

Planar cubic polynomial differential systems with the maximum number of invariant straight lines

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Abstract

We classify all cubic systems possessing the maximum number of invariant straight lines (real or complex) taking into account their multiplicities. We prove that there are exactly 23 topological different classes of such systems. For every class we provide the configuration of its invariant straight lines in the Poincaré disc. Moreover, every class is characterized by a set of affine invariant conditions.

1. Introduction and statement of the main results

We consider here real polynomial differential system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (1)$$

where P, Q are polynomials in x, y with real coefficients, i.e. $P, Q \in \mathbb{R}[x, y]$. We shall say that system (1) is *cubic* if $\max(\deg(P), \deg(Q)) = 3$.

A straight line $ux + vy + w = 0$ satisfies

$$u \frac{dx}{dt} + v \frac{dy}{dt} = uP(x, y) + vQ(x, y) = (ux + vy + w)R(x, y)$$

for some polynomial $R(x, y)$ if and only if it is *invariant* under the flow of the system. If some of the coefficients u, v, w of an invariant straight line belongs to $\mathbb{C} \setminus \mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real*. Note that, since system (1) is real, if it has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

Let

$$\mathbf{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$$

be the polynomial vector field corresponding to system (1).

An invariant straight line $f = 0$ for a cubic vector field \mathbf{X} has *geometric multiplicity* m if there exists a sequence of cubic vector fields \mathbf{X}_k converging to \mathbf{X} , such that each \mathbf{X}_k has

m distinct invariant straight lines $f_k^1 = 0, \dots, f_k^m = 0$, converging to $f = 0$ as $k \rightarrow \infty$, and this does not occur for $m + 1$.

An invariant straight line $f = 0$ for a cubic vector field \mathbf{X} has *algebraic multiplicity* m if m is the greatest positive integer such that f^m divides $P\mathbf{X}(Q) - Q\mathbf{X}(P)$. In [4] it is proved that both notions of multiplicity coincide. The algebraic definition of multiplicity is very useful for its computation.

We note that this definition of multiplicity can be applied to the infinite line $Z = 0$ in the case when this line is not full of singular points. So, including the infinite line according with [2] the maximum number of the invariant straight lines for cubic systems is 9.

In this paper we classify all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities.

Invariant straight lines for quadratic systems has been studied by Druzhkova [6] and Popa and Sibirskii [16, 17], for cubic systems by Liybimova [9, 10], for quartic systems by Sokulski [22] and Xiang Zhang [23], for some more general systems by Popa [13, 14] and Popa and Sibirskii [15]

The maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line, see [2]. This bound is always reached if we consider the real and the complex invariant straight lines, see [4].

Using geometric invariants as well as algebraic ones a classification of all quadratic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [19]

It is well known that for cubic system (1) there exist at most 4 different slopes for invariant affine straight lines, for more information about the slopes of invariant straight lines for polynomial vector fields, see [1].

If a cubic system (1) possesses 9 distinct invariant straight lines we say that these lines form a *configuration of type* (3,3,1,1) if there exist two triples of parallel lines and two additional lines every set with different slopes. And we shall say that these lines form a *configuration of type* (3,2,2,1) if there exist one triple and two couple of parallel lines and one additional line every set with different slopes. Note that in both configurations the straight line which is omitted is the infinite one.

If a cubic system (1) possesses 9 invariant straight lines taking into account their multiplicities we shall say that these lines form a *potential configuration of type* (3,3,1,1) (respectively, (3,2,2,1)) if there exists a sequence of vector fields \mathbf{X}_k as in the definition of

geometric multiplicity having 9 distinct line of type (3,3,1,1) (respectively, (3,2,2,1)).

Consider generic cubic systems of the form:

$$\begin{aligned}\frac{dx}{dt} &= p_0 + p_1(x, y) + p_2(x, y) + p_3(x, y) \equiv P(x, y), \\ \frac{dy}{dt} &= q_0 + q_1(x, y) + q_2(x, y) + q_3(x, y) \equiv Q(x, y),\end{aligned}\tag{2}$$

with real homogeneous polynomials p_i and q_i ($i = 0, 1, 2, 3$) of degree i in x, y . We introduce the following polynomials:

$$\begin{aligned}C_i &= yp_i(x, y) - xq_i(x, y), \\ D_j &= \frac{\partial p_j}{\partial x} + \frac{\partial q_j}{\partial y},\end{aligned}$$

for $i = 0, 1, 2, 3$ and $j = 1, 2, 3$ which in fact are GL -comitants, see [20].

In order to state our Main Theorem we need to construct some T -comitants and CT -comitants (see [18] for detailed definitions) which will be responsible for the existence of the maximum number of invariant straight lines for system (2). They are constructed by using the polynomials C_i and D_i via the differential operator $(f, g)^{(k)}$ called *transvectant of index k* (see, for example, [7]) which acts on $\mathbb{R}[\mathbf{a}, x, y]$ as follows:

$$(f, g)^{(k)} = \sum_{h=0}^k (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

Here $f(x, y)$ and $g(x, y)$ are polynomials in x and y of the degree r and s , respectively, and $\mathbf{a} \in \mathbb{R}^{20}$ is the 20-tuple formed by all the coefficients of system (2).

First we construct the following comitants of second degree with respect to the coefficients of the initial system:

$$\begin{aligned}T_1 &= (C_0, C_1)^{(1)}, & T_{10} &= (C_1, C_3)^{(1)}, & T_{19} &= (C_2, D_3)^{(1)}, \\ T_2 &= (C_0, C_2)^{(1)}, & T_{11} &= (C_1, C_3)^{(2)}, & T_{20} &= (C_2, D_3)^{(2)}, \\ T_3 &= (C_0, D_2)^{(1)}, & T_{12} &= (C_1, D_3)^{(1)}, & T_{21} &= (D_2, C_3)^{(1)}, \\ T_4 &= (C_0, C_3)^{(1)}, & T_{13} &= (C_1, D_3)^{(2)}, & T_{22} &= (D_2, D_3)^{(1)}, \\ T_5 &= (C_0, D_3)^{(1)}, & T_{14} &= (C_2, C_2)^{(2)}, & T_{23} &= (C_3, C_3)^{(2)}, \\ T_6 &= (C_1, C_1)^{(2)}, & T_{15} &= (C_2, D_2)^{(1)}, & T_{24} &= (C_3, C_3)^{(4)}, \\ T_7 &= (C_1, C_2)^{(1)}, & T_{16} &= (C_2, C_3)^{(1)}, & T_{25} &= (C_3, D_3)^{(1)}, \\ T_8 &= (C_1, C_2)^{(2)}, & T_{17} &= (C_2, C_3)^{(2)}, & T_{26} &= (C_3, D_3)^{(2)}, \\ T_9 &= (C_1, D_2)^{(1)}, & T_{18} &= (C_2, C_3)^{(3)}, & T_{27} &= (D_3, D_3)^{(2)}.\end{aligned}$$

Then we need the following polynomials:

$$\begin{aligned}\mathcal{D}_1(\mathbf{a}) &= 6T_{24}^3 - [(C_3, T_{23})^{(4)}]^2, \\ \mathcal{D}_2(\mathbf{a}, x, y) &= -T_{23}, \\ \mathcal{D}_3(\mathbf{a}, x, y) &= (T_{23}, T_{23})^{(2)} - 6C_3(C_3, T_{23})^{(4)},\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_4(\mathbf{a}) &= (C_3, \mathcal{D}_2)^{(4)}, \\
\mathcal{V}_1(\mathbf{a}, x, y) &= T_{23} + 2D_3^2, \\
\mathcal{V}_2(\mathbf{a}, x, y) &= T_{26}, \\
\mathcal{V}_3(\mathbf{a}, x, y) &= 6T_{25} - 3T_{23} - 2D_3^2, \\
\mathcal{V}_4(\mathbf{a}, x, y) &= C_3 \left[(C_3, T_{23})^{(4)} + 36 (D_3, T_{26})^{(2)} \right], \\
\mathcal{L}_1(\mathbf{a}, x, y) &= 9C_2 (T_{24} + 24T_{27}) - 12D_3 (T_{20} + 8T_{22}) - 12 (T_{16}, D_3)^{(2)} \\
&\quad - 3 (T_{23}, C_2)^{(2)} - 16 (T_{19}, C_3)^{(2)} + 12 (5T_{20} + 24T_{22}, C_3)^{(1)}, \\
\mathcal{L}_2(\mathbf{a}, x, y) &= 32 (13T_{19} + 33T_{21}, D_2)^{(1)} + 84 (9T_{11} - 2T_{14}, D_3)^{(1)} \\
&\quad + 8D_2 (12T_{22} + 35T_{18} - 73T_{20}) - 448 (T_{18}, C_2)^{(1)} \\
&\quad - 56 (T_{17}, C_2)^{(2)} - 63 (T_{23}, C_1)^{(2)} + 756D_3T_{13} - 1944D_1T_{26} \\
&\quad + 112 (T_{17}, D_2)^{(1)} - 378 (T_{26}, C_1)^{(1)} + 9C_1 (48T_{27} - 35T_{24}), \\
\mathcal{L}_3(\mathbf{a}, x, y) &= (T_{23}, D_3)^{(2)} \left[(D_2, T_{22})^{(1)} - D_1T_{27} \right], \\
\mathcal{L}_4(\mathbf{a}, x, y) &= T_{25}, \\
\mathcal{N}_1(\mathbf{a}, x, y) &= 4C_2(27D_1D_3 - 8D_2^2) + 2C_2(20T_{15} - 4T_{14} + 39T_{12}) \\
&\quad + 18C_1(3T_{21} - D_2D_3) + 54D_3(3T_4 - T_7) - 288C_3T_9 \\
&\quad + 54 (T_7, C_3)^{(1)} - 567 (T_4, C_3)^{(1)} + 135C_0D_3^2, \\
\mathcal{N}_2(\mathbf{a}, x, y) &= 2C_2D_3 - 3C_3D_2, \\
\mathcal{N}_3(\mathbf{a}, x, y) &= C_2D_3 + 3T_{16}, \\
\mathcal{N}_4(\mathbf{a}, x, y) &= D_2D_3 + 9T_{21} - 2T_{17}, \\
\mathcal{N}_5(\mathbf{a}, x, y) &= T_{17} + 2T_{19}, \\
\mathcal{N}_6(\mathbf{a}, x, y) &= 6C_3(T_{12} + 6T_{11}) - 9C_1(T_{23} + T_{25}) - 8 (T_{16}, C_2)^{(1)} - C_3D_2^2, \\
\mathcal{N}_7(\mathbf{a}, x, y) &= 6C_3(12T_{11} - T_{12} - 6D_1D_3) - 21C_1T_{23} - 24 (T_{16}, C_2)^{(1)} \\
&\quad + 3C_1T_{25} + 4D_2(T_{16} + 2D_2C_3 - C_2D_3), \\
\mathcal{N}_8(\mathbf{a}, x, y) &= D_2^2 - 4D_1D_3, \\
\mathcal{N}_9(\mathbf{a}, x, y) &= C_2^2 - 3C_1C_3, \\
\mathcal{N}_{10}(\mathbf{a}, x, y) &= 2C_2D_1 + 3T_4.
\end{aligned}$$

Main Theorem *Any cubic system having invariant straight lines with total multiplicity 9 via affine transformation and time rescaling can be written as one of the following 23 systems. In the figure associated to each system is presented the configuration of its invariant straight lines in the Poincaré disc. Real invariant straight lines are represented by continuous lines. Complex invariant straight lines are represented by dashed lines. If an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is more thick. Moreover, every system has associated a*

set of affine invariant conditions which characterize it.

- (1) $\begin{cases} \dot{x} = x(x^2 - 1), \\ \dot{y} = y(y^2 - 1) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{L}_3 < 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 1;}$
- (2) $\begin{cases} \dot{x} = x(x^2 + 1), \\ \dot{y} = y(y^2 + 1) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{L}_3 > 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 2;}$
- (3) $\begin{cases} \dot{x} = x^3, \\ \dot{y} = y^3 \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{L}_3 = 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 3;} \\ (30) \end{array}$
- (4) $\begin{cases} \dot{x} = 2x(x^2 - 1), \\ \dot{y} = (3x - y)(y^2 - 1) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{L}_3 > 0, \\ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 4;}$
- (5) $\begin{cases} \dot{x} = 2x(x^2 + 1), \\ \dot{y} = (3x - y)(y^2 + 1) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{L}_3 < 0, \\ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 5;}$
- (6) $\begin{cases} \dot{x} = 2x^3, \\ \dot{y} = y^2(3x - y) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{L}_3 = 0, \\ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 6;} \\ (44) \end{array}$
- (7) $\begin{cases} \dot{x} = x(1 + x^2), \\ \dot{y} = y(1 - y^2) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 \neq 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 7;}$
- (8) $\begin{cases} \dot{x} = x^3, \\ \dot{y} = -y^3, \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 = 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 8;} \\ (52) \end{array}$
- (9) $\begin{cases} \dot{x} = x(1 + x^2 - 3y^2), \\ \dot{y} = y(1 + 3x^2 - y^2) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 \neq 0, \mathcal{L}_4 > 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 9;}$
- (10) $\begin{cases} \dot{x} = x(x^2 - 3y^2), \\ \dot{y} = y(3x^2 - y^2) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 = 0, \mathcal{L}_4 > 0, \\ \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 10;} \\ (55) \end{array}$
- (11) $\begin{cases} \dot{x} = 2x(x^2 - 1), \\ \dot{y} = y(3x^2 + y^2 + 1) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 < 0, \\ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 11;}$
- (12) $\begin{cases} \dot{x} = 2x(x^2 + 1), \\ \dot{y} = y(3x^2 + y^2 - 1) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 > 0, \\ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \text{Fig. 12;}$
- (13) $\begin{cases} \dot{x} = 2x^3, \\ \dot{y} = y(3x^2 + y^2) \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 < 0, \mathcal{L}_3 = 0, \\ \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 13} \\ (62); \end{array}$
- (14) $\begin{cases} \dot{x} = x(x^2 - 1), \\ \dot{y} = 2y \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_7 = 0, \mathcal{N}_8 < 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 14} \\ (71), (72); \end{array}$
- (15) $\begin{cases} \dot{x} = x(x^2 + 1), \\ \dot{y} = -2y \end{cases} \Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_7 = 0, \mathcal{N}_8 > 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 15} \\ (73), (74); \end{array}$

$$\begin{aligned}
(16) \quad \begin{cases} \dot{x} = x(x^2 - 1), \\ \dot{y} = -y \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 = 0, \mathcal{N}_8 > 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 16} \\ (76),(77); \end{array} \\
(17) \quad \begin{cases} \dot{x} = x(x^2 + 1), \\ \dot{y} = y \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 = 0, \mathcal{N}_8 < 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 17} \\ (78),(79); \end{array} \\
(18) \quad \begin{cases} \dot{x} = x^3, \\ \dot{y} = 1 \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 < 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 = 0, \mathcal{N}_8 = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 18} \\ (80),(81); \end{array} \\
(19) \quad \begin{cases} \dot{x} = x(x^2 - 1), \\ \dot{y} = y(3x^2 - 1) \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 > 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 = 0, \mathcal{N}_8 > 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 19} \\ (88); \end{array} \\
(20) \quad \begin{cases} \dot{x} = x(x^2 + 1), \\ \dot{y} = y(3x^2 + 1) \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 > 0, \\ \mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 = 0, \mathcal{N}_8 < 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 20} \\ (89); \end{array} \\
(21) \quad \begin{cases} \dot{x} = 2x(x^2 - 1), \\ \dot{y} = y(3x^2 + 1) \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 > 0, \\ \mathcal{V}_3 = \mathcal{N}_1 = \mathcal{N}_4 = \mathcal{N}_5 = \mathcal{N}_7 = 0, \mathcal{N}_8 > 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 21} \\ (95); \end{array} \\
(22) \quad \begin{cases} \dot{x} = 2x(x^2 + 1), \\ \dot{y} = y(3x^2 - 1) \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0, \mathcal{D}_2 \neq 0, \mathcal{L}_4 > 0, \\ \mathcal{V}_3 = \mathcal{N}_1 = \mathcal{N}_4 = \mathcal{N}_5 = \mathcal{N}_7 = 0, \mathcal{N}_8 < 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 22} \\ (96); \end{array} \\
(23) \quad \begin{cases} \dot{x} = x, \\ \dot{y} = y - x^3 \end{cases} &\Leftrightarrow \left[\begin{array}{l} \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \mathcal{V}_1 = 0, \\ \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_9 = \mathcal{N}_{10} = 0 \end{array} \right] \Leftrightarrow \begin{array}{l} \text{Fig. 23} \\ (107),(108). \end{array}
\end{aligned}$$

Here, a condition of the type $S(\mathbf{a}, x, y) > 0$ (respectively, $S(\mathbf{a}, x, y) < 0$) means that the respective homogeneous polynomial of even degree in x and y is positive (respectively, negative) defined. And equality $S(\mathbf{a}, x, y) = 0$ must be understood in $\mathbb{R}[x, y]$.

Note that, only in the case that some invariant straight lines have multiplicity > 1 , in the last column of the statement of the Main Theorem appear some numbers under the figures indicating the corresponding perturbed systems which show the potential configurations of the considered cubic system.

2. Necessary conditions for the existence of parallel invariant straight lines

We define the auxiliary polynomial $\mathcal{U}_1(\mathbf{a}) = T_{24} - 4T_{27}$.

Lemma 1. *For cubic systems (2) the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for the existence of two triples of parallel invariant straight lines with different slope.*

Proof: Let $L_i(x, y) = \alpha x + \beta y + \gamma_i = 0$, ($i = 1, 2, 3$) be three parallel invariant straight lines for a cubic system (2). Then, we have

$$\alpha P(x, y) + \beta Q(x, y) = \xi(\alpha x + \beta y + \gamma_1)(\alpha x + \beta y + \gamma_2)(\alpha x + \beta y + \gamma_3),$$

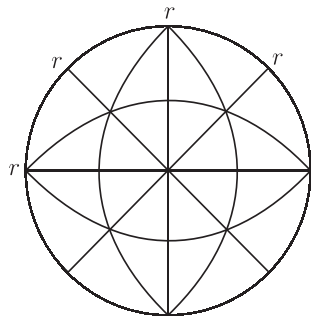


Figure 1

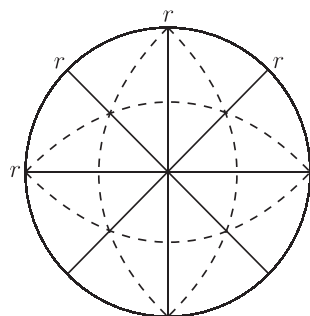


Figure 2

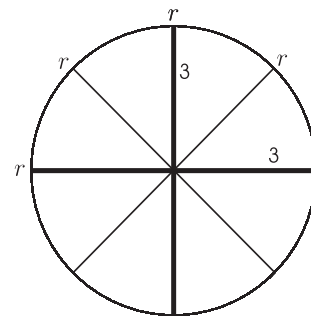


Figure 3

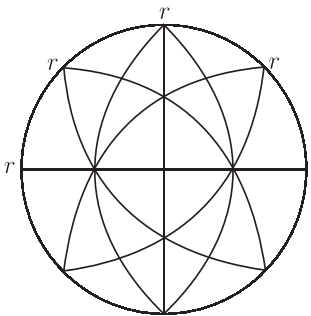


Figure 4

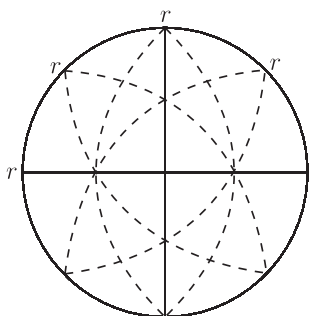


Figure 5

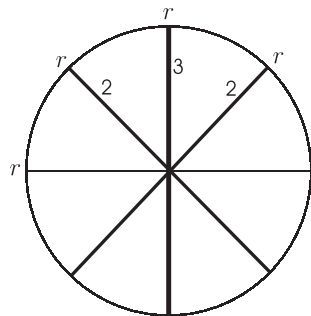


Figure 6

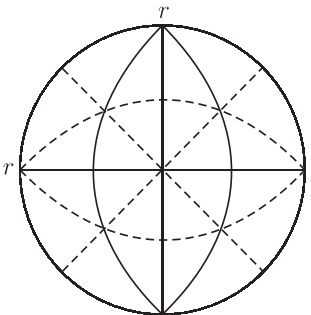


Figure 7

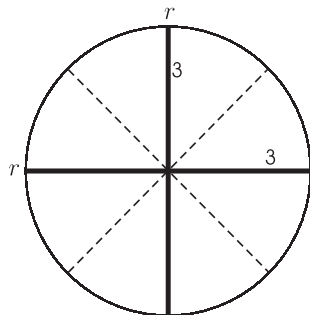


Figure 8

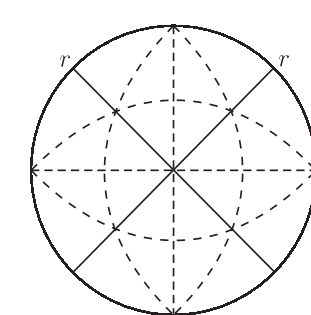


Figure 9

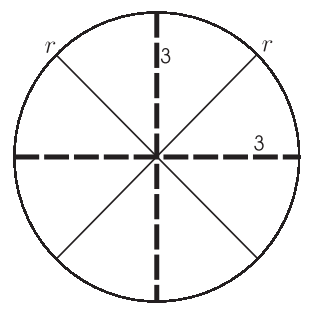


Figure 10

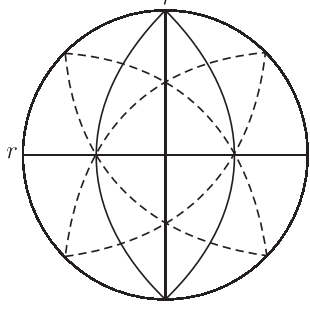


Figure 11

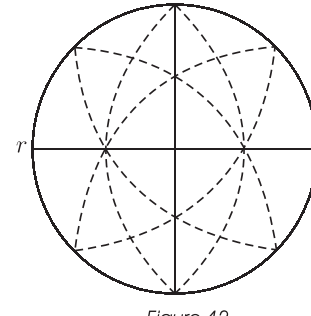


Figure 12

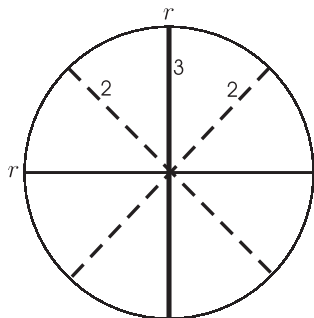


Figure 13

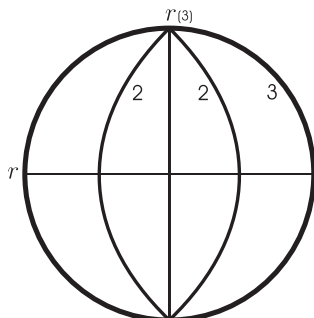


Figure 14

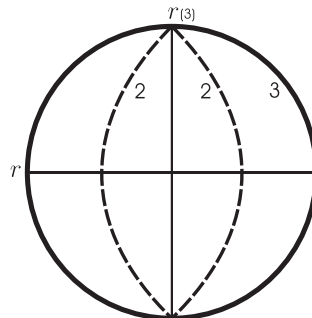


Figure 15

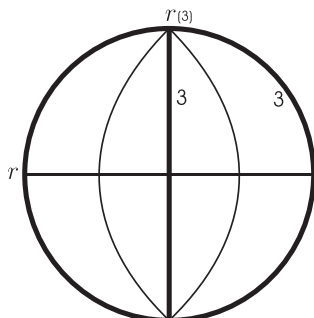


Figure 16

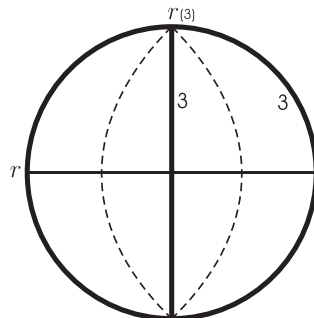


Figure 17

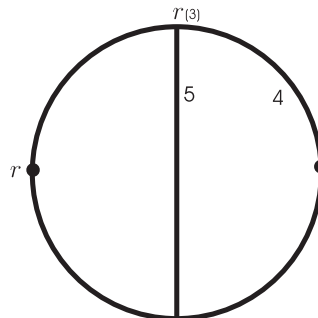


Figure 18

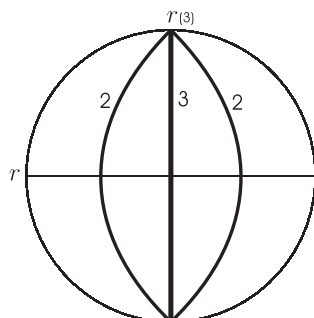


Figure 19

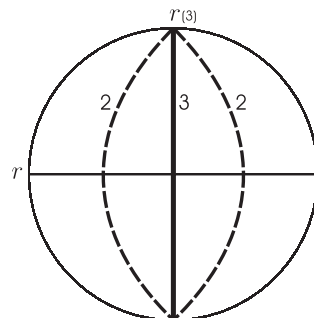


Figure 20

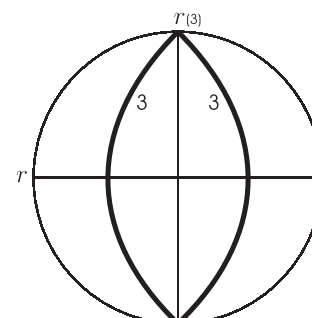


Figure 21

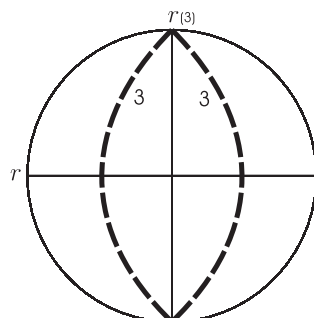


Figure 22

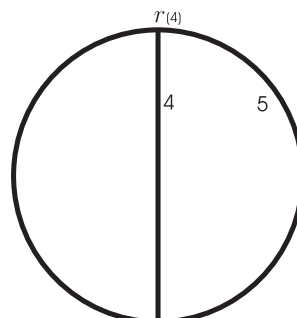


Figure 23

where the constant ξ can be considered 1 (rescaling the time, if necessary). Therefore, from the cubic terms we obtain $\alpha p_3(x, y) + \beta q_3(x, y) = (\alpha x + \beta y)^3$. If we denote

$$p_3(x, y) = px^3 + 3qx^2y + 3rxy^2 + sy^3, \quad q_3(x, y) = tx^3 + 3ux^2y + 3vxy^2 + wy^3.$$

then, for the existence of 3 parallel invariant straight lines it is necessary the solvability of the following systems of cubic equations with respect to the parameters α and β :

$$\begin{aligned} A_1 \equiv \alpha p + \beta t - \alpha^3 &= 0, & A_2 \equiv \alpha q + \beta u - \alpha^2\beta &= 0, \\ A_3 \equiv \alpha r + \beta v - \alpha\beta^2 &= 0, & A_4 \equiv \alpha s + \beta w - \beta^3 &= 0. \end{aligned} \quad (3)$$

Without loss of generality we can consider $\alpha\beta \neq 0$, otherwise a rotation of the phase plane can be done. We have:

$$\begin{aligned} B_1 \equiv \alpha A_2 - \beta A_1 &= q\alpha^2 + (u - p)\alpha\beta - t\beta^2 = 0, \\ B_2 \equiv \alpha A_3 - \beta A_2 &= r\alpha^2 + (v - q)\alpha\beta - u\beta^2 = 0, \\ B_3 \equiv \alpha A_4 - \beta A_3 &= s\alpha^2 + (w - r)\alpha\beta - v\beta^2 = 0. \end{aligned} \quad (4)$$

Clearly, for the existence of two directions (α_1, β_1) and (α_2, β_2) such that in each of them there are 3 parallel invariant straight lines of a system (2) it is necessary that the $\text{rank}(U) = 1$, where

$$U = \begin{pmatrix} q & u-p & -t \\ r & v-q & -u \\ s & w-r & -v \end{pmatrix}.$$

We denote by U_{kl}^{ij} the 2×2 -minor of the matrix U formed only by the columns i and j and by the rows k and l of U . We obtain

$$\begin{aligned} U_{12}^{12} &= \begin{vmatrix} q & u-p \\ r & v-q \end{vmatrix}, & U_{12}^{13} &= \begin{vmatrix} q & -t \\ r & -u \end{vmatrix}, & U_{12}^{23} &= \begin{vmatrix} u-p & -t \\ v-q & -u \end{vmatrix}, \\ U_{13}^{12} &= \begin{vmatrix} q & u-p \\ s & w-r \end{vmatrix}, & U_{13}^{13} &= \begin{vmatrix} q & -t \\ s & -v \end{vmatrix}, & U_{13}^{23} &= \begin{vmatrix} u-p & -t \\ w-r & -v \end{vmatrix}, \\ U_{23}^{12} &= \begin{vmatrix} r & v-q \\ s & w-r \end{vmatrix}, & U_{23}^{13} &= \begin{vmatrix} r & -u \\ s & -v \end{vmatrix}, & U_{23}^{23} &= \begin{vmatrix} v-q & -u \\ w-r & -v \end{vmatrix}. \end{aligned}$$

Hence, the $\text{rank}(U) = 1$ if and only if $U_{kl}^{ij} = 0$ for all $1 \leq i < j \leq 3$ and $1 \leq k < l \leq 3$.

On the other hand, it is easy to calculate the values of the T -comitants

$$\begin{aligned} \mathcal{V}_1 &= 16 [U_{12}^{23}x^4 + (U_{13}^{23} - 2U_{12}^{13})x^3y + (U_{12}^{12} - 2U_{13}^{13} + U_{23}^{23})x^2y^2 \\ &\quad + (U_{13}^{12} - 2U_{23}^{13})xy^3 + U_{23}^{12}y^4], \\ \mathcal{V}_2 &= 8 [- (2U_{12}^{13} + U_{13}^{23})x^2 + 2(U_{12}^{12} - U_{23}^{23})xy + (U_{13}^{12} + 2U_{23}^{13})y^2], \\ \mathcal{U}_1 &= 2^7 (U_{12}^{12} + U_{13}^{13} + U_{23}^{23}). \end{aligned}$$

Thus, it is obvious that $U_{kl}^{ij} = 0$ ($1 \leq i < j \leq 3$, $1 \leq k < l \leq 3$) if and only if $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$. This completes the proof of the lemma. \blacksquare

We assume that $\mathcal{V}_1^2 + \mathcal{V}_2^2 + \mathcal{U}_1^2 \neq 0$. Then, by Lemma 1, there cannot exist two triples of parallel invariant straight lines for system (2). Now, we shall examine the case when a system (2) possesses only one triple of parallel invariant straight lines. This means that system (4) can have at most one solution (α_0, β_0) . By using (4) and considering (3), we construct the following linear system with respect to the parameters α and β :

$$\begin{aligned}
\alpha B_1 &= q\alpha^3 + (u-p)\alpha^2\beta - t\alpha\beta^2 &= B_{11}\alpha + B_{12}\beta &= 0, \\
\beta B_1 &= q\alpha^2\beta + (u-p)\alpha\beta^2 - t\beta^3 &= B_{21}\alpha + B_{22}\beta &= 0, \\
\alpha B_2 &= r\alpha^3 + (v-q)\alpha^2\beta - u\alpha\beta^2 &= B_{31}\alpha + B_{32}\beta &= 0, \\
\beta B_2 &= r\alpha^2\beta + (v-q)\alpha\beta^2 - u\beta^3 &= B_{41}\alpha + B_{42}\beta &= 0, \\
\alpha B_3 &= s\alpha^3 + (w-r)\alpha^2\beta - v\alpha\beta^2 &= B_{51}\alpha + B_{52}\beta &= 0, \\
\beta B_3 &= s\alpha^2\beta + (w-r)\alpha\beta^2 - v\beta^3 &= B_{61}\alpha + B_{62}\beta &= 0,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
B_{11} &= -U_{12}^{13}, & B_{12} &= -U_{12}^{23}, & B_{21} &= -U_{12}^{12} - U_{13}^{13}, & B_{22} &= -U_{12}^{13} - U_{13}^{23}, \\
B_{31} &= U_{12}^{12}, & B_{32} &= U_{12}^{13}, & B_{41} &= -U_{23}^{13}, & B_{42} &= -U_{23}^{23}, \\
B_{51} &= U_{13}^{12} + U_{23}^{13}, & B_{52} &= U_{13}^{13} + U_{23}^{23}, & B_{61} &= U_{23}^{12}, & B_{62} &= U_{23}^{13}.
\end{aligned}$$

We denote by \mathcal{B} the 6×2 -matrix of the linear system (5) and by M_{ij} its corresponding 2×2 minors:

$$\mathcal{B} = (B_{ij})_{\{i=1,\dots,6; j=1,2\}}, \quad M_{ij} = \begin{vmatrix} B_{i1} & B_{i2} \\ B_{j1} & B_{j2} \end{vmatrix}, \quad 1 \leq i < j \leq 6.$$

It is clear that the linear system (5) has a non-trivial solution if and only if the $\text{rank}(\mathcal{B}) = 1$, i.e. $M_{ij} = 0$ for $1 \leq i < j \leq 6$.

On the other hand calculating the polynomial \mathcal{V}_3 as well as the auxiliary polynomials

$$\begin{aligned}
\mathcal{U}_2(\mathbf{a}, x, y) &= 6(T_{23} - 3T_{25}, T_{26})^{(1)} - 3T_{23}(T_{24} + 8T_{27}) - 24T_{26}^2 \\
&\quad + 2C_3(C_3, T_{23})^{(4)} + 24D_3(D_3, T_{26})^{(1)} + 24D_3^2T_{27}, \\
\mathcal{U}_3(\mathbf{a}, x, y) &= D_3 \left[(C_3, T_{23})^{(4)} + 36(D_3, T_{26})^{(2)} \right],
\end{aligned}$$

for system (2) we have

$$\begin{aligned}
\mathcal{V}_4 &= 2^{10} \cdot 3^5 [(M_{12} + M_{13})x^4 + 3(M_{16} + M_{34})x^2y^2 + (M_{46} + M_{56})y^4 \\
&\quad + (3M_{14} + M_{15} - M_{23})x^3y + (M_{26} + 3M_{36} - M_{45})xy^3], \\
\mathcal{U}_2 &= 2^{12} \cdot 3^5 \cdot 5 [M_{13}x^4 + (2M_{16} + M_{25} - M_{34})x^2y^2 + M_{46}y^4 \\
&\quad + (M_{14} + M_{15} + M_{23})x^3y + (M_{26} + M_{36} + M_{45})xy^3], \\
\mathcal{U}_3 &= 2^{10} \cdot 3^6 [(M_{15} - M_{14} - M_{23})x^2 + (M_{16} - 2M_{24} - M_{34})xy \\
&\quad + (M_{36} - M_{26} + M_{45})y^2].
\end{aligned}$$

It is not difficult to observe that conditions $\mathcal{V}_4 = \mathcal{U}_2 = \mathcal{U}_3 = 0$ are equivalent to $M_{ij} = 0$ for $1 \leq i < j \leq 6$. Moreover, taking into account the expressions of the polynomials \mathcal{V}_4 and \mathcal{U}_3 , we can conclude for $C_3 \neq 0$, and that condition $\mathcal{V}_4 = 0$ implies $\mathcal{U}_3 = 0$. Hence, the following lemma was proved:

Lemma 2. *For cubic system (2) the necessary conditions for the existence of one triple of parallel invariant straight lines are $\mathcal{V}_4 = \mathcal{U}_2 = 0$.*

The next step is to find some necessary conditions in order that system (2) possesses three couples of parallel invariant straight lines.

Let $L_i(x, y) = \alpha x + \beta y + \gamma_i = 0$, ($i = 1, 2$) be two parallel invariant straight lines for a cubic system (2). Then, we have

$$\alpha P(x, y) + \beta Q(x, y) = (\alpha x + \beta y + \gamma_1)(\alpha x + \beta y + \gamma_2)(\mu x + \eta y + \gamma_3).$$

Therefore, from the cubic terms we get

$$\alpha p_3(x, y) + \beta q_3(x, y) = (\alpha x + \beta y)^2(\mu x + \eta y).$$

Thus, for the existence of 2 parallel invariant straight lines it is necessary the solvability of the following system of cubic equations with respect to parameters α , β , μ and η :

$$\begin{aligned} E_1 &\equiv \alpha p + \beta t - \alpha^2 \mu = 0, & E_2 &\equiv 3\alpha q + 3\beta u - \alpha^2 \eta - 2\alpha \beta \mu = 0, \\ E_3 &\equiv 3\alpha r + 3\beta v - 2\alpha \beta \eta + \beta^2 \mu = 0, & E_4 &\equiv \alpha s + \beta w - \beta^2 \eta = 0. \end{aligned} \quad (6)$$

Without loss of generality we may consider $\alpha \beta \neq 0$, otherwise a rotation of the phase plane can be done. We have

$$\begin{aligned} F_1 &\equiv \text{Res}_\eta \left(\text{Res}_\mu (E_1, E_2), E_4 \right) / \alpha = s\alpha^4 + w\alpha^3\beta - 3q\alpha^2\beta^2 + (2p - 3u)\alpha\beta^3 + 2t\beta^3 = 0, \\ F_2 &\equiv \text{Res}_\eta \left(\text{Res}_\mu (E_1, E_3), E_4 \right) / \beta = 2s\alpha^4 + (2w - 3r)\alpha^3\beta - 3v\alpha^2\beta^2 + p\alpha\beta^3 + t\beta^3 = 0, \end{aligned}$$

where $\text{Res}_z(f, g)$ denotes the resultant of the polynomials f and g with respect the variable z , for more details on the resultant see [11]. Then

$$\begin{aligned} G_1(\alpha, \beta) &\equiv \frac{2F_2 - F_1}{3\alpha} = s\alpha^3 + (w - 2r)\alpha^2\beta + (q - 2v)\alpha\beta^2 + u\beta^3 = 0, \\ G_2(\alpha, \beta) &\equiv \frac{2F_1 - F_2}{3\beta} = r\alpha^3 + (v - 2q)\alpha^2\beta + (p - 2u)\alpha\beta^2 + t\beta^3 = 0. \end{aligned}$$

Now, it is clear that for the existence of three distinct solutions (α_i, β_i) ($i = 1, 2, 3$) of the system (6) it is necessary that the polynomials $G_1(\alpha, \beta)$ and $G_2(\alpha, \beta)$ be proportional, i.e. the following identity holds:

$$\mathcal{G} = \begin{vmatrix} \frac{\partial G_1}{\partial \alpha} & \frac{\partial G_1}{\partial \beta} \\ \frac{\partial G_2}{\partial \alpha} & \frac{\partial G_2}{\partial \beta} \end{vmatrix} = 3(K_0\alpha^4 + K_1\alpha^3\beta + K_2\alpha^2\beta^2 + K_3\alpha\beta^3 + K_4\beta^4) = 0,$$

where

$$\begin{aligned}
K_0 &= -rw + 2r^2 - 2qs + vs, \\
K_1 &= 2ps - 4us - 2qr + 4vr, \\
K_2 &= 2q^2 + 2v^2 + 3ts - 2pr + ur + pw - 2uw - 5qv, \\
K_3 &= -4tr + 2tw + 4uq - 2uv, \\
K_4 &= 2u^2 + tq - 2tv - pu.
\end{aligned}$$

On the other hand, the comitant \mathcal{V}_3 calculated for system (2) gives

$$\mathcal{V}_3 = 2^5 \cdot 3^2 (K_4x^4 - K_3x^3y + K_2x^2y^2 - K_1xy^3 + K_0y^4).$$

Consequently, condition $\mathcal{G} = 0$ is equivalent to $\mathcal{V}_3 = 0$. Hence, we get the next result.

Lemma 3. *For cubic systems (2) condition $\mathcal{V}_3 = 0$ is necessary for the existence of three distinct couples of parallel invariant straight lines.*

Taking into account Lemmas 2 and 3 we obtain the next result.

Lemma 4. *If a cubic system (2) possesses the configuration or the potential configuration of parallel invariant straight lines of the type (3, 2, 2), then it is necessary $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$.*

3. Infinite singular points and associated homogeneous cubic canonical systems

From [12] (see also [5]) we have the following result. Here $a \in \mathbb{C}$ is *imaginary* if $a \notin \mathbb{R}$.

Lemma 5. *The number of distinct roots (real and imaginary) of the polynomial $C_3 = yp_3(x, y) - xq_3(x, y) \neq 0$ is determined by the following conditions:*

- [i] 4 real if $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$;
- [ii] 2 real and 2 imaginary if $\mathcal{D}_1 < 0$;
- [iii] 4 imaginary if $\mathcal{D}_1 > 0$ and for every (x, y) where $\mathcal{D}_2\mathcal{D}_3 \neq 0$ either $\mathcal{D}_2 < 0$ or $\mathcal{D}_3 < 0$;
- [iv] 3 real (1 double, 2 simple) if $\mathcal{D}_1 = 0$, $\mathcal{D}_3 > 0$;
- [v] 1 real and 2 imaginary (1 real double) if $\mathcal{D}_1 = 0$, $\mathcal{D}_3 < 0$;
- [vi] 2 real (1 triple and 1 simple) if $\mathcal{D}_1 = \mathcal{D}_3 = 0$, $\mathcal{D}_2 \neq 0$, $\mathcal{D}_4 = 0$;
- [vii] 2 real (2 double) if $\mathcal{D}_1 = \mathcal{D}_3 = 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_4 \neq 0$;
- [viii] 2 imaginary (2 double) if $\mathcal{D}_1 = \mathcal{D}_3 = 0$, $\mathcal{D}_2 < 0$, $\mathcal{D}_4 \neq 0$;
- [ix] 1 real (of the multiplicity 4) if $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$.

where \mathcal{D}_i for $i = 1, 2, 3, 4$ are the T -comitants defined in the introduction.

We consider the polynomial $C_3(\mathbf{a}, x, y) \neq 0$ as a quartic binary form. It is well known that there exists $g \in GL(2, \mathbb{R})$, $g(x, y) = (u, v)$, such that the transformed binary form $gC_3(\mathbf{a}, x, y) = C_3(\mathbf{a}, g^{-1}(u, v))$ is one of the following 9 canonical forms:

$$\begin{array}{lll} (i) & xy(x-y)(rx+sy), \quad rs(r+s) \neq 0; & (iv) \quad x^2y(x-y); \quad (vii) \quad x^2y^2; \\ (ii) & x(sx+y)(x^2+y^2); & (v) \quad x^2(x^2+y^2); \quad (viii) \quad (x^2+y^2)^2; \\ (iii) & (px^2+qy^2)(x^2+y^2), \quad pq > 0; & (vi) \quad x^3y; \quad (ix) \quad x^4. \end{array}$$

We note that each of such canonical forms corresponds to one of the cases enumerated in the statement of Lemma 5.

On the other hand, applying the same transformation g to the initial system and calculating for the transformed system its polynomial $C_3(\mathbf{a}(g), u, v)$ the following relation hold:

$$C_3(\mathbf{a}(g), u, v) = \det(g) C_3(\mathbf{a}, x, y) = \det(g) C_3(\mathbf{a}, g^{-1}(u, v)) = \lambda C_3(\mathbf{a}, g^{-1}(u, v)),$$

where we may consider $\lambda = 1$ (via a time rescaling).

Taking into account that $C_3(x, y) = yP_3(x, y) - xQ_3(x, y)$, we construct the canonical forms of the cubic homogeneous systems having their polynomials C_3 the indicated

canonical forms (i) – (ix):

$$\begin{aligned} x' &= (p+r)x^3 + (s+v)x^2y + qxy^2, & C_3 &= xy(x-y)(rx+sy), \\ y' &= px^2y + (r+v)xy^2 + (q+s)y^3, & & rs(r+s) \neq 0 \end{aligned} \quad (7)$$

$$\begin{aligned} x' &= (u+1)x^3 + (s+v)x^2y + rxy^2, & C_3 &= x(sx+y)(x^2+y^2), \\ y' &= -sx^3 + ux^2y + vxy^2 + (r-1)y^3, \end{aligned} \quad (8)$$

$$\begin{aligned} x' &= ux^3 + (p+q+v)x^2y + rxy^2 + qy^3, & C_3 &= (px^2 + qy^2)(x^2 + y^2), \\ y' &= -px^3 + ux^2y + vxy^2 + ry^3, & & pq > 0 \end{aligned} \quad (9)$$

$$\begin{aligned} x' &= 3(u+1)x^3 + (v-1)x^2y + rxy^2, & C_3 &= x^2y(x-y), \\ y' &= ux^2y + vxy^2 + ry^3, \end{aligned} \quad (10)$$

$$\begin{aligned} x' &= ux^3 + (v+1)x^2y + rxy^2, & C_3 &= x^2(x^2 + y^2), \\ y' &= -x^3 + ux^2y + vxy^2 + ry^3, \end{aligned} \quad (11)$$

$$\begin{aligned} x' &= (u+1)x^3 + vx^2y + rxy^2, & C_3 &= x^3y, \\ y' &= ux^2y + vxy^2 + ry^3, \end{aligned} \quad (12)$$

$$\begin{aligned} x' &= ux^3 + qx^2y + rxy^2, & C_3 &= (q-v)x^2y^2, \\ y' &= ux^2y + vxy^2 + ry^3, & & q-v \neq 0 \end{aligned} \quad (13)$$

$$\begin{aligned} x' &= ux^3 + (v+1)x^2y + rxy^2 + y^3, & C_3 &= (x^2 + y^2)^2, \\ y' &= -x^3 + ux^2y + 3(v-1)xy^2 + ry^3, \end{aligned} \quad (14)$$

$$\begin{aligned} x' &= ux^3 + vx^2y + rxy^2, & C_3 &= x^4, \\ y' &= -x^3 + ux^2y + vxy^2 + ry^3. \end{aligned} \quad (15)$$

4. Criteria for the existence of an invariant straight line with a given multiplicity

We consider a cubic system (2) and the associated four polynomials $C_i(x, y)$ for $i = 0, 1, 2, 3$.

Proposition 6. *The straight line $\tilde{L}(x, y) = ux + vy = 0$ is invariant for a cubic system (2) with $p_0^2 + q_0^2 \neq 0$ if and only if for $i = 0, 1, 2, 3$ the following relations hold:*

$$\text{either } C_i(-v, u) = 0, \quad (16)$$

$$\text{or } \text{Res}_\gamma(C_0, C_i) = 0 \quad (\gamma = \frac{y}{x} \quad \text{or} \quad \gamma = \frac{x}{y}). \quad (17)$$

Proof: The line $\tilde{L}(x, y)=0$ is invariant for system (2) if and only if

$$u(p_0 + p_1 + p_2 + p_3) + v(q_0 + q_1 + q_2 + q_3) = (ux + vy)(S_0 + S_1 + S_2),$$

for some homogeneous polynomials S_i of degree i . The last equality is equivalent to

$$\begin{aligned} up_0 + vq_0 &= 0, \\ up_1(x, y) + vq_1(x, y) &= (ux + vy)S_0, \\ up_2(x, y) + vq_2(x, y) &= (ux + vy)S_1(x, y), \\ up_3(x, y) + vq_3(x, y) &= (ux + vy)S_2(x, y). \end{aligned}$$

If $x = -v, y = u$, then the left-hand sides of the previous equalities become $C_0(-v, u)$, $C_1(-v, u)$, $C_2(-v, u)$ and $C_3(-v, u)$, respectively. At the same time the right-hand sides of these identities vanish. Thus, we obtain equations (16) in which C_0 (respectively, C_1 ; C_2 ; C_3) is a homogeneous polynomial of degree 1 (respectively 2; 3; 4) in the parameters u and v , and $C_0(x, y) \neq 0$ because $p_0^2 + q_0^2 \neq 0$. Hence, the necessary and sufficient conditions for the existence of a common solution of systems (16) are conditions (17). \blacksquare

Let $(x_0, y_0) \in \mathbb{R}^2$ be an arbitrary point on the phase plane of systems (2). Consider a translation τ bringing the origin of coordinates to the point (x_0, y_0) . We denote by (2^τ) the system obtained after applying the transformation τ , and by $\tilde{\mathbf{a}} = \mathbf{a}(x_0, y_0) \in \mathbb{R}^{20}$ the 20-tuple of its coefficients. If $\gamma = y/x$ or $\gamma = x/y$ then, for $i = 1, 2, 3$ we denote

$$\begin{aligned} \Omega_i(\mathbf{a}, x_0, y_0) &= \text{Res}_\gamma \left(C_i(\tilde{\mathbf{a}}, x, y), C_0(\tilde{\mathbf{a}}, x, y) \right) \in \mathbb{R}[\mathbf{a}, x_0, y_0]; \\ \mathcal{G}_i(\mathbf{a}, x, y) &= \Omega_i(\mathbf{a}, x_0, y_0)|_{\{x_0=x, y_0=y\}} \in \mathbb{R}[\mathbf{a}, x, y]. \end{aligned} \quad (18)$$

Remark 7. For $j = 1, 2, 3$ the polynomials $\mathcal{G}_j(x, y) = \mathcal{G}_j(\mathbf{a}, x, y)$ are affine comitants and are homogeneous in the coefficients of system (2) and non-homogeneous in the variables x and y . Additionally,

$$\begin{array}{lll} \deg_{\mathbf{a}} \mathcal{G}_1 = 3, & \deg_{\mathbf{a}} \mathcal{G}_2 = 4, & \deg_{\mathbf{a}} \mathcal{G}_3 = 5, \\ \deg_{(x,y)} \mathcal{G}_1 = 8, & \deg_{(x,y)} \mathcal{G}_2 = 10, & \deg_{(x,y)} \mathcal{G}_3 = 12. \end{array}$$

The geometrical meaning of these affine comitants is given by the following lemma.

Lemma 8. *The straight line $L(x, y) = ux + vy + w = 0$ is invariant for a cubic system (2) if and only if the polynomial $L(x, y)$ is a common factor of the polynomials \mathcal{G}_1 , \mathcal{G}_2 and \mathcal{G}_3 over \mathbb{C} .*

Proof: Let $(x_0, y_0) \in \mathbb{R}^2$ be a non-singular point of system (2) (i.e. $P(x_0, y_0)^2 + Q(x_0, y_0)^2 \neq 0$) which lies on the line $L(x, y) = 0$, i.e. $ux_0 + vy_0 + w = 0$. Denote by $\tilde{L}(x, y) =$

$(L \circ \tau)(x, y) = ux + vy$ (τ is a translation) and consider the line $ux + vy = 0$. By Proposition 6, the straight line $\tilde{L}(x, y) = 0$ will be an invariant line of systems (2^τ) if and only if conditions (17) are satisfied for these systems, i.e. for $i = 1, 2, 3$, $\Omega_i(\mathbf{a}, x_0, y_0) = 0$, for each point (x_0, y_0) on the line $L(x, y) = ux + vy + w = 0$. Thus, we have $\Omega_i(\mathbf{a}, x_0, y_0) = (ux_0 + vy_0 + w)\tilde{\Omega}_i(\mathbf{a}, x_0, y_0)$. Taking into account relations (18) the lemma follows. ■

Lemma 9. *If $L(x, y) = ux + vy + w = 0$ is an invariant straight line of (geometric) multiplicity k for a cubic system (2) then, for $i = 1, 2, 3$, we have that*

$$\mathcal{G}_i = (ux + vy + w)^k W_i(x, y).$$

Proof: By the definition of geometric multiplicity, we denote by (2_δ) the perturbed system from the system (2), which has k invariant lines of multiplicity 1: $L_{i\delta}(x, y)$ for $i = 1, \dots, k$.

By Lemma 8, for $i = 1, 2, 3$, system (2_δ) satisfies $\mathcal{G}_{i\delta} = L_{1\delta} \cdots L_{k\delta} \widetilde{W}_i(x, y)$, and when $\delta \rightarrow 0$ then $L_{i\delta}(x, y) \rightarrow L(x, y)$. At the same time $\mathcal{G}_{i\delta} \rightarrow \mathcal{G}_i = L(x, y)^k W_i$. ■

Taking into account Remark 7 and Lemmas 8 and 9 we conclude the following result.

Lemma 10. *If a cubic system (2) possesses the maximum number of invariant straight lines (counted with their multiplicities) then $\mathcal{G}_1(x, y) \mid \mathcal{G}_2(x, y)$ and $\mathcal{G}_1(x, y) \mid \mathcal{G}_3(x, y)$.*

In order to determine the degree of the common factor of the polynomials $\mathcal{G}_i(x, y)$ for $i = 1, 2, 3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [8], [11]).

We consider two polynomials

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad g(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m,$$

in variable the z of degree n and m , respectively.

We say that the k -th *subresultant* with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m + n - 2k) \times (m + n - 2k)$ determinant

$$R_z^{(k)}(f, g) = \left. \begin{array}{cccccc} a_0 & a_1 & a_2 & \cdots & \cdots & a_{m+n-2k-1} \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{m+n-2k-2} \\ 0 & 0 & a_0 & \cdots & \cdots & a_{m+n-2k-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & b_0 & \cdots & \cdots & b_{m+n-2k-3} \\ 0 & b_0 & b_1 & \cdots & \cdots & b_{m+n-2k-2} \\ b_0 & b_1 & b_2 & \cdots & \cdots & b_{m+n-2k-1} \end{array} \right\} \begin{array}{l} (m-k) - \text{times} \\ (n-k) - \text{times} \end{array} \quad (19)$$

in which there are $m - k$ rows of a 's and $n - k$ rows of b 's, and $a_i = 0$ for $i > n$, and $b_j = 0$ for $j > m$.

For $k = 0$ we obtain the standard resultant of two polynomials. In other words we can say that the k -th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and columns from its resultant written in the form (19) when $k = 0$.

The geometrical meaning of the subresultants is based in the following lemma.

Lemma 11. (see [8], [11]). *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (counting their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f, g) = R_z^{(1)}(f, g) = R_z^{(2)}(f, g) = \dots = R_z^{(k-1)}(f, g) = 0 \neq R_z^{(k)}(f, g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 11 the following result.

Lemma 12. *Two polynomials $\tilde{f}(x_1, x_2, \dots, x_n)$ and $\tilde{g}(x_1, x_2, \dots, x_n)$ have a common factor of degree k with respect to variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f}, \tilde{g}) = R_{x_j}^{(1)}(\tilde{f}, \tilde{g}) = R_{x_j}^{(2)}(\tilde{f}, \tilde{g}) = \dots = R_{x_j}^{(k-1)}(\tilde{f}, \tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f}, \tilde{g}),$$

where $R_{x_j}^{(i)}(\tilde{f}, \tilde{g}) = 0$ in $\mathbb{R}[x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n]$.

5. Cubic systems with 4 real simple roots of C_3

As it was shown above a cubic homogeneous system having 4 real distinct infinite singular points via a linear transformation becomes in the canonical form (7). Therefore, in what follows we consider the system

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + (p + r)x^3 + (s + v)x^2y + qxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) + px^2y + (r + v)xy^2 + (q + s)y^3, \end{aligned} \tag{20}$$

where the parameters r and s satisfy the condition $rs(r + s) \neq 0$. For system (20) we obtain $C_3 = xy(x - y)(rx + sy)$, and hence, infinite singular points are situated at the “ends” of the following straight lines: $x = 0$, $y = 0$, $x - y = 0$ and $rx + sy = 0$.

The goal of this section is to construct the cubic systems of the form (20) which have 8 invariant straight lines with the configuration (3, 3, 1, 1) or (3, 2, 2, 1).

5.1. Systems with the configuration (3, 3, 1, 1)

In this subsection we construct the cubic system with 4 real infinite singular points which possesses 8 invariant affine straight lines in the configuration or potential configuration

(3, 3, 1, 1), having total multiplicity 9, as always the invariant straight line of the infinity is considered.

According to Lemma 1 if a cubic system possesses 8 invariant straight lines in the configuration (3, 3, 1, 1), then necessarily the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ hold.

A straightforward computation of the values of \mathcal{V}_1 and \mathcal{V}_2 for system (20) yields:

$$\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j, \quad \mathcal{V}_2 = 8 \sum_{j=0}^2 \mathcal{V}_{2j} x^{2-j} y^j,$$

where

$$\begin{aligned} \mathcal{V}_{10} &= p(2p + 3r), \\ \mathcal{V}_{11} &= 2ps + 4pv + 2pr + 3r^2 + 3rv, \\ \mathcal{V}_{12} &= 4rs + 4pq + 3ps + 3rq + 2sv + 2rv - s^2 - r^2 + 2v^2, \\ \mathcal{V}_{13} &= 2sq + 4qv + 3sv + 2rq + 3s^2, \\ \mathcal{V}_{14} &= q(2q + 3s), \\ \mathcal{V}_{20} &= -3rv - 3r^2 + 2ps - 2pr, \\ \mathcal{V}_{21} &= 6rq - 2s^2 - 4sv + 4rv + 2r^2 - 6ps, \\ \mathcal{V}_{22} &= 2qs + 3s^2 - 2rq + 3sv. \end{aligned} \tag{21}$$

Consequently, relations $\mathcal{V}_1 = \mathcal{V}_2 = 0$ provides the following equalities:

$$\begin{aligned} \mathcal{V}_{10} &= p(2p + 3r) = 0, & \mathcal{V}_{14} &= q(2q + 3s) = 0, \\ \mathcal{V}_{11} + \mathcal{V}_{20} &= 4p(s + v) = 0, & \mathcal{V}_{13} - \mathcal{V}_{22} &= 4q(r + v) = 0, \\ \mathcal{V}_{11} - \mathcal{V}_{20} &= 2(r + v)(2p + 3r) = 0, & \mathcal{V}_{13} + \mathcal{V}_{22} &= 2(s + v)(2q + 3s) = 0. \end{aligned} \tag{22}$$

Thus, we shall consider three cases: (1) $pq \neq 0$; (2) $pq = 0, p^2 + q^2 \neq 0$; (3) $p = q = 0$.

Case $pq \neq 0$. Then, from (22), we obtain $v = -s, r = s, p = q = -3s/2 \neq 0$, and consequently $\mathcal{V}_1 = \mathcal{V}_2 = 0$, and $\mathcal{U}_1 = 0$. Therefore, by changing the time ($t \rightarrow -2t/(3s)$) we obtain the following system:

$$x' = p_0 + p_1 + p_2 + x^3 + 3xy^2, \quad y' = q_0 + q_1 + q_2 + 3x^2y + y^3, \tag{23}$$

for which $\mathcal{U}_1(\mathbf{a}) = 0$.

Case $pq = 0, p^2 + q^2 \neq 0$. Then, without loss of generality, we can consider $p = 0$ and $q \neq 0$ via the transformation $x \leftrightarrow y$ and the changes $p \leftrightarrow q$ and $r \leftrightarrow s$. From (22) we have $v = -r, q = -3s/2 \neq 0$, and

$$\mathcal{V}_{1i} = 0 \ (i = 0, 1, 3, 4), \quad \mathcal{V}_{2j} = 0, \ (j = 0, 2), \quad \mathcal{V}_{12} = 4\mathcal{V}_{11} = -\mathcal{V}_{21} = -4(r + 2s)(2r + s).$$

Consequently, we obtain either $s = -2r$, or $r = -2s$. The first case after a suitable time rescaling writes the system as

$$x' = p_0 + p_1 + p_2 + x^3 - 3x^2y + 3xy^2, \quad y' = q_0 + q_1 + q_2 + y^3, \tag{24}$$

whereas the second one goes over to the system

$$x' = p_0 + p_1 + p_2 + 4x^3 - 6x^2y + 3xy^2, \quad y' = q_0 + q_1 + q_2 + y^3. \quad (25)$$

We note that for both systems we have $\mathcal{U}_1(\mathbf{a}) = 0$.

Case $p = q = 0$. Then, by (22), we have $r(r + v) = s(s + v) = 0$. We claim that $rs \neq 0$. Indeed, we suppose $r = 0$ (case $s = 0$ can be reduced to this one by changing $x \leftrightarrow y$). Then, taking into account (21), we obtain

$$\mathcal{V}_{13} + 6\mathcal{V}_{21} = 12s^2 = 0 \Rightarrow \mathcal{V}_{12} = 16v^2 = 0.$$

Thus $r = s = v = 0$ and we obtain $p_3(x, y) = q_3(x, y) = 0$. Hence, the claim is proved. Considering (22), we obtain $t = -r = -s$, and after a suitable time rescaling the system becomes

$$x' = p_0 + p_1 + p_2 + x^3, \quad y' = q_0 + q_1 + q_2 + y^3, \quad (26)$$

for which $\mathcal{U}_1(\mathbf{a}) = 0$.

Lemma 13. *Systems (23), (24), (25) and (26) are all affine equivalent.*

Proof: It is sufficient to check by straightforward computation that the transformation $x_1 = x - y$, $y_1 = y$ writes system (26) into system (24), and the transformation $x_1 = x$, $y_1 = x - y$ writes system (23) into system (25). It remains to observe that the transformation $x_1 = x$, $y_1 = y/2$ and $t_1 = 4t$ writes system (25) into system (24). ■

Let $L(x, y) = Ux + Vy + W = 0$ be an invariant straight line of system (2), which we write explicitly as:

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3. \end{aligned}$$

Then, we have

$$UP(x, y) + VQ(x, y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F),$$

and this identity provides the following 10 relations:

$$\begin{aligned} Eq_1 &= pU + tV = 0, & Eq_6 &= (2h - E)U + (2m - D)V - 2BW = 0, \\ Eq_2 &= (3q - 2B)U + (3u - A)V = 0, & Eq_7 &= kU + (n - E)V - CW = 0, \\ Eq_3 &= (3r - CU) + (3v - 2B)V = 0, & Eq_8 &= (c - F)U + eV - DW = 0 \\ Eq_4 &= (s - C)U + VW = 0, & Eq_9 &= dU + (f - F)V - EW = 0, \\ Eq_5 &= (g - D)U + lV - AW = 0, & Eq_{10} &= aU + bV - FW = 0. \end{aligned} \quad (27)$$

We concentrate our attention to the system with 4 real distinct infinite singular points. According to Lemma 13 we can only work with system (26). It is clear that via a translation of the origin of coordinates at the point $(-g/3, -n/3)$, we can consider the parameter $g = 0$ (respectively, $n = 0$) in the polynomial p_2 (respectively, q_2). Thus, we shall work with the following system

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + y^3, \quad (28)$$

for which $C_3(x, y) = xy(x + y)(x - y)$. Therefore, there are the following 4 directions for the possible invariant straight lines: $x = 0$, $y = 0$, $y = -x$, $y = x$.

We claim that in the direction $y = -x$ as well as in the direction $y = x$ there can be only one invariant straight line. Indeed, for the directions $y = -x$ and $y = x$ we have $U = 1, V = \pm 1$ and then, from the first 6 equations (27), we obtain

$$A^\pm = C^\pm = 1, \quad B^\pm = \mp 1, \quad D^\pm = \pm l - W, \quad E^\pm = \pm 2W + 2h - l \pm 2m,$$

and $Eq_7 = -3W \mp 2h + k \pm l - 2m = 0$. Here, the values with a superindex + (respectively -) correspond to $V = +1$ (respectively $V = -1$). So, from system (27), we can obtain at most one solution W_0^\pm . Consequently, if system (28) possesses two couples of triples of parallel invariant straight lines, then their directions only can be in the directions $x = 0$ and $y = 0$. So, the claim is proved. Now we shall investigate the conditions in order to have two couples of triples of parallel invariant straight lines.

Direction $x = 0$. Then, $U = 1$, $V = 0$ and, from (27), we obtain

$$A = 1, \quad B = C = 0, \quad D = -W, \quad E = 2h, \quad F = W^2 + c, \\ Eq_7 = k, \quad Eq_9 = 2hW + d, \quad Eq_{10} = -W^3 - cW + a.$$

Thus, for the existence of three solutions W_i counted with their multiplicity, it is necessary and sufficient that $k = h = d = 0$.

Direction $y = 0$. In this case $U = 0$, $V = 1$ and, from (27), we obtain

$$A = B = 0, \quad C = 1, \quad D = 2m, \quad E = -W, \quad F = W^2 + f, \\ Eq_5 = l, \quad Eq_8 = -2mW + e, \quad Eq_{10} = -W^3 - fW + b.$$

Hence, for the existence of three solutions W_i counted with their multiplicity it is necessary and sufficient that $l = m = e = 0$.

Taking into account the conditions obtained we have the following system

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + y^3. \quad (29)$$

Now, it remains to find out the conditions for the existence of one invariant straight line in each of the directions $y = -x$ and $y = x$.

Direction $y \pm x = 0$. Considering equations (27) for system (29) in the directions $y \pm x = 0$ (i.e., $U = 1, V = \pm 1$) we obtain:

$$\begin{aligned} A^\pm &= C^\pm = 1, \quad B^\pm = \mp 1, \quad D^\pm = -W, \quad E^\pm = \pm 2W, \quad F^\pm = W^2 + c, \\ Eq_7^\pm &= -3W, \quad Eq_9^\pm = \mp 3W^2 \pm (f - c), \quad Eq_{10}^\pm = -W^3 + a \pm b. \end{aligned}$$

Thus, for both directions the unique solution can be $W^\pm = 0$, and in order to have in each direction an invariant straight line it is necessary and sufficient that $f - c = a = b = 0$. Thus, we have obtained the system

$$\dot{x} = cx + x^3, \quad \dot{y} = cy + y^3, \quad (30)$$

which possesses the invariant straight lines $x = 0$, $x = \pm\sqrt{-c}$, $y = 0$, $y = \pm\sqrt{-c}$, $y = \pm x$. It is clear that the lines $x = \pm\sqrt{-c}$ (respectively, $y = \pm\sqrt{-c}$) are real for $c < 0$, imaginary for $c > 0$, and coincide with the axes for $c = 0$. Hence we obtain Figure 1 (respectively, 2; 3) for $c < 0$ (respectively, $c > 0$; $c = 0$).

Remark 14. Assume $\alpha \in \mathbb{R}$. Then the transformation $x = |\alpha|^{1/2}x_1$, $y = |\alpha|^{1/2}y_1$ and $t = |\alpha|^{-1}t_1$ does not change the coefficients of the cubic part of the generic cubic system. Whereas each coefficient of the quadratic (respectively, linear; constant) part will be multiplied by $|\alpha|^{-1/2}$ (respectively, by $|\alpha|^{-1}$; $|\alpha|^{-3/2}$).

By Remark 14 for system (30) we can consider $c \in \{-1, 0, 1\}$.

In order to obtain equivalent invariant conditions we shall use the constructed T-comitants \mathcal{L}_i ($i = 1, 2, 3$) and \mathcal{N}_1 . We note that the T-comitants \mathcal{L}_i ($i = 1, 2, 3$) were constructed by Calin [3].

For system (28) we have $\mathcal{L}_1 = -2^8 3^4 (lx^3 + 2mx^2y - 2hxy^2 - ky^3)$. Thus, conditions $k = h = l = m = 0$ are equivalent to $\mathcal{L}_1 = 0$. Moreover, if for system (28) condition $\mathcal{L}_1 = 0$ holds, then we obtain $\mathcal{L}_2 = 2^7 3^5 x^2 y^2 [-ex^2 + 6(f - c)xy + dy^2]$, $\mathcal{L}_3 = 2^9 3^5 (c + f)(x^2 + y^2)$. This means that condition $\mathcal{L}_2 = 0$ is equivalent to $d = e = c - f = 0$ and, hence, since $f = c$ we obtain $\text{sign}(\mathcal{L}_3) = \text{sign}(c)$. Therefore, for $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = 0$, we obtain the system

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + cy + y^3. \quad (31)$$

for which $\mathcal{N}_1 = -2^3 3^5 xy(x^2 + y^2)(ax - by)$.

Remark 15. It is necessary to underline that the GL-comitants \mathcal{L}_i , $i = 1, 2, 3$ in fact are T-comitants for the initial system (i.e. their coefficients are absolute invariants under translations). But this is not the case for the GL-comitant \mathcal{N}_1 . However, for the system (31) \mathcal{N}_1 is a CT-comitant (see [18] for detailed definitions).

Proof: Indeed, we consider the system:

$$\begin{aligned}\dot{x}_1 &= a + c\gamma + \gamma^3 + (c + 3\gamma^2)x_1 + 3\gamma x_1^2 + x_1^3, \\ \dot{y}_1 &= b + c\delta + \delta^3 + (c + 3\delta^2)y_1 + 3\delta y_1^2 + y_1^3\end{aligned}\tag{32}$$

which is obtained from system (31) via the translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane. For system (32) we calculate the value of the *GL*-comitant $\mathcal{N}_1 = -2^3 3^5 x_1 y_1 (x_1^2 + y_1^2)(ax_1 - by_1)$. As we can observe, the value of this polynomial does not depend on the coordinates of the arbitrary point (γ, δ) and, consequently for system (31) condition $\mathcal{N}_1 = 0$ is equivalent to $a = b = 0$ and this is an affine invariant condition. \blacksquare

In short, we have the following result.

Proposition 16. *A cubic system (28) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Moreover, the configuration or the potential configuration of the lines corresponds with $(3, 3, 1, 1)$ given in Figure 1 (respectively, $2; 3$) for \mathcal{L}_3 negative (respectively, positive; zero).*

5.2 Systems with configuration $(3, 2, 2, 1)$

In this subsection we construct the cubic systems with 4 real infinite singular points which possess 8 distinct invariant affine straight lines with configuration or potential configuration $(3, 2, 2, 1)$, having total multiplicity 9, as always the invariant straight line of the infinity is considered.

For having the configuration $(3, 2, 2, 1)$ a cubic system has to possess three couples of parallel invariant lines and, moreover, one couple must increase up to a triplet. Thus, according to Lemma 4, if a cubic system possesses 8 invariant straight lines in the configuration $(3, 2, 2, 1)$, then necessarily the conditions $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$ hold.

A straightforward computation of the value of \mathcal{V}_3 for system (20) yields: $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$,

where

$$\begin{aligned}\mathcal{V}_{30} &= -p(p + 3r), & \mathcal{V}_{31} &= 2p(r - 2s - v), \\ \mathcal{V}_{32} &= 4rs + 3rq - sv - vr + 3ps - 2pq + 2s^2 + 2r^2 - v^2, \\ \mathcal{V}_{33} &= -2q(2r - s + v), & \mathcal{V}_{34} &= -q(q + 3s).\end{aligned}\tag{33}$$

If $pq \neq 0$, by (33), conditions $\mathcal{V}_{3i} = 0$, $i = 0, 1, 3, 4$ yield $p = q = -3r = -3s = 3v \neq 0$, and then, condition $\mathcal{V}_{32} = -27v^2 = 0$ implies $v = 0$, a contradiction. Thus, condition $pq = 0$ occurs, and we can suppose $q = 0$; otherwise we interchange $x \leftrightarrow y$, $p \leftrightarrow q$ and $r \leftrightarrow s$. We consider two cases: $p = 0$ and $p \neq 0$.

Case $p = 0$. Then, we have $\mathcal{V}_{30} = \mathcal{V}_{31} = 0$, and from (33) we obtain $\mathcal{V}_{32} = (2s + 2r + v)(s + r - v) = 0$.

Subcase $v = s + r$. For system (20) we obtain

$$\mathcal{V}_4 = 2^{10} 3^2 (s - r)(2r + s)(2s + r)xy(x - y)(rx + ys) = -\frac{1}{20}\mathcal{U}_2. \quad (34)$$

Via the transformation $x \leftrightarrow y$, $r \leftrightarrow s$ and $v \leftrightarrow v$, we can consider only two subcases: $r = s$ and $r = -2s$.

(a) We assume $r = s$. Then, $v = 2s$, and after a suitable time rescaling we obtain the following system:

$$x' = p_0 + p_1 + p_2 + x^3 + 3x^2y, \quad y' = q_0 + q_1 + q_2 + 3xy^2 + y^3. \quad (35)$$

(b) If $r = -2s$, then $v = -s$, and as above we obtain the system

$$x' = p_0 + p_1 + p_2 + 2x^3, \quad y' = q_0 + q_1 + q_2 + 3xy^2 - y^3. \quad (36)$$

Subcase $v = -2(s+r)$. Then, for system (20) we again obtain the values of the comitants \mathcal{V}_4 and \mathcal{U}_2 indicated in (34). Thus, we consider only two subcases: $r = s$ and $r = -2s$.

(a) If $r = s$ then $v = -4s$, and after a suitable time rescaling we obtain the system:

$$x' = p_0 + p_1 + p_2 + x^3 - 3x^2y, \quad y' = q_0 + q_1 + q_2 - 3xy^2 + y^3. \quad (37)$$

(b) Assume that $r = -2s$. Then, $v = 2s$ and this leads to the system

$$x' = p_0 + p_1 + p_2 - 2x^3 + 3x^2y, \quad y' = q_0 + q_1 + q_2 + y^3. \quad (38)$$

Case $p \neq 0$. From (33) we obtain $p = -3r$, $v = r - 2s$. Then, for system (20) we again obtain the values of the comitants \mathcal{V}_4 and \mathcal{U}_2 indicated in (34). However, as $q = 0$ and $p \neq 0$ we consider three subcases : $r = s$, $r = -2s$ and $s = -2r$.

Subcase $r = s$. Then, we have $v = -s$, $p = -3s \neq 0$, and this provides the system:

$$x' = p_0 + p_1 + p_2 + 2x^3, \quad y' = q_0 + q_1 + q_2 + 3x^2y - y^3. \quad (39)$$

Subcase $r = -2s$. Then $v = -4s$, $p = 6s \neq 0$, and via a suitable change of the time we obtain the system:

$$x' = p_0 + p_1 + p_2 + 4x^3 - 3x^2y, \quad y' = q_0 + q_1 + q_2 + 3x^2y - y^3. \quad (40)$$

Subcase $s = -2r$. Then $v = 5r$, $p = -3r \neq 0$, and we get the system:

$$x' = p_0 + p_1 + p_2 + 2x^3 - 3x^2y, \quad y' = q_0 + q_1 + q_2 + 3x^2y - 6xy^2 + 2y^3. \quad (41)$$

Lemma 17. *Canonical systems (35)–(41) are all affine equivalent.*

Proof: To prove this assertion it is sufficient to verify that the following changes go over to the systems as it is indicated:

$$\begin{aligned}
[x = x_1 - y_1, y = -y_1] : & & (39) & \Rightarrow & (41); \\
[x = y_1, y = 2x_1, t = t_1/2] : & & (38) & \Rightarrow & (36); \\
[x = x_1 - 2y_1, y = -2y_1, t = t_1/2] : & & (35) & \Rightarrow & (41); \\
[x = y_1 - x_1, y = -x_1] : & & (36) & \Rightarrow & (41); \\
[x = x_1, y = -y_1] : & & (37) & \Rightarrow & (35); \\
[x = x_1, y = 2y_1, t = t_1/2] : & & (40) & \Rightarrow & (41).
\end{aligned}$$

■

Remark 18. *We note that for system (35) the comitant $\mathcal{V}_1 = xy(x+y)^2 \neq 0$, and according to Lemma 1 this system cannot possess two triples of parallel invariant lines.*

Remark 19. *As it follows from (33) and (34) for the system with 4 real distinct infinite singular points the conditions $\mathcal{V}_3 = \mathcal{V}_4 = 0$ imply $\mathcal{U}_2 = 0$, as well as $\mathcal{U}_3 = 0$, because for systems (35) – (41) we have $\mathcal{U}_3 = 0$.*

According to Lemma 17 we can work for studying the configuration (3, 2, 2, 1) with the system (36). It is clear that, via translation of the origin of coordinates at the point $(-g/6, -m/3)$, we can consider $g = m = 0$. Thus, we only need to consider the system

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + 2x^3, \quad \dot{y} = b + ex + fy + lx^2 + ny^2 + 3xy^2 - y^3, \quad (42)$$

for which $C_3(x, y) = xy(x - y)(2x - y)$. Therefore, there are the following 4 directions for the possible invariant straight lines: $x = 0$, $y = 0$, $y = x$, $y = 2x$.

Direction $y = x$. We show that this direction it can be only one invariant straight line. Indeed, for $U = 1$ and $V = -1$ from the first 6 equations (27) we obtain

$$A = B = 2, \quad C = -1, \quad D = -2W - l, \quad E = -4W + 2h - l,$$

and $Eq_7 = -3W + 2h + k - l - n$. So, from system (27) we can obtain at most one solution W_0 .

Direction $x = 0$. Then $U = 1$, $V = 0$ and, from (27), we obtain

$$\begin{aligned}
A = 2, \quad B = C = 0, \quad D = -W, \quad E = 2h, \quad F = 2W^2 + c, \\
Eq_7 = k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = -2W^3 - cW + a.
\end{aligned}$$

Thus, for the existence of at least two solutions W_i it is necessary $k = h = d = 0$. Then there exist 3 invariant straight lines (which can coincide) in the direction $x = 0$. So, in what follows we shall suppose $k = h = d = 0$.

Direction $y = 0$. In this case $U = 0$, $V = 1$ and, from (27), we obtain

$$\begin{aligned} A = 0, \quad B = 3, \quad C = -1, \quad D = -3W, \quad E = W + n, \quad F = -W^2 - nW + f, \\ Eq_5 = l, \quad Eq_9 = 3W^2 + e, \quad Eq_{10} = W^3 + nW^2 - fW + b. \end{aligned}$$

Thus, we obtain $l = 0$, and

$$Eq_9 = 3W^2 + e = 0, \quad Eq_{10} = W^3 + nW^2 - fW + b = 0.$$

For having two common solutions W_i , according to Lemma 11, we obtain the relations:

$$R_W^{(0)}(Eq_9, Eq_{10}) = -3(3b - en)^2 - e(e + 3f)^2 = 0, \quad R_W^{(1)}(Eq_9, Eq_{10}) = 3(e + 3f) = 0.$$

Hence, we obtain $e = -3f$ and $b = en/3$, and the system (we set $n_1 = n/3$)

$$\dot{x} = a + cx + 2x^3, \quad \dot{y} = en_1 - 3fx + fy + 3n_1y^2 + 3xy^2 - y^3. \quad (43)$$

Direction $2x - y = 0$. Then $U = 2$, $V = -1$ and from (27) we obtain

$$\begin{aligned} A = 2, \quad B = 1, \quad C = -1, \quad D = -W + 6n_1, \quad E = -W + 3n_1, \quad F = -W^2 + 3n_1W + f, \\ Eq_5 = -12n_1, \quad Eq_9 = 3W^2 - 12n_1W + 2c + f, \quad Eq_{10} = W^3 - 3n_1W^2 - fW + 2a - en_1. \end{aligned}$$

So, $n_1 = 0$ and we obtain the following system of equations

$$Eq_9 = 3W^2 + 2c + f = 0, \quad Eq_{10} = W^3 - fW + 2a = 0.$$

For having two common solutions W_i , by Lemma 11, we obtain the relations:

$$R_W^{(0)}(Eq_9, Eq_{10}) = -4[27a^2 + (2c + f)(c + 2f)^2] = 0, \quad R_W^{(1)}(Eq_9, Eq_{10}) = 6(c + 2f) = 0.$$

Hence, we obtain $c + 2f = a = 0$ and, from (43), the following system

$$\dot{x} = -2fx + 2x^3, \quad \dot{y} = -3fx + fy + 3xy^2 - y^3, \quad (44)$$

which possesses the invariant straight lines $x = 0$, $x = \pm\sqrt{f}$, $y = \pm\sqrt{f}$, $y = x$, $2x - y = \pm\sqrt{f}$.

We note, that by Remark 14 we can consider $f \in \{-1, 0, 1\}$. Thus, we obtain that system (42) possesses the indicated invariant straight lines if and only if $h = k = l = n = 0$, $d = c + 2f = e + 3f = 0$, $a = b = 0$. We shall prove that these conditions are equivalent to $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Indeed, for system (42) we have $\mathcal{L}_1 =$

$2^9 3^4 [lx^3 + 2(h+n)x^2y - 3hxy^2 - ky^3]$. Thus, conditions $k = h = l = n = 0$ are equivalent to $\mathcal{L}_1 = 0$. Next, for system (42) with $\mathcal{L}_1 = 0$ we obtain $\mathcal{L}_2 = 2^8 3^5 [(5e - 8c - f)x^2 + (3c + 7d + 6f)xy - 5dy^2]$ and $\mathcal{L}_3 = -2^{10} 3^6 (x - y)^2 (c + f)$. Therefore, condition $\mathcal{L}_2 = 0$ is equivalent to $d = c + 2f = e + 3f = 0$, and for $c = -2f$ we obtain that $\text{sign}(\mathcal{L}_3) = \text{sign}(f)$.

Finally, for system (42) with $\mathcal{L}_1 = \mathcal{L}_2 = 0$ we calculate

$$\mathcal{N}_1 = 2^2 3^5 xy(2x - y) [(2a + 5b)x^2 - (5a + 2b) + (b - a)y^2],$$

and hence, conditions $a = b = 0$ are equivalent to $\mathcal{N}_1 = 0$. In the same manner as above (see Remark 15) it can be verified that the value of the CT -comitant \mathcal{N}_1 will not be changed after a translation of the origin of coordinates of the phase plane of the system

$$\dot{x} = a - 2fx + 2x^3, \quad \dot{y} = b - 3fx + fy + 3xy^2 - y^3,$$

at an arbitrary point (γ, δ) . Thus, considering Remark 19 we obtain the next result.

Proposition 20. *Cubic system (42) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Moreover, the configuration or the potential configuration of the lines corresponds with $(3, 2, 2, 1)$ given in Figure 4 (respectively, 5; 6) for \mathcal{L}_3 positive (respectively, negative; zero).*

From Lemma 5 and Propositions 16 and 20 the next theorem follows.

Theorem 21. *For cubic system (2) we assume that the conditions $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$ hold (i.e. system has 4 real infinite singular points). Then, this system will possess the maximum number of invariant straight lines (with total multiplicity 9) if and only if one of the following sequences of conditions holds:*

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0; \quad (\mathcal{A}_1)$$

$$\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0. \quad (\mathcal{A}_2)$$

Moreover, the configuration of the straight lines corresponds to Figure 1, 2 or 3 for (\mathcal{A}_1) if \mathcal{L}_3 negative, positive or zero, respectively; and to Figure 4, 5 or 6 for (\mathcal{A}_2) if \mathcal{L}_3 positive, negative or zero, respectively.

6. Cubic systems with 2 simple real and 2 simple imaginary roots of C_3

In this case, from (8), the cubic system after a linear transformation becomes:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + (u + 1)x^3 + (s + v)x^2y + rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) - sx^3 + ux^2y + vxy^2 + (r - 1)y^3. \end{aligned} \quad (45)$$

For system (45) we obtain $C_3 = x(sx + y)(x^2 + y^2)$, and hence, infinite singular points are situated at the “ends” of the straight lines: $x = 0$, $y = -sx$ and $y = \pm ix$.

In this section we shall construct the cubic systems of the form (45) which can possess 8 invariant straight lines with the configuration (3, 3, 1, 1) or (3, 2, 2, 1).

6.1 Systems with the configuration (3, 3, 1, 1)

In this subsection we construct the cubic system with 2 real and 2 imaginary infinite singular points which possess 8 invariant affine straight lines with configuration or potential configuration (3, 3, 1, 1), having total multiplicity 9, as always the invariant straight line of the infinity is considered.

By Lemma 1, if a cubic system possesses 8 invariant straight lines in the configuration (3, 3, 1, 1), then the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ hold. A straightforward computation of the values of \mathcal{V}_1 and \mathcal{V}_2 for system (45) yields: $\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$,

$\mathcal{V}_2 = 8 \sum_{j=0}^2 \mathcal{V}_{2j} x^{2-j} y^j$, where

$$\begin{aligned} \mathcal{V}_{10} &= 3s^2 + 2u^2 + 3u, & \mathcal{V}_{14} &= r(2r - 3), \\ \mathcal{V}_{11} &= 2su + 4uv + 9s + 3v, & \mathcal{V}_{20} &= 12sr - 9s - 3v + 2su, \\ \mathcal{V}_{12} &= 2sv + 4ur - s^2 + 3r - 3u + 2v^2, & \mathcal{V}_{21} &= 6r - 2s^2 - 4sv + 6u, \\ \mathcal{V}_{13} &= 2r(s + 2v) - 3(s + v), & \mathcal{V}_{22} &= 2sr - 3s - 3v. \end{aligned} \quad (46)$$

Consequently, relations $\mathcal{V}_1 = \mathcal{V}_2 = 0$ imply

$$\mathcal{V}_{14} = r(2r - 3) = 0, \quad \mathcal{V}_{13} - \mathcal{V}_{22} = 4rv = 0, \quad \mathcal{V}_{11} + \mathcal{V}_{20} = 2(2r - 3)(s + v) = 0. \quad (47)$$

Thus, we shall consider two cases: $r = 0$ and $r \neq 0$.

Case $r = 0$. From (47) we obtain $v = -s$ and, then the equation $\mathcal{V}_{10} + \mathcal{V}_{12} = 2(u^2 + s^2) = 0$ implies $u = s = 0$, and after the time rescaling $t \rightarrow t/3$, we get the system:

$$x' = p_0 + p_1 + p_2 + x^3, \quad y' = q_0 + q_1 + q_2 - y^3, \quad (48)$$

for which $\mathcal{U}_1(\mathbf{a}) = 0$.

Case $r \neq 0$. Then, from (47) we obtain $v = 0$, $r = 3/2$ and then we have $2\mathcal{V}_{10} + \mathcal{V}_{21} = (2u + 3)^2 + s^2 = 0$. Hence, $s = 0$, $u = -3/2$, and after a time rescaling $t \rightarrow -2t/3$, we obtain the system

$$x' = p_0 + p_1 + p_2 + x^3 - 3xy^2, \quad y' = q_0 + q_1 + q_2 + 3x^2y - y^3, \quad (49)$$

for which $\mathcal{U}_1(\mathbf{a}) = 0$. It is easy to check that systems (48) and (49) are not linearly equivalent. This immediately follows from the following remark.

Remark 22. For systems (48) and (49) we have $\mathcal{L}_4 = \alpha(x^4 + 6x^2y^2 + y^4)$, where $\alpha = -6$ for system (48) and $\alpha = 24$ for system (49). Since the T -comitant \mathcal{L}_4 is of degree 2 in the coefficients of the system and is of zero weight (i.e. is an absolute T -comitant), when conditions $\mathcal{V}_1 = \mathcal{V}_2 = 0$ are satisfied we obtain system (48) for $\mathcal{L}_4 < 0$ and system (49) for $\mathcal{L}_4 > 0$.

As it has been proved above we must examine systems (48) and (49).

System (48) Evidently via translation of the origin of coordinates at the point $(-g/3, -n/3)$ we can consider $g = n = 0$. Thus, we examine the system

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy - y^3, \quad (50)$$

for which $C_3(x, y) = xy(x^2 + y^2)$. Therefore, there are the following 4 directions for the possible invariant straight lines: $x = 0$, $y = 0$, $y = \pm ix$. We show that in each imaginary direction $y = \pm ix$ there is only one invariant straight line. Indeed, to show this it is sufficient to examine only the direction $x + iy = 0$, i.e. $U = 1$ and $V = i$. From the first 6 equations (27) we obtain

$$A = 1, \quad B = -i, \quad C = -1, \quad D = -W - 1, \quad E = 2h + i(2W + 2m + 1),$$

and $Eq_7 = 3W + 2m + k + 1 - 2ih$. So, from (27) we obtain at most one solution W_0 . We examine other directions in each of which it has to be one triple of parallel lines.

Direction $x = 0$. Then, $U = 1$, $V = 0$ and, from (27), we obtain

$$\begin{aligned} A = 1, \quad B = C = 0, \quad D = -W, \quad E = 2h, \quad F = W^2 + c, \\ Eq_7 = k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = -W^3 - cW + a. \end{aligned}$$

Thus, for the existence of three solutions W_i we must have $k = h = d = 0$, these three solutions can coincide in the direction $x = 0$. So, in what follows we shall suppose $k = h = d = 0$.

Direction $y = 0$. In this case $U = 0$, $V = 1$ and, again from (27), we obtain

$$\begin{aligned} A = B = 0, \quad C = -1, \quad D = 2m, \quad E = W, \quad F = -W^2 + f, \\ Eq_5 = l, \quad Eq_8 = -2mW + e, \quad Eq_{10} = W^3 - fW + b. \end{aligned}$$

Hence, for the existence of one triple of parallel lines in this direction it is necessary $l = m = e = 0$. Thus, we obtain the system

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy - y^3, \quad (51)$$

for which is necessary to examine the imaginary directions.

Direction $y + ix = 0$. We have $U = 1$, $V = i$ and, from (27), we obtain

$$\begin{aligned} A &= 1, \quad B = -i, \quad C = -1, \quad D = -W, \quad E = 2iW, \quad F = W^2 + W + c, \\ Eq_7 &= 3W, \quad Eq_8 = 3W^2 + c - f, \quad Eq_{10} = 2W^3 - fW + a + ib. \end{aligned}$$

So, there can exist a unique solution $W_0 = 0$. Hence, since the system is real we get up the conditions $c - f = a = b = 0$. Thus, we obtain the system

$$\dot{x} = cx + x^3, \quad \dot{y} = cy - y^3, \quad (52)$$

which possesses the invariant lines: $x = 0$, $y = 0$, $x = \pm\sqrt{-c}$, $y = \pm\sqrt{c}$, $y = \pm ix$.

We can consider $c \in \{0, 1\}$. Indeed, firstly we can suppose $c \geq 0$, otherwise the transformation $x \leftrightarrow y$, $t \leftrightarrow -t$ can be used. Then, for $c > 0$, we can apply Remark 14. Thus, we obtain that system (50) possesses the indicated invariant straight lines if and only if the following conditions hold: $h = k = l = m = 0$, $d = e = c - f = 0$, $a = b = 0$. We prove that these conditions are equivalent to $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Indeed, for system (50) we have $\mathcal{L}_1 = 2^8 3^4 (lx^3 + 2mx^2y - 2hxy^2 - ky^3)$. Hence, conditions $k = h = l = m = 0$ are equivalent to $\mathcal{L}_1 = 0$. Next, for system (50) with $\mathcal{L}_1 = 0$ we obtain $\mathcal{L}_2 = 2^7 3^5 [ex^2 + 6(c - f)xy - dy^2]$, $\mathcal{L}_3 = 2^9 3^5 (c + f)(x^2 - y^2)$. Therefore, condition $\mathcal{L}_2 = 0$ is equivalent to $d = e = c - f = 0$, and for $f = c$ we obtain that condition $c = 0$ is equivalent to $\mathcal{L}_3 = 0$.

Finally, if for system (50) the conditions $\mathcal{L}_1 = \mathcal{L}_2 = 0$ hold, then we obtain the system

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + cy - y^3, \quad (53)$$

for which we calculate $\mathcal{N}_1 = 2^3 3^5 xy(x^2 - y^2)(ax - by)$. Moreover, for the system

$$\begin{aligned} \dot{x} &= a + c\gamma + \gamma^3 + (c + 3\gamma^2)x_1 + 3\gamma x_1^2 + x_1^3, \\ \dot{y} &= b + c\delta - \delta^3 + (c + 3\delta^2)y_1 - 3\delta y_1^2 - y_1^3, \end{aligned}$$

which is obtained from system (53) via translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane of system (53), we have

$$\mathcal{N}_1(\mathbf{a}(\gamma, \delta), x_1, y_1) = 2^3 3^5 x_1 y_1 (x_1^2 - y_1^2) (ax_1 - b_1 y_1).$$

As we can observe that the value of this polynomial does not depend on the coordinates of the arbitrary point (γ, δ) and, consequently for system (53) condition $\mathcal{N}_1 = 0$ is equivalent to $a = b = 0$, and this condition is affine invariant. Thus, we have the next result.

Proposition 23. *A cubic system (48) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Moreover, the configuration or the potential configuration of the lines corresponds with $(3, 3, 1, 1)$ given in Figure 7 for $\mathcal{L}_3 \neq 0$, and in Figure 8 for $\mathcal{L}_3 = 0$.*

System (49) Doing a translation of the origin of coordinates at the point $(-g/3, -n/3)$, we can consider $g = n = 0$. Thus, we have the system

$$\begin{aligned}\dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3 - 3xy^2, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3x^2y - y^3,\end{aligned}\tag{54}$$

for which $C_3(x, y) = -2xy(x^2 + y^2)$. Therefore, there are the following 4 directions for possible invariant straight lines: $x = 0$, $y = 0$, $y = \pm ix$. We shall show that in the direction $x = 0$ as well as in the direction $y = 0$ there can exist at most one invariant straight line. Indeed, for the direction $x = 0$ (i.e. $U = 1$, $V = 0$) from the first 7 equations (27) we obtain $A = 1$, $B = 0$, $C = -3$, $D = -W$, $E = 2h$, $Eq_7 = 3W + k$. Whereas for the direction $y = 0$ (i.e. $U = 0$, $V = 1$) we have $A = 3$, $B = 0$, $C = -1$, $D = 2m$, $E = W$, $Eq_5 = -3W + l$. Thus, in both cases there can exist only one line in each considered direction.

We examine the imaginary directions, and since the system is real it is sufficient to consider only one direction.

Direction $x + iy = 0$. Then $U = 1$, $V = i$ and, from (27) we obtain

$$\begin{aligned}A &= 1, \quad B = 2i, \quad C = -1, \quad D = -W + il, \quad E = -i(W + k), \quad F = W^2 - ilW + c + ie, \\ Eq_6 &= l + 2h + i(k + 2m), \quad Eq_9 = (ik - l)W + i(f - c) + d + e, \\ Eq_{10} &= -W^3 + ilW^2 - (c + ie)W + a + ib.\end{aligned}$$

Thus, for the existence of three solutions W_i it is necessary $l + 2h = k + 2m = k = l = c - f = d + e = 0$, i.e. $l = k = h = m = 0$, $f = c$ and $d = -e$. In this case we get the system

$$\dot{x} = a + cx - ey + x^3 - 3xy^2, \quad \dot{y} = b + ex + cy + 3x^2y - y^3,$$

and we have to examine the real directions.

Direction $x = 0$. In this case $U = 1$, $V = 0$, and from (27), we obtain

$$\begin{aligned}A &= 1, \quad B = 0, \quad C = -3, \quad D = -W, \quad E = 0, \quad F = W^2 + c, \\ Eq_7 &= 3W, \quad Eq_9 = -e, \quad Eq_{10} = -W^3 - cW + a.\end{aligned}$$

So, there can exist only one solution $W_0 = 0$, and for the existence of one invariant straight line with the direction $x = 0$ it is necessary and sufficient $e = a = 0$.

Direction $y = 0$. Then $U = 0$, $V = 1$ and, from (27), we obtain

$$\begin{aligned}A &= 3, \quad B = 0, \quad C = -1, \quad D = 2m, \quad E = W, \quad F = -W^2 + c, \\ Eq_5 &= -3W, \quad Eq_{10} = W^3 - cW + b.\end{aligned}$$

So, again there is only one solution $W_0 = 0$, and for the existence of the invariant straight line with the direction $y = 0$ it is necessary and sufficient $b = 0$. Thus, we obtain the system

$$\dot{x} = cx + x^3 - 3xy^2, \quad \dot{y} = cy + 3x^2y - y^3, \quad (55)$$

which possesses the invariant straight lines: $x = 0$, $y = 0$, $x + iy = \pm\sqrt{-c}$, $x - iy = \pm\sqrt{-c}$, $x \pm iy = 0$. We can consider $c \in \{0, 1\}$. Indeed, firstly we can suppose $c \geq 0$, otherwise the transformation $x \leftrightarrow y$, $t \leftrightarrow -t$ can be used. Then, for $c > 0$ we can apply Remark 14. Thus, we obtain that system (54) possesses the indicated invariant straight lines if and only if $h = k = l = m = 0$, $d = e = c - f = 0$, $a = b = 0$. We shall prove that these conditions are equivalent to $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Indeed, for system (54) we have $\mathcal{L}_1 = 2^8 3^4 [(2h + 3l)x^3 + (k + 6m)x^2y - (6h + l)xy^2 - (3k + 2m)y^3]$. Hence, conditions $k = h = l = m = 0$ are equivalent to $\mathcal{L}_1 = 0$. Next, for system (54) with $\mathcal{L}_1 = 0$ we obtain

$$\mathcal{L}_2 = 2^8 3^5 [(7d - 5e)x^2 + 2(c - f)xy + (5d - 7e)y^2], \quad \mathcal{L}_3 = 2^{14} 3^5 (c + f) (x^2 - y^2).$$

Therefore, condition $\mathcal{L}_2 = 0$ is equivalent to $d = e = c - f = 0$, and for $f = c$ condition $c = 0$ is equivalent to $\mathcal{L}_3 = 0$.

Finally, system (54) for $\mathcal{L}_1 = \mathcal{L}_2 = 0$ becomes the system

$$\dot{x} = a + cx + x^3 - 3xy^2, \quad \dot{y} = b + cy + 3x^2y - y^3, \quad (56)$$

for which we calculate $\mathcal{N}_1 = 2^4 3^5 (x^4 - y^4) (ax + by)$. Moreover, for the system

$$\begin{aligned} \dot{x} &= (a + c\gamma + \gamma^3 - 3\gamma\delta^2) + (c + 3\gamma^2 - 3\delta^2)x_1 - 6\gamma\delta + 3\gamma x_1^2 - 6\delta x_1 y_1 - 3\gamma y_1^2 + x_1^3 - 3x_1 y_1^2, \\ \dot{y} &= (b + c\delta + 3\gamma^2\delta - \delta^3) + (c + 3\gamma^2 - 3\delta^2)y_1 + 3\delta x_1^2 + 6\gamma x_1 y_1 - 3\delta y_1^2 + 3x_1^2 y_1 - y_1^3, \end{aligned}$$

which is obtained from system (56) via the translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane of system (56), we have $\mathcal{N}_1(\mathbf{a}(\gamma, \delta), x_1, y_1) = 2^4 3^5 (x_1^4 - y_1^4) (ax_1 + by_1)$. So, the value of this polynomial does not depend on the coordinates of the arbitrary point (γ, δ) and, consequently for system (56) condition $\mathcal{N}_1 = 0$ is equivalent to $a = b = 0$, and this condition is affine invariant. Thus, we get the following result.

Proposition 24. *A cubic system (49) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Moreover, the configuration or the potential configuration of the lines corresponds with $(3, 3, 1, 1)$ given in Figure 9 for $\mathcal{L}_3 \neq 0$ or in Figure 10 for $\mathcal{L}_3 = 0$.*

6.2 Systems with configuration $(3, 2, 2, 1)$

In this subsection we construct the cubic systems with 2 real and 2 imaginary infinite singular points which possess 8 invariant affine straight lines with configuration or potential configuration $(3, 2, 2, 1)$, having total multiplicity 9.

By Lemma 4 if a cubic system possesses 8 invariant straight lines with configuration $(3, 2, 2, 1)$, then the conditions $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$ hold.

We consider again system (45). A straightforward computation of the value of \mathcal{V}_3 for this system yields: $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, where

$$\begin{aligned} \mathcal{V}_{30} &= -3s^2 - 3u + 3sv - u^2, & \mathcal{V}_{31} &= 6sr - 4su - 2uv - 18s, \\ \mathcal{V}_{32} &= -sv - 2ur - 9 + 2s^2 + 3r - 3u - v^2, \\ \mathcal{V}_{33} &= 2r(s - v), & \mathcal{V}_{34} &= r(3 - r). \end{aligned} \tag{57}$$

So, we need to consider the cases: $r \neq 0$ and $r = 0$.

If $r \neq 0$ then, from (57), we have $v = s$, $r = 3$ and, then

$$\mathcal{V}_{30} = -u(u + 3) = 0, \quad \mathcal{V}_{31} = -6su = 0, \quad \mathcal{V}_{32} = -9u = 0.$$

Hence, $u = 0$ and for system (45) we have:

$$\mathcal{V}_4 = 2^{11} 3^2 s(s^2 + 9)x(x^2 + y^2)(sx + y) = -\frac{1}{20}\mathcal{U}_2. \tag{58}$$

Consequently condition $\mathcal{V}_4 = 0$ implies $s = 0$, and this provides, after the rescaling of the time $t \rightarrow t/3$, the system

$$x' = p_0 + p_1 + p_2 + x^3 + 3xy^2, \quad y' = q_0 + q_1 + q_2 + 2y^3. \tag{59}$$

For $r = 0$, from (57), we obtain $\mathcal{V}_{32} = -sv - 9 + 2s^2 - 3u - v^2 = 0$. Hence $u = (2s^2 - sv - v^2 - 9)/3$. Then, by (57), we obtain

$$\mathcal{V}_{30} = -\frac{1}{9}(s - v)^2 [(2s + v)^2 + 9] = 0, \quad \mathcal{V}_{31} = -\frac{2}{3}(s - v) [(2s + v)^2 + 9] = 0,$$

that implies $v = s$, and then $u = -3$. Now, calculating the polynomials $\mathcal{V}_4 = 2^{11} 3^2 s(s^2 + 9)x(x^2 + y^2)(sx + y) = -\mathcal{U}_2/20 = 0$ we obtain $s = 0$. Thus, via the time rescaling $t \rightarrow -t/3$ we obtain the system

$$x' = p_0 + p_1 + p_2 + 2x^3, \quad y' = q_0 + q_1 + q_2 + 3x^2y + y^3. \tag{60}$$

Evidently, systems (59) and (60) are affine equivalent via the transformation $x \leftrightarrow y$.

We consider system (60). It is clear that via translation of the origin of coordinates at the point $(-g/6, -n/3)$ we can consider $g = n = 0$. Thus, we shall examine the system

$$\begin{aligned}\dot{x} &= a + cx + dy + 2hxy + ky^2 + 2x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3x^2y + y^3,\end{aligned}\tag{61}$$

for which $C_3(x, y) = -xy(x^2 + y^2)$. Therefore, there are the following 4 directions for the possible invariant straight lines: $x = 0$, $y = 0$, $y = \pm ix$. Since we are looking for the configuration (3, 2, 2, 1) in system (61), it follows that the two couples of parallel invariant lines must be in the imaginary directions.

First we show that in the direction $y = 0$ there cannot be a triple of parallel lines. Indeed, for $U = 0$ and $V = 1$, from the first five equations of (27), we obtain

$$A = 3, \quad B = 0, \quad C = 1, \quad Eq_5 = -3W + l = 0$$

and, hence, there can exist at most one solution of system (27).

Direction $x = 0$. Then $U = 1$, $V = 0$ and, from (27), we obtain

$$\begin{aligned}A &= 2, \quad B = C = 0, \quad D = -2W, \quad E = 2h, \quad F = 2W^2 + c, \\ Eq_7 &= k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = -2W^3 - cW + a.\end{aligned}$$

Thus, for the existence of three solutions W_i (which can coincide) it is necessary and sufficient $k = h = d = 0$, and in what follows we shall assume that these conditions hold.

Direction $y + ix = 0$. We have $U = 1$, $V = i$ and, from (27), we obtain

$$\begin{aligned}A &= 2, \quad B = i, \quad C = 1, \quad D = -2W + il, \quad E = iW, \quad F = -W^2 + f, \\ Eq_6 &= l + 2im, \quad Eq_8 = 3W^2 - ilW + c - f + ie, \quad Eq_{10} = W^3 - fW + a + ib.\end{aligned}$$

Thus, since the considered cubic system is real, for the existence of two solutions W_i it is necessary that $l = m = 0$. Then, according to Lemma 11, for having two common solutions W_i it is necessary $R_W^{(1)}(Eq_8, Eq_{10}) = 3(c + 2f + ie) = 0$. Hence, since the system is real, the last condition yields $e = 0$, $c = -2f$. Therefore, $R_W^{(0)} = 27(a + ib)^2 = 0$, that gives $a = b = 0$. It remains to note that the obtained conditions are sufficient for the existence of an invariant straight line in the real direction $y = 0$, as it is observed below. Thus, we obtain the system

$$\dot{x} = -2fx + 2x^3, \quad \dot{y} = fy + 3x^2y + y^3,\tag{62}$$

which possesses the invariant straight lines: $x = 0$, $y = 0$, $x = \pm\sqrt{f}$, $x - iy = \pm\sqrt{f}$, $x + iy = \pm\sqrt{f}$. It is clear that the lines $x = \pm\sqrt{f}$ (respectively, $y = \pm\sqrt{f}$) are real for $f > 0$, imaginary for $f < 0$, and coincide for $f = 0$. Hence, we obtain Figure 11

(respectively, 12 and 13) for $f > 0$ (respectively for $f < 0$ and $f = 0$). We note, that by Remark 14 we can consider $f \in \{-1, 0, 1\}$.

In short, we obtain that system (61) possesses the indicated invariant straight lines if and only if $h = k = l = m = 0$, $d = e = c + 2f = 0$, $a = b = 0$. We shall prove that these conditions are equivalent to $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Indeed, for system (61) we have $\mathcal{L}_1 = 2^9 3^4 [(h - l)x^3 - 2mx^2y + 3hxy^2 + ky^3]$. Thus, conditions $k = h = l = m = 0$ are equivalent to $\mathcal{L}_1 = 0$. Next, for system (61) with $\mathcal{L}_1 = 0$, we obtain

$$\mathcal{L}_2 = 2^8 3^5 [(7d - 5e)x^2 - 3(c + 2f)xy + 5dy^2], \quad \mathcal{L}_3 = 2^{10} 3^6 (c + f)y^2.$$

Hence, condition $\mathcal{L}_2 = 0$ is equivalent to $d = c + 2f = e + 3f = 0$ and, for $c = -2f$, we obtain $\text{sign}(\mathcal{L}_3) = -\text{sign}(f)$.

Finally, for system (61) with $\mathcal{L}_1 = \mathcal{L}_2 = 0$ we calculate $\mathcal{N}_1 = -2^3 3^5 x(x^2 + y^2)(4bx^2 - 7axy - by^2)$, and hence, conditions $a = b = 0$ are equivalent to $\mathcal{N}_1 = 0$. Moreover, the GL -comitant \mathcal{N}_1 is a CT -comitant for system (61) when conditions $\mathcal{L}_1 = \mathcal{L}_2 = 0$ are satisfied. Thus, the following result holds.

Proposition 25. *A cubic system (61) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0$. Moreover, the configuration or the potential configuration of the lines corresponds with $(3, 2, 2, 1)$ given in Figure 11 (respectively, 12; 13) for \mathcal{L}_3 negative (respectively, positive; zero).*

From Lemma 5 and Propositions 23, 24 and 25 the next theorem follows.

Theorem 26. *We assume that for a cubic system (2) the condition $\mathcal{D}_1 < 0$ holds (i.e. there are 2 real and 2 imaginary infinite singular points). Then, this system possesses the maximum number of invariant straight lines (with total multiplicity 9) if and only if one of the following sequences of conditions holds:*

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0; \quad (\mathcal{B}_1)$$

$$\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{N}_1 = 0. \quad (\mathcal{B}_2)$$

Moreover, the configuration of the lines corresponds to Figures 7, 8, 9 or 10 for (\mathcal{B}_1) if $|\mathcal{L}_3|\mathcal{L}_4 < 0$, $\mathcal{L}_3 = 0$ and $\mathcal{L}_4 < 0$, $|\mathcal{L}_3|\mathcal{L}_4 > 0$, or $\mathcal{L}_3 = 0$ and $\mathcal{L}_4 > 0$, respectively; to Figures 11, 12, or 13 for (\mathcal{B}_2) if \mathcal{L}_3 negative, positive or zero, respectively.

From Theorems 21 and 26, and the fact that the GL -comitants \mathcal{L}_1 and \mathcal{L}_2 are T -comitants for the initial cubic system (2) we obtain the next result.

Lemma 27. *Conditions $\mathcal{L}_1 = \mathcal{L}_2 = 0$ are necessary in order that a cubic system possesses the maximum number of the invariant straight lines counted with their multiplicities.*

Proof: The lemma is obvious for cubic systems having a *generic behavior at infinity*, i.e. the multiplicity of all singular points at infinity as roots of the polynomial C_3 is one. Then, by continuity, it follows for the cubic systems having non-generic behavior at infinity. ■

7. Cubic systems with 1 triple and 1 simple real roots of C_3

By (12) a cubic system having one triple and one real distinct infinite singular points via a linear transformation becomes the system:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + 3(u+1)x^3 + 3vx^2y + 3rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) + 3ux^2y + 3vxy^2 + 3ry^3. \end{aligned} \quad (63)$$

For system (63) we have $C_3 = 3x^3y$. Hence, the infinite singular points are situated at the “ends” of the following straight lines: $x = 0$ and $y = 0$.

The aim of this section is to construct cubic systems of the form (63) which possess invariant straight lines with total multiplicity 9, having 8 affine lines with potential configuration (3, 3, 1, 1) or (3, 2, 2, 1).

7.1 Systems with the potential configuration (3, 3, 1, 1)

By Lemma 1, if a cubic system possesses 8 invariant straight lines with configuration or potential configuration (3, 3, 1, 1), then the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ hold.

A straightforward computation of the value of \mathcal{V}_1 for system (63) provides that \mathcal{V}_1 is equal to $16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$, where

$$\begin{aligned} \mathcal{V}_{10} &= u(2u+3), & \mathcal{V}_{12} &= 4ru + 3r + 2v^2, \\ \mathcal{V}_{11} &= v(4u+3), & \mathcal{V}_{13} &= 4vr, & \mathcal{V}_{14} &= 2r^2. \end{aligned}$$

Consequently, relations $\mathcal{V}_1 = 0$ imply $v = r = 0$ and $u(2u+3) = 0$, and we have to consider two subcases $u = 0$ and $u = -3/2$.

For $u = 0$, by the time rescaling $t \rightarrow t/3$, we obtain the system

$$x' = p_0 + p_1 + p_2 + x^3, \quad y' = q_0 + q_1 + q_2, \quad (64)$$

whereas for $u = -3/2$, after the time rescaling $t \rightarrow -2t/3$, we have the system

$$x' = p_0 + p_1 + p_2 + x^3, \quad y' = q_0 + q_1 + q_2 + 3x^2y. \quad (65)$$

It has to be underlined that for systems (64) and (65) the relation $\mathcal{V}_2(\mathbf{a}) = \mathcal{U}_1(\mathbf{a}) = 0$ holds. On the other hand, by calculating the value of the T -comitant \mathcal{L}_4 for system (64) (respectively, (65)), we obtain $\mathcal{L}_4 = -6x^4$ (respectively, $\mathcal{L}_4 = 24x^4$). Hence, we get the next result.

Proposition 28. *If for a cubic system the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$, $\mathcal{D}_2 \neq 0$ and $\mathcal{V}_1 = 0$ hold, then via a linear transformation and time rescaling this system becomes into the form (64) for $\mathcal{L}_4 < 0$, and into the system (65) for $\mathcal{L}_4 > 0$.*

Remark 29. *We note that for system (64) the relations $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$ are also satisfied. By Lemma 4 this system can present the potential configuration (3, 2, 2, 1).*

As it was proved above, we must examine systems (64) and (65).

System (64) Via the translation of the origin of coordinates at the point $(-g/3, 0)$, we can consider $g = 0$ and, hence, we get the system

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2. \quad (66)$$

For this system we have $C_3(x, y) = x^3y$ and therefore, there exist two directions for the possible invariant straight lines: $x = 0$ and $y = 0$.

Direction $x = 0$. In this case $U = 1$, $V = 0$ and, from (27), we obtain

$$\begin{aligned} A = 1, \quad B = C = 0, \quad D = -W, \quad E = 2h, \quad F = W^2 + c, \\ Eq_7 = k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = -W^3 - cW + a. \end{aligned}$$

So, in order to have the maximum number of invariant straight lines we obtain the conditions: $k = h = d = 0$.

Direction $y = 0$. In this case $U = 0$, $V = 1$ and, from (27), we obtain

$$\begin{aligned} A = B = C = 0, \quad D = 2m, \quad E = n, \quad F = -nW + f, \\ Eq_5 = l, \quad Eq_8 = -2mW + e, \quad Eq_{10} = nW^2 - fW + b. \end{aligned}$$

So, we get the conditions : $l = m = e = 0$. Thus, system (66) becomes

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + ny^2,$$

for which we calculate $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = -2^8 3^5 n^2 xy$. Hence, since in order to reach the total multiplicity 9, by Lemma 27, it is necessary $\mathcal{L}_2 = 0$, we obtain the additional condition: $n = 0$. Therefore, system (64) goes over to

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy. \quad (67)$$

By Lemmas 8 and 9, in order to determine the possible invariant straight lines we shall use the affine comitants \mathcal{G}_i ($i = 1, 2, 3$). We consider the homogenized system

$$\dot{X} = aZ^3 + cXZ^2 + X^3, \quad \dot{Y} = bZ^3 + fYZ^2, \quad (68)$$

associated to system (67) and calculate the following polynomial:

$$H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2Z^2(fY + Zb)(aZ^3 + cXZ^2 + X^3).$$

Therefore, by Lemmas 8 and 9, we obtain that system (68) has 7 invariant affine straight lines (counted with their multiplicities) $Z = 0$, $fY + Zb = 0$, $aZ^3 + cXZ^2 + X^3 = 0$. We observe that $Z = 0$ has multiplicity 3. So, for having the total multiplicity 9, by Lemmas 8 and 9, the polynomial $H(\mathbf{a}, X, Y, Z)$ must have degree 8. In order to find out the conditions to reach this situation we shall calculate for system (68) the polynomials:

$$\begin{aligned} \mathcal{G}_1/H &= 3X^2 + (c - f)Z^2 = T(X, Z), \\ \mathcal{G}_3/H &= 12(aZ^3 + cXZ^2 + X^3)^2 = 12S^2(X, Z). \end{aligned} \tag{69}$$

From Lemma 10, in order to have the maximum number of invariant straight lines it is necessary that $T(X, Z) \mid S^2(X, Z)$. We consider two subcases: $c - f \neq 0$ and $c - f = 0$.

Case $c - f \neq 0$. In this case polynomial $T(X, Z)$ has two distinct factors and hence, it is necessary $T(X, Z) \mid S(X, Z)$. By Lemma 12, we have

$$R_X^{(1)}(T, S) = 3(2c + f)Z^2 = 0, \quad R_X^{(0)}(T, S) = -[27a^2 + ((2c + f)^2(c - f))] = 0,$$

and these conditions yield $2c + f = a = 0$. Thus, $f = -2c \neq 0$ (by condition $c - f \neq 0$), and then we can consider $b = 0$ via a translation of the origin of coordinates at the point $(0, b/(2c))$. As a result we obtain the system

$$\dot{x} = cx + x^3, \quad \dot{y} = -2cy, \tag{70}$$

which possesses the invariant straight lines: $x = 0$, $y = 0$, $x = \pm\sqrt{-c}$. It is clear, that $x = \pm\sqrt{-c}$ are real for $c < 0$, and imaginary for $c > 0$. Hence, we obtain Figure 14 for $c < 0$ and Figure 15 for $c > 0$.

By Remark 14 we can consider $c \in \{-1, 1\}$. Since for system (70) the conditions $\mathcal{V}_1 = \mathcal{V}_2 = 0$ hold as well as the conditions $\mathcal{V}_3 = \mathcal{V}_4 = 0$, we conclude that this system has invariant straight lines with total multiplicity 9 and presents both potential configurations: $(3, 3, 1, 1)$ and $(3, 2, 2, 1)$. This is proved by the four perturbed systems constructed below.

Subcase $c = -1$. The system

$$\dot{x} = -x + x^3, \quad \dot{y} = 2y + 3\varepsilon y^2 + \varepsilon^2 y^3, \tag{71}$$

with the invariant straight lines $x = 0$, $y = 0$, $x = \pm 1$, $\varepsilon y + 1 = 0$, $\varepsilon y + 2 = 0$, $\varepsilon y \pm x + 1 = 0$, has the configuration $(3, 3, 1, 1)$; and the system

$$\dot{x} = -x + x^3 - 3\varepsilon^2 x y^2, \quad \dot{y} = 2y - 2\varepsilon^2 y^3, \tag{72}$$

with the invariant straight lines $x = 0$, $y = 0$, $\varepsilon y = \pm 1$, $x + \varepsilon y = \pm 1$, $x - \varepsilon y = \pm 1$, has the configuration $(3, 2, 2, 1)$.

Subcase $c = 1$. The system

$$\dot{x} = x + x^3, \quad \dot{y} = -2y + 3\varepsilon y^2 - \varepsilon^2 y^3, \quad (73)$$

with the invariant straight lines $x = 0$, $y = 0$, $\varepsilon y = 1$, $\varepsilon y = 2$, $\varepsilon y \pm ix = 1$, $x = \pm i$, has the configuration $(3, 3, 1, 1)$; and the system

$$\dot{x} = x + x^3 - 3\varepsilon^2 x y^2, \quad \dot{y} = -2y - 2\varepsilon^2 y^3, \quad (74)$$

with the invariant straight lines $x = 0$, $y = 0$, $\varepsilon y = \pm 1$, $\varepsilon y + ix = \pm i$, $\varepsilon y - ix = \pm i$ has the configuration $(3, 2, 2, 1)$.

Case $c - f = 0$. From (69) it follows that polynomial $R(X, Z)$ has the double root $X = 0$ and in virtue of condition $R(X, Z) \mid S^2(X, Z)$ it is necessary $a = 0$. Thus, we obtain the system

$$\dot{x} = cx + x^3, \quad \dot{y} = b + cy, \quad (75)$$

and by Remark 14 we can consider $c \in \{-1, 0, 1\}$. Moreover, if $c \neq 0$ then the translation of the origin of coordinates at the point $(0, -b/c)$ implies $b = 0$; and for $c = 0$, by replacing $y \rightarrow by$, we can set $b = 1$. So, we obtain Figure 16 (respectively, Figure 17; 18) for $c = -1$, $b = 0$ (respectively for $c = 1$, $b = 0$; $c = 0$, $b = 1$).

Since for system (75) the conditions $\mathcal{V}_1 = \mathcal{V}_2 = 0$ hold as well as the conditions $\mathcal{V}_3 = \mathcal{V}_4 = 0$, we conclude that this system has invariant straight lines with total multiplicity 9 and presents both potential configurations: $(3, 3, 1, 1)$ and $(3, 2, 2, 1)$. It remains to construct the six perturbed systems which will prove the realization of all possibilities.

Subcase $c = -1$. The system

$$\dot{x} = -x + x^3, \quad \dot{y} = -y + \varepsilon^2 y^3, \quad (76)$$

with the invariant straight lines $x = 0$, $y = 0$, $x = \pm 1$, $\varepsilon y = \pm 1$, $x - \varepsilon y = 0$, $x + \varepsilon y = 0$, has the configuration $(3, 3, 1, 1)$; and the system

$$\dot{x} = (4\varepsilon^2 - 1)x - 6\varepsilon xy + x^3 - 12\varepsilon^2 xy^2, \quad \dot{y} = -2\varepsilon - (8\varepsilon^2 + 1)y - 6\varepsilon y^2 - 8\varepsilon^2 y^3, \quad (77)$$

with the invariant straight lines $x = 0$, $4\varepsilon y = -1$, $2x + 4\varepsilon y + 1 = \pm\sqrt{1 - 16\varepsilon^2}$, $2\varepsilon y^2 + y = -2\varepsilon$, $2x - 4\varepsilon y - 1 = \pm\sqrt{1 - 16\varepsilon^2}$, has the configuration $(3, 2, 2, 1)$.

Subcase $c = 1$. The system

$$\dot{x} = x + x^3, \quad \dot{y} = y + \varepsilon^2 y^3, \quad (78)$$

with the invariant straight lines $x = 0$, $y = 0$, $x = \pm i$, $\varepsilon y = \pm i$, $x - \varepsilon y = 0$, $x + \varepsilon y = 0$, has the configuration (3, 3, 1, 1); and the system

$$\dot{x} = x + 6\varepsilon xy + x^3 + 12\varepsilon^2 xy^2, \quad \dot{y} = y + 6\varepsilon y^2 + 8\varepsilon^2 y^3, \quad (79)$$

with the invariant straight lines $x = 0$, $y = 0$, $4\varepsilon y = -1$, $2\varepsilon y = -1$, $2\varepsilon y \pm ix = 0$, $2\varepsilon y \pm ix = -1$, has the configuration (3, 2, 2, 1).

Subcase $c = 0$. The system

$$\dot{x} = 18\varepsilon^2 x + 9\varepsilon x^2 + x^3, \quad \dot{y} = 1 - 6\varepsilon^2 y - 24\varepsilon^4 y^2 + 64\varepsilon^6 y^3, \quad (80)$$

with the invariant straight lines $x = 0$, $x = -3\varepsilon$, $x = -6\varepsilon$, $x + 8\varepsilon^3 y = -2\varepsilon$, $8\varepsilon^2 y = 1$, $4\varepsilon^2 y = -1$, $2\varepsilon^2 y = 1$, $x - 8\varepsilon^3 y = -4\varepsilon$, has the configuration (3, 3, 1, 1); and the system

$$\dot{x} = \varepsilon(1 - 4\varepsilon)x + 12\varepsilon^3 xy + x^3 - 12\varepsilon^4 xy^2, \quad \dot{y} = 1 - 2\varepsilon(1 + 2\varepsilon)y + 12\varepsilon^3 y^2 - 8\varepsilon^4 y^3, \quad (81)$$

with the invariant straight lines $x = 0$, $2\varepsilon y = 1$, $x - 2\varepsilon^2 y + \varepsilon = \pm\sqrt{\varepsilon^2 - \varepsilon}$, $\varepsilon(2\varepsilon y - 1)^2 = \varepsilon - 1$, $x + 2\varepsilon^2 y - \varepsilon = \pm\sqrt{\varepsilon^2 - \varepsilon}$ has in configuration (3, 2, 2, 1).

We construct necessary and sufficient affine invariant conditions for the realization of each possible configuration given in Figures 14–18 for system (66).

As it was proved above system (66) possesses 9 invariant straight lines (counted with their multiplicities) if and only if $h = k = l = m = n = 0$, $d = e = (c - f)(2c + f) = 0$, $a = 0$. First, we shall show that conditions $h = k = l = m = n = 0$ are equivalent to conditions $\mathcal{N}_2 = \mathcal{N}_3 = 0$. To get this goal we consider the system:

$$\dot{x}_1 = p_0^\tau + p_1^\tau + 3\gamma x_1^2 + 2hx_1 y_1 + ky_1^2 + x_1^3, \quad \dot{y}_1 = q_0^\tau + q_1^\tau + lx_1^2 + 2mx_1 y_1 + ny_1^2, \quad (82)$$

which is obtained from system (66) via the translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane of system (66) and $p_i^\tau(a(\gamma, \delta), x_1, y_1)$, $q_i^\tau(a(\gamma, \delta), x_1, y_1)$ ($i = 0, 1$) are the corresponding homogeneous polynomials of degree i in x_1 and y_1 .

For system (82) we calculate the values of the GL -comitants \mathcal{N}_2 and \mathcal{N}_3 which depend only on the coefficients of the quadratic and cubic parts of the system:

$$\begin{aligned} \mathcal{N}_2 &= -6x_1^2 [lx_1^3 + 3mx_1^2 y_1 + (2n - h)x_1 y_1^2 - ky_1^3], \\ \mathcal{N}_3 &= -12x_1^2 [lx_1^3 + (2h - n)x_1 y_1^2 + 2ky_1^3]. \end{aligned}$$

As we can observe, the values of these polynomials do not depend on the coordinates of the arbitrary point (γ, δ) and, consequently conditions $\mathcal{N}_2 = 0$ and $\mathcal{N}_3 = 0$ are affine invariant ones. It is obvious to find out that these conditions yield $h = k = l = m = n = 0$, i.e. all quadratic coefficients vanish. Thus, system (82) becomes the system

$$\begin{aligned} \dot{x}_1 &= (a + c\gamma + d\delta + \gamma^3) + (c + 3\gamma^2)x_1 + dy_1 + 3\gamma x_1^2 + x_1^3, \\ \dot{y}_1 &= (b + e\gamma + f\delta) + ex_1 + fy_1, \end{aligned}$$

for which we have:

$$\begin{aligned}
\mathcal{N}_6 &= -36x_1^4 [6ex_1^2 + (c-f)x_1y_1 - 16dy_1^2], \\
\mathcal{N}_7 &= -72x_1^4 [5ex_1^2 + (2c+f)x_1y_1 - 18dy_1^2], \\
\mathcal{N}_8 &= -12(c+f)x_1^2. \\
\mathcal{N}_1 &= -2^3 3^5 x_1^3 y_1 [ax_1 + d(\delta x_1 - 3\gamma y_1)].
\end{aligned}$$

As above condition $\mathcal{N}_6 = 0$ (respectively, $\mathcal{N}_7 = 0$) is an affine invariant one and implies $d = e = c - f = 0$ (respectively, $d = e = 2c + f = 0$). Moreover, in both cases the GL -comitant \mathcal{N}_1 will not depend on the coordinates of the arbitrary point (γ, δ) ; i.e. it becomes a T -comitant and the condition $\mathcal{N}_1 = 0$ yields $a = 0$. On the other hand, for $f = c$ (respectively, for $f = -2c$) we obtain that $\text{sign}(\mathcal{N}_8) = -\text{sign}(c)$ (respectively, $\text{sign}(\mathcal{N}_8) = \text{sign}(c)$). Thus, we obtain the next result.

Lemma 30. *System (64) possesses invariant straight lines with total multiplicity 9 if and only if*

$$\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 \mathcal{N}_7 = 0. \quad (\mathcal{C})$$

Moreover, the configurations of the lines correspond to Figures 14 or 15 for (\mathcal{C}) with $\mathcal{N}_6 \neq 0$ and $\mathcal{N}_8 < 0$ or $\mathcal{N}_8 > 0$, respectively; to Figures 16, 17 or 18 for (\mathcal{C}) with $\mathcal{N}_6 = 0$ and \mathcal{N}_8 positive, negative or zero, respectively.

System (65) It is clear that via a translation of the origin of coordinates at the point $(-g/3, -l/3)$, we can consider $g = l = 0$ and, hence, we must examine the system

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + 2mxy + ny^2 + 3x^2y. \quad (83)$$

For this system we have $C_3(x, y) = -2x^3y$ and therefore, there exists two directions for the possible invariant straight lines: $x = 0$ and $y = 0$.

Direction $x = 0$. In this case $U = 1$, $V = 0$ and from (27) we obtain

$$\begin{aligned}
A &= 1, \quad B = C = 0, \quad D = -W, \quad E = 2h, \quad F = W^2 + c, \\
Eq_7 &= k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = -W^3 - cW + a.
\end{aligned}$$

So, in order to have the maximum number of invariant straight lines we obtain the conditions: $k = h = d = 0$.

Direction $y = 0$. In this case $U = 0$, $V = 1$ and from (27) we obtain

$$\begin{aligned}
A &= 3, \quad B = C = 0, \quad D = 2m, \quad E = n, \quad F = -nW + f, \\
Eq_5 &= 3W, \quad Eq_8 = -2mW + e, \quad Eq_{10} = nW^2 - fW + b.
\end{aligned}$$

So, in this direction can be only one simple invariant straight line with $W_0 = 0$, and the necessary conditions are $e = b = 0$.

In short, system (83) becomes

$$\dot{x} = a + cx + x^3, \quad \dot{y} = fy + 2mxy + ny^2 + 3x^2y,$$

for which we calculate $\mathcal{L}_1 = 2^8 3^4 nx^3$, $\mathcal{L}_2 = 2^8 3^4 nx(mx + 9ny)$. Hence, in order to reach the total multiplicity 9, by Lemma 27, it is necessary $\mathcal{L}_1 = \mathcal{L}_2 = 0$, and we obtain the additional condition: $n = 0$. This provides the system:

$$\dot{x} = a + cx + x^3, \quad \dot{y} = fy + 2mxy + 3x^2y. \quad (84)$$

By Lemmas 8 and 9, in order to determine the possible invariant straight lines we shall use the affine comitants \mathcal{G}_i ($i = 1, 2, 3$). We consider the homogenized system

$$\dot{X} = aZ^3 + cXZ^2 + X^3, \quad \dot{Y} = fYZ^2 + 2mXYZ + 3X^2Y, \quad (85)$$

corresponding to system (84), and calculate the following polynomial:

$$H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2Y(aZ^3 + cXZ^2 + X^3),$$

and

$$\begin{aligned} \mathcal{G}_1/H &= -6X^4 - 8mX^3Z - (3f + 4m^2 + 3c)X^2Z^2 \\ &\quad - 2(3a + 2fm)XZ^3 - (2am + f^2 - cf)Z^4 = T(X, Z), \\ \mathcal{G}_2/H &= -3(3aZ^3 + 2mfZ^3 + 4Z^2m^2X + 3Z^2Xf + 3cXZ^2 \\ &\quad + 12mX^2Z + 12X^3)(aZ^3 + cXZ^2 + X^3) = S(X, Z). \end{aligned} \quad (86)$$

By Lemma 10, in order to have the maximum number of invariant straight lines it is necessary that $R(X, Z) \mid S(X, Z)$. Hence, by Lemma 12, the conditions $R_X^{(i)}(T, S) = 0$ ($i = 0, 1, 2, 3$), have to be satisfied, where $R_X^{(0)}(T, S) = \text{Res}_X(T, S)$. So, we can calculate: $R_X^{(3)}(T, S) = 72(27a + 18cm + 8m^3)Z^3 = 0$, and, hence, we obtain the condition: $a = -(18cm + 8m^3)/27$. Then, we have

$$R_X^{(2)}(R, S) = -36(9c - 3f + 4m^2)^2(3c - 3f + 20m^2)^2Z^8 = 0,$$

and this implies the necessity to examine the two subcases: $9c - 3f + 4m^2 = 0$ and $9c - 3f + 4m^2 \neq 0$.

Case $9c - 3f + 4m^2 = 0$. Then, $f = (9c + 4m^2)/3$, and we have $R_X^{(1)}(T, S) = R_X^{(0)}(T, S) = 0$, but at the same time this provides the degenerated system:

$$\dot{x} = -(2m - 3x)(4m^2 + 6xm + 9c + 9x^2)/27, \quad \dot{y} = y(4m^2 + 6xm + 9c + 9x^2)/3.$$

Case $9c - 3f + 4m^2 \neq 0$. Then, condition $R_X^{(2)}(T, S) = 0$ implies $f = (20m^2 + 3c)/3$, and we can calculate $R_X^{(1)}(T, S) = 2^{11}3^2m^3(3c + 28m^2)^3(3c - 8m^2)^3Z^{15} = 0$, and since condition $9c - 3f + 4m^2 \neq 0$ yields $3c - 8m^2 \neq 0$, we obtain condition $m(3c + 28m^2) = 0$.

If $m \neq 0$ then $c = -28m^2/3 \neq 0$, and this provides the degenerated system:

$$\dot{x} = (3x + 10m)(-3x + 8m)(-3x + 2m)/27, \quad \dot{y} = -y(4m + 3x)(-3x + 2m)/3.$$

Thus, $m = 0$ and we get the system :

$$\dot{x} = cx + x^3, \quad \dot{y} = cy + 3x^2y, \quad (87)$$

with $c \neq 0$ (otherwise system becomes degenerated). We note that by Remark 14 we can consider $c \in \{-1, 1\}$. So, we obtain Figure 19 for $c = -1$, and Figure 20 for $c = 1$. It remains to construct two perturbed systems.

For $c = -1$ the perturbed system

$$\dot{x} = -x + x^3, \quad \dot{y} = -y + 3x^2y - 18\varepsilon xy^2 + 36\varepsilon^2 y^3, \quad (88)$$

possesses the invariant straight lines $x = 0$, $x = \pm 1$, $y = 0$, $x - 3\varepsilon y = 0$, $x - 6\varepsilon y = 0$, $x - 6\varepsilon y = \pm 1$ in the configuration $(3, 3, 1, 1)$; and for $c = 1$ the system

$$\dot{x} = x + x^3, \quad \dot{y} = y + 3x^2y - 18\varepsilon xy^2 + 36\varepsilon^2 y^3, \quad (89)$$

possesses the invariant straight lines $x = 0$, $x = \pm i$, $y = 0$, $x - 3\varepsilon y = 0$, $x - 6\varepsilon y = 0$, $x - 6\varepsilon y = \pm i$ in the same configuration $(3, 3, 1, 1)$.

We construct necessary and sufficient affine invariant conditions for the realization of the two possible configurations for system (83), see Figures 19 and 20.

As it was proved above system (83) possesses 9 invariant straight lines (counted with their multiplicities) if and only if $h = k = m = n = 0$, $d = e = c - f = 0$, $a = b = 0$. First, we shall show that conditions $h = k = m = n = 0$ are equivalent to conditions $\mathcal{N}_4 = \mathcal{N}_5 = 0$. To get this goal we shall consider the system:

$$\begin{aligned} \dot{x}_1 &= p_0^\tau + p_1^\tau + 3\gamma x_1^2 + 2hx_1y_1 + ky_1^2 + x_1^3, \\ \dot{y}_1 &= q_0^\tau + q_1^\tau + 3\delta x_1^2 + 2(m + 3\gamma)x_1y_1 + ny_1^2 + 3x_1^1y_1, \end{aligned} \quad (90)$$

which is obtained from system (83) via the translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane of system (83) and $p_i^\tau(a(\gamma, \delta), x_1, y_1)$, $q_i^\tau(a(\gamma, \delta), x_1, y_1)$ ($i = 0, 1$) are the corresponding homogeneous polynomials of degree i in x_1 and y_1 .

For system (90) we calculate the values of the GL -comitants \mathcal{N}_4 and \mathcal{N}_5 which depend only on the coefficients of the quadratic and cubic parts of the system:

$$\mathcal{N}_4 = 24x_1 [3mx_1^2 + 5(h + n)x_1y_1 + 6ky_1^2], \quad \mathcal{N}_5 = -48x_1y_1 [(2h - n)x_1 + 3ky_1],$$

The values of these polynomials do not depend on the coordinates of the arbitrary point (γ, δ) and, consequently conditions $\mathcal{N}_4 = 0$ and $\mathcal{N}_5 = 0$ are affine invariant. These conditions yield $h = k = m = n = 0$, i.e. all quadratic coefficients vanish. Thus, system (90) becomes

$$\begin{aligned}\dot{x}_1 &= (a + c\gamma + d\delta + \gamma^3) + (c + 3\gamma^2)x_1 + dy_1 + 3\gamma x_1^2 + x_1^3, \\ \dot{y}_1 &= (b + e\gamma + f\delta + 3\gamma^2\delta) + (e + 6\gamma\delta)x_1 + (f + 3\gamma^2)y_1 + 3\delta x_1^2 + 6\gamma x_1 y_1 + 3x_1^2 y_1,\end{aligned}$$

for which we have:

$$\begin{aligned}\mathcal{N}_6 &= -144x_1^4 [3ex_1^2 + 2(c - f)x_1 y_1 - 17dy_1^2], \\ \mathcal{N}_8 &= -24(c + f)x_1^2, \\ \mathcal{N}_1 &= -216x_1^3 [-18x_1(bx_1 + ay_1) + (c - f)(23\delta x_1 + 25\gamma y_1)x_1 \\ &\quad - 12e\gamma x_1^2 + 2dy_1(19\gamma y_1 - 13\delta x_1)].\end{aligned}$$

As above $\mathcal{N}_6 = 0$ is an affine invariant condition and implies $d = e = c - f = 0$. Moreover, for $\mathcal{N}_6 = 0$ we obtain that the GL -comitant \mathcal{N}_1 does not depend on the coordinates of the arbitrary point (γ, δ) , i.e. it becomes a T -comitant and condition $\mathcal{N}_1 = 0$ yields $a = b = 0$. On the other hand for $f = c$ we obtain that $\text{sign}(\mathcal{N}_8) = -\text{sign}(c)$. Thus, the next lemma follows.

Lemma 31. *Canonical system (65) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{N}_1 = \mathcal{N}_4 = \mathcal{N}_5 = \mathcal{N}_6 = 0$. Moreover, the potential configuration of the lines corresponds with $(3, 3, 1, 1)$ given in Figure 19 for $\mathcal{N}_8 > 0$ and in Figure 20 for $\mathcal{N}_8 < 0$.*

7.2 Systems with the potential configuration $(3, 2, 2, 1)$

By Lemma 4 if a cubic system possesses 8 invariant straight lines in the potential configuration $(3, 2, 2, 1)$, then the conditions $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$ hold.

We consider again (63). A straightforward computation of the value of \mathcal{V}_3 yields:

$$\mathcal{V}_3 = 2^5 3^2 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j, \text{ where}$$

$$\begin{aligned}\mathcal{V}_{30} &= -u(u + 3), & \mathcal{V}_{32} &= -2ru + 3r - v^2, \\ \mathcal{V}_{31} &= -2uv, & \mathcal{V}_{33} &= -2vr, & \mathcal{V}_{34} &= -r^2.\end{aligned}$$

Hence $r = v = 0$ and $u(u + 3) = 0$. So, we have to examine two cases: $u = -3$ and $u = 0$.

For $u = -3$, by time rescaling $t \rightarrow -t/3$, we obtain the system

$$x' = p_0 + p_1(x, y) + p_2(x, y) + 2x^3, \quad y' = q_0 + q_1(x, y) + q_2(x, y) + 3x^2 y. \quad (91)$$

Whereas for $u = 0$, after the time rescaling $t \rightarrow t/3$, we get system (64) (see Remark 29). It remains to note that for system (91) we have $\mathcal{V}_4 = \mathcal{U}_2 = 0$. So, for additional investigations we must consider only system (91).

It is clear that via the translation of the origin of coordinates at the point $(-g/6, -l/3)$, we can consider $g = l = 0$ and, hence, we must examine the system

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + 2x^3, \quad \dot{y} = b + ex + fy + 2mxy + ny^2 + 3x^2y. \quad (92)$$

For this system we have $C_3(x, y) = -x^3y$. Therefore, there exists two directions for possible invariant straight lines: $x = 0$ and $y = 0$.

Direction $x = 0$. In this case $U = 1$, $V = 0$ and, from (27), we obtain

$$\begin{aligned} A = 2, \quad B = C = 0, \quad D = -2W, \quad E = 2h, \quad F = 2W^2 + c, \\ Eq_7 = k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = -2W^3 - cW + a. \end{aligned}$$

So, in order to have the maximum number of invariant straight lines we obtain the conditions: $k = h = d = 0$.

Direction $y = 0$. In this case $U = 0$, $V = 1$ and, from (27), we obtain

$$\begin{aligned} A = 3, \quad B = C = 0, \quad D = 2m, \quad E = n, \quad F = -nW + f, \\ Eq_5 = -3W, \quad Eq_8 = -2mW + e, \quad Eq_{10} = nW^2 - fW + b. \end{aligned}$$

Hence, in this direction can be only one simple invariant straight line with $W_0 = 0$, and the necessary conditions are $e = b = 0$. Thus, system (92) becomes

$$\dot{x} = a + cx + 2x^3, \quad \dot{y} = fy + 2mxy + ny^2 + 3x^2y,$$

for which we calculate $\mathcal{L}_1 = 2^9 3^4 nx^3$, $\mathcal{L}_2 = 2^9 3^4 nx(mx + 3ny)$. Hence, in order to reach the total multiplicity 9, by Lemma 27, it is necessary $\mathcal{L}_1 = \mathcal{L}_2 = 0$. Then we obtain the additional condition: $n = 0$. This provides the system:

$$\dot{x} = a + cx + 2x^3, \quad \dot{y} = fy + 2mxy + 3x^2y. \quad (93)$$

We consider the homogenized system

$$\dot{X} = aZ^3 + cXZ^2 + 2X^3, \quad \dot{Y} = fYZ^2 + 2mXYZ + 3X^2Y,$$

corresponding to system (93) and calculate the polynomial:

$$H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2Y(aZ^3 + cXZ^2 + 2X^3),$$

and

$$\begin{aligned}\mathcal{G}_1/H &= -3X^4 - 4mX^3Z - (3c + 4m^2)X^2Z^2 - 2(3a + 2fm)XZ^3 \\ &\quad - (2am - cf + f^2)Z^4 = \tilde{T}(X, Z), \\ \mathcal{G}_2/H &= -3(3aZ^3 + 2mfZ^3 + 3cXZ^2 + 4Z^2m^2X + 6mX^2Z + 6X^3) \\ &\quad \times (aZ^3 + cXZ^2 + 2X^3) = \tilde{S}(X, Z).\end{aligned}$$

By Lemma 10, in order to have the maximum number of invariant straight lines it is necessary that $\tilde{T}(X, Z) \mid \tilde{S}(X, Z)$, i.e. the conditions $R_X^{(i)}(\tilde{T}, \tilde{S}) = 0$, ($i = 0, 1, 2, 3$), have to be satisfied. So, we can calculate:

$$R_X^{(3)}(\tilde{T}, \tilde{S}) = -18(54a - 45cm + 54fm - 40m^3)Z^3 = 0,$$

and obtain that $a = (45cm - 54fm + 40m^3)/54$. Then, we have

$$R_X^{(2)}(\tilde{T}, \tilde{S}) = -81(9c - 6f + 8m^2)^2(c + 2f + 3m^2)^2Z^8 = 0,$$

and this implies the necessity to examine two cases: $9c - 6f + 8m^2 = 0$ and $9c - 6f + 8m^2 \neq 0$.

Case $9c - 6f + 8m^2 = 0$. Then $f = (9c + 8m^2)/6$, and we have $R_X^{(1)}(\tilde{T}, \tilde{S}) = R_X^{(0)}(\tilde{T}, \tilde{S}) = 0$, but at the same time this provides the degenerated system:

$$\dot{x} = -(2m - 3x)(8m^2 + 12xm + 9c + 18x^2)/27, \quad \dot{y} = y(8m^2 + 12xm + 9c + 18x^2)/6.$$

Case $9c - 6f + 8m^2 \neq 0$. Then condition $R_X^{(2)}(\tilde{T}, \tilde{S}) = 0$ implies $f = -(c + 3m^2)/2$, and we have $R_X^{(1)}(\tilde{T}, \tilde{S}) = -9m^3(6c + 13m^2)^3(12c + 17m^2)^3Z^{15} = 0$, and since condition $9c - 6f + 8m^2 \neq 0$ yields $12c + 17m^2 \neq 0$, we obtain condition $m(6c + 13m^2) = 0$.

If $m \neq 0$ then $c = -13m^2/6 \neq 0$, and this provides the degenerated system:

$$\dot{x} = -(6x + 5m)(-6x + 7m)(3x + m)/54, \quad \dot{y} = -y(6x + 5m)(m - 6x)/12.$$

Thus, $m = 0$ and we obtain the system (we set $c = 2p$):

$$\dot{x} = 2px + 2x^3, \quad \dot{y} = -py + 3x^2y, \tag{94}$$

with $p \neq 0$ (otherwise system becomes degenerated). We note that by Remark 14 we can consider $p \in \{-1, 1\}$. So, we obtain Figure 21 for $p = -1$ and Figure 22 for $p = 1$. It remains to construct two perturbed systems.

For $p = -1$ the perturbed system

$$\dot{x} = -2x + 2x^3, \quad \dot{y} = y + 3x^2y - \varepsilon^2y^3, \tag{95}$$

possesses the invariant straight lines $x = 0$, $y = 0$, $x = \pm 1$, $x + \varepsilon y = \pm 1$ and $x - \varepsilon y = \pm 1$ in the configuration $(3, 2, 2, 1)$; and for $p = 1$ the system

$$\dot{x} = 2x + 2x^3, \quad \dot{y} = -y + 3x^2y + \varepsilon^2y^3, \tag{96}$$

possesses the invariant straight lines $x = 0$, $y = 0$, $x = \pm i$, $\varepsilon y + ix = \pm 1$ and $\varepsilon y - ix = \pm 1$ in the same configuration $(3, 2, 2, 1)$.

We construct necessary and sufficient affine invariant conditions for the realization of Figures 21 and 22 for system (92). As it was proved above, system (92) possesses 9 invariant straight lines (counted with their multiplicities) if and only if $h = k = m = n = 0$, $d = e = c + 2f = 0$, $a = b = 0$. First, we shall show that conditions $h = k = m = n = 0$ are equivalent to conditions $\mathcal{N}_4 = \mathcal{N}_5 = 0$. To get this goal we consider the system:

$$\begin{aligned}\dot{x}_1 &= p_0^\tau + p_1^\tau + 6\gamma x_1^2 + 2hx_1y_1 + ky_1^2 + 2x_1^3, \\ \dot{y}_1 &= q_0^\tau + q_1^\tau + 3\delta x_1^2 + 2(m + 3\gamma)x_1y_1 + ny_1^2 + 3x_1^1y_1,\end{aligned}\tag{97}$$

which is obtained from system (92) via the translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane of system (92), and $p_i^\tau(a(\gamma, \delta), x_1, y_1)$, $q_i^\tau(a(\gamma, \delta), x_1, y_1)$ ($i = 0, 1$) are the corresponding homogeneous polynomials of degree i in x_1 and y_1 .

For system (97) we calculate the values of the GL -comitants

$$\mathcal{N}_4 = 24x_1 [2mx_1^2 + 3(h + n)x_1y_1 + 3ky_1^2], \quad \mathcal{N}_5 = 24x_1 [2mx_1^2 + 3(n - 2h)x_1y_1 - 6ky_1^2].$$

As we can observe, the values of these polynomials do not depend on the coordinates of the arbitrary point (γ, δ) and, consequently $\mathcal{N}_4 = 0$ and $\mathcal{N}_5 = 0$ are affine invariant conditions. These conditions yield $h = k = m = n = 0$, i.e. all quadratic coefficients vanish. Thus, system (97) becomes

$$\begin{aligned}\dot{x}_1 &= (a + c\gamma + d\delta + 2\gamma^3) + (c + 6\gamma^2)x_1 + dy_1 + 6\gamma x_1^2 + 2x_1^3, \\ \dot{y}_1 &= (b + e\gamma + f\delta + 3\gamma^2\delta) + (e + 6\gamma\delta)x_1 + (f + 3\gamma^2)y_1 + 3\delta x_1^2 + 6\gamma x_1y_1 + 3x_1^2y_1,\end{aligned}$$

for which we have:

$$\begin{aligned}\mathcal{N}_7 &= -216x_1^4 [2ex_1^2 - (c + 2f)x_1y_1 - 5dy_1^2], \\ \mathcal{N}_8 &= -36(c + f)x_1^2, \\ \mathcal{N}_1 &= -648x_1^3 [3x_1(4bx_1 - 7ay_1) + 7(c + 2f)(\delta x_1 - \gamma y_1)x_1 \\ &\quad + 6e\gamma x_1^2 + 22dy_1(\gamma y_1 - \delta x_1)].\end{aligned}$$

As above condition $\mathcal{N}_7 = 0$ is affine invariant and implies $d = e = c + 2f = 0$. Moreover, for $\mathcal{N}_7 = 0$ we obtain that the GL -comitant \mathcal{N}_1 does not depend on the coordinates of the arbitrary point (γ, δ) , i.e. it becomes a T -comitant and condition $\mathcal{N}_1 = 0$ yields $a = b = 0$. On the other hand, for $c = -2f$ we obtain that $\text{sign}(\mathcal{N}_8) = \text{sign}(f)$. Thus, the following lemma holds.

Lemma 32. *Canonical system (92) possesses invariant straight lines with total multiplicity 9 if and only if $\mathcal{N}_1 = \mathcal{N}_4 = \mathcal{N}_5 = \mathcal{N}_7 = 0$. Moreover, the potential configuration of the lines corresponds with $(3, 2, 2, 1)$ given in Figure 21 for $\mathcal{N}_8 > 0$ and in Figure 22 for $\mathcal{N}_8 < 0$.*

By Lemmas 5, 30, 31 and 32 the next theorem follows.

Theorem 33. *We assume that for cubic system (2) the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$, $\mathcal{D}_2 \neq 0$ hold (i.e. C_3 has 1 triple and 1 simple real roots). Then, this system will possess the maximum number of invariant straight lines (with total multiplicity 9) if and only if at least one of the following sets of conditions is fulfilled:*

$$\mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_7 = 0, \quad \mathcal{L}_4 < 0; \quad (\mathcal{D}_1)$$

$$\mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_6 = 0, \quad \mathcal{L}_4 < 0; \quad (\mathcal{D}_2)$$

$$\mathcal{V}_1 = \mathcal{N}_1 = \mathcal{N}_4 = \mathcal{N}_5 = \mathcal{N}_6 = 0, \quad \mathcal{L}_4 > 0; \quad (\mathcal{D}_3)$$

$$\mathcal{V}_3 = \mathcal{N}_1 = \mathcal{N}_4 = \mathcal{N}_5 = \mathcal{N}_7 = 0. \quad (\mathcal{D}_4)$$

Moreover, the configuration of the lines corresponds to Figures 14 or 15 for (\mathcal{D}_1) if \mathcal{N}_8 negative or positive, respectively; to Figures 16, 17 or 18 for (\mathcal{D}_2) if \mathcal{N}_8 positive, negative or zero, respectively; to Figures 19 or 20 for (\mathcal{D}_3) if \mathcal{N}_8 positive or negative, respectively; to Figures 21 or 22 for (\mathcal{D}_4) if \mathcal{N}_8 positive or negative, respectively.

8. Cubic systems with 1 real root of C_3 with multiplicity 4

The objective of this section is to construct the cubic systems with one real infinite singular point of multiplicity 4 which have invariant straight lines with total multiplicity 9.

First we obtain the form for the homogeneous part of degree 3 for these cubic systems.

Lemma 34. *Every cubic system with one real infinite singular point of multiplicity 4 which can admit invariant straight lines with total multiplicity 9 via a linear transformation can be written as*

$$x' = p_0 + p_1(x, y) + p_2(x, y), \quad y' = q_0 + q_1(x, y) + q_2(x, y) - x^3. \quad (98)$$

Proof: As it was shown in Section 3 the cubic system having one real infinite singular point of multiplicity 4 via a linear transformation becomes:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + ux^3 + vx^2y + rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) - x^3 + ux^2y + vxy^2 + ry^3, \end{aligned} \quad (99)$$

For system (99) we obtain $C_3 = x^4$ and hence, the infinite singular point is situated on the “end” of the line $x = 0$.

By Lemmas 1 and 4, we need to consider two cases.

Case 1: $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$. Then, a straightforward computation of the value of \mathcal{V}_1 for system (99) yields: $\mathcal{V}_1 = 2^5 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$, where $\mathcal{V}_{10} = u^2$, $\mathcal{V}_{11} = 2uv$, $\mathcal{V}_{12} = 2ur + v^2$,

$\mathcal{V}_{13} = 2rv$, $\mathcal{V}_{14} = r^2$. Consequently, relation $\mathcal{V}_1 = 0$ yield $v = r = u = 0$ and hence, we obtain system (98) which must satisfy the relations: $\mathcal{V}_2 = \mathcal{U}_1 = 0$.

Case 2: $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$. Now calculating the value of \mathcal{V}_3 for system (99) we obtain:
 $\mathcal{V}_3 = 2^5 3^2 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, where $\mathcal{V}_{30} = -u^2 + 3v$, $\mathcal{V}_{31} = -2uv + 6r$, $\mathcal{V}_{32} = -2ur - v^2$,
 $\mathcal{V}_{33} = -2rv$, $\mathcal{V}_{34} = -r^2$. Hence, condition $\mathcal{V}_3 = 0$ yields $r = v = u = 0$ (then $\mathcal{V}_4 = \mathcal{U}_2 = 0$) and, consequently we obtain again system (98). ■

We examine system (98). It is clear that via the translation of the origin of coordinates at the point $(0, l/3)$, we can consider parameter $l = 0$ in the polynomial $q_2(x, y)$. Thus, we must examine the system

$$\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2, \quad \dot{y} = b + ex + fy + 2mxy + ny^2 - x^3, \quad (100)$$

for which $C_3(x, y) = x^4$. Therefore, there exists only one direction for the possible invariant straight lines: $x = 0$. In this case $U = 1$, $V = 0$ and from (27) we obtain

$$\begin{aligned} A = B = C = 0, \quad D = g, \quad E = 2h, \quad F = -gW + c, \\ Eq_7 = k, \quad Eq_9 = -2hW + d, \quad Eq_{10} = gW^2 - cW + a. \end{aligned}$$

So, for having the maximum number of invariant straight lines it is necessary $k = h = d = 0$ and then, we have $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = -2^{10} 3^4 n^2 x^2$. Hence, in order to reach the total multiplicity 9, by Lemma 27, it is necessary $\mathcal{L}_1 = \mathcal{L}_2 = 0$. Therefore, we obtain the additional condition: $n = 0$. This provides the system:

$$\dot{x} = a + cx + 2gx^2, \quad \dot{y} = b + ex + fy + 2mxy - x^3. \quad (101)$$

We consider the homogenized system

$$\dot{X} = aZ^3 + cXZ^2 + 2gX^2Z, \quad \dot{Y} = bZ^3 + eXZ^2 + fYZ^2 + 2mXYZ - X^3, \quad (102)$$

corresponding to system (101) and calculate the following polynomial:

$$H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2Z^2(aZ^2 + cXZ + 2gX^2).$$

By Lemmas 8 and 9, we obtain that system (102) has 4 invariant straight lines (counted with their multiplicities) $Z = 0$ and $aZ^2 + cXZ + 2gX^2 = 0$. So, for having the total multiplicity 9, by Lemmas 8 and 9, the polynomial $H(\mathbf{a}, X, Y, Z)$ must be of the degree 8. In order to find out the conditions to reach this situation we shall calculate for system (102) the following polynomials:

$$\begin{aligned} \mathcal{G}_1/H &= Y\tilde{U}(X, Z) + \tilde{V}(X, Y) = T(X, Y, Z), \\ \mathcal{G}_3/H &= 12(Z^2a + cZX + 2gX^2)^3 Z^2 = 12Z^2 S^2(X, Z)^3, \end{aligned} \quad (103)$$

where

$$\begin{aligned}\tilde{U}(X, Z) &= Z [4m(g - m)X^2 + 4f(g - m)XZ - (2am + f^2 - fc)Z^2], \\ \tilde{V}(X, Z) &= 2(g + m)X^4 + (f + 2c)X^3Z + (3a + 2ge - 2me)X^2Z^2 \\ &\quad + (-2mb + 4gb - fe)XZ^3 - (ae - bc + fb)Z^4.\end{aligned}$$

By Lemma 10, in order to have the maximum number of invariant straight lines it is necessary $T(X, Y, Z) \mid Z^2S^2(X, Z)^3$. However, the second polynomial does not depend on the variable Y , so we obtain condition $\tilde{U}(X, Z) = 0$, i.e.

$$m(g - m) = 0, \quad f(g - m) = 0, \quad 2am + f^2 - fc = 0. \quad (104)$$

Thus, we have to consider two cases: $g - m \neq 0$ and $g - m = 0$.

Case $g - m \neq 0$. Then, by (104), we obtain $m = f = 0$, $g \neq 0$, and then

$$\begin{aligned}T(X, Z) &= 2gX^4 + 2cX^3Z + (3a + 2ge)X^2Z^2 + 4gbXZ^3 - (ae - bc)Z^4, \\ S^2(X, Z) &= Z^2a + cZX + 2gX^2.\end{aligned}$$

Since $g \neq 0$, for obtaining the maximum number of invariant straight lines it is necessary that polynomial T be proportional to polynomial S^2 .

By Lemma 12, we have $R_X^{(3)}(T, S^2) = 0$, $R_X^{(2)}(T, S^2) = -4g^2(2ag - c^2 + 4eg^2)^2Z^4 = 0$, and since $g \neq 0$, we can consider $g = 1$ (via the transformation $x = x_1$, $y = y_1/g$ and $t = t_1/g$ to system (101)). Thus, we obtain $2a = c^2 - 4e$, and by setting $c = 2p$, $a = 2(p^2 - e)$, we get $R_X^{(1)}(T, S^2) = 512(pe - p^3 + b)^3Z^9 = 0$. Therefore, the condition $b = p^3 - pe$ holds. Then we obtain:

$$\begin{aligned}\mathcal{G}_1 &= 8Z^2(Z^2p^2 + XZp - eZ^2 + X^2)^3, \\ \mathcal{G}_2 &= 48Z^3(2X + Zp)(Z^2p^2 + XZp - eZ^2 + X^2)^3, \\ \mathcal{G}_3 &= 384Z^4(Z^2p^2 + XZp - eZ^2 + X^2)^4.\end{aligned}$$

Polynomials \mathcal{G}_i ($i = 1, 2, 3$) have the common factor of degree 8, but system (101) becomes degenerated:

$$\dot{x} = 2(p^2 - e + px + x^2), \quad \dot{y} = (p - x)(p^2 - e + px + x^2).$$

So, in the case $g - m \neq 0$ the system cannot possess invariant straight lines with total multiplicity 9.

Case $g - m = 0$. We take $m = g$. Then, conditions (104) yield $2ag + f(f - c) = 0$, and we need to examine two subcases: $g \neq 0$ and $g = 0$.

Subcase $g \neq 0$. Then, via the transformation $x = x_1$, $y = y_1/g$ and $t = t_1/g$ to system (101), we can consider $g = 1$ (hence, $m = 1$). Therefore, by setting $f = 2p$, we obtain the condition $a = p(c - 2p)$. Thus, we can calculate:

$$\begin{aligned} R_X^{(3)}(T, S^2) &= 8(c - p)Z, \\ R_X^{(2)}(T, S^2) &= 64(p - c)(-ep + p^3 + b)Z^4, \\ R_X^{(1)}(T, S^2) &= 128(-ep + p^3 + b)^2(4b + 20p^3 - c^3 - 24cp^2 - 4ep + 9c^2p)Z^9. \end{aligned}$$

Hence, $c = p$ and then

$$R_X^{(3)}(T, S^2) = R_X^{(2)}(T, S^2) = 0, \quad R_X^{(1)}(T, S^2) = 512(-ep + p^3 + b)^3Z^9 = 0,$$

that implies $b = pe - p^3$. In this case we obtain:

$$\begin{aligned} \mathcal{G}_1 &= -2Z^2(X + Zp)^3(-2X + Zp)^3, \\ \mathcal{G}_2 &= -18Z^3X(X + Zp)^3(-2X + Zp)^3, \\ \mathcal{G}_3 &= 24Z^4(X + Zp)^4(-2X + Zp)^4. \end{aligned}$$

i.e. $\mathcal{G}_1 \mid \mathcal{G}_2$ and $\mathcal{G}_1 \mid \mathcal{G}_3$ but the system becomes degenerated:

$$\dot{x} = (p + x)(2x - p), \quad \dot{y} = -(p + x)(p^2 - px - e - 2y + x^2).$$

Subcase $g = 0$. Then, conditions (104) yield $m = 0$, $f(f - c) = 0$, and we have to examine 2 subcases: $f = 0$ and $f \neq 0$.

For $f = 0$ system (102) becomes

$$\dot{X} = aZ^3 + cXZ^2, \quad \dot{Y} = bZ^3 + eXZ^2 - X^3, \quad (105)$$

and we obtain

$$H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2(Za + cX)Z^4,$$

and

$$\begin{aligned} \mathcal{G}_1/H &= 2cX^3 + 3aX^2Z + (bc - ae)Z^3 = \hat{T}(X, Z), \\ \mathcal{G}_3/H &= 12(Za + cX)^3Z^4 = 12Z^4\hat{S}^3(X, Z). \end{aligned}$$

Thus, for obtaining the maximum number of invariant straight lines it is necessary that the polynomial \hat{T} divides $Z^4\hat{S}^3$. This implies $c \neq 0$, otherwise the second polynomial does not depend on X and, hence, it would be necessary $a = 0$, that yields the degenerated system (105) with $c = a = 0$. So, $c \neq 0$ and then the polynomial \hat{T} must be proportional to \hat{S}^3 . As above we calculate

$$\begin{aligned} R_X^{(2)}(\hat{T}, \hat{S}^3) &= 36ac^3Z = 0, \\ R_X^{(1)}(\hat{T}, \hat{S}^3) &= -432ac^4(a^3 - c^2ae + c^3b)Z^4 = 0, \\ R_X^{(0)}(\hat{T}, \hat{S}^3) &= 1728c^3(a^3 - c^2ae + c^3b)^3Z^9 = 0, \end{aligned}$$

and, hence, by $c \neq 0$ we obtain $a = b = 0$ that also provides a degenerated system.

For $f \neq 0$, from conditions (104), we obtain $f = c \neq 0$, and we can consider $c = 1$ via the transformation $x = x_1$, $y = y_1/c$ and $t = t_1/c$. Then, system (102) becomes

$$\dot{X} = aZ^3 + XZ^2, \quad \dot{Y} = bZ^3 + eXZ^2 + YZ^2 - X^3,$$

for which we have

$$H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2(Za + X)^2Z^4,$$

and

$$\mathcal{G}_1/H = 3X^2 - eZ^2 = \bar{T}(X, Z), \quad \mathcal{G}_3/H = 12(Za + X)^2Z^4 = 12Z^4\bar{S}^2(X, Z).$$

We calculate the resultant of the polynomials \bar{T} and \bar{S}^2 and its first subresultant:

$$\begin{aligned} R_X^{(1)}(\bar{T}, \bar{S}^2) &= 6aZ = 0, \\ R_X^{(0)}(\bar{T}, \bar{S}^2) &= -(e - 3a^2)^2Z^4 = 0. \end{aligned}$$

Consequently, we obtain $a = e = 0$, and after translation of the origin of coordinates at the point $(0, -b)$, we obtain the system:

$$\dot{x} = x, \quad \dot{y} = y - x^3, \tag{106}$$

which is not degenerated. For the respective homogenized system we have $H(\mathbf{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^4Z^4$. Thus, we obtain Figure 23. Since for system (106) the conditions $\mathcal{V}_1 = \mathcal{V}_2 = 0$ hold as well as the conditions $\mathcal{V}_3 = \mathcal{V}_4 = 0$, we conclude that this system has invariant straight lines with total multiplicity 9 of both potential configurations: $(3, 3, 1, 1)$ and $(3, 2, 2, 1)$. Indeed, the system

$$\dot{x} = x - 4\varepsilon^2x^3, \quad \dot{y} = y - x^3 - 3\varepsilon^2x^2y + 9\varepsilon^4xy^2 - 9\varepsilon^6y^3, \tag{107}$$

with the invariant straight lines $x = 0$, $2\varepsilon x = \pm 1$, $x - \varepsilon^2y = 0$, $\varepsilon x - 3\varepsilon^3y = \pm 1$, $3\varepsilon^2y \pm x = 0$, has the configuration $(3, 3, 1, 1)$; and the system

$$\dot{x} = x + 6\varepsilon x^2 + 8\varepsilon^2x^3, \quad \dot{y} = y + 6\varepsilon xy - x^3 + 9\varepsilon^2x^2y + 9\varepsilon^4xy^2 - 9\varepsilon^6y^3, \tag{108}$$

with the invariant straight lines

$$\begin{aligned} x = 0, \quad 2\varepsilon x = -1, \quad \varepsilon^2y = x, \quad 3\varepsilon^2y \pm x = 0, \\ 4\varepsilon x = -1, \quad 3\varepsilon(\varepsilon^2y - x) = 1, \quad \varepsilon(3\varepsilon^2y + x) = -1, \end{aligned}$$

has the configuration $(3, 2, 2, 1)$. As above, system (100) has the maximum number of invariant straight lines with the configuration given by Figure 23 if and only if

$$h = k = m = n = g = 0, \quad d = f = c - f = 0, \quad a = 0. \tag{109}$$

We give the respective affine invariant conditions in order to be able to distinguish this class of cubic systems directly in the space \mathbb{R}^{20} of all cubic systems. To get this goal we consider the system:

$$\dot{x}_1 = p_0^\tau + p_1^\tau + gx_1^2 + 2hx_1y_1 + ky_1^2, \quad \dot{y}_1 = q_0^\tau + q_1^\tau - 3\gamma x_1^2 + 2mx_1y_1 + ny_1^2 - x_1^3, \quad (110)$$

which is obtained from system (100) via the translation $x = x_1 + \gamma$, $y = y_1 + \delta$, where (γ, δ) is an arbitrary point of the phase plane of system (100) and $p_i^\tau(a(\gamma, \delta), x_1, y_1)$, $q_i^\tau(a(\gamma, \delta), x_1, y_1)$ ($i = 0, 1$) are the corresponding homogeneous polynomials of degree i in x_1 and y_1 .

For system (110) we calculate the GL -comitants

$$\mathcal{N}_2 = -6x_1^4 [(g + m)x_1 + (h + n)y_1], \quad \mathcal{N}_3 = -12x_1^3 [(g - 2m)x_1^2 + 2(2h - n)x_1y_1 + 3ky_1^2].$$

The values of these polynomials do not depend on the coordinates of the arbitrary point (γ, δ) and, consequently conditions $\mathcal{N}_2 = 0$ and $\mathcal{N}_3 = 0$ are affine invariant conditions. These conditions yield $g = h = k = m = n = 0$, i.e. all quadratic coefficients vanish. Thus, system (110) becomes the system

$$\begin{aligned} \dot{x}_1 &= (a + c\gamma + d\delta) + cx_1 + dy_1, \\ \dot{y}_1 &= (b + e\gamma + f\delta - \gamma^3) + (e - 3\gamma^2)x_1 + fy_1 - 3\gamma x_1^2 - x_1^3, \end{aligned}$$

for which we have:

$$\mathcal{N}_9 = 3x_1^4 [ex_1^2 + (f - c)x_1y_1 - dy_1^2], \quad \mathcal{N}_{10} = -12ax_1^3 + 6(f - c)\gamma x_1^3 - 12d\delta x_1^3.$$

As above $\mathcal{N}_9 = 0$ is affine invariant condition and implies $d = e = c - f = 0$. Moreover, in this case the GL -comitant \mathcal{N}_{10} is independent of the coordinates of the arbitrary point (γ, δ) , i.e. it becomes a T -comitant and condition $\mathcal{N}_{10} = 0$ yields $a = 0$. Thus, it was proved, that for system (100) conditions (109) are equivalent to $\mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_9 = \mathcal{N}_{10} = 0$. Hence, taking into account Lemma 5 we obtain:

Theorem 35. *We assume that for a cubic system (2) the conditions $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$, $C_3 \neq 0$ hold (i.e. there exists only one (real) infinite singular point). Then, this system will possess the maximum number of invariant straight lines (with total multiplicity 9) if and only if $\mathcal{V}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_9 = \mathcal{N}_{10} = 0$. Moreover, there exists a unique configuration given in Figure 23.*

9. Cubic systems whose infinite point configuration do not allow to possess invariant straight lines with total multiplicity 9

The goal of this section is to prove that all others classes of cubic systems enumerated in the statement of Lemma 5 cannot have invariant straight lines with total multiplicity 9.

9.1 Systems with 4 imaginary simple roots of C_3

If a cubic system has 4 imaginary infinite singular points via a linear transformation can written into the form (see, Section 3):

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + ux^3 + (p + q + v)x^2y + rxy^2 + qy^3, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) - px^3 + ux^2y + vxy^2 + ry^3, \end{aligned} \quad (111)$$

for which $C_3(x, y) = (px^2 + qy^2)(x^2 + y^2)$, $pq > 0$. By Lemma 1, in order to apply the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$, it is sufficient to calculate only the T-comitant: $\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$, and to examine the relation $\mathcal{V}_{10} = 2u^2 + 3p^2 + 3pq = 0$. Indeed, since $pq > 0$ the relation $\mathcal{V}_{10} = 0$ cannot be satisfied and, therefore the conditions of Lemma 1 cannot be satisfied for system (111).

We examine the conditions of Lemma 4: $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$. For system (111) we have $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, and we examine the following two coefficients: $\mathcal{V}_{30} = -3p^2 - 3pq + 3pv - u^2$, $\mathcal{V}_{34} = -6q^2 - 6pq - 3qv - r^2$. Thus, conditions $\mathcal{V}_{30} = \mathcal{V}_{34} = 0$ imply $3pv = 3p^2 + u^2 + 3pq > 0$, $3qv = -(6q^2 + r^2 + 6pq) < 0$, and hence, $9pqv^2 < 0$ in contradiction with $pq > 0$. So, the conditions of Lemma 4 cannot be verified for system (111).

9.2 Systems with 1 double and 2 simple real roots of C_3

As it was shown in Section 3 in this case the cubic system via a linear transformation can be written into the form:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + (u + 1)x^3 + (v - 1)x^2y + rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) + ux^2y + vxy^2 + ry^3, \end{aligned} \quad (112)$$

for which $C_3(x, y) = x^2y(x - y)$. By Lemma 1, we calculate:

$$\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j, \quad \mathcal{V}_2 = 8 \sum_{j=0}^2 \mathcal{V}_{2j} x^{2-j} y^j, \quad \text{where}$$

$$\begin{aligned} \mathcal{V}_{10} &= u(3 + 2u), & \mathcal{V}_{14} &= 2r^2, \\ \mathcal{V}_{11} &= -2u + 4uv + 3v, & \mathcal{V}_{20} &= -3v - 2u, \\ \mathcal{V}_{12} &= -1 - 2v + 4ru + 3r + 2v^2, & \mathcal{V}_{21} &= 6r + 4v - 2, \\ \mathcal{V}_{13} &= 2r(-1 + 2v), & \mathcal{V}_{22} &= -2r. \end{aligned}$$

Consequently, condition $\mathcal{V}_{14} = 2r^2 = 0$ implies $r = 0$, and then we obtain the following contradictory relations: $\mathcal{V}_{10} = u(2u + 3) = 0$, $4\mathcal{V}_{20} + 3\mathcal{V}_{21} = -2(4u + 3) = 0$. So, the conditions of Lemma 1 cannot be satisfied for system (112).

We examine the conditions of Lemma 4. For system (112) we have $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, where $\mathcal{V}_{30} = -u(3 + u)$, $\mathcal{V}_{32} = 2 + v - 2ru + 3r - v^2$, $\mathcal{V}_{31} = 2u(2 - v)$, $\mathcal{V}_{33} = -2r(1 + v)$, $\mathcal{V}_{34} = -r^2$. Thus, condition $\mathcal{V}_{34} = 0$ implies $r = 0$, and then we again obtain the following contradictory relations: $\mathcal{V}_{32} = -(v + 1)(v - 2) = 0$, $\mathcal{V}_4 = 2^{10} 3^2 v(v - 1)x^2 y(y - x) = 0$. So, the conditions of Lemma 4 cannot be satisfied for system (111).

9.3 Systems with 1 real double and 2 imaginary simple roots of C_3

In this case the cubic system via a linear transformation can be written into the form:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + ux^3 + (v + 1)x^2y + rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) - x^3 + ux^2y + vxy^2 + ry^3, \end{aligned} \quad (113)$$

for which $C_3(x, y) = x^2(x^2 + y^2)$. By Lemma 1, in order to apply the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ it is sufficient to calculate only the T-comitant: $\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$, and to observe the contradiction $\mathcal{V}_{10} = 2u^2 + 3 = 0$.

We examine the conditions of Lemma 4. For system (113) we have $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, where $\mathcal{V}_{30} = -u^2 - 3 + 3v$, $\mathcal{V}_{31} = -2uv - 4u + 6r$, $\mathcal{V}_{32} = -2ur - v^2 - v + 2$, $\mathcal{V}_{33} = -2r(v - 1)$, $\mathcal{V}_{34} = -r^2$. Thus, condition $\mathcal{V}_{34} = 0$ implies $r = 0$, and then we again obtain the following contradictory relations: $\mathcal{V}_{32} = -(v + 2)(v - 1) = 0$, $\mathcal{V}_4 = 2^{10} 3^2 v(v + 1)x^2(x^2 + y^2) = 0$. So, the conditions of Lemma 4 cannot be satisfied for system (113).

9.4 Systems with 2 double real roots of C_3

In this case the cubic system via a linear transformation becomes:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + ux^3 + qx^2y + rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) + ux^2y + vxy^2 + ry^3, \end{aligned} \quad (114)$$

for which $C_3(x, y) = (q - v)x^2y^2$, $q - v \neq 0$. By Lemma 1, in order to apply the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ we calculate: $\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$, $\mathcal{V}_2 = 16 \sum_{j=0}^2 \mathcal{V}_{2j} x^{2-j} y^j$, where $\mathcal{V}_{10} = 2u^2$, $\mathcal{V}_{11} = 2u(q + v)$, $\mathcal{V}_{12} = 4ur - q^2 + 4qv - v^2$, $\mathcal{V}_{13} = 2r(q + v)$, $\mathcal{V}_{14} = 2r^2$, $\mathcal{V}_{20} = u(q - v)$, $\mathcal{V}_{21} = (v - q)(v + q)$, $\mathcal{V}_{22} = r(q - v)$. Consequently, conditions $\mathcal{V}_{10} = \mathcal{V}_{14} = 0$

imply $u = r = 0$, and then from $q - v \neq 0$, we obtain the following contradictory relations: $\mathcal{V}_{21} = (v - q)(v + q) = 0$, $\mathcal{V}_{12} - 3\mathcal{V}_{21} = 2v(2q - v) = 0$. So, the conditions of Lemma 1 cannot be satisfied for system (114).

We examine the conditions of Lemma 4. For system (114) we have $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, where $\mathcal{V}_{30} = -u^2$, $\mathcal{V}_{31} = 2u(v - 2q)$, $\mathcal{V}_{32} = -2ur + 2q^2 - 5qv + 2v^2$, $\mathcal{V}_{33} = 2r(q - 2v)$, $\mathcal{V}_{34} = -r^2$. Thus, conditions $\mathcal{V}_{30} = \mathcal{V}_{34} = 0$ imply $u = r = 0$ and then by condition $q - v \neq 0$ we again obtain the following contradictory relations:

$$\mathcal{V}_{32} = (2q - v)(q - 2v) = 0, \quad \mathcal{V}_4 = 2^{10} 3^2 qv(q - v)^2 x^2 y^2 = 0.$$

So, the conditions of Lemma 4 also cannot be verified for system (114).

9.5 Systems with 2 double imaginary roots of C_3

In this case the cubic system via a linear transformation goes over to

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + ux^3 + 3(v + 1)x^2y + rxy^2 + y^3, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) - x^3 + ux^2y + (v - 1)xy^2 + ry^3, \end{aligned} \quad (115)$$

for which $C_3(x, y) = (x^2 + y^2)^2$. By Lemma 1, in order to apply the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ it is sufficient to calculate only the T-comitant: $\mathcal{V}_1 = 16 \sum_{j=0}^4 \mathcal{V}_{1j} x^{4-j} y^j$, and to observe the contradiction $\mathcal{V}_{10} = u^2 + 3 = 0$.

We examine the conditions of Lemma 4. For system (115) we have $\mathcal{V}_3 = 32 \sum_{j=0}^4 \mathcal{V}_{3j} x^{4-j} y^j$, and we examine only the following two coefficients: $\mathcal{V}_{30} = -u^2 - 9 + 3v$, $\mathcal{V}_{34} = -r^2 - 9 - 3v$. Therefore, we obtain the following contradictory condition: $\mathcal{V}_{30} + \mathcal{V}_{34} = -(u^2 + r^2 + 18) = 0$, and, hence, the conditions of Lemma 4 do not hold for system (115).

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