\sigma]_\Gamma|$ is equal to the union on the right side. Since we used the double subdivision of the original object, it follows that σ and $\gamma\sigma$ are equal or there is no chain of simplices containing both of them. That proves Part (1).

The first equality for Part (2) follows from (1), and the last homeomorphism is standard. \square

Now we have all the basic tools to apply Segal's Theorem on the functor U .

Lemma 4.4. *Assume that G satisfies the \mathbb{S} -IC. Then for each $t \in \text{cat}(S^1)^{op}$, there is a natural homotopy equivalence:*

$$P_*\rho(t) = \text{hocolim}_{\text{cat}(B\mathcal{G}_\Gamma)^{op}} (\mathbb{S}\circ\text{bar}(q_\Gamma)(t, -)) \simeq \mathbb{S}(BG).$$

Proof. We will show that

$$\text{hocolim}_{\text{cat}(B\mathcal{G})^{op}} (\mathbb{S}\circ\text{bar}(p_G)) \simeq \text{hocolim}_{\text{cat}(B\mathcal{G}_\Gamma)^{op}} (\mathbb{S}\circ\text{bar}(q_\Gamma)(t, -)).$$

Since G satisfies the \mathbb{S} -IC, the result will follow.

Segal's Theorem implies that

$$\text{hocolim}_{\text{cat}(B\mathcal{G})_G^{op}} (\mathbb{S}\circ\text{bar}(p_G)) \simeq \text{hocolim}_{\text{cat}(B\mathcal{G})_\Gamma^{op}} (h)$$

where, for each simplex σ of $\mathcal{E}\mathcal{G}_\Gamma$,

$$h([\sigma]_\Gamma) = \text{hocolim}_{\text{cat}(|U\downarrow[\sigma]_\Gamma|)^{op}} (\mathbb{S}\circ\text{bar}(p_G|))$$

where $[\sigma]_\Gamma$ is the Γ -orbit of σ and

$$p_G| : (p_G)^{-1}(|U\downarrow[\sigma]_\Gamma|) \rightarrow |U\downarrow[\sigma]_\Gamma|$$

is the restriction of p_G . Using Lemma 4.3,

$$\begin{aligned}
h([\sigma]_\Gamma) &= \text{hocolim}_{\text{cat}(|U_\downarrow[\sigma]_\Gamma|)^{op}}(\mathbb{S}\text{obar}(p_G|)) \\
&\simeq \bigvee_{[m_k]} \text{hocolim}_{\text{cat}(|\text{Id}_\downarrow[(1, m_k)\sigma]_G|)^{op}}(\mathbb{S}\text{obar}(p_G|)), && \text{from Lemma 4.3, Part (1)} \\
&\simeq \bigvee_{[m_k]} \text{hocolim}_{\text{cat}(|[(1, m_k)\sigma]_G|)^{op}}(\mathbb{S}\text{obar}(p_G|)), && \text{because } |\text{Id}_\downarrow[(1, m_k)\sigma]_\Gamma| \simeq [(1, m_k)\sigma]_G \\
&\simeq \bigvee_{[m_k]} \mathbb{S}(p_G^{-1}([(1, m_k)\hat{\sigma}]_G)), && \text{from the definition} \\
&\simeq \mathbb{S}\left(\prod_{[m_k]} p_G^{-1}([(1, m_k)\hat{\sigma}]_G)\right) \\
&= \mathbb{S}\left(\prod_{[m_k]} EG \times_G G(1, m_k)\hat{\sigma}\right) \\
&\simeq \mathbb{S}\left(\prod_{[m_k]} \text{Im}(i_k)\right), && \text{from Lemma 4.2} \\
&\simeq \mathbb{S}((EG \times \{\hat{t}\} \times \mathbb{Z}) \times_\Gamma \Gamma \hat{\sigma}), && \text{from Lemma 4.1} \\
&= \mathbb{S}\text{obar}(q_\Gamma)(t, [\sigma]_\Gamma)
\end{aligned}$$

(the wedge is the coproduct in the category of spectra). Therefore

$$\mathbb{S}(BG) \simeq \text{hocolim}_{\text{cat}(\mathcal{BG})_G^{op}}(\mathbb{S}\text{obar}(p_G)) \simeq \text{hocolim}_{\text{cat}(\mathcal{BG})_G^{op}}(h) \simeq \text{hocolim}_{\text{cat}(\mathcal{BG})_\Gamma^{op}}(\mathbb{S}\text{obar}(q_\Gamma)(t, -))$$

completing the proof. \square

Proposition 4.5. *If the \mathbb{S} -IC holds for G , then the homotopy cofiber of $1 - \phi_*$ is $\mathbb{H}.(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma))$. Thus there is a long exact sequence:*

$$\dots \rightarrow \pi_i(\mathbb{S}(BG)) \xrightarrow{1 - \phi_*} \pi_i(\mathbb{S}(BG)) \rightarrow H_i(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \rightarrow \pi_{i-1}(\mathbb{S}(BG)) \rightarrow \dots$$

Proof. We use the fact that q_Γ has a homotopy colimit structure and

$$H_i(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \cong \pi_i(\text{hocolim}_{\text{cat}(S^1 \times B\mathcal{G}_\Gamma)^{op}}(\mathbb{S}\text{obar}(q_\Gamma)))$$

Using the multiplicative properties of homotopy colimits and Lemma 4.4,

$$H_i(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) = \pi_i(\text{hocolim}_{\text{cat}(S^1)^{op}}(\mathbb{S}(BG)))$$

The result follows as in Section 3 in [17]. \square

The commutative diagram

$$\begin{array}{ccccc}
E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\
q_\Gamma \downarrow & & \downarrow p_\Gamma & & \downarrow \\
S^1 \times B\mathcal{G}_\Gamma & \longrightarrow & B\mathcal{G}_\Gamma & \longrightarrow & *
\end{array}$$

induces a map of spectra

$$f : \mathbb{H}.(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \rightarrow \mathbb{S}(B\Gamma).$$

Corollary 4.6. *The following diagram commutes, up to homotopy:*

$$\begin{array}{ccc} \mathbb{S}(BG) & \longrightarrow & \mathbb{H}.(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \\ \text{id} \downarrow & & \downarrow f \\ \mathbb{S}(BG) & \longrightarrow & \mathbb{S}(B\Gamma) \end{array}$$

Proof. This is because the inclusion induced map $\mathbb{S}(BG) \rightarrow \mathbb{S}(B\Gamma)$ factors through the spectrum $\mathbb{H}.(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma))$ because of the commutative diagram

$$\begin{array}{ccccccc} BG & \longleftarrow & EG \times_G \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\ \downarrow & & p_G \downarrow & & q_\Gamma \downarrow & & \downarrow p_\Gamma & & \downarrow \\ * & \longleftarrow & B\mathcal{G}_G & \longrightarrow & S^1 \times B\mathcal{G}_\Gamma & \longrightarrow & B\mathcal{G}_\Gamma & \longrightarrow & * \end{array}$$

Notice that the map in the second square is not natural because it depends on the choice of an element of S^1 but the composition induced by the third square is not affected by that choice. \square

We set up the notation for the Bundle version of Proposition 4.5. Start with a commutative diagram:

$$\begin{array}{ccccccc} \bar{Y} & \longrightarrow & \bar{Y} & \longrightarrow & \bar{Y} & \longrightarrow & Y \\ \bar{\rho} \downarrow & & \bar{\rho} \downarrow & & \bar{\rho} \downarrow & & \downarrow \rho \\ EG \times_G \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & EG \times_G \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\ q_G \downarrow & & p_G \downarrow & & p_\Gamma \downarrow & & \downarrow \\ S^1 \times B\mathcal{G}_G & \longrightarrow & B\mathcal{G}_G & \longrightarrow & B\mathcal{G}_\Gamma & \longrightarrow & * \end{array}$$

where ρ is a bundle and the top diagrams are pull-back diagrams. Then $\bar{\rho}$ is also a bundle. The proofs of Lemma 4.4 and Proposition 4.5 also work in this case.

Theorem 4.7. *Assume that the Bundle \mathbb{S} -IC holds for G . Then there is an exact sequence:*

$$\dots \rightarrow \pi_i(\mathbb{S}(\bar{Y})) \xrightarrow{1-\phi_*} \pi_i(\mathbb{S}(\bar{Y})) \rightarrow H_i(S^1 \times B\mathcal{G}_\Gamma, \mathbb{S}(\bar{\rho} \circ q_\Gamma)) \rightarrow \pi_{i-1}(\mathbb{S}(\bar{Y})) \rightarrow \dots$$

where $\phi : \bar{Y} \rightarrow \bar{Y}$ is the homeomorphism induced by α .

We specialize to the case $\mathbb{S} = \mathbb{K}_R$, the K -theory spectrum with coefficients in a ring R . Then Corollary 4.6 implies that there is a commutative diagram of exact sequences for each i :

$$\begin{array}{ccccc}
 & & H_i(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(q_\Gamma)) & & \\
 & \nearrow & \downarrow f & \searrow & \\
 K_i(RG) & \xrightarrow{1-\phi_*} & K_i(RG) & & K_{i-1}(RG) \xrightarrow{1-\phi_*} K_{i-1}(RG) \\
 & \searrow & \downarrow f & \nearrow & \\
 & & K_i(R\Gamma) & &
 \end{array}$$

Proposition 4.8. *Let RG be a regular coherent ring. Then*

$$f : H_i(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(q_\Gamma)) \rightarrow K_i(R\Gamma)$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proof. This follows from the analogue of the Bass–Heller–Swan formula for semidirect products ([22], [23]). The top sequence is exact by Proposition 4.5. The bottom sequence is exact because the assumption on $R\Gamma$ implies that the Nil-groups vanish. \square

From the definition of f we have that f factors as:

$$f : H_i(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(q_\Gamma)) \xrightarrow{p} H_i(\mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(p_\Gamma)) \xrightarrow{A} K_i(R\Gamma)$$

where p is induced by the projection to the second coordinate and A is the assembly map.

Proposition 4.9. *Let RG be a regular coherent ring. Then*

$$A : H_i(\mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(p_\Gamma)) \rightarrow K_i(R\Gamma)$$

is an epimorphism.

Proof. It follows from Proposition 4.8. \square

5. A SPECIAL CASE

Let $\Gamma = G \times \mathbb{Z}$. Then a model for $E\Gamma$ can be chosen to be $EG \times \mathbb{R}$, where EG is any model for the classifying space of G . In this case, Proposition 4.5 has a much simpler interpretation. The assumption for this section is that:

Assumption (IC): The \mathbb{S} -IC holds for groups of the form $G_0 \times \mathbb{Z}$ where G_0 is a virtually infinite cyclic subgroup of G .

Remark 5.1. For \mathbb{S} the pseudoisotopy spectrum the result follows from [10] because the $G_0 \times \mathbb{Z}$ is virtually abelian. For the \mathbb{K} -theory spectrum it is not known if the \mathbb{K} -IC holds for such groups. Partial results in this direction are in [5] and [12].

Proposition 5.2. *Let $\Gamma = G \times \mathbb{Z}$. Assume that the \mathbb{S} -IC holds for G . Then, in the notation of the last section, there is a homotopy equivalence:*

$$\mathbb{H}.(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \simeq \mathbb{S}(BG) \times \Omega^{-1}(\mathbb{S}(BG)).$$

The projection map $\Gamma \rightarrow G$ induces a commutative diagram

$$\begin{array}{ccc} (EG \times \mathbb{R}) \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & (EG \times \mathbb{R}) \times_\Gamma \mathcal{E}\mathcal{G}_G \\ p_\Gamma \downarrow & & \downarrow q \\ \mathcal{B}\mathcal{G}_\Gamma & \xrightarrow{u} & \mathcal{B}\mathcal{G}_G \end{array}$$

where Γ acts on $\mathcal{E}\mathcal{G}_G$ through the projection $\Gamma \rightarrow G$.

Lemma 5.3. *Let $[x]$ be a point in $\mathcal{B}\mathcal{G}_G$. Then $q^{-1}([x]) \cong p_G^{-1}([x]) \times S^1$.*

Proof. Let $[x] \in \mathcal{B}\mathcal{G}_G$, with x the G -orbit of a point $x \in \mathcal{E}\mathcal{G}_G$. Then a point $[(e, r), y] \in (EG \times \mathbb{R}) \times_\Gamma \mathcal{E}\mathcal{G}_G$ is in $q^{-1}(x)$, if there is $g \in G$ such that $gy = x$. Define a map

$$\alpha : q^{-1}([x]) \rightarrow p_G^{-1}([x]) \times S^1, [(e, r), y] \mapsto ([e, y], [r])$$

where $[r]$ is the \mathbb{Z} -orbit of r . The map is a well-defined homeomorphism. \square

The map u induces a functor $U : \text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op} \rightarrow \text{cat}(\mathcal{B}\mathcal{G}_G)^{op}$. Let σ be a simplex $\mathcal{B}\mathcal{G}_G$ with isotropy group G_0 . We will describe the category $U \downarrow \sigma$. Let $[\sigma]_G$ be the G -orbit of a simplex σ of $\mathcal{E}\mathcal{G}_G$ with isotropy group $G_0 < G$. A simplex $[\bar{\sigma}]_\Gamma$ of $\mathcal{B}\mathcal{G}_\Gamma$ will be an object of $U \downarrow \sigma$ if $u([\bar{\sigma}]_\Gamma) \geq [\sigma]_G$. The homotopy invariant functor \mathbb{S} induces a functor

$$\mathbb{S}[S^1] : \mathbf{Top} \rightarrow \mathbf{Spectra}, \mathbb{S}[S^1](X) = \mathbb{S}(X \times S^1)$$

which is also homotopy invariant.

Proposition 5.4. *Assume that G satisfies Assumption (IC). With the above notation, there is a homotopy equivalence of spectra:*

$$\text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}} (\mathbb{S} \circ \text{bar}(p_\Gamma)) \simeq \text{hocolim}_{\text{cat}(\mathcal{B}\mathcal{G}_G)^{op}} (\mathbb{S}[S^1] \circ \text{bar}(p_G)).$$

Proof. We will use Segal's Pushdown Theorem for comparing the two homotopy colimits. The map u induces a functor $U : \text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op} \rightarrow \text{cat}(\mathcal{B}\mathcal{G}_G)^{op}$. Let $[\sigma]_G$ be a simplex in $\mathcal{B}\mathcal{G}_G$, with isotropy group G_0 (defined up to conjugation). We choose the classifying spaces of G_0 and $G_0 \times \mathbb{Z}$:

- (1) Since G_0 is in \mathcal{G}_G , $\mathcal{B}\mathcal{G}_{G_0}$ can be chosen to be a representative σ and $\mathcal{E}\mathcal{G}_{G_0} = |\sigma|$.
- (2) Since $G_0 \times \mathbb{Z} < \Gamma$, we choose $\mathcal{E}\mathcal{G}_{G_0 \times \mathbb{Z}}$ to be the subcomplex of $\mathcal{E}\mathcal{G}_\Gamma$ consisting of all simplices with isotropy group a subgroup of a $G_0 \times \mathbb{Z}$ -conjugate of $G_0 \times \mathbb{Z}$ i.e. of the form $\gamma H \gamma^{-1}$, with $\gamma \in G_0 \times \mathbb{Z}$ and $H < G_0 \times \mathbb{Z}$. Set $\mathcal{B}\mathcal{G}_{G_0 \times \mathbb{Z}} = \mathcal{E}\mathcal{G}_{G_0 \times \mathbb{Z}} / G_0 \times \mathbb{Z}$.

With the above choices, we get a commutative diagram:

$$\begin{array}{ccccc}
(EG \times \mathbb{R}) \times_{G_0 \times \mathbb{Z}} \mathcal{E}\mathcal{G}_{G_0 \times \mathbb{Z}} & \longrightarrow & (EG \times \mathbb{R}) \times_{G_0 \times \mathbb{Z}} |\sigma| & \xrightarrow{\simeq} & B(G_0 \times \mathbb{Z}) \\
p_{G_0 \times \mathbb{Z}} \downarrow & & q_{G_0 \times \mathbb{Z}} \downarrow & & \downarrow \\
\mathcal{B}\mathcal{G}_{G_0 \times \mathbb{Z}} & \xrightarrow{u_0} & |[\sigma]_G| & \longrightarrow & *
\end{array}$$

Since the \mathbb{S} -IC holds for $G_0 \times \mathbb{Z}$, the induced map

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_{G_0 \times \mathbb{Z}})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}})) \rightarrow \mathrm{hocolim}_*(\mathbb{S}\circ*) = \mathbb{S}(B(G_0 \times \mathbb{Z}))$$

is a homotopy equivalence. Since in the right square the horizontal maps are homotopy equivalences, the induced maps

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_{G_0 \times \mathbb{Z}})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}})) \rightarrow \mathrm{hocolim}_{\mathrm{cat}([\sigma]_G)^{op}}(\mathbb{S}\circ\mathrm{bar}(q_{G_0 \times \mathbb{Z}})) \simeq \mathbb{S}(B(G_0 \times \mathbb{Z})) \quad (*)$$

are also homotopy equivalences. The map u_0 induces a functor U_0 between the corresponding categories. We will apply Segal's Pushdown Theorem on the functor U_0 . For each $[\tau]_G < [\sigma]_G$, define the following map, which is the restriction of $p_{G_0 \times \mathbb{Z}}$:

$$p_\tau : (EG \times \mathbb{R}) \times_{G_0 \times \mathbb{Z}} p_{G_0 \times \mathbb{Z}}^{-1}(|U_0 \downarrow [\tau]_G|) \rightarrow |U_0 \downarrow [\tau]_G|$$

Define a functor

$$h_\sigma : \mathrm{cat}([\sigma]_G)^{op} \rightarrow \mathbb{S}pectra, \quad h(\tau) = \mathrm{hocolim}_{U_0 \downarrow [\tau]_G}(\mathbb{S}\circ\mathrm{bar}(p_\tau)).$$

Then Segal's Pushdown Construction implies that

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_{G_0 \times \mathbb{Z}})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}})) \simeq \mathrm{hocolim}_{\mathrm{cat}([\sigma]_G)^{op}} h.$$

But the category $\mathrm{cat}([\sigma]_G)^{op}$ has a unique minimal element, namely $[\sigma]_G$. Thus

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_{G_0 \times \mathbb{Z}})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}})) \simeq \mathrm{hocolim}_{\mathrm{cat}([\sigma]_G)^{op}} h \simeq \mathrm{hocolim}_{U_0 \downarrow [\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(q_{G_0 \times \mathbb{Z}})).$$

Each simplex of $U \downarrow [\sigma]_G$ has isotropy group that it is contained in $G_0 \times \mathbb{Z}$. Thus, the natural map induces a commutative diagram

$$\begin{array}{ccc}
(EG \times \mathbb{R}) \times_{G_0 \times \mathbb{Z}} p_{G_0 \times \mathbb{Z}}^{-1}(|U_0 \downarrow [\sigma]_G|) & \longrightarrow & (EG \times \mathbb{R}) \times_{\Gamma} p_{\Gamma}^{-1}(|U \downarrow [\sigma]_G|) \\
p_{G_0 \times \mathbb{Z}} \downarrow & & \downarrow p_{\Gamma} \\
|U_0 \downarrow [\sigma]_G| & \longrightarrow & |U \downarrow [\sigma]_G|
\end{array}$$

and the horizontal maps are homeomorphisms. Therefore,

$$\mathrm{hocolim}_{U_0 \downarrow [\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}})) \simeq \mathrm{hocolim}_{U \downarrow [\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{\Gamma})).$$

Combining with (*),

$$\mathbb{S}(B(G_0 \times \mathbb{Z})) \simeq \mathrm{hocolim}_{U \downarrow [\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}}))$$

which implies that

$$\mathbb{S}(B(G_0 \times \mathbb{Z})) \simeq \mathrm{hocolim}_{U \downarrow [\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{G_0 \times \mathbb{Z}})) \quad (**)$$

and the homotopy equivalence is natural for $[\sigma]_G$ a simplex in \mathcal{BG}_G .

Now we apply Segal's Pushdown Construction to the functor U . Then

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(\chi)$$

where, for each simplex $[\sigma]_G$ of \mathcal{BG}_G ,

$$\chi(\sigma) = \mathrm{hocolim}_{U\downarrow[\sigma]_G}(\mathbb{S}\circ\mathrm{bar}(p_{G_0\times\mathbb{Z}})).$$

From (**), we have that

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(\psi)$$

where $\psi([\sigma]_G) = \mathbb{S}(B(G_\sigma\times\mathbb{Z}))$. The result follows from the definition and the homotopy invariance of the functor \mathbb{S} . \square

We now specialize to the case that $\mathbb{S} = \mathbb{K}_R$ is the K -theory spectrum with coefficients in a ring R . Then Proposition 5.4 provides an alternative description of the homology term in the IC, corresponding to the Bass-Heller-Swan splitting of K -theory ([3], [14], [19]). For the definition of the Nil-spectrum functor we use ideas from the Bass-Heller-Swan Formula:

$$\mathrm{Nil}_R(-) = \mathrm{Cofiber}[\mathbb{K}_R(-) \rightarrow \mathbb{K}_{R[t]}(-)]$$

where $R[t]$ is the polynomial ring. The Bass-Heller-Swan Formula implies that there are natural homotopy equivalences:

$$\begin{aligned} \mathbb{K}_{R[t]}(-) &\simeq \mathbb{K}_R(-) \times \mathrm{Nil}_R(-) \\ \mathbb{K}_{R[t^{-1}]}(-) &\simeq \mathbb{K}_R(-) \times \mathrm{Nil}_R(-) \\ \mathbb{K}_{R[t,t^{-1}]}(-) &\simeq \mathbb{K}_R(-) \times \Omega^{-1}\mathbb{K}_R(-) \times \mathrm{Nil}_R(-) \times \mathrm{Nil}_R(-) \end{aligned}$$

The following result is the homological analogue of the Bass-Heller-Swan Formula.

Lemma 5.5 (Bass-Heller-Swan Formula). *Let G be a group that satisfies Assumption (IC). Then there is a homotopy equivalence:*

$$\begin{aligned} \mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R(p_\Gamma)) &\simeq \\ \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)) &\times \mathbb{H}(\mathcal{BG}_G, \Omega^{-1}\mathbb{K}_R(p_G)) \times \mathbb{H}(\mathcal{BG}_G, \mathrm{Nil}_R(p_G)) \times \mathbb{H}(\mathcal{BG}_R, \mathrm{Nil}_R(p_G)). \end{aligned}$$

Proof. Proposition 5.4 shows that

$$\mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R(p_\Gamma)) \simeq \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_{R[t,t^{-1}]}(p_G)).$$

The Bass-Heller-Swan Formula applies to the coefficient spectrum. The result follows. \square

The homotopy groups of the homology spectra can be computed using spectral sequences. Using this description, we immediately have that:

$$\mathbb{H}(\mathcal{BG}_G, \Omega^{-1}\mathbb{K}_R(p_G)) \simeq \Omega^{-1}\mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)).$$

Corollary 5.6. *Let G be a group as in Lemma 5.5. Then there is a homotopy equivalence:*

$$\begin{aligned} \mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R(p_\Gamma)) &\simeq \\ \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)) \times \Omega^{-1} \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)) \times \mathbb{H}(\mathcal{BG}_G, \text{Nil}_R(p_G)) \times \mathbb{H}(\mathcal{BG}_R, \text{Nil}_R(p_G)). \end{aligned}$$

The following summarizes certain immediate consequences of the calculations above.

Corollary 5.7. *Let G be a torsion free group that satisfies the \mathbb{K}_R -IC and Assumption (IC). Let $\Gamma = G \times \mathbb{Z}$. If R is a regular coherent ring, then there are homotopy equivalences:*

$$\mathbb{K}_R(BG) \times \Omega^{-1} \mathbb{K}_R(BG) \simeq \mathbb{H}(S^1 \times \mathcal{BG}_\Gamma, \mathbb{K}_R \circ \text{bar}(q'_\Gamma)) \simeq \mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R \circ \text{bar}(p_\Gamma)).$$

Furthermore, the commutative diagram

$$\begin{array}{ccc} E\Gamma \times \mathcal{EG}_\Gamma & \longrightarrow & E\Gamma \times \mathcal{EG}_\Gamma \\ q'_\Gamma \downarrow & & \downarrow p_\Gamma \\ S^1 \times \mathcal{BG}_\Gamma & \longrightarrow & \mathcal{BG}_\Gamma \end{array}$$

induces the second homotopy equivalence.

Proof. Since G is torsion free, the virtually cyclic subgroups of G are infinite cyclic. But, since R is regular coherent the Nil-groups of R and $R[\mathbb{Z}]$ vanish ([18], [23]). Therefore the Nil-spectrum of the corresponding spaces is contractible. Lemma 5.5 and Corollary 5.6 imply that

$$\begin{aligned} \mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R(p_\Gamma)) &\simeq \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)) \times \mathbb{H}(\mathcal{BG}_G, \Omega^{-1} \mathbb{K}_R(p_G)) \simeq \\ \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)) \times \Omega^{-1} \mathbb{H}(\mathcal{BG}_G, \mathbb{K}_R(p_G)) &\simeq \mathbb{K}_R(BG) \times \Omega^{-1} \mathbb{K}_R(BG). \end{aligned}$$

The result follows from Proposition 5.2.

For the second part of the Corollary, let ϕ be the map induced from the commutative diagram. Notice that there are commutative diagrams:

$$\begin{array}{ccccccc} EG \times_G \mathcal{EG}_\Gamma & \longrightarrow & (EG \times \mathbb{Z}t) \times_\Gamma \mathcal{EG}_\Gamma & \longrightarrow & E\Gamma \times \mathcal{EG}_\Gamma & \longrightarrow & E\Gamma \times \mathcal{EG}_\Gamma \\ p_G \downarrow & & q'_\Gamma \downarrow & & q'_\Gamma \downarrow & & p_\Gamma \downarrow \\ \mathcal{BG}_G & \longrightarrow & \{t\} \times \mathcal{BG}_\Gamma & \longrightarrow & S^1 \times \mathcal{BG}_\Gamma & \longrightarrow & \mathcal{BG}_\Gamma \end{array}$$

The composition map induced by the commutative diagrams induce the inclusion

$$\mathbb{K}_R(BG) \rightarrow \mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R(p_\Gamma)).$$

The map induced from the first two diagrams induce the map

$$\mathbb{K}_R(BG) \rightarrow \mathbb{H}(S^1 \times \mathcal{BG}_\Gamma, \mathbb{K}_R(q'_\Gamma)).$$

Thus the diagram commutes:

$$\begin{array}{ccc} \mathbb{K}_R(BG) & \longrightarrow & \mathbb{H}(\mathcal{BG}_\Gamma, \mathbb{K}_R(p_\Gamma)) \\ \parallel & & \downarrow \phi \\ \mathbb{K}_R(BG) & \longrightarrow & \mathbb{H}(S^1 \times \mathcal{BG}_\Gamma, \mathbb{K}_R(q'_\Gamma)) \end{array} .$$

The result follows. \square

Theorem 5.8. *Let RG be a regular coherent ring, the \mathbb{K}_R -IC holds for G , and G satisfies Assumption (IC). Then the \mathbb{K}_R -IC holds for Γ .*

Proof. That follows from the Bass-Heller-Swan Formula for \mathbb{K}_R ([18]):

$$\mathbb{K}_R(B\Gamma) \simeq \mathbb{K}_R(BG) \times \Omega^{-1} \mathbb{K}_R(BG)$$

because RG is regular coherent. The result follows from Corollary 5.7 because G must be torsion free for RG to have finite cohomological dimension. \square

For the Bundle \mathbb{K}_R -IC the regularity of $R\Gamma$ is not needed.

Proposition 5.9. *Let G a group that satisfies the bundle \mathbb{K}_R -IC and the bundle version of Assumption (IC). Then the bundle \mathbb{K}_R -IC holds for Γ .*

Proof. The proof follows from Corollary 3.6. \square

Another application of the methods coming from the Bass-Heller-Swan Formula is the IC for the Nil_R -spectrum.

Theorem 5.10. *Let G be a group that satisfies the Bundle \mathbb{K}_R -IC and the bundle version of Assumption (IC). Then G satisfies the Bundle Nil_R -IC.*

Proof. Again we will give the proof when the bundle map is the identity. The general case follows similarly. For $F < G$ a virtually cyclic subgroup, its inverse image under the projection map $\Gamma \rightarrow G$ are of the form $F \times \mathbb{Z}$. Since G and the inverse image of its virtually cyclic subgroups satisfy the Bundle \mathbb{K}_R -IC, Γ does too (Theorem 3.5). Since \mathbb{K}_R -IC holds for G , Proposition 4.5 and Lemma 5.5 imply that

$$\begin{array}{ccc} \mathbb{H}.(S^1 \times \mathcal{BG}_\Gamma, \mathbb{K}_R(q_\Gamma)) \times \mathbb{H}.(B\Gamma, \text{Nil}_R(p_\Gamma)) \times \mathbb{H}.(B\Gamma, \text{Nil}_R(p_\Gamma)) & \xrightarrow{\simeq} & \mathbb{H}.(B\Gamma, \mathbb{K}_R(p_\Gamma)) \\ \downarrow & & \downarrow \\ \mathbb{H}.(S^1 \times \mathcal{BG}_\Gamma, \mathbb{K}_R(q_\Gamma)) \times \text{Nil}_R(B\Gamma) \times \text{Nil}_R(B\Gamma) & \xrightarrow{\simeq} & \mathbb{K}_R(B\Gamma) \end{array}$$

where the bottom horizontal map is a homotopy equivalence by the Bass-Heller-Swan Splitting. The vertical maps are induced by the assembly map and the right one is a homotopy equivalence because the \mathbb{K}_R -IC holds for Γ . That implies that the left map, which is the product of the identity and two assembly maps, is an isomorphism. Thus, the assembly map

$$\mathbb{H}.(B\Gamma, \text{Nil}_R(p_\Gamma)) \rightarrow \text{Nil}_R(B\Gamma)$$

is a homotopy equivalence. \square

Remark 5.11. In Theorem 5.10 we can not remove the bundle assumption even if only the Nil_R -IC is to be proved.

6. CONTROLLED GROUPS OVER THE INTERVAL

Let $\Gamma = G_1 *_{G_0} G_2$, where H is a subgroup of $G_1 \cap G_2$. Let BG_i , $i = 0, 1, 2$, be classifying spaces for the corresponding groups with BG_0 a subcomplex of $BG_1 \cap BG_2$. Choose $B\Gamma$ to be the double mapping cylinder of the inclusion maps. Then there is a natural map $\rho : B\Gamma \rightarrow I$, where I is the unit interval. Let $E\Gamma$ be the universal cover of $B\Gamma$. Let

$$E\Gamma \times_{\Gamma} \mathcal{E}\mathcal{G}_{\Gamma} \xrightarrow{q_{\Gamma}} I \times B\mathcal{G}_{\Gamma}$$

be maps induced by the natural projection.

We work as in Section 4. Choose coset representatives:

$$\Delta_i = \{\gamma_{i,j} : \gamma_{i,j} \in \mathcal{A}_i\}, \quad i = 0, 1, 2.$$

In other words,

$$\Gamma = \coprod_{j \in \mathcal{A}_i} G_i \gamma_{i,j}, \quad i = 0, 1, 2.$$

With the above notation:

- Every simplex σ of $\mathcal{E}\mathcal{G}_{\Gamma}$ defines an equivalence relation on Δ_i ,

$$\gamma_{i,j} \sim_{\sigma} \gamma'_{i,j} \iff \Gamma_{\sigma} \cap \gamma_{i,j} G_i (\gamma'_{i,j})^{-1} \neq \emptyset.$$

- Let $t \in I$. Set $i = 0$ if $t \in \text{Int}(I)$ and $i = t$ if $t \in \partial I$. For each $\gamma_{i,j} \in \Delta_i$, a map

$$\iota_{\gamma_{i,j}} : EG_i \times_{\Gamma_{\sigma} \cap G_i} g_{i,j} \hat{\sigma} \rightarrow EG_i \Gamma \times_{\Gamma} \Gamma \hat{\sigma}, \quad [x, \gamma_{i,j} \hat{\sigma}] \mapsto [x, \gamma_{i,j} \hat{\sigma}]$$

here $EG_i \Gamma \subset E\Gamma$. Notice that the image of $\iota_{\gamma_{i,j}}$ depends on the choice of $t \in I$.

The analogues of Lemmata 4.1 and 4.2 hold in this case. Thus there is a homeomorphism:

$$\chi_i : EG_i \times_{G_i} G_i \gamma_{i,j} \hat{\sigma} \rightarrow E\Gamma \times_{\Gamma} \mathcal{E}\mathcal{G}_{\Gamma}$$

onto $\text{Im}(\iota_{\gamma_{i,j}})$.

Equip I with the structure of a simplicial complex with one 1-simplex I , and two 0-simplices $0, 1$. The projection map induces a functor:

$$\rho : \text{cat}(I \times B\mathcal{G}_{\Gamma})^{op} \rightarrow \text{cat}(I)^{op}$$

We will use Segal's Pushdown Construction. We start by giving a description of the functor $P_*\rho$ on $\text{cat}(I)^{op}$. For each simplex t of I , define a functor:

$$\text{Sobar}(q_{\Gamma})(t, -) : \text{cat}(B\mathcal{G}_{\Gamma})^{op} \rightarrow \text{Spectra}, \quad \sigma \mapsto \text{Sobar}(q_{\Gamma})(t, \sigma)$$

Then the functor $P_*\rho$ is defined as:

$$P_*\rho : \text{cat}(I)^{op} \rightarrow \text{Spectra}, \quad P_*\rho(t) = \text{hocolim}_{\text{cat}(B\mathcal{G}_{\Gamma})^{op}} (\text{Sobar}(q_{\Gamma})).$$

So if we fix t , the functor $\text{bar}(q_{\Gamma})$ associates to σ , the space

$$q_{\Gamma}^{-1}(\hat{t} \times \hat{\sigma}) = (EG_i) \Gamma \times_{\Gamma} \Gamma \hat{\sigma} \cong (EG_i) \Gamma \times_{\Gamma_{\sigma}} \hat{\sigma},$$

after subdividing, where $i = 0$ if t is an interior point and $G_i = G_{t+1}$ when $t \in \partial I$. Actually, the inverse image for $t \in \text{Int}(I)$, is

$$q_\Gamma^{-1}(\hat{t} \times \hat{\sigma}) = (EG_i \times \{t\}) \Gamma \times_\Gamma \Gamma \hat{\sigma}$$

The inclusion map $G_i \rightarrow \Gamma$ induces a commutative diagram:

$$\begin{array}{ccc} EG_i \times_{G_i} \mathcal{E}\mathcal{G}_\Gamma & \xrightarrow{f} & EG_i \Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma \\ p_G \downarrow & & \downarrow q_\Gamma \\ \mathcal{E}\mathcal{G}_\Gamma / G_i = \mathcal{B}\mathcal{G}_{G_i} & \xrightarrow{u} & \hat{t} \times \mathcal{B}\mathcal{G}_\Gamma = \hat{t} \times \mathcal{E}\mathcal{G}_\Gamma / \Gamma \end{array}$$

where u is just the quotient map. The analogue of Lemma 4.4 also works here.

Lemma 6.1. *Suppose that the \mathbb{S} -IC holds for G_i , $i = 0, 1, 2$. For each $t \in \text{cat}(I)^{\text{op}}$, there is a natural homotopy equivalence:*

$$P_* \rho(t) = \text{hocolim}_{\text{cat}(BG_\Gamma)^{\text{op}}} (\mathbb{S} \circ \text{bar}(q_\Gamma)(t, -)) \simeq \mathbb{S}(BG_i).$$

The analogue of Proposition 4.5 works in this case too.

Proposition 6.2. *Assume that the \mathbb{S} -IC holds for G_i , $i = 0, 1, 2$. Then the following is a homotopy cartesian diagram:*

$$\begin{array}{ccc} \mathbb{S}(BG_0) & \longrightarrow & \mathbb{S}(BG_1) \\ \downarrow & & \downarrow \\ \mathbb{S}(BG_1) & \longrightarrow & \mathbb{H}.(I \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \end{array}$$

Proof. The proof works as in Proposition 4.5. The only difference is that for the final result [17], Section 2 is used. \square

We also have the analogue of Propositions 4.8 and 4.9 when $\mathbb{S} = \mathbb{K}_R$.

Proposition 6.3. *Let R be a ring such that RG_0 is regular coherent. Then*

(1) *the forgetful map*

$$f : \mathbb{H}.(I \times \mathcal{B}\mathcal{G}_\Gamma : \mathbb{K}_R(q_\Gamma)) \rightarrow \mathbb{K}_R(B\Gamma)$$

is a homotopy equivalence.

(2) *the assembly map*

$$A : \mathbb{H}.(B\mathcal{G}_\Gamma, \mathbb{K}_R(p_\Gamma)) \rightarrow \mathbb{K}_R(B\Gamma)$$

induces an epimorphism on homotopy.

Proof. The proof follows from the splitting theorem in [22] and [23]. The assumption on the ring guarantees that the Waldhausen's Nil- groups vanish. \square

7. APPLICATIONS

We will apply the results to prove that $\mathbb{K}_R\text{-IC}$ is true for certain types of groups. Before we start we state the Novikov Conjecture for a discrete group Γ . Let \mathcal{C}_Γ be the classifying space for the class of finite subgroups of Γ . Again, there is a commutative diagram:

$$\begin{array}{ccc} E\Gamma \times_\Gamma \mathcal{E}\mathcal{C}_\Gamma & \longrightarrow & B\Gamma \\ p \downarrow & & \downarrow \\ \mathcal{B}\mathcal{C}_\Gamma & \longrightarrow & * \end{array}$$

A group Γ satisfies the **integral \mathbb{S} -Novikov Conjecture** if the assembly map

$$A_{\mathcal{C}} : \mathbb{H}(\mathcal{B}\mathcal{C}_\Gamma, \mathbb{S}(p)) \rightarrow \mathbb{S}(B\Gamma)$$

induces a split injection on the homotopy groups. It is an open question if all the torsion free groups satisfy the integral \mathbb{K}_R -Novikov Conjecture. It was proved that groups of finite cohomological dimension that also have finite asymptotic dimension satisfy the integral \mathbb{K}_R -Novikov Conjecture ([4]), generalizing the calculations in [7].

A connection between the integral Novikov conjecture and the Isomorphism conjecture is given in the following statement.

Lemma 7.1. *Let R be a regular coherent ring. Let G be a torsion free group that satisfies the integral \mathbb{K}_R -Novikov Conjecture. Then the assembly map*

$$A_{\mathcal{C}} : \mathbb{H}(\mathcal{B}\mathcal{C}_G, \mathbb{K}_R(p)) \rightarrow \mathbb{K}_R(B\Gamma)$$

induces a monomorphism on the homotopy groups.

Proof. Following the ideas in the Appendix in [10], notice that $A_{\mathcal{C}} = A \circ A_{\mathcal{C},G}$ because of the commutative diagram

$$\begin{array}{ccccc} EG \times_G \mathcal{E}\mathcal{C}_G & \longrightarrow & EG \times_G \mathcal{E}\mathcal{G}_G & \longrightarrow & BG \\ p \downarrow & & \downarrow p_G & & \downarrow \\ \mathcal{B}\mathcal{C}_G & \longrightarrow & \mathcal{B}\mathcal{G}_G & \longrightarrow & * \end{array}$$

The relative assembly map is the map induced by the first commutative square. Theorem A.10 in [10] states that $A_{\mathcal{C},G}$ is an equivalence if the assembly map $A_{\mathcal{C}_S}$ is an equivalence for all virtually cyclic subgroups S of G (a similar calculation appears in [8]). Since G is torsion free, the only finite subgroup of G is the trivial group and the only virtually cyclic subgroups are infinite cyclic subgroups. Thus $A_{\mathcal{C}_S}$ is the assembly map in the integral \mathbb{K}_R -Novikov conjecture for $S \cong \mathbb{Z}$. Since R is regular coherent, this map is an equivalence ([10], Remark A.11). Therefore, since $A_{\mathcal{C}}$ induces a monomorphism on homotopy groups, so does A . \square

Theorem 7.2. *Let G be a torsion free group and R a regular coherent ring. Assume that G satisfies the $\mathbb{K}_R\text{-IC}$, and that RG is a regular ring. Let Γ be a torsion free group defined by:*

- (i) $\Gamma = G \rtimes \mathbb{Z}$, or
- (ii) $\Gamma = G_1 *_G G_2$, such that G_i , $i = 1, 2$, satisfy the \mathbb{K}_R -IC,

such that Γ satisfies the integral \mathbb{K}_R -Novikov conjecture. Then Γ satisfies the \mathbb{K}_R -IC.

Proof. Using Lemma 7.1, we see that the assembly map A induces a monomorphism on the homotopy groups. Using Proposition 4.9 for case (i) and Proposition 6.3 for case (ii), we see that A is an epimorphism. Thus A is an equivalence. \square

Now we give applications of the Theorem.

Corollary 7.3. *Let G be a torsion free group and R a regular coherent ring. Assume that:*

- (1) G satisfies the \mathbb{K}_R -IC,
- (2) G has finite asymptotic dimension,
- (3) RG is a regular ring.

Let Γ be a torsion free group defined by:

- (i) $\Gamma = G \rtimes \mathbb{Z}$, or
- (ii) $\Gamma = G_1 *_G G_2$, such that G_i , $i = 1, 2$, satisfy the \mathbb{K}_R -IC and they have finite cohomological and asymptotic dimensions.

Then Γ satisfies the \mathbb{K}_R -IC.

Proof. The assumptions on the groups imply that Γ has finite asymptotic dimension ([6]). Also, Γ has finite cohomological dimension and thus it admits a finite dimensional $B\Gamma$. By [4], Γ satisfies the integral \mathbb{K}_R -Novikov conjecture. The result follows from Theorem 7.2. \square

Corollary 7.4. *Let F be a finitely generated free group and R a regular coherent ring. Then*

- (1) F satisfies the \mathbb{K}_R -IC.
- (2) $F \rtimes \mathbb{Z}$ satisfies the \mathbb{K}_R -IC

Proof. The first statement follows by induction on the number k of generators of F : If $k = 2$, then $F = \mathbb{Z} * \mathbb{Z}$ and the result follows from 7.3. For $k > 2$, $F = F_{k-1} * \mathbb{Z}$, where F_{k-1} is the free group on $(k - 1)$ generators. All the assumptions of Corollary 7.3 are satisfied and thus F satisfies the \mathbb{K}_R -IC.

For the second statement, we use again Corollary 7.3. Part (1) implies that assumption (1) is satisfied. Also, the free group has finite asymptotic dimension and thus assumption (2) is satisfied. Assumption (3) follows because R is regular Noetherian ring. \square

Remark 7.5. In [2] and [13], there was a special assumption for groups of type $F \rtimes \mathbb{Z}$ to satisfy the fibered pseudoisotopy IC. Essentially the assumption was that the action of \mathbb{Z} on F has certain geometric properties. Corollary 7.4 is more general because such an assumption is not needed but it only gives the \mathbb{K}_R -IC and not the fibered version. A general fibered version could not follow

along the same lines because the assumption in Corollary 7.3 guarantee that all the Nil-groups that appear vanish.

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