

ON THE ESTIMATION OF SOME MELLIN TRANSFORMS  
CONNECTED WITH THE FOURTH MOMENT OF  $|\zeta(\frac{1}{2} + it)|$

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ABSTRACT

Mean square estimates for  $\mathcal{Z}_2(s) = \int_1^\infty |\zeta(\frac{1}{2} + ix)|^4 x^{-s} dx$  ( $\Re s > 1$ ) are discussed, and some related Mellin transforms of quantities connected with the fourth power moment of  $|\zeta(\frac{1}{2} + ix)|$ .

## 1. Introduction

Let  $\mathcal{Z}_2(s)$  be the analytic continuation of the function

$$\mathcal{Z}_2(s) = \int_1^\infty |\zeta(\frac{1}{2} + ix)|^4 x^{-s} dx \quad (\Re s > 1),$$

which represents the (modified) Mellin transform of  $|\zeta(\frac{1}{2} + ix)|^4$ . It was introduced by Y. Motohashi [14] (see also [7], [9], [12] and [15]), who showed that it has meromorphic continuation over  $\mathbb{C}$ . In the half-plane  $\sigma = \Re s > 0$  it has the following singularities: the pole  $s = 1$  of order five, simple poles at  $s = \frac{1}{2} \pm i\kappa_j$  ( $\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$ ) and poles at  $s = \frac{1}{2}\rho$ , where  $\rho$  denotes complex zeros of  $\zeta(s)$ . Here as usual  $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$  is the discrete spectrum of the non-Euclidean Laplacian acting on  $SL(2, \mathbb{Z})$ -automorphic forms (see [15, Chapters 1–3] for a comprehensive account of spectral theory and the Hecke  $L$ -functions).

The aim of this note is to study the estimation  $\mathcal{Z}_2(s)$  in mean square and the (modified) Mellin transforms of certain other quantities related to the fourth power moment of  $|\zeta(\frac{1}{2} + it)|$ . This research was begun in [12], and continued in [7] and [9]. It was shown there that we have

$$(1.1) \quad \int_0^T |\mathcal{Z}_2(\sigma + it)|^2 dt \ll_\varepsilon T^\varepsilon \left( T + T^{\frac{2-2\sigma}{1-\sigma}} \right) \quad \left( \frac{1}{2} < \sigma < 1 \right),$$

and we also have unconditionally

$$(1.2) \quad \int_0^T |\mathcal{Z}_2(\sigma + it)|^2 dt \ll T^{\frac{10-8\sigma}{3}} \log^C T \quad \left(\frac{1}{2} < \sigma < 1, C > 0\right).$$

Here and later  $\varepsilon$  denotes arbitrarily small, positive constants, which are not necessarily the same ones at each occurrence, while  $\sigma$  is assumed to be fixed. The constant  $c$  appearing in (1.1) is defined by

$$(1.3) \quad E_2(T) \ll_{\varepsilon} T^{c+\varepsilon},$$

where the function  $E_2(T)$  denotes the error term in the asymptotic formula for the mean fourth power of  $|\zeta(\frac{1}{2} + it)|$ . It is customarily defined by the relation

$$(1.4) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = TP_4(\log T) + E_2(T),$$

with

$$(1.5) \quad P_4(x) = \sum_{j=0}^4 a_j x^j, \quad a_4 = \frac{1}{2\pi^2}.$$

For an explicit evaluation of the  $a_j$ 's in (1.5), see the author's work [3]. The best known value of  $c$  in (1.3) is  $c = 2/3$  (see e.g., [11] or [15]), and it is conjectured that  $c = 1/2$  holds, which would be optimal. Namely (see [2], [4], [14] and [15]) one has

$$(1.6) \quad E_2(T) = \Omega_{\pm}(T^{1/2}).$$

Mean value estimates for  $\mathcal{Z}_2(s)$  are a natural tool to investigate the eighth power moment of  $|\zeta(\frac{1}{2} + it)|$ . Indeed, one has (see [7, eq. (4.7)])

$$(1.7) \quad \int_T^{2T} |\zeta(\frac{1}{2} + it)|^8 dt \ll_{\varepsilon} T^{2\sigma-1} \int_1^{T^{1+\varepsilon}} |\mathcal{Z}_2(\sigma + it)|^2 dt \quad \left(\frac{1}{2} < \sigma < 1\right).$$

In [9] the pointwise estimate for  $\mathcal{Z}_2(s)$  was given by

$$(1.8) \quad \mathcal{Z}_2(\sigma + it) \ll_{\varepsilon} t^{\frac{4}{3}(1-\sigma)+\varepsilon},$$

for  $\frac{1}{2} < \sigma \leq 1$  fixed and  $t \geq t_0 > 0$ . This result is still much weaker than the bound conjectured in [7] by the author, namely that for any given  $\varepsilon > 0$  and fixed  $\sigma$  satisfying  $\frac{1}{2} < \sigma < 1$ , one has

$$\mathcal{Z}_2(\sigma + it) \ll_{\varepsilon} t^{\frac{1}{2}-\sigma+\varepsilon} \quad (t \geq t_0 > 0).$$

To define another Mellin transform related to  $\mathcal{Z}_2(s)$ , let  $P_4(x)$  be defined by (1.4), let

$$(1.9) \quad Q_4(x) := P_4(x) + P_4'(x)$$

and set

$$(1.10) \quad \mathcal{K}(s) := \int_1^\infty (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x)) E_2(x) x^{-s} dx.$$

The integral on the right-hand side of (1.10) converges absolutely at least for  $\sigma > 5/3$ , in view of (1.3) with  $c = 2/3$  and the bound for the fourth moment of  $|\zeta(\frac{1}{2} + ix)|$ . However, the interest in  $\mathcal{K}(s)$  lies in the fact that (1.4) and (1.9) yield

$$(1.11) \quad E_2'(x) = |\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x).$$

Thus an integration by parts shows that

$$(1.12) \quad \mathcal{K}(s) = -\frac{1}{2} E_2^2(1) - \frac{1}{2} s \int_1^\infty E_2^2(x) x^{-s-1} dx.$$

In view of the mean square bound (see e.g., [10] and [15])

$$(1.13) \quad \int_1^T E_2^2(t) dt \ll T^2 \log^C T,$$

it follows that (1.12) furnishes analytic continuation of  $\mathcal{K}(s)$  to the region  $\sigma > 1$ . A true asymptotic formula for the integral in (1.13) would provide further analytic continuation of  $\mathcal{K}(s)$ . For example, a strong conjecture is that

$$(1.14) \quad \int_1^T E_2^2(t) dt = T^2 p(\log T) + R(T), \quad R(T) \ll_\varepsilon T^{\rho+\varepsilon}$$

with  $p(x)$  a suitable polynomial (perhaps of degree zero) and  $\frac{3}{2} \leq \rho < 2$ . Namely from [2, Theorem 4.1] with  $|H| = T^{\rho/3}$  it follows that (1.14) implies (1.3) with  $c \leq \rho/3$ , hence  $\rho \geq \frac{3}{2}$  must hold in view of (1.7). Then the integral in (1.12) becomes

$$\int_1^\infty (2p(\log x) + p'(\log x)) x^{-s} dx + O(1) + (s+1) \int_1^\infty R(x) x^{-s-1} dx.$$

The first integral above is easily evaluated as  $\sum_{j=1}^{m+1} b_j (s-1)^{-j}$ , where  $m$  is the degree of  $p(x)$ . The second integral is regular for  $\sigma > \rho - 1$  if  $R(x) \ll_\varepsilon x^{\rho+\varepsilon}$ . Thus,

on (1.14), it is seen that  $\mathcal{K}(s)$  is regular for  $\sigma > \rho - 1$  except for a pole at  $s = 1$  of order  $1 + \deg p(x)$ .

Finally we define a Mellin transform related to the spectral theory of the non-Euclidean Laplacian. Let, as usual,  $\alpha_j = |\rho_j(1)|^2 (\cosh \pi \kappa_j)^{-1}$ , where  $\rho_j(1)$  is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue  $\lambda_j$  to which the Hecke series  $H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s}$  is attached. For  $0 < \xi \leq 1$  we define

$$(1.15) \quad \mathcal{I}(t; \xi) = \frac{1}{\sqrt{\pi t \xi}} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it + iu)|^4 \exp(-(u/t^\xi)^2) du.$$

The importance of the function  $\mathcal{I}(t; \xi)$  in the theory of the fourth power moment of  $|\zeta(\frac{1}{2} + it)|$  comes from the fact that, for  $\frac{1}{2} \leq \xi < 1$  and suitable  $C > 0$ , we have (see Y. Motohashi [13] and [15]) the explicit formula

$$(1.16) \quad \begin{aligned} \mathcal{I}(t; \xi) &= \frac{\pi}{\sqrt{2t}} \sum_{j=1}^{\infty} \alpha_j H_j^3(\frac{1}{2}) \kappa_j^{-1/2} \sin\left(\kappa_j \log \frac{\kappa_j}{4et}\right) \exp(-\frac{1}{4}(t^{\xi-1} \kappa_j)^2) + O(\log^C t) \\ &= I(t; \xi) + O(\log^C t), \end{aligned}$$

say. Then we let

$$(1.17) \quad J(s, \xi) = \int_1^{\infty} I(x; \xi) x^{-s} dx$$

denote the (modified) Mellin transform of  $I(t; \xi)$ . In view of the bound (see e.g. [15])

$$(1.18) \quad \sum_{\kappa_j \leq T} \alpha_j H_j^3(\frac{1}{2}) \ll T^2 \log^C T \quad (C > 0)$$

it easily follows that  $J(s, \xi)$  is regular for  $\sigma > 2 - \frac{3}{2}\xi$ .

The plan of the paper is as follows. The function  $\mathcal{Z}_2(s)$  will be studied in Section 2,  $\mathcal{K}(s)$  in Section 3, while Section 4 is devoted to  $J(s, \xi)$ .

## 2. The function $\mathcal{Z}_2(s)$

The new result concerning mean square bounds for  $\mathcal{Z}_2(s)$  is contained in

**THEOREM 1.** *For  $\frac{5}{6} \leq \sigma \leq \frac{5}{4}$  we have*

$$(2.1) \quad \int_1^T |\mathcal{Z}_2(\sigma + it)|^2 dt \ll_{\varepsilon} T^{\frac{15-12\sigma}{5} + \varepsilon}.$$

**Proof.** To prove (2.1) we first introduce, as in [7], the function

$$(2.2) \quad F_K(s) := \int_{K/2}^{5K'/2} \varphi(x) |\zeta(\tfrac{1}{2} + ix)|^4 x^{-s} dx \quad (K < K' \leq 2K),$$

where  $\varphi(x) \in C^\infty$  is a nonnegative function supported in  $[K/2, 5K'/2]$  such that  $\varphi(x) = 1$  for  $K < K' \leq 2K$ , and

$$(2.3) \quad \varphi^{(r)}(x) \ll_r K^{-r} \quad (r = 0, 1, 2, \dots).$$

To connect  $F_K(s)$  and  $\mathcal{Z}_2(s)$  note that from the Mellin inversion formula (e.g., [7, eq. (2.6)] we have

$$|\zeta(\tfrac{1}{2} + ix)|^4 = \frac{1}{2\pi i} \int_{(1+\varepsilon)} \mathcal{Z}_2(s) x^{s-1} ds \quad (x > 1),$$

where the  $\int_{(c)}$  denotes integration over the line  $\Re s = c$ . Here we replace the line of integration by the contour  $\mathcal{L}$ , consisting of the same straight line from which the segment  $[1 + \varepsilon - i, 1 + \varepsilon + i]$  is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole  $s = 1$  of the integrand. By the residue theorem we have

$$(2.4) \quad |\zeta(\tfrac{1}{2} + ix)|^4 = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{Z}_2(s) x^{s-1} ds + Q_4(\log x) \quad (x > 1),$$

where  $Q_4$  is defined by (1.9). Hence by using (2.4) we obtain

$$(2.5) \quad \begin{aligned} F_K(s) &= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{Z}_2(w) \left( \int_{K/2}^{5K'/2} \varphi(x) x^{w-s-1} dx \right) dw \\ &\quad + \int_{K/2}^{5K'/2} \varphi(x) Q_4(\log x) x^{-s} dx. \end{aligned}$$

In view of (2.3) we infer, by repeated integration by parts, that the last integral in (2.5) is  $\ll T^{-A}$  for any given  $A > 0$ . Similarly we note that

$$\begin{aligned} &\int_{K/2}^{5K'/2} \varphi(x) x^{w-s-1} dx \\ &= (-1)^r \int_{K/2}^{5K'/2} \varphi^{(r)}(x) \frac{x^{w-s+r-1}}{(w-s) \cdots (w-s+r-1)} dx \ll T^{-A} \end{aligned}$$

for any given  $A > 0$ , provided that  $|\Im w - \Im s| > T^\varepsilon$  and  $r = r(A, \varepsilon)$  is sufficiently large. Thus if in the  $w$ -integral in (2.5) we replace the contour  $\mathcal{L}$  by the straight line  $\Re w = d$  ( $\frac{1}{2} < d < 1$ ), we shall obtain

$$(2.6) \quad F_K(s) \ll_\varepsilon K^{d-\sigma} \int_{t-T^\varepsilon}^{t+T^\varepsilon} |\mathcal{Z}_2(d+iv)| dv + T^{-2}.$$

Squaring (2.6) and integrating it follows that

$$(2.7) \quad \int_T^{2T} |F_K(s)|^2 dt \ll_\varepsilon T^{-1} + K^{2d-2\sigma} T^\varepsilon \int_{T-T^\varepsilon}^{2T+T^\varepsilon} |\mathcal{Z}_2(d+iv)|^2 dv \quad (\sigma > d).$$

However, from (1.1) and (1.3) with  $c = 2/3$  it follows that

$$(2.8) \quad \int_T^{2T} |\mathcal{Z}_2(\frac{5}{6} + it)|^2 dt \ll_\varepsilon T^{1+\varepsilon},$$

so that (2.7) yields

$$(2.9) \quad \int_T^{2T} |F_K(s)|^2 dt \ll_\varepsilon T^{-1} + K^{5/3-2\sigma} T^{1+\varepsilon} \quad (\frac{5}{6} \leq \sigma \leq 1).$$

Now we write

$$\mathcal{Z}_2(s) = \int_1^X |\zeta(\frac{1}{2} + ix)|^4 x^{-s} dx + \int_X^\infty |\zeta(\frac{1}{2} + ix)|^4 x^{-s} dx = I_1 + I_2,$$

say, which initially holds for  $\sigma > 1$ . To estimate the mean square of  $I_1$ , we use the bound (which, up to ‘ $\varepsilon$ ’, is the strongest one known; see e.g., [1])

$$(2.10) \quad \int_1^X |\zeta(\frac{1}{2} + ix)|^8 dx \ll_\varepsilon X^{3/2+\varepsilon},$$

and the following lemma, whose proof can be found in [7].

LEMMA 1. *Suppose that  $g(x)$  is a real-valued, integrable function on  $[a, b]$ , a subinterval of  $[2, \infty)$ , which is not necessarily finite. Then*

$$(2.11) \quad \int_0^T \left| \int_a^b g(x)x^{-s} dx \right|^2 dt \leq 2\pi \int_a^b g^2(x)x^{1-2\sigma} dx \quad (s = \sigma + it, T > 0, a < b).$$

Then, from (2.10) and (2.11), we find that

$$(2.12) \quad \int_T^{2T} I_1^2 dt \ll_\varepsilon X^{5/2-2\sigma+\varepsilon}.$$

From (2.6) and (2.7) with  $d = 5/6$  we obtain the analytic continuation of  $I_2 = I_2(s)$  to the region  $\sigma > 5/6$ , taking first  $K = 2X$ , writing  $1 = \varphi(x) + (1 - \varphi(x))$  in the integral over  $[\frac{1}{2}X, X]$ , and estimating the mean square of

$$\int_{X/2}^X (1 - \varphi(x)) |\zeta(\frac{1}{2} + ix)|^4 x^{-s} dx$$

by the bound in (2.11). For the remaining integrals we use, after the integrals are split into subintegrals of the type  $F_K(s)$ , the bound given by (2.9). We obtain

$$(2.13) \quad \int_T^{2T} |\mathcal{Z}_2(\sigma + it)|^2 dt \ll_\varepsilon X^{5/2-2\sigma+\varepsilon} + T^{1+\varepsilon} X^{5/3-2\sigma} \ll_\varepsilon T^{(15-12\sigma)/5+\varepsilon}$$

with the choice  $X = T^{6/5}$ . Replacing  $T$  by  $T2^{-j}$  and adding up all the results, we obtain (2.1) in the range  $\frac{5}{6} \leq \sigma \leq 1$ .

To obtain (2.1) in the remaining range  $1 < \sigma \leq \frac{5}{4}$ , first we note that by a slight change of proof we see that (2.7) holds for  $d \geq 1$ . Thus invoking (2.1) with  $\sigma = 1$  it is seen that for  $1 < \sigma \leq \frac{5}{4}$  (when the exponent in (2.12) is non-negative) we obtain

$$(2.14) \quad \int_T^{2T} |\mathcal{Z}_2(\sigma + it)|^2 dt \ll_\varepsilon X^{5/2-2\sigma+\varepsilon} + T^{3/5+\varepsilon} X^{2-2\sigma} \ll_\varepsilon T^{(15-12\sigma)/5+\varepsilon}$$

again with the choice  $X = T^{6/5}$ . The proof of Theorem 1 is complete.

As a corollary of (2.1) we can obtain (2.10), although this is somewhat going round in a circle, since we actually used (2.10) in the course of proof of (2.1). Recall that we have (1.7), but the analysis of its proof clearly shows that it remains valid for  $\sigma \geq 1$  as well. If we use (2.1) with  $\sigma = 5/4$  in (1.7), then (2.10) immediately follows. The essentially new result provided by Theorem 1 is the bound

$$(2.15) \quad \int_1^T |\mathcal{Z}_2(1 + it)|^2 dt \ll_\varepsilon T^{3/5+\varepsilon},$$

and it would be of great interest to decrease the exponent of  $T$  on the right-hand side of (2.15). In fact, the hypothetical estimate

$$(2.16) \quad \int_1^X |\zeta(\frac{1}{2} + ix)|^8 dx \ll_\varepsilon X^{1+\varepsilon}$$

is equivalent to

$$(2.17) \quad \int_1^T |\mathcal{Z}_2(1+it)|^2 dt \ll_{\varepsilon} T^{\varepsilon}.$$

From (1.7) with  $\sigma = 1$  it follows at once that (2.17) implies (2.16), and the other implication follows by the method of proof of (1.7) in [7]. This fact stresses out once again the importance of mean square bounds for  $\mathcal{Z}_2(s)$ .

### 3. The function $\mathcal{K}(s)$

In this section we shall deal with the function  $\mathcal{K}(s)$ , defined by (1.10) or (1.12). Our result is the following

**THEOREM 2.** *The function  $\mathcal{K}(s)$ , defined by (1.10), admits analytic continuation which is regular for  $\Re s > 1$ . It satisfies*

$$(3.1) \quad \mathcal{K}(\sigma + it) \ll_{\varepsilon} |t|^{\varepsilon} (|t|^{3-2\sigma} + 1) \quad (\sigma > 1)$$

and

$$(3.2) \quad \int_0^T |\mathcal{K}(\sigma + it)|^2 dt \ll_{\varepsilon} T^{\frac{13-6\sigma}{3} + \varepsilon} \quad \left(\frac{7}{6} \leq \sigma \leq \frac{13}{6}\right).$$

**Proof.** To prove (3.1) note first that, by the Cauchy-Schwarz inequality for integrals, we have ( $C > 0$ )

$$(3.3) \quad \begin{aligned} & \int_Y^{2Y} \left| |\zeta(\tfrac{1}{2} + ix)|^4 - Q_4(\log x) \right| |E_2(x)| x^{-\sigma} dx \ll Y^{3/2-\sigma} \log^C Y + \\ & + Y^{-\sigma} \left( \int_Y^{2Y} |\zeta(\tfrac{1}{2} + ix)|^4 dx \right)^{1/2} \left( \int_Y^{2Y} (|\zeta(\tfrac{1}{2} + ix)|^4 - Q_4(\log x)) E_2^2(x) dx \right)^{1/2} \\ & \ll Y^{3/2-\sigma} \log^C Y. \end{aligned}$$

In the last integral we integrated by parts, recalling that that (1.11) holds, as well as (1.13) and (1.3) with  $c = 2/3$ . The above bound shows then that  $\mathcal{K}(s) \ll 1$  for  $\sigma > 3/2$ . Suppose now that  $1 < \sigma \leq 3/2$ . Similarly to (1.12) we have

$$(3.4) \quad \begin{aligned} \mathcal{K}(s) &= \int_1^X (|\zeta(\tfrac{1}{2} + ix)|^4 - Q_4(\log x)) E_2(x) x^{-s} dx \\ &\quad - \tfrac{1}{2} E_2^2(X) X^{-s} + \tfrac{1}{2} s \int_X^{\infty} E_2^2(x) x^{-s-1} dx. \end{aligned}$$



From (3.3) it follows that the first integral above is  $\ll X^{3/2-\sigma} \log^C X$ , and the second (by (1.13)) is  $\ll |t|X^{1-\sigma}$ . The choice  $X = t^2$  easily leads then to (3.1).

To prove (3.2) we start from (3.4) and use Lemma 1. We obtain

$$(3.5) \quad \int_T^{2T} \left| \int_1^X (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x)) E_2(x) x^{-s} dx \right|^2 dt \\ \ll \int_1^X (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x))^2 E_2^2(x) x^{1-2\sigma} dx.$$

Defining the Lindelöf function

$$\mu(\sigma) = \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t} \quad (\sigma \in \mathbb{R})$$

in the customary way and letting  $\varphi(x)$  be as in (2.2), we see that

$$\int_{K/2}^{5K'/2} \varphi(x) (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x))^2 E_2^2(x) x^{1-2\sigma} dx \\ \ll_\varepsilon K^{1-2\sigma+4\mu(\frac{1}{2})+\varepsilon} \int_{K/2}^{5K'/2} \varphi(x) (|\zeta(\frac{1}{2} + ix)|^4 + \log^8 x) E_2^2(x) dx \\ = K^{1-2\sigma+4\mu(\frac{1}{2})+\varepsilon} \int_{K/2}^{5K'/2} \varphi(x) (E_2'(x) + Q_4(\log x) + \log^8 x) E_2^2(x) dx \\ = K^{1-2\sigma+4\mu(\frac{1}{2})+\varepsilon} \int_{K/2}^{5K'/2} \varphi(x) (\frac{1}{3} E_2^3(x))' dx + O_\varepsilon(K^{3-2\sigma+4\mu(\frac{1}{2})+\varepsilon}) \\ \ll_\varepsilon K^{3-2\sigma+4\mu(\frac{1}{2})+\varepsilon}.$$

Therefore the expression on the right-hand side of (3.5) is, for  $X = T^C$ ,  $C > 0$ ,

$$(3.6) \quad \ll_\varepsilon T^\varepsilon (1 + X^{3-2\sigma+4\mu(\frac{1}{2})}).$$

Next we have, by Lemma 1, (1.3) with  $c = 2/3$  and (1.13),

$$(3.7) \quad \int_T^{2T} \left| s \int_X^\infty E_2^2(x) x^{-s-1} dx \right|^2 dt \ll T^2 \int_X^\infty E_2^4(x) x^{-2\sigma-1} dx \\ \ll_\varepsilon T^2 \int_X^\infty E_2^2(x) x^{1/3-2\sigma+\varepsilon} dx \ll_\varepsilon T^2 X^{7/3-2\sigma+\varepsilon},$$

provided that  $\sigma > \frac{7}{6}$ . From (3.4), (3.6) and (3.7) we infer that

$$\int_T^{2T} |\mathcal{K}(\sigma + it)|^2 dt \ll_\varepsilon T^\varepsilon (1 + X^{3-2\sigma+4\mu(\frac{1}{2})} + T^2 X^{7/3-2\sigma}) \\ \ll_\varepsilon T^{\frac{13-6\sigma}{3}+\varepsilon} \quad (\frac{7}{6} \leq \sigma \leq \frac{13}{6})$$

with the trivial bound  $\mu(\frac{1}{2}) < \frac{1}{6}$  and  $X = T$ . This easily gives (3.2), and slight improvements are possible with a better value of  $\mu(\frac{1}{2})$ . A mean square bound can also be obtained for the whole range  $\sigma > 1$ , by using the trivial bound  $tX^{1-\sigma+\varepsilon}$  for the second integral in (3.4). This will lead to

$$\int_1^T |\mathcal{K}(\sigma + it)|^2 dt \ll_\varepsilon \begin{cases} T^{1+\varepsilon} & (\sigma > 3/2), \\ T^{\frac{33-18\sigma}{5}+\varepsilon} & (1 < \sigma \leq 3/2). \end{cases}$$

Mean square estimates for  $\mathcal{K}(s)$  can be used to bound the fourth moment of  $E_2(t)$ , much in the same way that mean square estimates for  $\mathcal{Z}_2(s)$  can be used (cf. (1.7)) to bound the eighth moment of  $|\zeta(\frac{1}{2} + it)|$ . We have

**THEOREM 3.** *For  $\sigma > 1$  fixed*

$$(3.8) \quad \int_T^{2T} E_2^4(t) dt \ll_\varepsilon T^{2\sigma+1} \left( 1 + \int_0^{T^{1+\varepsilon}} \frac{|\mathcal{K}(\sigma + it)|^2}{1+t^2} dt \right).$$

**Proof.** Write (1.12) as

$$(3.9) \quad k(s) := \int_1^\infty E_2^2(x) x^{-s-1} dx = \frac{2}{s} (\mathcal{K}(s) + \frac{1}{2} E_2^2(1)),$$

so that  $k(s)$  is regular for  $\sigma > 1$ . From the Mellin inversion formula for the (modified) Mellin transform (see [7, Lemma 1]) we have

$$E_2^2(x) = \frac{1}{2\pi i} \int_{(c)} k(s) x^s ds \quad (x > 1, c > 1).$$

If  $\psi(t)$  is a smooth, nonnegative function supported in  $[T/2, 5T/2]$  such that  $\psi(t) = 1$  for  $T \leq t \leq 2T$ , then

$$(3.10) \quad \int_T^{2T} E_2^4(x) dx \leq \int_{T/2}^{5T/2} \psi(x) E_2^4(x) dx = \frac{1}{2\pi i} \int_{(c)} k(s) \left( \int_{T/2}^{5T/2} \psi(x) E_2^2(x) x^s dx \right) ds.$$

In the last integral over  $x$  we perform a large number of integrations by parts, keeping in mind that  $\psi^{(j)}(x) \ll_j T^{-j}$  ( $j = 0, 1, \dots$ ). It transpires that only the

values of  $|t| \leq T^{1+\varepsilon}$  in the integral over  $s = \sigma + it$  will make a non-negligible contribution. Hence (3.10) (with  $c = \sigma > 1$ ) and Lemma 1 yield

$$\begin{aligned} I &:= \int_{T/2}^{5T/2} \psi(x) E_2^4(x) dx \ll_{\varepsilon} 1 + \int_{-T^{1+\varepsilon}}^{T^{1+\varepsilon}} |k(\sigma + it)| \left| \int_{T/2}^{5T/2} \psi(x) E_2^2(x) x^s dx \right| dt \\ &\ll_{\varepsilon} 1 + \left( \int_{-T^{1+\varepsilon}}^{T^{1+\varepsilon}} |k(\sigma + it)|^2 dt \right)^{1/2} \left( \int_{T/2}^{5T/2} \psi^2(x) E_2^4(x) x^{2\sigma+1} dx \right)^{1/2} \\ &\ll_{\varepsilon} 1 + \left( \int_0^{T^{1+\varepsilon}} |k(\sigma + it)|^2 dt \right)^{1/2} T^{\sigma+\frac{1}{2}} I^{1/2}. \end{aligned}$$

Simplifying the above expression and using (3.9) we arrive at (3.8).

One expects, in conjunction with the conjecture  $E_2(x) \ll_{\varepsilon} x^{1/2+\varepsilon}$ , the bound

$$(3.11) \quad \int_0^T E_2^4(t) dt \ll_{\varepsilon} T^{3+\varepsilon}$$

to hold as well. In fact, from the author's work [5, Theorem 2] with  $a = 4$ , one sees that the lower bound

$$\int_0^T E_2^4(t) dt \gg T^3$$

does indeed hold. The upper bound in (3.11) nevertheless seems unattainable at present. If true, it implies (by e.g., [7, eq. (4.4)] and Hölder's inequality) the hitherto unproved bounds  $E_2(T) \ll_{\varepsilon} T^{3/5+\varepsilon}$  and ([2, Lemma 4.1])  $\zeta(\frac{1}{2} + it) \ll_{\varepsilon} |t|^{3/20+\varepsilon}$ . From (3.2) of Theorem 2 with  $\sigma = 7/6$  we obtain

$$(3.12) \quad \int_0^T E_2^4(t) dt \ll_{\varepsilon} T^{10/3+\varepsilon}.$$

However, the bound (3.12) was already used in proving Theorem 3 via (3.7). It is (up to ' $\varepsilon$ ') the strongest known bound for the integral in question. From (3.8) it is seen that the conjectural bound (3.11) holds if

$$(3.13) \quad \int_1^T |\mathcal{K}(\sigma + it)|^2 dt \ll_{\varepsilon} T^{2+\varepsilon} \quad (\sigma > 1)$$

holds. Conversely, if (3.11) holds, then the bound in (3.7) is to be replaced by  $\ll_{\varepsilon} T^2 X^{2-2\sigma+\varepsilon}$  ( $\sigma > 1$ ), and (3.13) follows from this bound and (3.6) (with  $X = T^{6/5}$ ). Therefore (3.11) is equivalent to the mean square bound (3.13).

#### 4. The function $J(s, \xi)$

The result on the function  $J(s, \xi)$  ( $0 \leq \xi < 1$ ) is contained in

**THEOREM 4.** *The function  $J(s, \xi)$  admits analytic continuation to the region  $\Re s > \frac{1}{2}$ , where it represents a regular function. Moreover*

$$(4.1) \quad J(\sigma + it, \xi) \ll_{\varepsilon} t^{-1} + t^{\frac{1-\frac{1}{2}\xi-\sigma}{1-\xi}+\varepsilon} \quad (\sigma > \frac{1}{2}, t \geq t_0, 0 \leq \xi < 1).$$

**Proof.** Let  $X = t^{1/(1-\xi)-\delta}$  for a small, fixed  $\delta > 0$ . We define a sequence of non-negative, smooth functions  $\rho_j(x)$  ( $j \in \mathbb{N}$ ) in the following way. Let  $\rho_1(x) (\geq 0)$  be a smooth function supported in  $[1, 2X]$  such that  $\rho_1(x) = 1$  for  $1 \leq x \leq X$ , and  $\rho_1(x)$  monotonically decreases from 1 to 0 in  $[X, 2X]$ . The function  $\rho_2(x)$  is supported in  $[X, 6X]$ , where  $\rho_2(x) = 1 - \rho_1(x)$  for  $X \leq x \leq 2X$ ,  $\rho_2(x) = 1$  for  $2X \leq x \leq 4X$  and  $\rho_2(x)$  monotonically decreases from 1 to 0 in  $[4X, 6X]$ . In general, the function  $\rho_j(x)$ , supported in  $[2^{j-1}X, 3 \cdot 2^j X]$ , satisfies  $\rho_j(x) = 1 - \rho_{j-1}(x)$  for  $2^{j-1}X \leq x \leq 3 \cdot 2^{j-1}X$ ,  $\rho_j(x) = 1$  for  $[2^{j-1}X, 2^j X]$  and then decreases monotonically from 1 to 0 in  $[2^j X, 3 \cdot 2^j X]$ . In this way we obtain that

$$(4.2) \quad \rho_j^{(r)}(x) \ll_{j,r} (2^j X)^{-r} \quad (j, r \in \mathbb{N}).$$

Now we write (cf. (1.16))

$$(4.3) \quad J(s, \xi) = \int_1^{2X} \rho_1(x) I(x; \xi) x^{-s} dx + \sum_{j \geq 2} \int_{2^{j-1}X}^{3 \cdot 2^j X} \rho_j(x) I(x; \xi) x^{-s} dx.$$

In the first integral in (4.3) we insert the expression (1.16) for  $I(x; \xi)$  and integrate repeatedly by parts the factor  $x^{-1/2-i\kappa_j}$ . The integrated terms, after  $r$  integrations by parts, will be

$$\sum_{j=1}^r \frac{A_j}{(s - \frac{1}{2})^j}$$

for suitable constants  $A_j$ . The remaining integral, in view of (4.2), will be  $\ll t^{-B}$  for any given  $B > 0$ , provided that  $r = r(B)$  is sufficiently large. There remain the integrals

$$I(K) := \int_{K/2}^{3K} \rho(x) I(x; \xi) x^{-s} dx \quad (\rho(x) = \rho_j(x), K = 2^j X).$$

Writing the sine in (1.16) as a sum of exponentials, it follows that  $I(K)$  is a linear combination of expressions the type

$$J_{\pm}(K) := \sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right) \kappa_j^{-1/2} e^{\pm \kappa_j \log \frac{\kappa_j}{4e}} \int_{K/2}^{3K} \rho(x) x^{-\frac{1}{2} \mp i \kappa_j - s} \exp\left(-\frac{1}{4}(x^{\xi-1} \kappa_j)^2\right) dx,$$

and we may consider only the case of the ‘+’ sign, since the other case is treated analogously. The above series may be truncated at  $\kappa_j = K^{1-\xi} \log K$  with a negligible error. After an integration by parts the integral in  $J_{\pm}(K)$  becomes

$$\begin{aligned} & \frac{1}{s - i \kappa_j + \frac{1}{2}} \int_{K/2}^{3K} x^{\frac{1}{2} + i \kappa_j - s} \exp\left(-\frac{1}{4}(x^{\xi-1} \kappa_j)^2\right) \times \\ & \times \left(\rho'(x) + \frac{1}{2}(1 - \xi)\rho(x)x^{2\xi-3}\kappa_j^2\right) dx. \end{aligned}$$

In the range  $\kappa_j \leq K^{1-\xi} \log K$  the above expression in parentheses is

$$\ll K^{-1} + K^{2\xi-3} K^{2-2\xi} \log^2 K \ll K^{-1} \log^2 K.$$

It transpires that, performing sufficiently many integrations by parts, only the values of  $\kappa_j$  for which  $|\kappa_j - t| \leq K^\varepsilon$  will make a non-negligible contribution. For the estimation of  $\alpha_j H_j^3\left(\frac{1}{2}\right)$  in short intervals we shall need (see the author’s work [6])

LEMMA 2. *We have*

$$(4.4) \quad \sum_{K-G \leq \kappa_j \leq K+G} \alpha_j H_j^3\left(\frac{1}{2}\right) \ll_{\varepsilon} G K^{1+\varepsilon} \quad (K^\varepsilon \leq G \leq K).$$

Note that (4.4) implies the bound

$$H_j\left(\frac{1}{2}\right) \ll_{\varepsilon} \kappa_j^{1/3+\varepsilon},$$

which breaks the convexity bound  $H_j\left(\frac{1}{2}\right) \ll_{\varepsilon} \kappa_j^{1/2+\varepsilon}$ , but is still far away from the conjectural bound

$$H_j\left(\frac{1}{2} + it\right) \ll_{\varepsilon} (\kappa_j + |t|)^{\varepsilon},$$

which may be thought of as the analogue of the classical Lindelöf hypothesis ( $\zeta\left(\frac{1}{2} + it\right) \ll_{\varepsilon} |t|^{\varepsilon}$ ) for the Hecke series.

To complete the proof of Theorem 4, note that with the use of Lemma 2 we obtain

$$\begin{aligned} J_{\pm}(K) & \ll_{\varepsilon} K^{-1/2-\sigma} K \sum_{|\kappa_j - t| \leq K^\varepsilon} \alpha_j H_j^3\left(\frac{1}{2}\right) \kappa_j^{-1/2} \\ & \ll_{\varepsilon} K^{1/2-\sigma} t^{1/2+\varepsilon} \ll_{\varepsilon} t^{\frac{1-\frac{1}{2}\xi-\sigma}{1-\xi}+\varepsilon} \end{aligned}$$

since  $K \gg X (= t^{1/(1-\xi)-\delta})$ . This leads to (4.1) in view of (4.3) and the preceding discussion.

## REFERENCES

- [1] A. Ivić, The Riemann zeta-function, *John Wiley and Sons*, New York, 1985.
- [2] A. Ivić, Mean values of the Riemann zeta-function, LN's **82**, *Tata Institute of Fundamental Research*, Bombay, 1991 (distr. by Springer Verlag, Berlin etc.).
- [3] A. Ivić, On the fourth moment of the Riemann zeta-function, *Publs. Inst. Math. (Belgrade)* **57(71)** (1995), 101-110.
- [4] A. Ivić, The Mellin transform and the Riemann zeta-function, *Proceedings of the Conference on Elementary and Analytic Number Theory (Vienna, July 18-20, 1996)*, Universität Wien & Universität für Bodenkultur, Eds. W.G. Nowak and J. Schoißengeier, Vienna 1996, 112-127.
- [5] A. Ivić, On the error term for the fourth moment of the Riemann zeta-function, *J. London Math. Soc.* **60(2)**(1999), 21-32.
- [6] A. Ivić, On sums of Hecke series in short intervals, *J. de Théorie des Nombres Bordeaux* **13**(2001), 1-16.
- [7] A. Ivić, On some conjectures and results for the Riemann zeta-function, *Acta Arith.* **109**(2001), 115-145.
- [8] A. Ivić, Some mean value results for the Riemann zeta-function, in 'Number Theory. Proc. Turku Symposium 1999' (M. Jutila et al. eds.), de Gruyter, 2001, Berlin, 145-161.
- [9] A. Ivić, On the estimation of  $Z_2(s)$ , in 'Anal. Probab. Number Theory' (A. Dubickas et al. eds.), TEV, 2002, Vilnius, 83-98.
- [10] A. Ivić and Y. Motohashi, The mean square of the error term for the fourth moment of the zeta-function, *Proc. London Math. Soc.* (3)**66**(1994), 309-329.
- [11] A. Ivić and Y. Motohashi, The fourth moment of the Riemann zeta-function, *J. Number Theory* **51**(1995), 16-45.
- [12] A. Ivić, M. Jutila and Y. Motohashi, The Mellin transform of powers of the Riemann zeta-function, *Acta Arith.* **95**(2000), 305-342.
- [13] Y. Motohashi, An explicit formula for the fourth power mean of the Riemann zeta-function, *Acta Math.* **170**(1993), 181-220.
- [14] Y. Motohashi, A relation between the Riemann zeta-function and the hyperbolic Laplacian, *Annali Scuola Norm. Sup. Pisa, Cl. Sci. IV ser.* **22**(1995), 299-313.
- [15] Y. Motohashi, Spectral theory of the Riemann zeta-function, *Cambridge University Press*, Cambridge, 1997.

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